Proof of Convergence for Correct-Decoding Exponent Computation

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Abstract—For a discrete memoryless channel with finite input and output alphabets, we prove convergence of a parametric family of iterative computations of the optimal correct-decoding exponent. The exponent, as a function of communication rate, is computed for a fixed rate and for a fixed slope.

I. INTRODUCTION

Consider a standard information theoretic setting of transmission through a discrete memoryless channel (DMC), with finite input and output alphabets, using block codes. For communication rates above capacity, the average probability of correct decoding in a block code tends to zero exponentially fast as a function of the block length. In the limit of a large block length, the lowest possible exponent corresponding to the probability of correct decoding, also called the reliability function above capacity, for all¹ rates $R \geq 0$ is given by [1]

$$E_c(R) = \min_{\substack{Q(x), \\ W(y \mid x)}} \left\{ D(W \parallel P \mid Q) + \left| R - I(Q, W) \right|^+ \right\}, \quad (1)$$

where P denotes the channel's transition probability $P(y \mid x)$, $D(W \parallel P \mid Q)$ is the Kullback-Leibler divergence between the conditional distributions W and P, averaged over Q, I(Q,W) is the mutual information of a pair of random variables with a joint distribution $Q(x)W(y \mid x)$, and $|t|^+ = \max\{0,t\}$.

For certain applications, it is important to be able to know the actual value of $E_c(R)$ when it is positive. For example, in applications of secrecy, it might be interesting to know the correct-decoding exponent of an eavesdropper. Several algorithms have been proposed for computation of $E_c(R)$.

In the algorithm by Arimoto [2] the computation of $E_c(R)$ is facilitated by an alternative expression for it [3], [1], [4]:

$$E_c(R) = \sup_{0 \le \rho < 1} \min_{Q} \left\{ E_0(-\rho, Q) + \rho R \right\}, \tag{2}$$

where $E_0(-\rho,Q)$ is the Gallager exponent function [6, Eq. 5.6.14]. In [2], $\min_Q E_0(-\rho,Q)$ is computed for a fixed slope parameter ρ . The computation is performed iteratively as alternating minimization, based on the property that $\min_Q E_0(-\rho,Q)$ can be written as a double minimum:

$$\min_{Q} \min_{V} \left\{ -\log \sum_{x, y} Q^{1-\rho}(x) V^{\rho}(x \mid y) P(y \mid x) \right\}, \quad (3)$$

where the inner minimum is in fact equal to $E_0(-\rho,Q)$. In [4], [5] a different alternating-minimization algorithm is introduced, based on the property, that $\min_Q E_0(-\rho,Q)$ can be written as another double minimum over distributions:

$$\min_{T, V} \min_{T_1, V_1} \left\{ -\sum_{x, y} T(y) V(x \mid y) \log \frac{V_1^{\rho}(x \mid y) P(y \mid x)}{U_1^{\rho - 1}(x) T(y) V(x \mid y)} \right\}, \tag{4}$$

where $U_1(x) = \sum_y T_1(y) V_1(x \mid y)$. As with (3), the computation of $E_c(R)$ with (4) is also performed for a fixed ρ .

Sometimes, however, it is suitable or desirable to compute $E_c(R)$ directly for a given rate R. For example, when $E_c(R)=0$, and we would like to find such a distribution Q, for which the minimum (1) is zero, as a by-product of the computation. Such distribution Q has a practical meaning of a channel input distribution achieving reliable communication. In [7], an iterative minimization procedure for computation of $E_c(R)$ at fixed R is proposed, using the property that $E_c(R)$ can be written as a double minimum [8]:

$$\min_{Q(x)} \min_{\substack{T(y), \\ V(x \mid y)}} \left\{ D(TV \parallel QP) + \left| R - D(V \parallel Q \mid T) \right|^{+} \right\},$$

where the inner min equals $\sup_{0 \le \rho < 1} \big\{ E_0(-\rho,Q) + \rho R \big\}$. In [7], the inner minimum of (5) is computed stochastically by virtue of a correct-decoding *event* itself, yielding the minimizing solution T^*V^* . The computation is then repeated iteratively, by assigning $Q(x) = \sum_y T^*(y)V^*(x \mid y)$. It is shown in [7, Theorem 1], that the iterative procedure using the inner minimum of (5) leads to convergence of this minimum to the double minimum (5), which is evaluated at least over some *subset* of the support of the initial distribution Q_0 . In addition, a sufficient condition on Q_0 is provided, which guarantees convergence of the inner minimum in (5) to zero. This condition on Q_0 in [7, Lemma 6] is rather limiting, and is hard to verify.

In the current work, we improve the result of [7]. We modify the method of Csiszár and Tusnády [9] to prove that the iterative minimization procedure of [7] converges to the global minimum (5) over the *support* of the initial distribution Q_0 itself, for any R (i.e., not only if the global minimum is zero), and without any additional condition. In particular, use of a strictly positive Q_0 guarantees convergence to $E_c(R)$.

By a similar method, we also show convergence of the fixed-slope counterpart of the minimization (5), which is

 $^{^{1}}$ The expression gives zero for the rates $R \leq \max_{Q} I(Q, P)$.

an alternating minimization at fixed ρ , based on the double minimum [10]

$$\min_{Q} \min_{T, V} \left\{ -\sum_{x, y} T(y) V(x \mid y) \log \frac{Q^{1-\rho}(x) P(y \mid x)}{T(y) V^{1-\rho}(x \mid y)} \right\}, (6)$$

where the inner minimum is in fact equal to $E_0(-\rho, Q)$.

Furthermore, in the current paper we extend the analysis, presented in the shorter version of the paper [11]. Here we slightly generalize the expression (5). Using this generalization, we prove convergence of a parametric family of iterative computations, of which the computation according to (5) from [7], as well as the computations according to (6), [10], and according to (4), [4], become special cases.

As in the shorter version of the paper [11], besides the variable R, we take into account also a possible channelinput constraint, denoted by α . In Section II we examine the expression for the correct-decoding exponent. In Section III we prove convergence of the iterative minimization for fixed (R, α) . In Section IV we prove convergence of the iterative minimization for fixed gradient w.r.t. (R, α) . In Sections V and VI we prove convergence of mixed scenarios: for fixed α and slope ρ in the direction of R, and vice versa.

II. CORRECT-DECODING EXPONENT

Let $P(y \mid x)$ denote transition probabilities in a DMC from $x \in \mathcal{X}$ to $y \in \mathcal{Y}$, where \mathcal{X} and \mathcal{Y} are finite channel input and output alphabets, respectively. Suppose also that the channel input x with an additive cost function $f: \mathcal{X} \to \mathbb{R}$ satisfies on average an input constraint $\alpha \in \mathbb{R}$, chosen large enough, such that $\alpha \geq \min_{x} f(x)$. The maximum-likelihood correctdecoding exponent ([1], [12]) of this channel, as a function of the rate $R \ge 0$ and the input constraint α , is given by

$$\begin{split} E_{c}(R,\alpha) &= \\ \min_{\substack{Q(x): \\ \mathbb{E}_{Q}[f(X)] \leq \alpha}} \min_{W(y \mid x)} \left\{ D(W \parallel P \mid Q) + \left| R - I(Q,W) \right|^{+} \right\}, \end{split}$$

where $\mathbb{E}_Q[f(X)]$ denotes the expectation of f(x) w.r.t. the distribution Q(x) over \mathcal{X} .

Let $Q(x)W(y|x) \equiv T(y)V(x|y)$, or QW, denote a distribution over $\mathcal{X} \times \mathcal{Y}$, and let QW be another such distribution. We can think of 4 different divergences from QW to QW: $D(Q \parallel Q)$, $D(W \parallel W \mid Q)$, $D(T \parallel T)$, and $D(V \parallel V \mid T)$. Using 4 non-negative parameters $t_i \ge 0$, i = 1, 2, 3, 4, we define a non-negative linear combination of these divergences:

$$D^{\mathbf{t}}(QW, \widetilde{Q}\widetilde{W}) \triangleq t_1 D(Q \parallel \widetilde{Q}) + t_2 D(W \parallel \widetilde{W} \mid Q) + t_3 D(T \parallel \widetilde{T}) + t_4 D(V \parallel \widetilde{V} \mid T),$$
(8)

where $\mathbf{t} \triangleq (t_1, t_2, t_3, t_4)$ is an index. With the help of $D^{\mathbf{t}}(QW, \widetilde{QW})$, the expression (7) can be rewritten as follows:

$$\min_{\substack{Q, W: \\ \mathbb{E}_{Q}[f(X)] \leq \alpha}} \left\{ D(W \parallel P \mid Q) + \left| R - I(Q, W) \right|^{+} \right\}$$

$$= \qquad \min_{\substack{Q, W: \\ \mathbb{E}_{Q}[f(X)] \leq \alpha}} \max \left\{ D(W \parallel P \mid Q), \right.$$

$$D(W \parallel P \mid Q) + R - I(Q, W) \right\}$$

$$= \min_{\widetilde{Q}, \, \widetilde{W}} \min_{\substack{Q, \, W: \\ \mathbb{E}_{Q}[f(X)] \leq \alpha}} \max \Big\{ D(W \parallel P \mid Q) + D^{\mathbf{t}}(QW, \, \widetilde{Q}\widetilde{W}), \\ D(W \parallel P \mid Q) + R - I(Q, W) \Big\},$$

$$D(W || P || Q) + R - I(Q, W)$$
,

where the first equality holds because $|a|^+ = \max\{0, a\}$, and the second equality follows since $\min_{\widetilde{Q}\widetilde{W}} D^{\mathbf{t}}(QW,\widetilde{\widetilde{Q}}\widetilde{W}) = 0$ and the minima can be interchanged. In [7] a special case (t =(1,0,0,0)) of the inner minimum of (9) was used as a basis of an iterative procedure to find minimizing solutions of (7). In what follows, we modify the method of Csiszár and Tusnády [9] to show convergence of that minimization procedure. The method allows us to prove convergence in a slightly more general setting (9), (8), with arbitrary non-negative parameters $(t_1, t_2, t_3, t_4).$

III. CONVERGENCE OF THE ITERATIVE MINIMIZATION FOR FIXED (R, α)

Let us define a short notation for the maximum in (9):

$$\begin{split} F_1^{\mathbf{t}}(QW,\,\widetilde{Q}\widetilde{W}) \, &\triangleq \, D(W \parallel P \, | \, Q) + D^{\mathbf{t}}(QW,\,\widetilde{Q}\widetilde{W}), \ (10) \\ F_2(QW,R) \, &\triangleq \, D(W \parallel P \, | \, Q) - I(Q,W) + R, \quad (11) \\ F^{\mathbf{t}}(QW,\,\widetilde{Q}\widetilde{W},\,R) \, &\triangleq \, \max \Big\{ F_1^{\mathbf{t}}(QW,\,\widetilde{Q}\widetilde{W}),\, F_2(QW,R) \Big\}. \end{split}$$

Define notation for the inner minimum in (9):

$$E_c^{\mathbf{t}}(\widetilde{Q}\widetilde{W}, R, \alpha) \triangleq \min_{\substack{Q, W: \\ \mathbb{E}_Q[f(X)] \leq \alpha}} F^{\mathbf{t}}(QW, \widetilde{Q}\widetilde{W}, R) \quad (13)$$

The iterative minimization procedure from [7], consisting of two steps in each iteration², in a more general form is given

$$\begin{array}{rcl} Q_{\ell}W_{\ell} & \in & \underset{Q,\,W:}{\arg\min} \quad F^{\mathbf{t}}(QW,\,\widetilde{Q}_{\ell}\widetilde{W}_{\ell},\,R), \\ & & & \\ \widetilde{Q}_{\ell+1}\widetilde{W}_{\ell+1} & = & Q_{\ell}W_{\ell}, \end{array} \tag{14}$$

We assume that $\widetilde{Q}_0\widetilde{W}_0$ in (14) is chosen such that the set $\left\{QW: \sum_x Q(x)f(x) \leq \alpha, \ F_1^{\mathbf{t}}(QW, \widetilde{Q}_0\widetilde{W}_0) < +\infty\right\}$ is non-empty, which guarantees $F^{\mathbf{t}}(Q_0W_0, \widetilde{Q}_0\widetilde{W}_0, R) =$ $E_c^{\mathbf{t}}(\widetilde{Q}_0\widetilde{W}_0, R, \alpha) < +\infty$. By (10) it is clear that (14) produces a monotonically non-increasing sequence $E_c^{\mathbf{t}}(Q_{\ell}W_{\ell}, R, \alpha)$, $\ell = 0, 1, 2, \dots$ Our main result is given by the following theorem, which is an improvement on [7, Theorem 1] and [7, Lemma 6]:

Theorem 1: Let $\{Q_{\ell}W_{\ell}\}_{\ell=0}^{+\infty}$ be a sequence of iterative solutions produced by (14). Then

$$E_{c}^{\mathbf{t}}(\widetilde{Q}_{\ell}\widetilde{W}_{\ell}, R, \alpha) \stackrel{\ell \to \infty}{\searrow} \min_{\substack{\widetilde{Q}, \widetilde{W}: \\ D^{\mathbf{t}}(\widetilde{Q}\widetilde{W}, \widetilde{Q}_{0}\widetilde{W}_{0}) < \infty}} E_{c}^{\mathbf{t}}(\widetilde{Q}\widetilde{W}, R, \alpha),$$

$$(15)$$

²Note that (14) is not just an alternating minimization procedure w.r.t. $F^{\mathbf{t}}(QW,\widetilde{Q}\widetilde{W},R)$, or not the only one possible, in a sense that other choices of $\widetilde{Q}_{\ell+1}\widetilde{W}_{\ell+1}$ may also minimize $F^{\mathbf{t}}(Q_{\ell}W_{\ell},\cdot,R)$. where $E_c^{\mathbf{t}}(\widetilde{Q}\widetilde{W},R,\alpha)$ is defined in (13) and $D^{\mathbf{t}}(\cdot\,,\cdot)$ in (8).

Suppose Q^*W^* is a minimizing solution of (7). If the initial distribution $\widetilde{Q}_0\widetilde{W}_0$ in the iterations (14) is chosen such that $D^{\mathbf{t}}(Q^*W^*,\widetilde{Q}_0\widetilde{W}_0)<+\infty$ (for example, if $\operatorname{support}(\widetilde{Q}_0\widetilde{W}_0)=\mathcal{X}\times\mathcal{Y}$), then by (9) the RHS of (15) gives (7). The choice of $\mathbf{t}=(1,0,0,0)$ in (8) corresponds to the iterative minimization in [7]. In order to prove Theorem 1, we use a lemma, which is similar to "the five points property" from [9].

Lemma 1: Let $\hat{Q}\hat{W}$ be such, that $\sum_x \hat{Q}(x)f(x) \leq \alpha$ and $F_1^{\mathbf{t}}(\hat{Q}\hat{W}, \widetilde{Q}_0\widetilde{W}_0) < +\infty$. Then

$$F^{\mathbf{t}}(Q_0W_0, \widetilde{Q}_0\widetilde{W}_0, R) \leq F^{\mathbf{t}}(\hat{Q}\hat{W}, \hat{Q}\hat{W}, R) \\ + \left| F_1^{\mathbf{t}}(\hat{Q}\hat{W}, \widetilde{Q}_0\widetilde{W}_0) - F_1^{\mathbf{t}}(\hat{Q}\hat{W}, \widetilde{Q}_1\widetilde{W}_1) \right|^+. \tag{16}$$

Proof: Let us define a set of distributions QW:

$$\mathcal{S} \triangleq \bigg\{ QW: \ \sum_{x} Q(x) f(x) \leq \alpha, \ F_1^{\mathbf{t}}(QW, \ \widetilde{Q}_0 \widetilde{W}_0) < +\infty \bigg\}.$$

Observe that $\mathcal S$ is a closed convex set. Since $Q\widetilde W \in \mathcal S$, then $\mathcal S$ is non-empty and by (14) we have also that $Q_0W_0 \in \mathcal S$. Observe further that the two terms in the maximization of (12), $F_1^{\mathbf t}(QW,\,\widetilde Q_0\widetilde W_0)$ and $F_2(QW,R)$, as functions of QW, are convex (\cup) and continuous in $\mathcal S$.

Consider the case $F_1^{\mathbf{t}}(Q_0W_0,\widetilde{Q}_0\widetilde{W}_0)>F_2(Q_0W_0,R)$ first. Then $F^{\mathbf{t}}(Q_0W_0,\widetilde{Q}_0\widetilde{W}_0,R)=F_1^{\mathbf{t}}(Q_0W_0,\widetilde{Q}_0\widetilde{W}_0)$ by (12). By (14), we conclude that $F_1^{\mathbf{t}}(Q_0W_0,\widetilde{Q}_0\widetilde{W}_0)$ cannot be decreased in the vicinity of $QW=Q_0W_0$ inside the convex set \mathcal{S} . Let us define a point inside \mathcal{S} :

$$Q^{(\lambda)}(x)W^{(\lambda)}(y\mid x) \triangleq$$

$$\lambda \hat{Q}(x)\hat{W}(y\mid x) + (1-\lambda)Q_0(x)W_0(y\mid x), \qquad \lambda \in (0,1).$$

$$(17)$$

We have that $Q^{(\lambda)}W^{(\lambda)}\in\mathcal{S}$, and the function $f_1(\lambda)\triangleq F_1^{\mathbf{t}}(Q^{(\lambda)}W^{(\lambda)},\widetilde{Q}_0\widetilde{W}_0)$ is convex (\cup) and differentiable w.r.t. $\lambda\in(0,1)$. Since $f_1(\lambda)$ has to be non-decreasing at $\lambda=0$, the following condition must hold:

$$\lim_{\lambda \to 0} \frac{df_1(\lambda)}{d\lambda} \ge 0. \tag{18}$$

Differentiating $f_1(\lambda)$, similarly as in the proof of the "Pythagorean" theorem for divergence [13] (proved as "the three points property" in [9, Lemma 2]), we obtain:

$$F_{1}^{\mathbf{t}}(Q_{0}W_{0}, \widetilde{Q}_{0}\widetilde{W}_{0}) + D(\hat{W} \| W_{0} | \hat{Q}) + D^{\mathbf{t}}(\hat{Q}\hat{W}, Q_{0}W_{0}) \\ \leq F_{1}^{\mathbf{t}}(\hat{Q}\hat{W}, \widetilde{Q}_{0}\widetilde{W}_{0}). \tag{19}$$

Since $F_1^{\mathbf{t}}(\hat{Q}\hat{W}, \tilde{Q}_0\tilde{W}_0) < +\infty$, then the divergences on the LHS of (19) are also finite. By the definition (10),

$$F_1^{\mathbf{t}}(\hat{Q}\hat{W},\,\widetilde{Q}_0\widetilde{W}_0) \; = \; F_1^{\mathbf{t}}(\hat{Q}\hat{W},\,\hat{Q}\hat{W}) + D^{\mathbf{t}}(\hat{Q}\hat{W},\,\widetilde{Q}_0\widetilde{W}_0).$$

Omitting $D(\hat{W} \parallel W_0 \mid \hat{Q}) \geq 0$ from (19), noting that $Q_0W_0 = \widetilde{Q}_1\widetilde{W}_1$, and combining (19) with (20), we get

$$F_{1}^{\mathbf{t}}(Q_{0}W_{0}, \widetilde{Q}_{0}\widetilde{W}_{0}) \leq F_{1}^{\mathbf{t}}(\hat{Q}\hat{W}, \hat{Q}\hat{W}) + D^{\mathbf{t}}(\hat{Q}\hat{W}, \widetilde{Q}_{0}\widetilde{W}_{0}) - D^{\mathbf{t}}(\hat{Q}\hat{W}, \widetilde{Q}_{1}\widetilde{W}_{1}). \tag{21}$$

Now, (16) follows because $F^{\mathbf{t}}(Q_0W_0,\,\widetilde{Q}_0\widetilde{W}_0,\,R)=F_1^{\mathbf{t}}(Q_0W_0,\,\widetilde{Q}_0\widetilde{W}_0)$ and $F_1^{\mathbf{t}}(\hat{Q}\hat{W},\,\hat{Q}\hat{W})\leq F^{\mathbf{t}}(\hat{Q}\hat{W},\,\hat{Q}\hat{W},R).$ Consider the case $F_1^{\mathbf{t}}(Q_0W_0,\,\widetilde{Q}_0\widetilde{W}_0)< F_2(Q_0W_0,\,R)$

Consider the case $F_1^{\mathbf{t}}(Q_0W_0,Q_0W_0) < F_2(Q_0W_0,R)$ next. Then $F^{\mathbf{t}}(Q_0W_0,\widetilde{Q}_0\widetilde{W}_0,R) = F_2(Q_0W_0,R)$ by (12). By (14), we conclude that $F_2(Q_0W_0,R)$ cannot be decreased in the vicinity of $QW=Q_0W_0$ inside the convex set \mathcal{S} , and by convexity (\cup) of $F_2(QW,R)$ it follows that

$$\begin{array}{rcl} \boldsymbol{F}_{2}(Q_{0}W_{0},\,R) & = & \min_{QW\in\,\mathcal{S}}\boldsymbol{F}_{2}(QW,R) \\ & \stackrel{(a)}{\leq} & \boldsymbol{F}_{2}(\hat{Q}\hat{W},R) & \stackrel{(b)}{\leq} & \boldsymbol{F^{t}}(\hat{Q}\hat{W},\,\hat{Q}\hat{W},\,R), \end{array}$$

where (a) follows because $\hat{Q}\hat{W} \in \mathcal{S}$, and (b) follows by (12). This again gives (16).

Finally, assume now the equality $F_1^{\mathbf{t}}(Q_0W_0, \widetilde{Q}_0\widetilde{W}_0) = F_2(Q_0W_0, R)$. In this case, using the definition (17), we look at two functions: $f_1(\lambda)$ and $f_2(\lambda) \triangleq F_2(Q^{(\lambda)}W^{(\lambda)}, R)$, both of which are convex (\cup) and differentiable w.r.t. $\lambda \in (0, 1)$. At least one of these two functions has to be non-decreasing at $\lambda = 0$. This implies either (18) or

$$\lim_{\lambda \to 0} \frac{df_2(\lambda)}{d\lambda} \ge 0. \tag{22}$$

The condition (18) results in (16) as before, while (22) by convexity (\cup) of $f_2(\lambda)$ implies

$$F_2(Q_0W_0,R) \; \leq \; F_2(\hat{Q}\hat{W},R) \; \leq \; F^{\mathbf{t}}(\hat{Q}\hat{W},\hat{Q}\hat{W},R),$$

where the second inequality is by definition (12). Since $F_2(Q_0W_0,\,R)=F^{\mathbf{t}}(Q_0W_0,\,\widetilde{Q}_0\widetilde{W}_0,\,R)$, this gives (16). \square

A similar, alternative, lemma can be proved if we add $D^{\mathbf{t}}(QW,\,\widetilde{Q}\widetilde{W})$ to the second term of the maximum in (9), and not to the first.

Proof of Theorem 1: By (9) we can rewrite the RHS of (15) as

$$\min_{\substack{\widetilde{Q}, \widetilde{W}: \\ D^{\mathbf{t}}(\widetilde{Q}\widetilde{W}, \widetilde{Q}_{0}\widetilde{W}_{0}) < \infty}} E_{c}^{\mathbf{t}}(\widetilde{Q}\widetilde{W}, R, \alpha) = \min_{\substack{Q, W: \\ \mathbb{E}_{Q}[f(X)] \leq \alpha \\ D^{\mathbf{t}}(QW, \widetilde{Q}_{0}\widetilde{W}_{0}) < \infty}} F^{\mathbf{t}}(QW, QW, R).$$
(23)

Suppose (23) is finite, and let $\hat{Q}\hat{W}$ achieve the RHS min in (23). Then $F_1^{\mathbf{t}}(\hat{Q}\hat{W},\,\widetilde{Q}_0\widetilde{W}_0)<+\infty$ and $\sum_x\hat{Q}(x)f(x)\leq\alpha$. Then Lemma 1 implies that there exist only two possibilities for the outcome of the iterations in (14). One possibility is that at some iteration ℓ it holds that

$$F^{\mathbf{t}}(Q_{\ell}W_{\ell},\, \widetilde{Q}_{\ell}\widetilde{W}_{\ell},\, R) \ \leq \ F^{\mathbf{t}}(\hat{Q}\hat{W},\hat{Q}\hat{W},R),$$

meaning that the monotonically non-increasing sequence of $F^{\mathbf{t}}(Q_{\ell}W_{\ell},\widetilde{Q}_{\ell}\widetilde{W}_{\ell},R)=E_{c}^{\mathbf{t}}(\widetilde{Q}_{\ell}\widetilde{W}_{\ell},R,\alpha)$ has converged to (23). The alternative possibility is that for *all* iterations $\ell=0,1,2,\ldots$, it holds that

$$\begin{split} F^{\mathbf{t}}(Q_{\ell}W_{\ell},\,\widetilde{Q}_{\ell}\widetilde{W}_{\ell},\,R) & \leq F^{\mathbf{t}}(\hat{Q}\hat{W},\hat{Q}\hat{W},R) \\ & + F_{1}^{\mathbf{t}}(\hat{Q}\hat{W},\,\widetilde{Q}_{\ell}\widetilde{W}_{\ell}) - F_{1}^{\mathbf{t}}(\hat{Q}\hat{W},\,\widetilde{Q}_{\ell+1}\widetilde{W}_{\ell+1}), \end{split}$$

with all terms finite. Now, just like in [9, Lemma 1], it has to be true that

$$\liminf_{\ell \to \infty} \left\{ F_1^{\mathbf{t}}(\hat{Q}\hat{W}, \, \widetilde{Q}_{\ell}\widetilde{W}_{\ell}) - F_1^{\mathbf{t}}(\hat{Q}\hat{W}, \, \widetilde{Q}_{\ell+1}\widetilde{W}_{\ell+1}) \right\} \leq 0,$$

because the divergences in (10) are non-negative (i.e., bounded from below). Therefore $F^{\mathbf{t}}(Q_{\ell}W_{\ell},\,\widetilde{Q}_{\ell}\widetilde{W}_{\ell},\,R)$ must converge to $F^{\mathbf{t}}(\hat{Q}\hat{W},\hat{Q}\hat{W},R)$, yielding (23), and this concludes the proof of Theorem 1. \square

IV. CONVERGENCE OF THE ITERATIVE MINIMIZATION FOR FIXED GRADIENT

Let us define for two real numbers $0 \le \rho < 1$ and $\eta \ge 0$

$$F^{\mathbf{t}}(\rho, \eta, QW, \widetilde{Q}\widetilde{W}) \triangleq D(W \parallel P \mid Q) - \rho I(Q, W) + \eta \mathbb{E}_{Q}[f(X)] + (1 - \rho)D^{\mathbf{t}}(QW, \widetilde{Q}\widetilde{W}), \quad (24)$$

$$E_{0}^{\mathbf{t}}(\rho, \eta, \widetilde{Q}\widetilde{W}) \triangleq \min_{Q, W} F^{\mathbf{t}}(\rho, \eta, QW, \widetilde{Q}\widetilde{W}). \quad (25)$$

If finite, the quantity $E_0^{\mathbf{t}}(\rho,\eta,\widetilde{Q}\widetilde{W})$ has a meaning of the vertical axis intercept (" E_0 ") of a lower supporting plane in the variables (R,α) for the function $E(R,\alpha)=E_c^{\mathbf{t}}(\widetilde{Q}\widetilde{W},R,\alpha)$, defined in (13), as the following lemma shows.

Lemma 2: For any $0 < \rho < 1$ and $\eta > 0$ it holds that

$$E_c^{\mathbf{t}}(\widetilde{Q}\widetilde{W}, R, \alpha) \geq E_0^{\mathbf{t}}(\rho, \eta, \widetilde{Q}\widetilde{W}) + \rho R - \eta \alpha,$$
 (26)

and there exist $R \ge 0$ and $\alpha \ge \min_x f(x)$ which satisfy (26) with equality.

Proof: By definition (13)

$$\min_{\substack{Q,W:\\ \mathbb{E}_{Q}[f(X)] \leq \alpha}} \left\{ D(W \parallel P \mid Q) + D^{\mathbf{t}}(QW, \widetilde{Q}\widetilde{W}) + \right.$$

$$\left| R - I(Q,W) - D^{\mathbf{t}}(QW, \widetilde{Q}\widetilde{W}) \right|^{+} \right\}$$

$$\geq \min_{\substack{Q,W:\\ \mathbb{E}_{Q}[f(X)] \leq \alpha}} \left\{ D(W \parallel P \mid Q) + D^{\mathbf{t}}(QW, \widetilde{Q}\widetilde{W}) + \right.$$

$$\rho \left[R - I(Q,W) - D^{\mathbf{t}}(QW, \widetilde{Q}\widetilde{W}) \right] + \eta \left[\mathbb{E}_{Q}[f(X)] - \alpha \right] \right\},$$

$$\geq \min_{\substack{Q,W:\\ \mathbb{E}_{Q}[f(X)] \leq \alpha}} \left\{ D(W \parallel P \mid Q) + D^{\mathbf{t}}(QW, \widetilde{Q}\widetilde{W}) + \right.$$

$$\rho \left[R - I(Q,W) - D^{\mathbf{t}}(QW, \widetilde{Q}\widetilde{W}) \right] + \eta \left[\mathbb{E}_{Q}[f(X)] - \alpha \right] \right\},$$
(28)

where (a) holds for any $0 \le \rho < 1$ and $\eta \ge 0$. Using (24) and (25), we see that the lower bound expression (28) is equal to the RHS of (26). Suppose (28) is finite. Let $Q_{\rho,\eta},W_{\rho,\eta}$ denote distributions Q,W, respectively, which jointly minimize (28). Observe that for each $0 \le \rho < 1$ and $\eta \ge 0$ we can find $R \ge 0$ and $\alpha \ge \min_x f(x)$, such that the differences in the square brackets are zero. In this case, $Q_{\rho,\eta}$ will satisfy the input constraint and there will be equality between (28) and (27). \square

Lemma 3: Suppose $\widetilde{Q}\widetilde{W} \equiv \widetilde{T}\widetilde{V}$ is such that the minimum (25) is finite. If $t_1 = t_4 + 1$ in (8), then, with definitions of $a \triangleq (t_2 + t_4)(1 - \rho)$ and $b \triangleq (t_3 + t_4)(1 - \rho)$, $0 \leq \rho < 1$ and

 $\eta \geq 0$, the unique minimizing solution of the minimum (25) can be written as

$$Q^{*}(x)W^{*}(y \mid x) = \frac{1}{K} \left[\widetilde{Q}^{1-\rho}(x) \widetilde{V}^{b}(x \mid y) P_{\eta}(x, y) \right]^{\frac{1}{b+1-\rho}} \times \widetilde{T}^{\frac{a}{a+1}}(y) \left\{ \sum_{\tilde{x}} \left[\widetilde{Q}^{1-\rho}(\tilde{x}) \widetilde{V}^{b}(\tilde{x} \mid y) P_{\eta}(\tilde{x}, y) \right]^{\frac{1}{b+1-\rho}} \right\}^{\frac{b-a-\rho}{a+1}},$$
(29)

where $P_{\eta}(x,y) \triangleq e^{-\eta f(x)} P(y \mid x)$ and K is a normalization constant, resulting in

$$E_0^{\mathbf{t}}(\rho, \eta, \widetilde{Q}\widetilde{W}) = -(a+1)\log \sum_{y} \widetilde{T}^{\frac{a}{a+1}}(y) \times \left\{ \sum_{y} \left[\widetilde{Q}^{1-\rho}(x) \widetilde{V}^{b}(x \mid y) P_{\eta}(x, y) \right]^{\frac{1}{b+1-\rho}} \right\}^{\frac{b+1-\rho}{a+1}}.$$
 (30)

If $t_3 = t_2 + \frac{\rho}{1-\rho}$ in (8), then, with $c \triangleq (t_1 + t_2)(1-\rho)$ and a as defined above, $0 < \rho < 1$ and $\eta \geq 0$, the unique minimizing solution of the minimum (25) can be written as

$$Q^{*}(x)W^{*}(y \mid x) = \frac{1}{K} \left[\widetilde{W}^{a}(y \mid x) \widetilde{V}^{\rho}(x \mid y) P_{\eta}(x, y) \right]^{\frac{1}{a+1}} \times \widetilde{Q}^{\frac{c}{c+\rho}}(x) \left\{ \sum_{\tilde{y}} \left[\widetilde{W}^{a}(\tilde{y} \mid x) \widetilde{V}^{\rho}(x \mid \tilde{y}) P_{\eta}(x, \tilde{y}) \right]^{\frac{1}{a+1}} \right\}^{\frac{a+1-c-\rho}{c+\rho}},$$
(31)

where $P_{\eta}(x,y)$ is defined as above and K is a normalization constant, resulting in

$$E_0^{\mathbf{t}}(\rho, \eta, \widetilde{Q}\widetilde{W}) = -(c+\rho) \log \sum_{x} \widetilde{Q}^{\frac{c}{c+\rho}}(x) \times \left\{ \sum_{y} \left[\widetilde{W}^{a}(y \mid x) \widetilde{V}^{\rho}(x \mid y) P_{\eta}(x, y) \right]^{\frac{1}{a+1}} \right\}^{\frac{a+1}{c+\rho}}.$$
 (32)

Proof: Similarly to [7, Lemma 3]. \square

An iterative minimization procedure at a fixed gradient (ρ, η) , $0 < \rho < 1$, $\eta \ge 0$, is given by

$$Q_{\ell}W_{\ell} = \underset{Q,W}{\operatorname{arg\,min}} F^{\mathbf{t}}(\rho, \eta, QW, \widetilde{Q}_{\ell}\widetilde{W}_{\ell}),$$

$$\widetilde{Q}_{\ell+1}\widetilde{W}_{\ell+1} = Q_{\ell}W_{\ell},$$

$$\ell = 0, 1, 2, \dots.$$
(33)

We assume that the initial distribution $\widetilde{Q}_0\widetilde{W}_0$ in (33) is chosen such that the set $\left\{QW:F_1^{\mathbf{t}}(QW,\widetilde{Q}_0\widetilde{W}_0)<+\infty\right\}$ is non-empty, which guarantees $F^{\mathbf{t}}(\rho,\eta,Q_0W_0,\widetilde{Q}_0\widetilde{W}_0)=E_0^{\mathbf{t}}(\rho,\eta,\widetilde{Q}_0\widetilde{W}_0)<+\infty$. By (24) it is clear that (33) produces a monotonically non-increasing sequence $E_0^{\mathbf{t}}(\rho,\eta,\widetilde{Q}_\ell\widetilde{W}_\ell)$, $\ell=0,1,2,\ldots$. Depending on the choice of the non-negative parameters (t_1,t_2,t_3,t_4) in (8), the update of $Q_\ell W_\ell$ in (33) can be done according to the expression (29) with any $a\geq 0$ and $b\geq 0$, or according to (31) with any $a\geq 0$ and $c\geq 0$, with $\widetilde{Q},\widetilde{V},\widetilde{T},\widetilde{W}$ replaced by $\widetilde{Q}_\ell,\widetilde{V}_\ell,\widetilde{T}_\ell,\widetilde{W}_\ell$, correspondingly. The choice of a=b=0 in (29) gives the fixed-slope counterpart of the algorithm in [7], analysed in [10]. The choice (a,c)=(0,1) in (31) gives the fixed-slope counterpart

of the algorithm in [14]. The choice $(a,b)=(0,\rho)$ in (29), or, alternatively, $(a,c)=(0,1-\rho)$ in (31) gives the algorithm in [4], [5]. The main result of the section is given by the following theorem:

Theorem 2: Let $\{Q_\ell W_\ell\}_{\ell=0}^{+\infty}$ be a sequence of iterative solutions produced by (33). Then

$$E_0^{\mathbf{t}}(\rho,\eta,\widetilde{Q}_{\ell}\widetilde{W}_{\ell}) \overset{\ell \to \infty}{\searrow} \min_{\substack{\widetilde{Q},\,\widetilde{W}:\\ D^{\mathbf{t}}(\widetilde{Q}\widetilde{W},\,\widetilde{Q}_0\widetilde{W}_0) < \infty}} E_0^{\mathbf{t}}(\rho,\eta,\widetilde{Q}\widetilde{W}),$$

where $E_0^{\mathbf{t}}(\rho, \eta, \widetilde{Q}\widetilde{W})$ is defined in (25) and $D^{\mathbf{t}}(\cdot, \cdot)$ in (8).

In order to prove Theorem 2, we use the following lemma:

Lemma 4: Let $\hat{Q}\hat{W}$ be such that $F_1^{\mathbf{t}}(\hat{Q}\hat{W},\,\widetilde{Q}_0\widetilde{W}_0)<+\infty.$ Then

$$F^{\mathbf{t}}(\rho, \eta, Q_{0}W_{0}, \widetilde{Q}_{0}\widetilde{W}_{0}) \leq F^{\mathbf{t}}(\rho, \eta, \hat{Q}\hat{W}, \hat{Q}\hat{W}) \\ + (1 - \rho) \Big[F_{1}^{\mathbf{t}}(\hat{Q}\hat{W}, \widetilde{Q}_{0}\widetilde{W}_{0}) - F_{1}^{\mathbf{t}}(\hat{Q}\hat{W}, \widetilde{Q}_{1}\widetilde{W}_{1}) \Big]. \tag{35}$$

Proof: Since $+\infty > F_1^{\mathbf{t}}(\hat{Q}\hat{W}, \widetilde{Q}_0\widetilde{W}_0)$, then also $+\infty > F_1^{\mathbf{t}}(Q_0W_0, \widetilde{Q}_0\widetilde{W}_0)$. Let $Q^{(\lambda)}W^{(\lambda)}$ be a convex combination of $\hat{Q}\hat{W}$ and Q_0W_0 , as in (17). Then the function $g(\lambda) = F^{\mathbf{t}}(\rho, \eta, Q^{(\lambda)}W^{(\lambda)}, \widetilde{Q}_0\widetilde{W}_0)$ is convex (\cup) and differentiable in $\lambda \in (0,1)$. Since Q_0W_0 achieves the minimum of $F^{\mathbf{t}}(\rho, \eta, QW, \widetilde{Q}_0\widetilde{W}_0)$ over QW, then necessarily

$$\lim_{\lambda \to 0} \frac{dg(\lambda)}{d\lambda} \ge 0.$$

Differentiation results in the following condition in the limit:

$$F^{\mathbf{t}}(\rho, \eta, Q_{0}W_{0}, \widetilde{Q}_{0}\widetilde{W}_{0}) + \rho D(\hat{T} \| T_{0})$$

$$+ (1 - \rho) \Big[D(\hat{W} \| W_{0} | \hat{Q}) + D^{\mathbf{t}}(\hat{Q}\hat{W} \| Q_{0}W_{0}) \Big]$$

$$\leq F^{\mathbf{t}}(\rho, \eta, \hat{Q}\hat{W}, \widetilde{Q}_{0}\widetilde{W}_{0}),$$
(36)

where \hat{T} and T_0 denote the y-marginal distributions of $\hat{Q}\hat{W}$ and Q_0W_0 , respectively. Since $F_1^{\mathbf{t}}(\hat{Q}\hat{W},\,\widetilde{Q}_0\widetilde{W}_0)<+\infty$, then all terms in (36) are finite. On the other hand, by (24)

$$F^{\mathbf{t}}(\rho, \eta, \hat{Q}\hat{W}, \widetilde{Q}_{0}\widetilde{W}_{0}) = F^{\mathbf{t}}(\rho, \eta, \hat{Q}\hat{W}, \hat{Q}\hat{W}) + (1 - \rho)D^{\mathbf{t}}(\hat{Q}\hat{W} \parallel \widetilde{Q}_{0}\widetilde{W}_{0}).$$
(37)

Combining (37) with (36), noting that $Q_0W_0=\widetilde{Q}_1\widetilde{W}_1$, and omitting non-negative terms $(1-\rho)D(\hat{W}\parallel W_0\mid \hat{Q})\geq 0$ and $\rho D(\hat{T}\parallel T_0)\geq 0$, we obtain a weaker inequality (35). \square

Proof of Theorem 2: Using (24), (25), it can be verified, that the RHS of (34) can be rewritten as

$$\min_{\substack{\widetilde{Q},\widetilde{W}:\\D^{\mathbf{t}}(\widetilde{Q}\widetilde{W},\widetilde{Q}_{0}\widetilde{W}_{0})<\infty}} E_{0}^{\mathbf{t}}(\rho,\eta,\widetilde{Q}\widetilde{W}) = \min_{\substack{Q,W:\\D^{\mathbf{t}}(QW,\widetilde{Q}_{0}\widetilde{W}_{0})<\infty}} F^{\mathbf{t}}(\rho,\eta,QW,QW).$$

Suppose (38) is finite and let $\hat{Q}\hat{W}$ achieve the minimum on the RHS of (38). Then by Lemma 4 we conclude that for all iterations $\ell=0,1,2,\ldots$, it holds that

$$\begin{split} F^{\mathbf{t}}(\rho,\,\eta,\,Q_{\ell}W_{\ell},\,\widetilde{Q}_{\ell}\widetilde{W}_{\ell}) & \leq & F^{\mathbf{t}}(\rho,\eta,\hat{Q}\hat{W},\hat{Q}\hat{W}) \\ & + & (1-\rho)\Big[F_{1}^{\mathbf{t}}(\hat{Q}\hat{W},\,\widetilde{Q}_{\ell}\widetilde{W}_{\ell}) \, - \, F_{1}^{\mathbf{t}}(\hat{Q}\hat{W},\,\widetilde{Q}_{\ell+1}\widetilde{W}_{\ell+1})\Big]. \end{split}$$

The conclusion of the proof is the same as in Theorem 1. \square

The next two sections show convergence of fixed-slope computation in the directions of R and α , respectively. They are similar in structure to Section IV.

V. Convergence for fixed α and ρ

In this section we show convergence of an iterative minimization at a fixed slope ρ in the direction of R, i.e., for a given α . With the help of (24) let us define $F^{\mathbf{t}}(\rho, QW, \widetilde{Q}\widetilde{W}) \triangleq F^{\mathbf{t}}(\rho, \eta, QW, \widetilde{Q}\widetilde{W})\Big|_{\eta=0}$ and

$$E_0^{\mathbf{t}}(\rho, \widetilde{Q}\widetilde{W}, \alpha) \triangleq \min_{\substack{Q, W: \\ \mathbb{E}_Q[f(X)] \leq \alpha}} F^{\mathbf{t}}(\rho, QW, \widetilde{Q}\widetilde{W}).$$
 (39)

Here $E_0^{\mathbf{t}}(\rho,\widetilde{Q}\widetilde{W},\alpha)$ plays a role of " E_0 " of a supporting line in the variable R of the function $E(R)=E_c^{\mathbf{t}}(\widetilde{Q}\widetilde{W},R,\alpha)$, defined in (13), as shown by the following lemma.

Lemma 5: For any $0 \le \rho < 1$ *it holds that*

$$E_c^{\mathbf{t}}(\widetilde{Q}\widetilde{W}, R, \alpha) \geq E_0^{\mathbf{t}}(\rho, \widetilde{Q}\widetilde{W}, \alpha) + \rho R,$$
 (40)

and there exists $R \ge 0$ which satisfies (40) with equality. *Proof:* Similar to Lemma 2. \square

An iterative minimization procedure at a fixed slope ρ is given by

$$Q_{\ell}W_{\ell} \in \underset{\mathbb{E}_{Q}[f(X)] \leq \alpha}{\operatorname{arg\,min}} F^{\mathbf{t}}(\rho, QW, \widetilde{Q}_{\ell}\widetilde{W}_{\ell}),$$

$$\widetilde{Q}_{\ell+1}\widetilde{W}_{\ell+1} = Q_{\ell}W_{\ell},$$

$$\ell = 0, 1, 2, \dots.$$

$$(41)$$

It is assumed that $\widetilde{Q}_0\widetilde{W}_0$ in (41) is chosen such that the set $\left\{QW:\sum_x Q(x)f(x)\leq \alpha, F_1^{\mathbf{t}}(QW,\widetilde{Q}_0\widetilde{W}_0)<+\infty\right\}$ is nonempty, so that $F^{\mathbf{t}}(\rho,Q_0W_0,\widetilde{Q}_0\widetilde{W}_0)=E_c^{\mathbf{t}}(\rho,\widetilde{Q}_0\widetilde{W}_0,\alpha)<+\infty.$ By the definition of $F^{\mathbf{t}}(\rho,QW,\widetilde{Q}\widetilde{W})$ according to (24), this procedure results in a monotonically non-increasing sequence $E_0^{\mathbf{t}}(\rho,\widetilde{Q}_\ell\widetilde{W}_\ell,\alpha),\ \ell=0,1,2,\dots$. The main result of this section is stated in the following theorem.

Theorem 3: Let $\{Q_\ell W_\ell\}_{\ell=0}^{+\infty}$ be a sequence of iterative solutions produced by (41). Then

$$E_{0}^{\mathbf{t}}(\rho, \widetilde{Q}_{\ell}\widetilde{W}_{\ell}, \alpha) \stackrel{\ell \to \infty}{\searrow} \min_{\substack{\widetilde{Q}, \widetilde{W}: \\ D^{\mathbf{t}}(\widetilde{Q}\widetilde{W}, \widetilde{Q}_{0}\widetilde{W}_{0}) < \infty}} E_{0}^{\mathbf{t}}(\rho, \widetilde{Q}\widetilde{W}, \alpha),$$

$$(42)$$

(38) where $E_0^{\mathbf{t}}(\rho, \widetilde{Q}\widetilde{W}, \alpha)$ is defined in (39) and $D^{\mathbf{t}}(\cdot, \cdot)$ in (8).

To prove Theorem 3, we use a lemma, similar to Lemma 4:

Lemma 6: Let $\hat{Q}\hat{W}$ be such, that $\sum_{x}\hat{Q}(x)f(x) \leq \alpha$ and $F_1^{\mathbf{t}}(\hat{Q}\hat{W}, \, \widetilde{Q}_0\widetilde{W}_0) < +\infty$. Then

$$\begin{split} F^{\mathbf{t}}(\rho,\,Q_0W_0,\,\widetilde{Q}_0\widetilde{W}_0) &\leq F^{\mathbf{t}}(\rho,\hat{Q}\hat{W},\hat{Q}\hat{W}) \\ &+ (1-\rho)\Big[F_1^{\mathbf{t}}(\hat{Q}\hat{W},\,\widetilde{Q}_0\widetilde{W}_0) \,-\,F_1^{\mathbf{t}}(\hat{Q}\hat{W},\,\widetilde{Q}_1\widetilde{W}_1)\Big]. \end{split} \tag{43}$$

Proof: Analogous to Lemma 4. □

Proof of Theorem 3: The RHS of (42) can be rewritten in terms of $F^{\mathbf{t}}(\rho, QW, \widetilde{Q}\widetilde{W})$ as:

$$\min_{\substack{\widetilde{Q},\,\widetilde{W}:\\ D^{\mathbf{t}}(\widetilde{Q}\widetilde{W},\,\widetilde{Q}_{0}\widetilde{W}_{0}) < \infty}} E_{0}^{\mathbf{t}}(\rho,\widetilde{Q}\widetilde{W},\alpha) \ = \ \min_{\substack{Q,\,W:\\ \mathbb{E}_{Q}[f(X)] \leq \alpha\\ D^{\mathbf{t}}(QW,\,\widetilde{Q}_{0}\widetilde{W}_{0}) < \infty}} F^{\mathbf{t}}(\rho,QW,QW).$$

Suppose (44) is finite and $\hat{Q}\hat{W}$ achieves the minimum on the RHS. Then we can use Lemma 6 with $\hat{Q}\hat{W}$. The rest of the proof is the same as for Theorem 2. \square

VI. Convergence for fixed R and η

In this section we show convergence of an iterative minimization at a fixed slope η in the direction of α , i.e., for a given R. Let us define

$$F^{\mathbf{t}}(\eta, QW, \widetilde{Q}\widetilde{W}, R) \triangleq \max \left\{ F_1^{\mathbf{t}}(QW, \widetilde{Q}\widetilde{W}), F_2(QW, R) \right\} + \eta \mathbb{E}_Q[f(X)], \tag{45}$$

where $\boldsymbol{F}_1^{\mathbf{t}}(QW,\,\widetilde{Q}\widetilde{W})$ and $\boldsymbol{F}_2(QW,R)$ are as defined in (10) and (11), respectively.

$$E_0^{\mathbf{t}}(\eta, \widetilde{Q}\widetilde{W}, R) \triangleq \min_{Q, W} F^{\mathbf{t}}(\eta, QW, \widetilde{Q}\widetilde{W}, R).$$
 (46)

Here $E_0^{\bf t}(\eta,\widetilde{Q}\widetilde{W},R)$ plays a role of " E_0 " of a supporting line in the variable α of the function $E(\alpha) = E_c^{\mathbf{t}}(\widetilde{Q}\widetilde{W}, R, \alpha)$, defined in (13), as shown by the following lemma.

Lemma 7: For any $\eta > 0$ *it holds that*

$$E_c^{\mathbf{t}}(\widetilde{Q}\widetilde{W}, R, \alpha) \geq E_0^{\mathbf{t}}(\eta, \widetilde{Q}\widetilde{W}, R) - \eta\alpha,$$
 (47)

and there exists $\alpha \geq \min_{x} f(x)$ which satisfies (47) with equality.

Proof: Similar to Lemma 2. \square

An iterative minimization procedure at a fixed slope η is defined as follows.

$$Q_{\ell}W_{\ell} \in \underset{Q,W}{\operatorname{arg\,min}} F^{\mathbf{t}}(\eta, QW, \widetilde{Q}_{\ell}\widetilde{W}_{\ell}, R),$$

$$\widetilde{Q}_{\ell+1}\widetilde{W}_{\ell+1} = Q_{\ell}W_{\ell},$$

$$\ell = 0, 1, 2, \dots.$$
(48)

It is assumed that the set $\{QW: F_1^{\mathbf{t}}(QW, \widetilde{Q}_0\widetilde{W}_0) < +\infty\}$ is non-empty, which guarantees $F^{\mathbf{t}}(\eta,\,Q_0W_0,\,\widetilde{Q}_0\widetilde{W}_0,\,R)=E_0^{\mathbf{t}}(\eta,\,\widetilde{Q}_0\widetilde{W}_0,\,R)<+\infty.$ The iterative procedure results in a monotonically non-increasing sequence $E_0^{\mathbf{t}}(\eta,\,\widetilde{Q}_\ell\widetilde{W}_\ell,\,R)$, $\ell = 0, 1, 2, ...$, as can be seen from (45), (46). The sequence converges to the global minimum in the set $\{QW:$ $D^{\mathbf{t}}(\widetilde{Q}\widetilde{W},\widetilde{Q}_{0}\widetilde{W}_{0})<+\infty\}$, as stated in the following theo-

Theorem 4: Let $\{Q_\ell W_\ell\}_{\ell=0}^{+\infty}$ be a sequence of iterative solutions produced by (48). Then

$$E_{0}^{\mathbf{t}}(\eta, \widetilde{Q}_{\ell}\widetilde{W}_{\ell}, R) \overset{\ell \to \infty}{\searrow} \min_{\substack{\widetilde{Q}, \widetilde{W}: \\ D^{\mathbf{t}}(\widetilde{Q}\widetilde{W}, \widetilde{Q}_{0}\widetilde{W}_{0}) < \infty}} E_{0}^{\mathbf{t}}(\eta, \widetilde{Q}\widetilde{W}, R),$$

$$(49)$$

where $E_0^{\mathbf{t}}(\eta, \widetilde{Q}\widetilde{W}, R)$ is defined in (46) and $D^{\mathbf{t}}(\cdot, \cdot)$ in (8).

To prove this theorem, we use a lemma, which is similar to Lemma 1:

Lemma 8: Let $\hat{Q}\hat{W}$ be such that $F_1^{\mathbf{t}}(\hat{Q}\hat{W}, \tilde{Q}_0\tilde{W}_0) < +\infty$.

$$F^{\mathbf{t}}(\eta, Q_{0}W_{0}, \tilde{Q}_{0}\tilde{W}_{0}, R) \leq F^{\mathbf{t}}(\eta, \hat{Q}\hat{W}, \hat{Q}\hat{W}, R) + \left|F_{1}^{\mathbf{t}}(\hat{Q}\hat{W}, \tilde{Q}_{0}\tilde{W}_{0}) - F_{1}^{\mathbf{t}}(\hat{Q}\hat{W}, \tilde{Q}_{1}\tilde{W}_{1})\right|^{+}. (50)$$

Proof: Similar to Lemma 1. \square

Proof of Theorem 4: The RHS of (49) can be rewritten in terms of $F^{\mathbf{t}}(\eta, QW, \widetilde{QW}, R)$ as:

$$\min_{\substack{\widetilde{Q},\,\widetilde{W}:\\D^{\mathbf{t}}(\widetilde{Q}\widetilde{W},\,\widetilde{Q}_{0}\widetilde{W}_{0})<\infty}} E_{0}^{\mathbf{t}}(\eta,\widetilde{Q}\widetilde{W},R) = \min_{\substack{Q,\,W:\\D^{\mathbf{t}}(QW,\,\widetilde{Q}_{0}\widetilde{W}_{0})<\infty}} F^{\mathbf{t}}(\eta,QW,QW,R).$$

Suppose (51) is finite, and let $\hat{Q}\hat{W}$ achieve the minimum on the RHS. Then we can use Lemma 8 with $\hat{Q}\hat{W}$. The rest of the proof is the same as for Theorem 1. \square

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