# Proof of Convergence for Correct-Decoding Exponent Computation 

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#### Abstract

For a discrete memoryless channel with finite input and output alphabets, we prove convergence of a parametric family of iterative computations of the optimal correct-decoding exponent. The exponent, as a function of communication rate, is computed for a fixed rate and for a fixed slope.


## I. Introduction

Consider a standard information theoretic setting of transmission through a discrete memoryless channel (DMC), with finite input and output alphabets, using block codes. For communication rates above capacity, the average probability of correct decoding in a block code tends to zero exponentially fast as a function of the block length. In the limit of a large block length, the lowest possible exponent corresponding to the probability of correct decoding, also called the reliability function above capacity, for all 1 rates $R \geq 0$ is given by [1]

$$
\begin{equation*}
E_{c}(R)=\min _{\substack{Q(x), W(y \mid x)}}\left\{D(W \| P \mid Q)+|R-I(Q, W)|^{+}\right\} \tag{1}
\end{equation*}
$$

where $P$ denotes the channel's transition probability $P(y \mid x)$, $D(W \| P \mid Q)$ is the Kullback-Leibler divergence between the conditional distributions $W$ and $P$, averaged over $Q, I(Q, W)$ is the mutual information of a pair of random variables with a joint distribution $Q(x) W(y \mid x)$, and $|t|^{+}=\max \{0, t\}$.

For certain applications, it is important to be able to know the actual value of $E_{c}(R)$ when it is positive. For example, in applications of secrecy, it might be interesting to know the correct-decoding exponent of an eavesdropper. Several algorithms have been proposed for computation of $E_{c}(R)$.

In the algorithm by Arimoto [2] the computation of $E_{c}(R)$ is facilitated by an alternative expression for it [3], [1], [4]:

$$
\begin{equation*}
E_{c}(R)=\sup _{0 \leq \rho<1} \min _{Q}\left\{E_{0}(-\rho, Q)+\rho R\right\} \tag{2}
\end{equation*}
$$

where $E_{0}(-\rho, Q)$ is the Gallager exponent function [6, Eq. 5.6.14]. In [2], $\min _{Q} E_{0}(-\rho, Q)$ is computed for a fixed slope parameter $\rho$. The computation is performed iteratively as alternating minimization, based on the property that $\min _{Q} E_{0}(-\rho, Q)$ can be written as a double minimum:

$$
\begin{equation*}
\min _{Q} \min _{V}\left\{-\log \sum_{x, y} Q^{1-\rho}(x) V^{\rho}(x \mid y) P(y \mid x)\right\} \tag{3}
\end{equation*}
$$

[^0]where the inner minimum is in fact equal to $E_{0}(-\rho, Q)$. In [4], [5] a different alternating-minimization algorithm is introduced, based on the property, that $\min _{Q} E_{0}(-\rho, Q)$ can be written as another double minimum over distributions:
\[

$$
\begin{equation*}
\min _{T, V} \min _{T_{1}, V_{1}}\left\{-\sum_{x, y} T(y) V(x \mid y) \log \frac{V_{1}^{\rho}(x \mid y) P(y \mid x)}{U_{1}^{\rho-1}(x) T(y) V(x \mid y)}\right\} \tag{4}
\end{equation*}
$$

\]

where $U_{1}(x)=\sum_{y} T_{1}(y) V_{1}(x \mid y)$. As with (3), the computation of $E_{c}(R)$ with (4) is also performed for a fixed $\rho$.

Sometimes, however, it is suitable or desirable to compute $E_{c}(R)$ directly for a given rate $R$. For example, when $E_{c}(R)=0$, and we would like to find such a distribution $Q$, for which the minimum (1) is zero, as a by-product of the computation. Such distribution $Q$ has a practical meaning of a channel input distribution achieving reliable communication. In [7], an iterative minimization procedure for computation of $E_{c}(R)$ at fixed $R$ is proposed, using the property that $E_{c}(R)$ can be written as a double minimum [8]:

$$
\begin{equation*}
\min _{Q(x)} \min _{\substack{T(y), V(x \mid y)}}\left\{D(T V \| Q P)+|R-D(V \| Q \mid T)|^{+}\right\} \tag{5}
\end{equation*}
$$

where the inner min equals $\sup _{0 \leq \rho<1}\left\{E_{0}(-\rho, Q)+\rho R\right\}$. In [7], the inner minimum of (5) is computed stochastically by virtue of a correct-decoding event itself, yielding the minimizing solution $T^{*} V^{*}$. The computation is then repeated iteratively, by assigning $Q(x)=\sum_{y} T^{*}(y) V^{*}(x \mid y)$. It is shown in [7, Theorem 1], that the iterative procedure using the inner minimum of (5) leads to convergence of this minimum to the double minimum (5), which is evaluated at least over some subset of the support of the initial distribution $Q_{0}$. In addition, a sufficient condition on $Q_{0}$ is provided, which guarantees convergence of the inner minimum in (5) to zero. This condition on $Q_{0}$ in [7] Lemma 6] is rather limiting, and is hard to verify.

In the current work, we improve the result of [7]. We modify the method of Csiszár and Tusnády [9] to prove that the iterative minimization procedure of [7] converges to the global minimum (5) over the support of the initial distribution $Q_{0}$ itself, for any $R$ (i.e., not only if the global minimum is zero), and without any additional condition. In particular, use of a strictly positive $Q_{0}$ guarantees convergence to $E_{c}(R)$.

By a similar method, we also show convergence of the fixed-slope counterpart of the minimization (5), which is
an alternating minimization at fixed $\rho$, based on the double minimum [10]

$$
\begin{equation*}
\min _{Q} \min _{T, V}\left\{-\sum_{x, y} T(y) V(x \mid y) \log \frac{Q^{1-\rho}(x) P(y \mid x)}{T(y) V^{1-\rho}(x \mid y)}\right\} \tag{6}
\end{equation*}
$$

where the inner minimum is in fact equal to $E_{0}(-\rho, Q)$.
Furthermore, in the current paper we extend the analysis, presented in the shorter version of the paper [11]. Here we slightly generalize the expression (5). Using this generalization, we prove convergence of a parametric family of iterative computations, of which the computation according to (5) from [7], as well as the computations according to (6), [10], and according to (4), [4], become special cases.

As in the shorter version of the paper [11], besides the variable $R$, we take into account also a possible channelinput constraint, denoted by $\alpha$. In Section $\Pi$ we examine the expression for the correct-decoding exponent. In Section III we prove convergence of the iterative minimization for fixed $(R, \alpha)$. In Section IV we prove convergence of the iterative minimization for fixed gradient w.r.t. $(R, \alpha)$. In Sections $\square$ and VI we prove convergence of mixed scenarios: for fixed $\alpha$ and slope $\rho$ in the direction of $R$, and vice versa.

## II. Correct-decoding exponent

Let $P(y \mid x)$ denote transition probabilities in a DMC from $x \in \mathcal{X}$ to $y \in \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are finite channel input and output alphabets, respectively. Suppose also that the channel input $x$ with an additive cost function $f: \mathcal{X} \rightarrow \mathbb{R}$ satisfies on average an input constraint $\alpha \in \mathbb{R}$, chosen large enough, such that $\alpha \geq \min _{x} f(x)$. The maximum-likelihood correctdecoding exponent ( [1], [12]) of this channel, as a function of the rate $R \geq 0$ and the input constraint $\alpha$, is given by

$$
\begin{aligned}
& E_{c}(R, \alpha)= \\
& \min _{\substack{Q(x): \\
\mathbb{E}_{Q}[f(X)] \leq \alpha}} \min _{W(y \mid x)}\left\{D(W \| P \mid Q)+|R-I(Q, W)|^{+}\right\},
\end{aligned}
$$

where $\mathbb{E}_{Q}[f(X)]$ denotes the expectation of $f(x)$ w.r.t. the distribution $Q(x)$ over $\mathcal{X}$.

Let $Q(x) W(y \mid x) \equiv T(y) V(x \downarrow y)$, or $Q W$, denote a distribution over $\mathcal{X} \times \mathcal{Y}$, and let $\widetilde{Q} \widetilde{W}$ be another such distribution. We can think of 4 different divergences from $\widetilde{Q} \widetilde{W}$ to $Q W: D(Q \| \widetilde{Q}), D(W \| \widetilde{W} \mid Q), D(T \| \widetilde{T})$, and $D(V \| \widetilde{V} \mid T)$. Using 4 non-negative parameters $t_{i} \geq 0, i=1,2,3,4$, we define a non-negative linear combination of these divergences:

$$
\begin{align*}
D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W}) \triangleq & t_{1} D(Q \| \widetilde{Q})+t_{2} D(W \| \widetilde{W} \mid Q)+ \\
& t_{3} D(T \| \widetilde{T})+t_{4} D(V \| \widetilde{V} \mid T) \tag{8}
\end{align*}
$$

where $\mathbf{t} \triangleq\left(t_{1}, t_{2}, t_{3}, t_{4}\right)$ is an index. With the help of $D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W})$, the expression (7) can be rewritten as follows:

$$
\begin{aligned}
& \min _{\substack{Q, W: \\
\mathbb{E}_{Q}[f(X)] \leq \alpha}}\left\{D(W \| P \mid Q)+|R-I(Q, W)|^{+}\right\} \\
= & \min _{\substack{Q, W: \\
\mathbb{E}_{Q}[f(X)] \leq \alpha}} \max \{D(W \| P \mid Q), \\
& D(W \| P \mid Q)+R-I(Q, W)\}
\end{aligned}
$$

$$
\begin{array}{r}
=\min _{\widetilde{Q}, \widetilde{W}}^{\substack{Q, W: \\
\mathbb{E}_{Q}[f(X)] \leq \alpha}} \min ^{\max \{ } \operatorname{m(W\| P|Q)+D^{\mathbf {t}}(QW,\widetilde {Q}\widetilde {W}),} \\
D(W \| P \mid Q)+R-I(Q, W)\}, \tag{9}
\end{array}
$$

where the first equality holds because $|a|^{+}=\max \{\underset{\sim}{0}, a\}$, and the second equality follows since $\min _{\widetilde{Q} \widetilde{W}} D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W})=0$ and the minima can be interchanged. In [7] a special case $(\mathbf{t}=$ $(1,0,0,0)$ ) of the inner minimum of (9) was used as a basis of an iterative procedure to find minimizing solutions of (7). In what follows, we modify the method of Csiszár and Tusnády [9] to show convergence of that minimization procedure. The method allows us to prove convergence in a slightly more general setting (9), (8), with arbitrary non-negative parameters $\left(t_{1}, t_{2}, t_{3}, t_{4}\right)$.

## III. CONVERGENCE OF THE ITERATIVE MINIMIZATION FOR FIXED $(R, \alpha)$

Let us define a short notation for the maximum in (9):

$$
\begin{align*}
F_{1}^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W}) & \triangleq D(W \| P \mid Q)+D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W}),  \tag{10}\\
F_{2}(Q W, R) & \triangleq D(W \| P \mid Q)-I(Q, W)+R,  \tag{11}\\
F^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W}, R) & \triangleq \max \left\{F_{1}^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W}), F_{2}(Q W, R)\right\} \tag{12}
\end{align*}
$$

Define notation for the inner minimum in (9):

$$
\begin{equation*}
E_{c}^{\mathbf{t}}(\widetilde{Q} \widetilde{W}, R, \alpha) \triangleq \min _{\substack{Q, W: \\ \mathbb{E}_{Q}[f(X)] \leq \alpha}} F^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W}, R) \tag{13}
\end{equation*}
$$

The iterative minimization procedure from [7], consisting of two steps in each iteration $\sqrt{2}$, in a more general form is given by

$$
\begin{align*}
Q_{\ell} W_{\ell} & \in \underset{\substack{Q, W: \\
\mathbb{E}_{Q}[f(X)] \leq \alpha}}{\arg \min } F^{\mathbf{t}}\left(Q W, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}, R\right) \\
\widetilde{Q}_{\ell+1} \widetilde{W}_{\ell+1} & =\quad Q_{\ell} W_{\ell} \tag{14}
\end{align*}
$$

We assume that $\widetilde{Q}_{0} \widetilde{W}_{0}$ in (14) is chosen such that the set $\left\{Q W: \sum_{x} Q(x) f(x) \leq \alpha, F_{1}^{\mathbf{t}}\left(Q W, \widetilde{Q}_{0} \widetilde{W}_{0}\right)_{\sim_{\sim}^{e}}<+\infty\right\}$ is non-empty, which guarantees $F^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}, R\right)=$ $E_{c}^{\mathbf{t}}\left(\widetilde{Q}_{0} \widetilde{W}_{0}, R, \alpha\right)<+\infty$. By (10) it is clear that (14) produces a monotonically non-increasing sequence $E_{c}^{\mathrm{t}}\left(\widetilde{Q}_{\ell} \widetilde{W}_{\ell}, R, \alpha\right)$, $\ell=0,1,2, \ldots$. Our main result is given by the following theorem, which is an improvement on [7] Theorem 1] and [7, Lemma 6]:

Theorem 1: Let $\left\{Q_{\ell} W_{\ell}\right\}_{\ell=0}^{+\infty}$ be a sequence of iterative solutions produced by (14). Then

$$
\begin{equation*}
E_{c}^{\mathbf{t}}\left(\widetilde{Q}_{\ell} \widetilde{W}_{\ell}, R, \alpha\right) \stackrel{\ell \rightarrow \infty}{\searrow} \min _{\substack{\widetilde{Q}, \widetilde{W}_{i} \\ D^{\mathbf{t}}\left(\widetilde{Q} \tilde{W}, \widetilde{Q}_{0} \tilde{W}_{0}\right)<\infty}} E_{c}^{\mathbf{t}}(\widetilde{Q} \tilde{W}, R, \alpha), \tag{15}
\end{equation*}
$$

[^1]where $E_{c}^{\mathbf{t}}(\widetilde{Q} \widetilde{W}, R, \alpha)$ is defined in (13) and $D^{\mathbf{t}}(\cdot, \cdot)$ in (8).
Suppose $Q_{\widetilde{\sim}}^{*} W_{\sim}^{*}$ is a minimizing solution of (7). If the initial distribution $\widetilde{Q}_{0} \widetilde{W}_{0}$ in the iterations (14) is chosen such that $D^{\mathbf{t}}\left(Q^{*} W^{*}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty\left(\right.$ for example, if $\operatorname{support}\left(\widetilde{Q}_{0} \widetilde{W}_{0}\right)=$ $\mathcal{X} \times \mathcal{Y}$ ), then by (9) the RHS of (15) gives (7). The choice of $\mathbf{t}=(1,0,0,0)$ in (8) corresponds to the iterative minimization in [7]. In order to prove Theorem 1] we use a lemma, which is similar to "the five points property" from [9].

Lemma 1: Let $\hat{Q} \hat{W}$ be such, that $\sum_{x} \hat{Q}(x) f(x) \leq \alpha$ and $F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty$. Then

$$
\begin{align*}
F^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}, R\right) & \leq F^{\mathbf{t}}(\hat{Q} \hat{W}, \hat{Q} \hat{W}, R) \\
& +\left|F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)-F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{1} \widetilde{W}_{1}\right)\right|^{+} \tag{16}
\end{align*}
$$

Proof: Let us define a set of distributions $Q W$ :
$\mathcal{S} \triangleq\left\{Q W: \sum_{x} Q(x) f(x) \leq \alpha, F_{1}^{\mathbf{t}}\left(Q W, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty\right\}$.
Observe that $\mathcal{S}$ is a closed convex set. Since $\hat{Q} \hat{W} \in \mathcal{S}$, then $\mathcal{S}$ is non-empty and by (14) we have also that $Q_{0} W_{0} \in \mathcal{S}$. Observe further that the two terms in the maximization of (12), $F_{1}^{\mathbf{t}}\left(Q W, \widetilde{Q}_{0} \widetilde{W}_{0}\right)$ and $F_{2}(Q W, R)$, as functions of $Q W$, are convex ( $\cup$ ) and continuous in $\mathcal{S}$.

Consider the case $F_{1}^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)>F_{2}\left(\underline{Q}_{0}{\underset{W}{W}}_{0}, R\right)$ first. Then $F^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}, R\right)=F_{1}^{\mathbf{t}}\left(Q_{Q} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)$ by (12). By (14), we conclude that $F_{1}^{\mathrm{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)$ cannot be decreased in the vicinity of $Q W=Q_{0} W_{0}$ inside the convex set $\mathcal{S}$. Let us define a point inside $\mathcal{S}$ :

$$
\begin{align*}
& Q^{(\lambda)}(x) W^{(\lambda)}(y \mid x) \triangleq  \tag{17}\\
& \lambda \hat{Q}(x) \hat{W}(y \mid x)+(1-\lambda) Q_{0}(x) W_{0}(y \mid x), \quad \lambda \in(0,1) .
\end{align*}
$$

We have that ${\underset{Q}{Q}}^{(\lambda)} W^{(\lambda)} \in \mathcal{S}$, and the function $f_{1}(\lambda) \triangleq$ $F_{1}^{\mathbf{t}}\left(Q^{(\lambda)} W^{(\lambda)}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)$ is convex $(\cup)$ and differentiable w.r.t. $\lambda \in(0,1)$. Since $f_{1}(\lambda)$ has to be non-decreasing at $\lambda=0$, the following condition must hold:

$$
\begin{equation*}
\lim _{\lambda \rightarrow 0} \frac{d f_{1}(\lambda)}{d \lambda} \geq 0 \tag{18}
\end{equation*}
$$

Differentiating $f_{1}(\lambda)$, similarly as in the proof of the "Pythagorean" theorem for divergence [13] (proved as "the three points property" in [9, Lemma 2]), we obtain:

$$
\begin{align*}
& F_{1}^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)+D\left(\hat{W} \| W_{0} \mid \hat{Q}\right)+D^{\mathbf{t}}\left(\hat{Q} \hat{W}, Q_{0} W_{0}\right) \\
& \leq F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right) . \tag{19}
\end{align*}
$$

Since $F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty$, then the divergences on the LHS of (19) are also finite. By the definition (10),

$$
\begin{equation*}
F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)=F_{1}^{\mathbf{t}}(\hat{Q} \hat{W}, \hat{Q} \hat{W})+D^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right) \tag{20}
\end{equation*}
$$

$\underset{\widetilde{Q}}{ }$ Omitting $D\left(\hat{W} \| W_{0} \mid \hat{Q}\right) \geq 0$ from (19), noting that $Q_{0} W_{0}=$ $\widetilde{Q}_{1} \widetilde{W}_{1}$, and combining (19) with (20), we get

$$
\begin{align*}
& F_{1}^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right) \leq F_{1}^{\mathbf{t}}(\hat{Q} \hat{W}, \hat{Q} \hat{W}) \\
& \quad+D^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)-D^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{1} \widetilde{W}_{1}\right) \tag{21}
\end{align*}
$$

Now, (16) follows because $F^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}, R\right)=$ $F_{1}^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)$ and $F_{1}^{\mathbf{t}}(\hat{Q} \hat{W}, \hat{Q} \hat{W}) \leq F^{\mathbf{t}}(\hat{Q} \hat{W}, \hat{Q} \hat{W}, R)$.

Consider the case $F_{1}^{\mathrm{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<F_{2}\left(Q_{0} W_{0}, R\right)$ next. Then $F^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}, R\right)=F_{2}\left(Q_{0} W_{0}, R\right)$ by (12). By (14), we conclude that $F_{2}\left(Q_{0} W_{0}, R\right)$ cannot be decreased in the vicinity of $Q W=Q_{0} W_{0}$ inside the convex set $\mathcal{S}$, and by convexity $(\cup)$ of $F_{2}(Q W, R)$ it follows that

$$
F_{2}\left(Q_{0} W_{0}, R\right)=\min _{Q W \in \mathcal{S}} F_{2}(Q W, R)
$$

$$
\stackrel{(a)}{\leq} F_{2}(\hat{Q} \hat{W}, R) \stackrel{(b)}{\leq} F^{\mathbf{t}}(\hat{Q} \hat{W}, \hat{Q} \hat{W}, R)
$$

where ( $a$ ) follows because $\hat{Q} \hat{W} \in \mathcal{S}$, and (b) follows by (12). This again gives (16).

Finally, assume now the equality $F_{1}^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)=$ $F_{2}\left(Q_{0} W_{0}, R\right)$. In this case, using the definition (17), we look at two functions: $f_{1}(\lambda)$ and $f_{2}(\lambda) \triangleq F_{2}\left(Q^{(\lambda)} W^{(\lambda)}, R\right)$, both of which are convex $(\cup)$ and differentiable w.r.t. $\lambda \in(0,1)$. At least one of these two functions has to be non-decreasing at $\lambda=0$. This implies either (18) or

$$
\begin{equation*}
\lim _{\lambda \rightarrow 0} \frac{d f_{2}(\lambda)}{d \lambda} \geq 0 \tag{22}
\end{equation*}
$$

The condition (18) results in (16) as before, while (22) by convexity $(\cup)$ of $f_{2}(\lambda)$ implies

$$
F_{2}\left(Q_{0} W_{0}, R\right) \leq F_{2}(\hat{Q} \hat{W}, R) \leq F^{\mathbf{t}}(\hat{Q} \hat{W}, \hat{Q} \hat{W}, R)
$$

where the second inequality is by definition (12). Since $F_{2}\left(Q_{0} W_{0}, R\right)=F^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}, R\right)$, this gives (16).

A similar, alternative, lemma can be proved if we add $D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W})$ to the second term of the maximum in (9), and not to the first.

Proof of Theorem 7. By (9) we can rewrite the RHS of (15) as

$$
\min _{\substack{\widetilde{Q}, \widetilde{W} \\ D^{\mathbf{t}}\left(\widetilde{Q} \widetilde{W}, \widetilde{Q_{0}} \tilde{W}_{0}\right)<\infty}} E_{c}^{\mathbf{t}}(\widetilde{Q} \widetilde{W}, R, \alpha)=\min _{\substack{Q, W: \\ \mathbb{E}_{Q}[f(X)] \leq \alpha \\ D^{\mathbf{t}}\left(Q W, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<\infty}} F^{\mathbf{t}}(Q W, Q W, R) .
$$

Suppose (23) is finite, and let $\hat{Q} \hat{W}$ achieve the RHS min in (23). Then $F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty$ and $\sum_{x} \hat{Q}(x) f(x) \leq \alpha$. Then Lemma 1 implies that there exist only two possibilities for the outcome of the iterations in (14). One possibility is that at some iteration $\ell$ it holds that

$$
F^{\mathbf{t}}\left(Q_{\ell} W_{\ell}, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}, R\right) \leq F^{\mathbf{t}}(\hat{Q} \hat{W}, \hat{Q} \hat{W}, R)
$$

meaning that the monotonically non-increasing sequence of $F^{\mathbf{t}}\left(Q_{\ell} W_{\ell}, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}, R\right)=E_{c}^{\mathbf{t}}\left(\widetilde{Q}_{\ell} \widetilde{W}_{\ell}, R, \alpha\right)$ has converged to (23). The alternative possibility is that for all iterations $\ell=$ $0,1,2, \ldots$, it holds that

$$
\begin{aligned}
& F^{\mathbf{t}}\left(Q_{\ell} W_{\ell}, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}, R\right) \leq F^{\mathbf{t}}(\hat{Q} \hat{W}, \hat{Q} \hat{W}, R) \\
& \quad+F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}\right)-F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{\ell+1} \widetilde{W}_{\ell+1}\right)
\end{aligned}
$$

with all terms finite. Now, just like in [9, Lemma 1], it has to be true that
$\liminf _{\ell \rightarrow \infty}\left\{F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}\right)-F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{\ell+1} \widetilde{W}_{\ell+1}\right)\right\} \leq 0$,
because the divergences in (10) are non-negative (i.e., bounded from below). Therefore $F^{\mathbf{t}}\left(Q_{\ell} W_{\ell}, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}, R\right)$ must converge to $F^{\mathbf{t}}(\hat{Q} \hat{W}, \hat{Q} \hat{W}, R)$, yielding (23), and this concludes the proof of Theorem 1

## IV. Convergence of the iterative minimization for FIXED GRADIENT

Let us define for two real numbers $0 \leq \rho<1$ and $\eta \geq 0$

$$
\begin{array}{r}
F^{\mathbf{t}}(\rho, \eta, Q W, \widetilde{Q} \widetilde{W}) \triangleq D(W \| P \mid Q)-\rho I(Q, W) \\
+\eta \mathbb{E}_{Q}[f(X)]+(1-\rho) D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W}), \\
E_{0}^{\mathbf{t}}(\rho, \eta, \widetilde{Q} \widetilde{W}) \triangleq \min _{Q, W} F^{\mathbf{t}}(\rho, \eta, Q W, \widetilde{Q} \widetilde{W}) \tag{25}
\end{array}
$$

If finite, the quantity $E_{0}^{\mathbf{t}}(\rho, \eta, \widetilde{Q} \widetilde{W})$ has a meaning of the vertical axis intercept (" $E_{0}$ ") of a lower supporting plane in the variables $(R, \alpha)$ for the function $E(R, \alpha)=E_{c}^{\mathbf{t}}(\widetilde{Q} \widetilde{W}, R, \alpha)$, defined in (13), as the following lemma shows.

Lemma 2: For any $0 \leq \rho<1$ and $\eta \geq 0$ it holds that

$$
\begin{equation*}
E_{c}^{\mathbf{t}}(\widetilde{Q} \widetilde{W}, R, \alpha) \geq E_{0}^{\mathbf{t}}(\rho, \eta, \widetilde{Q} \widetilde{W})+\rho R-\eta \alpha \tag{26}
\end{equation*}
$$

and there exist $R \geq 0$ and $\alpha \geq \min _{x} f(x)$ which satisfy (26) with equality.

Proof: By definition (13)

$$
\begin{align*}
& \min _{\substack{Q, W: \\
\mathbb{E}_{Q}[f(X)] \leq \alpha}}\left\{D(W \| P \mid Q)+D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W})+\right. \\
& \left.\left|R-I(Q, W)-D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W})\right|^{+}\right\} \tag{27}
\end{align*}
$$

$$
\begin{align*}
& \stackrel{(a)}{\geq} \min _{\substack{Q, W: \\
\mathbb{E}_{Q}[f(X)] \leq \alpha}}\left\{D(W \| P \mid Q)+D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W})+\right. \\
& \left.\rho\left[R-I(Q, W)-D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W})\right]+\eta\left[\mathbb{E}_{Q}[f(X)]-\alpha\right]\right\} \\
& \geq \quad \min _{Q, W}\left\{D(W \| P \mid Q)+D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W})+\right. \\
& \left.\rho\left[R-I(Q, W)-D^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W})\right]+\eta\left[\mathbb{E}_{Q}[f(X)]-\alpha\right]\right\} \tag{28}
\end{align*}
$$

where (a) holds for any $0 \leq \rho<1$ and $\eta \geq 0$. Using (24) and (25), we see that the lower bound expression (28) is equal to the RHS of (26). Suppose (28) is finite. Let $Q_{\rho, \eta}, W_{\rho, \eta}$ denote distributions $Q, W$, respectively, which jointly minimize (28). Observe that for each $0 \leq \rho<1$ and $\eta \geq 0$ we can find $R \geq 0$ and $\alpha \geq \min _{x} f(x)$, such that the differences in the square brackets are zero. In this case, $Q_{\rho, \eta}$ will satisfy the input constraint and there will be equality between (28) and (27).

Lemma 3: Suppose $\widetilde{Q} \widetilde{W} \equiv \widetilde{T} \widetilde{V}$ is such that the minimum (25) is finite. If $t_{1}=t_{4}+1$ in (8), then, with definitions of $a \triangleq\left(t_{2}+t_{4}\right)(1-\rho)$ and $b \triangleq\left(t_{3}+t_{4}\right)(1-\rho), 0 \leq \rho<1$ and
$\eta \geq 0$, the unique minimizing solution of the minimum (25) can be written as

$$
\begin{align*}
& Q^{*}(x) W^{*}(y \mid x)=\frac{1}{K}\left[\widetilde{Q}^{1-\rho}(x) \widetilde{V}^{b}(x \mid y) P_{\eta}(x, y)\right]^{\frac{1}{b+1-\rho}} \\
& \times \widetilde{T}^{\frac{a}{a+1}}(y)\left\{\sum_{\tilde{x}}\left[\widetilde{Q}^{1-\rho}(\tilde{x}) \widetilde{V}^{b}(\tilde{x} \mid y) P_{\eta}(\tilde{x}, y)\right]^{\frac{1}{b+1-\rho}}\right\}^{\frac{b-a-\rho}{a+1}} \tag{29}
\end{align*}
$$

where $P_{\eta}(x, y) \triangleq e^{-\eta f(x)} P(y \mid x)$ and $K$ is a normalization constant, resulting in

$$
\begin{align*}
& E_{0}^{\mathbf{t}}(\rho, \eta, \widetilde{Q} \widetilde{W})=-(a+1) \log \sum_{y} \widetilde{T}^{\frac{a}{a+1}}(y) \times \\
& \left\{\sum_{x}\left[\widetilde{Q}^{1-\rho}(x) \widetilde{V}^{b}(x \mid y) P_{\eta}(x, y)\right]^{\frac{1}{b+1-\rho}}\right\}^{\frac{b+1-\rho}{a+1}} \tag{30}
\end{align*}
$$

If $t_{3}=t_{2}+\frac{\rho}{1-\rho}$ in (8), then, with $c \triangleq\left(t_{1}+t_{2}\right)(1-\rho)$ and a as defined above, $0<\rho<1$ and $\eta \geq 0$, the unique minimizing solution of the minimum (25) can be written as

$$
\begin{align*}
& Q^{*}(x) W^{*}(y \mid x)=\frac{1}{K}\left[\widetilde{W}^{a}(y \mid x) \widetilde{V}^{\rho}(x \mid y) P_{\eta}(x, y)\right]^{\frac{1}{a+1}} \\
& \times \widetilde{Q}^{\frac{c}{c+\rho}}(x)\left\{\sum_{\tilde{y}}\left[\widetilde{W}^{a}(\tilde{y} \mid x) \widetilde{V}^{\rho}(x \mid \tilde{y}) P_{\eta}(x, \tilde{y})\right]^{\frac{1}{a+1}}\right\}^{\frac{a+1-c-\rho}{c+\rho}} \tag{31}
\end{align*}
$$

where $P_{\eta}(x, y)$ is defined as above and $K$ is a normalization constant, resulting in

$$
\begin{align*}
& E_{0}^{\mathbf{t}}(\rho, \eta, \widetilde{Q} \widetilde{W})=-(c+\rho) \log \sum_{x} \widetilde{Q}^{\frac{c}{c+\rho}}(x) \times \\
& \left\{\sum_{y}\left[\widetilde{W}^{a}(y \mid x) \widetilde{V}^{\rho}(x \mid y) P_{\eta}(x, y)\right]^{\frac{1}{a+1}}\right\}^{\frac{a+1}{c+\rho}} \tag{32}
\end{align*}
$$

Proof: Similarly to [7, Lemma 3].
An iterative minimization procedure at a fixed gradient $(\rho, \eta), 0<\rho<1, \eta \geq 0$, is given by

$$
\begin{align*}
& Q_{\ell} W_{\ell}=\underset{Q, W}{\arg \min } F^{\mathbf{t}}\left(\rho, \eta, Q W, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}\right),  \tag{33}\\
& \widetilde{Q}_{\ell+1} \widetilde{W}_{\ell+1}=Q_{\ell} W_{\ell}, \quad \\
& \quad \ell=0,1,2, \ldots
\end{align*}
$$

We assume that the initial distribution $\widetilde{Q}_{Q} \widetilde{W}_{Q}$ in (33) is chosen such that the set $\left\{Q W: F_{1}^{\mathrm{t}}\left(Q W, Q_{0} W_{0}\right)_{\sim}<{ }_{\sim}+\infty\right\}$ is non-empty, which guarantees $F^{\mathbf{t}}\left(\rho, \eta, Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)=$ $E_{0}^{\mathbf{t}}\left(\rho, \eta, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty$. By (24) it is clear that (33) produces a monotonically non-increasing sequence $E_{0}^{\mathbf{t}}\left(\rho, \eta, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}\right)$, $\ell=0,1,2, \ldots$. Depending on the choice of the non-negative parameters $\left(t_{1}, t_{2}, t_{3}, t_{4}\right)$ in (8), the update of $Q_{\ell} W_{\ell}$ in (33) can be done according to the expression (29) with any $a \geq 0$ and $b \geq 0$, or according to (31) with any $a \geq 0$ and $c \geq 0$, with $\widetilde{Q}, \widetilde{V}, \widetilde{T}, \widetilde{W}$ replaced by $\widetilde{Q}_{\ell}, \widetilde{V}_{\ell}, \widetilde{T}_{\ell}, \widetilde{W}_{\ell}$, correspondingly. The choice of $a=b=0$ in (29) gives the fixed-slope counterpart of the algorithm in [7], analysed in [10]. The choice $(a, c)=(0,1)$ in (31) gives the fixed-slope counterpart
of the algorithm in [14]. The choice $(a, b)=(0, \rho)$ in (29), or, alternatively, $(a, c)=(0,1-\rho)$ in (31) gives the algorithm in [4], [5]. The main result of the section is given by the following theorem:

Theorem 2: Let $\left\{Q_{\ell} W_{\ell}\right\}_{\ell=0}^{+\infty}$ be a sequence of iterative solutions produced by (33). Then
where $E_{0}^{\mathrm{t}}(\rho, \eta, \widetilde{Q} \widetilde{W})$ is defined in (25) and $D^{\mathrm{t}}(\cdot, \cdot)$ in (8).

In order to prove Theorem 2, we use the following lemma:
Lemma 4: Let $\hat{Q} \hat{W}$ be such that $F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty$. Then

$$
\begin{align*}
& F^{\mathbf{t}}\left(\rho, \eta, Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right) \leq F^{\mathbf{t}}(\rho, \eta, \hat{Q} \hat{W}, \hat{Q} \hat{W}) \\
& \quad+(1-\rho)\left[F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)-F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{1} \widetilde{W}_{1}\right)\right] \tag{35}
\end{align*}
$$

Proof: Since $+\infty>F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)$, then also $+\infty>$ $F_{1}^{\mathbf{t}}\left(Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)$. Let $Q^{(\lambda)} W^{(\lambda)}$ be a convex combination of $\hat{Q} W$ and $Q_{0} W_{0}$, as in (17). Then the function $g(\lambda)=$ $F^{\mathbf{t}}\left(\rho, \eta, Q^{(\lambda)} W^{(\lambda)}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)$ is convex $(\cup)$ and differentiable in $\lambda \in(0,1)$. Since $Q_{0} W_{0}$ achieves the minimum of $F^{\mathbf{t}}\left(\rho, \eta, Q W, \widetilde{Q}_{0} \widetilde{W}_{0}\right)$ over $Q W$, then necessarily

$$
\lim _{\lambda \rightarrow 0} \frac{d g(\lambda)}{d \lambda} \geq 0
$$

Differentiation results in the following condition in the limit:

$$
\begin{align*}
& \quad F^{\mathbf{t}}\left(\rho, \eta, Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)+\rho D\left(\hat{T} \| T_{0}\right) \\
& \quad+(1-\rho)\left[D\left(\hat{W} \| W_{0} \mid \hat{Q}\right)+D^{\mathbf{t}}\left(\hat{Q} \hat{W} \| Q_{0} W_{0}\right)\right] \\
& \leq F^{\mathbf{t}}\left(\rho, \eta, \hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right) \tag{36}
\end{align*}
$$

where $\hat{T}$ and $T_{0}$ denote the $y$-marginal distributions of $\hat{Q} \hat{W}$ and $Q_{0} W_{0}$, respectively. Since $F_{1}^{\mathrm{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty$, then all terms in (36) are finite. On the other hand, by (24)

$$
\begin{align*}
& F^{\mathbf{t}}\left(\rho, \eta, \hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)= \\
& F^{\mathbf{t}}(\rho, \eta, \hat{Q} \hat{W}, \hat{Q} \hat{W})+(1-\rho) D^{\mathbf{t}}\left(\hat{Q} \hat{W} \| \widetilde{Q}_{0} \widetilde{W}_{0}\right) \tag{37}
\end{align*}
$$

Combining (37) with (36), noting that $Q_{0} W_{0}=\widetilde{Q}_{1} \widetilde{W}_{1}$, and omitting non-negative terms $(1-\rho) D\left(\hat{W} \| W_{0} \mid \hat{Q}\right) \geq 0$ and $\rho D\left(\hat{T} \| T_{0}\right) \geq 0$, we obtain a weaker inequality (35).

Proof of Theorem 2. Using (24), (25), it can be verified, that the RHS of (34) can be rewritten as

$$
\begin{align*}
& \min _{\widetilde{\widetilde{Q}}, \widetilde{W}_{:}} E_{0}^{\mathbf{t}}(\rho, \eta, \widetilde{Q} \widetilde{W})=\min _{Q, W:} F^{\mathbf{t}}(\rho, \eta, Q W, Q W) . \\
& D^{\mathbf{t}}\left(\widetilde{Q} \widetilde{W}, \widetilde{Q} \widetilde{Q}_{0} \widetilde{W}_{0}\right)<\infty \tag{38}
\end{align*}
$$

Suppose (38) is finite and let $\hat{Q} \hat{W}$ achieve the minimum on the RHS of (38). Then by Lemma 4 we conclude that for all iterations $\ell=0,1,2, \ldots$, it holds that

$$
\begin{aligned}
& F^{\mathbf{t}}\left(\rho, \eta, Q_{\ell} W_{\ell}, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}\right) \leq F^{\mathbf{t}}(\rho, \eta, \hat{Q} \hat{W}, \hat{Q} \hat{W}) \\
& +(1-\rho)\left[F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}\right)-F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{\ell+1} \widetilde{W}_{\ell+1}\right)\right]
\end{aligned}
$$

The conclusion of the proof is the same as in Theorem $1 \square$

The next two sections show convergence of fixed-slope computation in the directions of $R$ and $\alpha$, respectively. They are similar in structure to Section IV.

## V. CONVERGENCE FOR FIXED $\alpha$ AND $\rho$

In this section we show convergence of an iterative minimization at a fixed slope $\rho$ in the direction of $R$, i.e., for a given $\alpha$. With the help of (24) let us define $F^{\mathbf{t}}(\rho, Q W, \widetilde{Q} \widetilde{W}) \triangleq$ $\left.F^{\mathbf{t}}(\rho, \eta, Q W, \widetilde{Q} \widetilde{W})\right|_{\eta=0}$ and

$$
\begin{equation*}
E_{0}^{\mathbf{t}}(\rho, \widetilde{Q} \widetilde{W}, \alpha) \triangleq \min _{\substack{Q, W: \\ \mathbb{E}_{Q}[f(X)] \leq \alpha}} F^{\mathbf{t}}(\rho, Q W, \widetilde{Q} \widetilde{W}) \tag{39}
\end{equation*}
$$

Here $E_{0}^{\mathbf{t}}(\rho, \widetilde{Q} \widetilde{W}, \alpha)$ plays a role of " $E_{0}$ " of a supporting line in the variable $R$ of the function $E(R)=E_{c}^{\mathbf{t}}(\widetilde{Q} \widetilde{W}, R, \alpha)$, defined in (13), as shown by the following lemma.

Lemma 5: For any $0 \leq \rho<1$ it holds that

$$
\begin{equation*}
E_{c}^{\mathbf{t}}(\widetilde{Q} \widetilde{W}, R, \alpha) \geq E_{0}^{\mathbf{t}}(\rho, \widetilde{Q} \widetilde{W}, \alpha)+\rho R \tag{40}
\end{equation*}
$$

and there exists $R \geq 0$ which satisfies (40) with equality. Proof: Similar to Lemma 2

An iterative minimization procedure at a fixed slope $\rho$ is given by

$$
\begin{align*}
Q_{\ell} W_{\ell} & \in \quad \underset{\substack{Q, W: \\
\mathbb{E}_{Q}[f(X)] \leq \alpha}}{\arg \min } F^{\mathbf{t}}\left(\rho, Q W, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}\right), \\
\widetilde{Q}_{\ell+1} \widetilde{W}_{\ell+1} & =\quad Q_{\ell} W_{\ell}, \quad  \tag{41}\\
& \quad \ell=0,1,2, \ldots
\end{align*}
$$

It is assumed that $\widetilde{Q}_{0} \widetilde{W}_{0}$ in (41) is chosen such that the set $\left\{Q W: \sum_{x} Q(x) f(x) \leq \alpha, F_{1}^{\mathbf{t}}\left(Q \underset{\sim}{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty\right\}$ is nonempty, so that $F^{\mathbf{t}}\left(\rho, Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)=\widetilde{Q}_{c}^{\mathbf{t}}\left(\rho, \widetilde{Q}_{0} \widetilde{W}_{0}, \alpha\right)<$ $+\infty$. By the definition of $F^{\mathbf{t}}(\rho, Q W, \widetilde{Q} \widetilde{W})$ according to (24), this procedure results in a monotonically non-increasing sequence $E_{0}^{\mathbf{t}}\left(\rho, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}, \alpha\right), \ell=0,1,2, \ldots$. The main result of this section is stated in the following theorem.

Theorem 3: Let $\left\{Q_{\ell} W_{\ell}\right\}_{\ell=0}^{+\infty}$ be a sequence of iterative solutions produced by (47). Then

$$
\begin{equation*}
E_{0}^{\mathbf{t}}\left(\rho, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}, \alpha\right){ }^{\ell \rightarrow \infty} \min _{\substack{\widetilde{Q} \\ D^{\mathbf{t}}\left(\widetilde{Q} \tilde{W}, \widetilde{W}_{0} \\ \tilde{W}_{0} \\ \hline\right.}} E_{0}^{\mathbf{t}}(\rho, \widetilde{Q} \widetilde{W}, \alpha), \tag{42}
\end{equation*}
$$

where $E_{0}^{\mathbf{t}}(\rho, \widetilde{Q} \widetilde{W}, \alpha)$ is defined in (39) and $D^{\mathbf{t}}(\cdot, \cdot)$ in (8).

To prove Theorem 3, we use a lemma, similar to Lemma 4
Lemma 6: Let $\hat{Q} \hat{W}$ be such, that $\sum_{x} \hat{Q}(x) f(x) \leq \alpha$ and $F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty$. Then

$$
\begin{align*}
& F^{\mathbf{t}}\left(\rho, Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}\right) \leq F^{\mathbf{t}}(\rho, \hat{Q} \hat{W}, \hat{Q} \hat{W}) \\
& \quad+(1-\rho)\left[F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)-F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{1} \widetilde{W}_{1}\right)\right] . \tag{43}
\end{align*}
$$

Proof: Analogous to Lemma 4.
Proof of Theorem (3): The RHS of (42) can be rewritten in terms of $F^{\mathbf{t}}(\rho, Q W, \widetilde{Q} \widetilde{W})$ as:

$$
\quad \min _{\substack{\widetilde{Q}, \widetilde{W}: \\ D^{\mathbf{t}}\left(\widetilde{Q} \widetilde{W}, \widetilde{Q_{0}} \tilde{W}_{0}\right)<\infty}} E_{0}^{\mathbf{t}}(\rho, \widetilde{Q} \widetilde{W}, \alpha)=\min _{\substack{Q, W: \\ \mathbb{E}_{Q}[f(X)] \leq \alpha \\ \\ \\ D^{\mathbf{t}}\left(Q W, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<\infty}} F^{\mathbf{t}}(\rho, Q W, Q W) .
$$

Suppose (44) is finite and $\hat{Q} \hat{W}$ achieves the minimum on the RHS. Then we can use Lemma 6 with $\hat{Q} \hat{W}$. The rest of the proof is the same as for Theorem 2

## VI. Convergence for fixed $R$ and $\eta$

In this section we show convergence of an iterative minimization at a fixed slope $\eta$ in the direction of $\alpha$, i.e., for a given $R$. Let us define

$$
\begin{align*}
F^{\mathbf{t}}(\eta, Q W, \widetilde{Q} \widetilde{W}, R) \triangleq & \max \left\{F_{1}^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W}), F_{2}(Q W, R)\right\} \\
& +\eta \mathbb{E}_{Q}[f(X)] \tag{45}
\end{align*}
$$

where $F_{1}^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W})$ and $F_{2}(Q W, R)$ are as defined in (10) and (11), respectively.

$$
\begin{equation*}
E_{0}^{\mathbf{t}}(\eta, \widetilde{Q} \widetilde{W}, R) \triangleq \min _{Q, W} F^{\mathbf{t}}(\eta, Q W, \widetilde{Q} \widetilde{W}, R) \tag{46}
\end{equation*}
$$

Here $E_{0}^{\mathbf{t}}(\eta, \widetilde{Q} \widetilde{W}, R)$ plays a role of " $E_{0}$ " of a supporting line in the variable $\alpha$ of the function $E(\alpha)=E_{c}^{\mathbf{t}}(\widetilde{Q} \widetilde{W}, R, \alpha)$, defined in (13), as shown by the following lemma.

Lemma 7: For any $\eta \geq 0$ it holds that

$$
\begin{equation*}
E_{c}^{\mathbf{t}}(\widetilde{Q} \widetilde{W}, R, \alpha) \geq E_{0}^{\mathbf{t}}(\eta, \widetilde{Q} \widetilde{W}, R)-\eta \alpha \tag{47}
\end{equation*}
$$

and there exists $\alpha \geq \min _{x} f(x)$ which satisfies (47) with equality.
Proof: Similar to Lemma 2,
An iterative minimization procedure at a fixed slope $\eta$ is defined as follows.

$$
\begin{align*}
& Q_{\ell} W_{\ell} \in \underset{Q, W}{\arg \min } F^{\mathbf{t}}\left(\eta, Q W, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}, R\right)  \tag{48}\\
& \widetilde{Q}_{\ell+1} \widetilde{W}_{\ell+1}=Q_{\ell} W_{\ell}, \\
& \ell=0,1,2, \ldots
\end{align*}
$$

It is assumed that the set $\left\{Q W: F_{1}^{\mathbf{t}}\left(Q W, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty\right\}$ is non-empty, which guarantees $F^{\mathbf{t}}\left(\eta, Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}, R\right)=$ $E_{0}^{\mathbf{t}}\left(\eta, \widetilde{Q}_{0} \widetilde{W}_{0}, R\right)<+\infty$. The iterative procedure results in a monotonically non-increasing sequence $E_{0}^{\mathbf{t}}\left(\eta, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}, R\right)$,
$\ell=0,1,2, \ldots$, as can be seen from (45), (46). The sequence converges to the global minimum in the set $\{\widetilde{Q} \widetilde{W}$ : $\left.D^{\mathbf{t}}\left(\widetilde{Q} \widetilde{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty\right\}$, as stated in the following theorem.

Theorem 4: Let $\left\{Q_{\ell} W_{\ell}\right\}_{\ell=0}^{+\infty}$ be a sequence of iterative solutions produced by (48). Then

$$
\begin{equation*}
E_{0}^{\mathbf{t}}\left(\eta, \widetilde{Q}_{\ell} \widetilde{W}_{\ell}, R\right) \stackrel{\ell \rightarrow \infty}{\searrow} \min _{\substack{\widetilde{Q}, \widetilde{W}_{i} \\ D^{\mathbf{t}}\left(\widetilde{Q} \tilde{W}, \widetilde{Q}_{0} \tilde{W}_{0}\right)<\infty}} E_{0}^{\mathbf{t}}(\eta, \widetilde{Q} \widetilde{W}, R) \tag{49}
\end{equation*}
$$

where $E_{0}^{\mathbf{t}}(\eta, \widetilde{Q} \widetilde{W}, R)$ is defined in (46) and $D^{\mathbf{t}}(\cdot, \cdot)$ in (8).
To prove this theorem, we use a lemma, which is similar to Lemma 1

Lemma 8: Let $\hat{Q} \hat{W}$ be such that $F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)<+\infty$. Then

$$
\begin{align*}
& F^{\mathbf{t}}\left(\eta, Q_{0} W_{0}, \widetilde{Q}_{0} \widetilde{W}_{0}, R\right) \leq F^{\mathbf{t}}(\eta, \hat{Q} \hat{W}, \hat{Q} \hat{W}, R) \\
& +\left|F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{0} \widetilde{W}_{0}\right)-F_{1}^{\mathbf{t}}\left(\hat{Q} \hat{W}, \widetilde{Q}_{1} \widetilde{W}_{1}\right)\right|^{+} \tag{50}
\end{align*}
$$

Proof: Similar to Lemma 1
Proof of Theorem (4. The RHS of (49) can be rewritten in terms of $F^{\mathbf{t}}(\eta, Q W, \widetilde{Q} \widetilde{W}, R)$ as:

Suppose (51) is finite, and let $\hat{Q} \hat{W}$ achieve the minimum on the RHS. Then we can use Lemma 8 with $\hat{Q} \hat{W}$. The rest of the proof is the same as for Theorem 1

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[^0]:    ${ }^{1}$ The expression gives zero for the rates $R \leq \max _{Q} I(Q, P)$.

[^1]:    ${ }^{2}$ Note that (14) is not just an alternating minimization procedure w.r.t. $F^{\mathbf{t}}(Q W, \widetilde{Q} \widetilde{W}, R)$, or not the only one possible, in a sense that other choices of $\widetilde{Q}_{\ell+1} \widetilde{W}_{\ell+1}$ may also minimize $F^{\mathbf{t}}\left(Q_{\ell} W_{\ell}, \cdot, R\right)$.

