Characterization of Conditional Independence and Weak Realizations of Multivariate Gaussian Random Variables: Applications to Networks

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Abstract— The Gray and Wyner lossy source coding for a simple network for sources that generate a tuple of jointly Gaussian random variables (RVs) $X_1 : \Omega \to \mathbb{R}^{p_1}$ and $X_2 : \Omega \to \mathbb{R}^{p_2}$, with respect to square-error distortion at the two decoders is reexamined using (1) Hotelling's geometric approach of Gaussian RVs-the canonical variable form, and (2) van Putten's and van Schuppen's parametrization of joint distributions $P_{X_1,X_2,W}$ by Gaussian RVs $W: \Omega \to \mathbb{R}^n$ which make (X_1, X_2) conditionally independent, and the weak stochastic realization of (X_1, X_2) . Item (2) is used to parametrize the lossy rate region of the Gray and Wyner source coding problem for joint decoding with mean-square error distortions $\mathbf{E}\{||X_i - \hat{X}_i||_{\mathbb{R}^{p_i}}^2\} \leq \Delta_i \in$ $[0,\infty], i=1,2$, by the covariance matrix of RV W. From this then follows Wyner's common information $C_W(X_1, X_2)$ (information definition) is achieved by W with identity covariance matrix, while a formula for Wyner's lossy common information (operational definition) is derived, given by $C_{WL}(X_1, X_2) = C_W(X_1, X_2) =$ $\frac{1}{2}\sum_{j=1}^{n}\ln\left(\frac{1+d_{j}}{1-d_{j}}\right), \text{ for the distortion region } 0 \leq \Delta_{1} \leq \sum_{j=1}^{n}(1-d_{j}),$ $0 \le \Delta_2 \le \sum_{j=1}^{n} (1-d_j)$, and where $1 > d_1 \ge d_2 \ge ... \ge d_n > 0$ in (0,1) are the canonical correlation coefficients computed from the canonical variable form of the tuple (X_1, X_2) .

The methods are of fundamental importance to other problems of multi-user communication, where conditional independence is imposed as a constraint.

I. INTRODUCTION, MAIN CONCEPTS, LITERATURE, MAIN RESULTS

In information theory and communications an important class of theoretical and practical problems is of a multi-user nature, such as, lossless and lossy network source coding for data compression over noiseless channels, network channel coding for data transmission over noisy channels [1], and secure communication [2]. A sub-class of network source coding problems deals with two sources that generate at each time instant, symbols that are stationary memoryless, multivariate, and jointly Gaussian distributed, and similarly for network channel coding problems, i.e., Gaussian multiple access channels (MAC) with two or more multivariate correlated sources and a multivariate output.

In this paper we show the relevance of *three fundamental* concepts of statistics and probability to the network problems discussed above found in the report by Charalambous and van Schuppen [3] that involve a tuple of multivariate jointly independent and identically distributed multivariate Gaussian random variables (RVs) $(X_1^N, X_2^N) = \{(X_{1,i}, X_{2,i}):$

$$i = 1, 2, \dots, N \},$$

$$X_{1,i} : \Omega \to \mathbb{R}^{p_1} = \mathbb{X}_1, \quad X_{2,i} : \Omega \to \mathbb{R}^{p_2} = \mathbb{X}_2, \quad \forall i, \quad (1)$$

$$\mathbf{P}_{X_{1,i}X_{2,i}} = \mathbf{P}_{X_1,X_2} \quad \text{jointly Gaussian and}$$

$$(X_{1,i},X_{2,i}) \quad \text{indep. of } (X_{1,j},X_{2,j}), \quad \forall i \neq j$$
(2)

We illustrate their application to the calculation of rates that lie in the Gray and Wyner rate region [4] of *the simple network* shown in Fig. 1, with respect to the average squareerror distortions at the two decoders

$$\mathbf{E}\left\{D_{X_i}(X_i^N, \hat{X}_i^N)\right\} \le \Delta_i, \quad \Delta_i \in [0, \infty], \quad i = 1, 2, \qquad (3)$$

$$D_{X_i}(x_i^N, \hat{x}_i^N) \stackrel{\triangle}{=} \frac{1}{N} \sum_{j=1}^N ||x_{i,j} - \hat{x}_{i,j}||_{\mathbb{R}^{p_i}}^2, \quad i = 1, 2,$$
(4)

and where $|| \cdot ||_{\mathbb{R}^{p_i}}^2$ are Euclidean distances on \mathbb{R}^{p_i} , i = 1, 2. The rest of this section and the remaining of the paper is organized as follows.

In Section I-A we introduced the three concepts which are further described in Charalambous and van Schuppen [3], in Sections I-B-I-C we recall the Gray and Wyner characterization of the rate region [4], and the characterization of the minimum lossy common message rate on the Gray and Wyner rate region due to Viswanatha, Akyol and Rose [5], and Xu, Liu, and Chen [6]. In Section II we present our main results in the form of theorems. In Section III we give the proofs of the main theorems, while citing [3] if necessary.

A. Three Concepts of Statistics and Probability

Notation. An \mathbb{R}^n -valued Gaussian RV, denoted by $X \in G(m_X, Q_X)$, with as parameters the mean value $m_X \in \mathbb{R}^n$ and the variance $Q_X \in \mathbb{R}^{n \times n}$, $Q_X = Q_X^T \ge 0$, is a function $X : \Omega \to \mathbb{R}^n$ which is a RV and such that the measure of this RV equals a Gaussian measure described by its characteristic function. This definition includes $Q_X = 0$.

The *effective dimension* of the RV is denoted by $\dim(X) = \operatorname{rank}(Q_X)$. An $n \times n$ identity matrix is denoted by I_n .

A tuple of Gaussian RVs (X_1, X_2) will be denoted this way to save space, rather than by

$$\left(\begin{array}{c}X_1\\X_2\end{array}\right).$$

Then the variance matrix of this tuple is denoted by

$$\begin{aligned} & (X_1, X_2) \in G(0, \mathcal{Q}_{(X_1, X_2)}), \\ & \mathcal{Q}_{(X_1, X_2)} = \begin{pmatrix} \mathcal{Q}_{X_1} & \mathcal{Q}_{X_1, X_2} \\ \mathcal{Q}_{X_1, X_2}^T & \mathcal{Q}_{X_2} \end{pmatrix} \in \mathbb{R}^{(p_1 + p_2) \times (p_1 + p_2)}. \end{aligned}$$

The variance $Q_{(Y_1,Y_2)}$ is distinguished from $Q_{Y_1,Y_2} \in \mathbb{R}^{p_1 \times p_2}$.

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Fig. 1: The Gray and Wyner source coding for a simple network [4] $(X_{1,i}, X_{2,i}) \sim \mathbf{P}_{X_1, X_2}, i = 1, \dots, N$.

The first concept is Hotelling's [7] geometric approach to Gaussian RVs [8], [9], where the underlying geometric object of a Gaussian RV $Y : \Omega \to \mathbb{R}^p$ is the σ -algebra \mathscr{F}^Y generated by Y. A basis transformation of such a RV is then the transformation defined by a non-singular matrix $S \in \mathbb{R}^{p \times p}$, and it then directly follows that $\mathscr{F}^Y = \mathscr{F}^{SY}$. For the tuple of jointly Gaussian multivariate RVs (X_1, X_2) , a basis transformation of this tuple consists of a matrix composed of two square and non-singular matrices, (S_1, S_2) (see [3, Algorithm 2.10]),

$$S \stackrel{\triangle}{=} \text{Block-diag}(S_1, S_2), \quad X_1^c \stackrel{\triangle}{=} S_1 X, \quad X_2^c \stackrel{\triangle}{=} S_2 X_2, \quad (5)$$

$$\mathscr{F}^{X_1} = \mathscr{F}^{S_1 X_1}, \quad \mathscr{F}^{X_2} = \mathscr{F}^{S_2 X_2}. \tag{6}$$

S maps (X_1, X_2) into the so-called canonical form of the tuple of RVs (the full specification is given in [3, Section 2.2, Definition 2.2]), which identifies identical, correlated, and private information, as interpreted in the table below,

$X_{11}^c = X_{21}^c - a.s.$	<i>identical information</i> of X_1^c and X_2^c
X_{12}^{c}	<i>correlated information</i> of X_1^c w.r.t X_2^c
X_{13}^{c}	private information of X_1^c w.r.t X_2^c
$X_{21}^c = X_{11}^c - a.s.$	<i>identical information</i> of X_1^c and X_2^c
X_{22}^{c}	<i>correlated information</i> of X_2^c w.r.t X_1^c

where

$$X_{ii}^c: \Omega \to \mathbb{R}^{p_{ij}}, \ i = 1, 2, \ j = 1, 2, 3,$$
 (7)

$$p_{11} = p_{21}, \quad p_{12} = p_{22} = n,$$
 (8)

$$p_1 = p_{11} + p_{12} + p_{13}, \quad p_2 = p_{21} + p_{22} + p_{23},$$
 (9)

$$S_1 X_1 = (X_{11}^c, X_{12}^c, X_{13}^c), \quad S_2 X_2 = (X_{21}^c, X_{22}^c, X_{23}^c), \tag{10}$$

$$X_{11}^c = X_{21}^c - a.s., \quad X_{11}^c, X_{21}^c \in G(0, I_{p_{11}}), \tag{11}$$

$$X_{13}^c \in G(0, I_{p_{13}})$$
 and $X_{23}^c \in G(0, I_{p_{23}})$ are independent (12)

$$X_{12}^c \in G(0, I_{p_{12}})$$
 and $X_{22}^c \in G(0, I_{p_{22}})$ are correlated, (13)

$$\mathbf{E}[X_{12}^c(X_{22}^c)^T] = D = \text{Diag}(d_1, \dots, d_{p_{12}}), d_i \in (0, 1) \ \forall i. \ (14)$$

The entries of *D* are called *the canonical correlation coefficients*. For $X_{11}^c = X_{21}^c - a.s.$ the term *identical information* is used. The linear transformation $S = \text{Block-diag}(S_1, S_2)$ is equivalent to a pre-processing of (X_1, X_2) by a linear preencoder (see [3] for applications to network problems).

The expression of mutual information between X_1 and X_2 , denoted by $I(X_1;X_2)$, as a function of the canonical correlation coefficients, discussed in [10] is given in Theorem 2.1.

The second concept is van Putten's and van Schuppen's [11] parametrization of the family of all jointly Gaussian probability distributions $\mathbf{P}_{X_1,X_2,W}$ by an auxiliary Gaussian RV $W: \Omega \to \mathbb{R}^k = \mathbb{W}$ that makes X_1 and X_2 conditional independent, defined by

$$\mathcal{P}^{CIG} \stackrel{\triangle}{=} \left\{ \mathbf{P}_{X_1, X_2, W} \middle| \quad \mathbf{P}_{X_1, X_2 | W} = \mathbf{P}_{X_1 | W} \mathbf{P}_{X_2 | W}, \\ \text{the } \mathbb{X}_1 \times \mathbb{X}_2 - \text{marginal dist. of } \mathbf{P}_{X_1, X_2, W} \text{ is the fixed dist.} \\ \mathbf{P}_{X_1, X_2}, \text{ and } (X_1, X_2, W) \text{ is jointly Gaussian} \right\}$$
(15)

and its subset \mathscr{P}_{min}^{CIG} of the set \mathscr{P}^{CIG} , with the additional constraint that the dimension of the RV *W* is minimal while all other conditions hold. The parametrization is in terms of a set of matrices. Consequences are found in [3, Section 2.3].

The third concept is the weak stochastic realization of RVs (X_1, X_2, W) that induces distributions $\mathbf{P}_{X_1, X_2, W}$ in the sets \mathcal{P}^{CIG} and \mathcal{P}^{CIG}_{min} (see [11, Def. 2.17 and Prop. 2.18] and [3, Def. 2.17 and Prop. 2.18]).

Theorem 2.2 (our main theorem) gives as a special case (part (d)) an achievable lower bound on Wyner's single letter *in-formation theoretic characterization of common information*:

$$C_W(X_1, X_2) \stackrel{\triangle}{=} \inf_{\mathbf{P}_{X_1, X_2, W}: \mathbf{P}_{X_1, X_2 \mid W} = \mathbf{P}_{X_1 \mid W} \mathbf{P}_{X_2 \mid W}} I(X_1, X_2; W) \quad (16)$$

and the weak stochastic realization of RVs (X_1, X_2, W) that induce distributions $\mathbf{P}_{X_1, X_2, W}$ in the sets \mathscr{P}^{CIG} and \mathscr{P}^{CIG}_{min} .

B. The Gray and Wyner Lossy Rate Region

Now, we describe our results with respect to the fundamental question posed by Gray and Wyner [4] for the simple network shown in Fig. 1, which is: determine which channel capacitity triples (C_0, C_1, C_2) are necessary and sufficient for each sequence (X_1^N, X_2^N) to be reliably reproduced at the intended decoders, while satisfying the average distortions with respect to single letter distortion functions $D_{X_i}(x_i^N, \hat{x}_i^N) \stackrel{\triangle}{=} \frac{1}{N} \sum_{t=1}^n d_{X_i}(x_{i,t} \hat{x}_{i,t}), i = 1, 2$. Gray and Wyner characterized the *operational rate region*, denoted by $\mathscr{R}_{GW}(\Delta_1, \Delta_2)$ by a coding scheme that uses the auxiliary RV $W : \Omega \to W$, as described below. Define the family of probability distributions

$$\mathscr{P} \triangleq \left\{ \begin{array}{l} \mathbf{P}_{X_1, X_2, W}, \quad x_1 \in \mathbb{X}_1, \ x_2 \in \mathbb{X}_2, \ w \in \mathbb{W} \\ \mathbf{P}_{X_1, X_2, W}(x_1, x_2, \infty) = \mathbf{P}_{X_1, X_2} \end{array} \right\}$$

for some auxiliary random variable W.

Theorem 8 in [4]: Let $\mathscr{R}_{GW}(\Delta_1, \Delta_2)$ denote the Gray and Wyner rate region. Suppose there exists $\hat{x}_i \in \hat{\mathbb{X}}_i$ such that $\mathbf{E}\{d_{X_i}(X_i, \hat{x}_i)\} < \infty, i = 1, 2$. For each $\mathbf{P}_{X_1, X_2, W} \in \mathscr{P}$ and $\Delta_1 \ge 0, \Delta_2 \ge 0$, define the subset of Euclidean 3–D space

$$\mathscr{R}_{GW}^{\mathbf{P}_{X_1,X_2,W}}(\Delta_1,\Delta_2) = \left\{ \left(R_0, R_1, R_2 \right) : R_0 \ge I(X_1, X_2; W), \\ R_1 \ge R_{X_1|W}(\Delta_1), R_2 \ge R_{X_2|W}(\Delta_2) \right\}$$
(17)

where $R_{X_i|W}(\Delta_i)$ is rate distortion function (RDF) of X_i , conditioned on W, at decoder i, i = 1, 2, and $R_{X_1,X_2}(\Delta_1, \Delta_2)$ is the joint RDF of joint decoding of (X_1, X_2) . Let

$$\mathscr{R}_{GW}^{*}(\Delta_{1},\Delta_{2}) \stackrel{\triangle}{=} \left(\bigcup_{\mathbf{P}_{X_{1},X_{2},W} \in \mathscr{P}} \mathscr{R}_{GW}^{\mathbf{P}_{X_{1},X_{2},W}}(\Delta_{1},\Delta_{2})\right)^{c}$$
(18)

where $(\cdot)^{c}$ denotes the closure of the indicated set. Then the achievable Gray-Wyner lossy rate region is given by

$$\mathscr{R}_{GW}(\Delta_1, \Delta_2) = \mathscr{R}_{GW}^*(\Delta_1, \Delta_2).$$
(19)

By [4, Theorem 6] if $(R_0, R_1, R_2) \in \mathscr{R}_{GW}(\Delta_1, \Delta_2)$, then

$$R_0 + R_1 + R_2 \ge R_{X_1, X_2}(\Delta_1, \Delta_2), \tag{20}$$

$$R_0 + R_1 \ge R_{X_1}(\Delta_1), \quad R_0 + R_2 \ge R_{X_2}(\Delta_2)$$
 (21)

(20) is called the *Pangloss Bound* of $\mathscr{R}_{GW}(\Delta_1, \Delta_2)$, and the set of triples $(R_0, R_1, R_2) \in \mathscr{R}_{GW}(\Delta_1, \Delta_2)$ that satisfy $R_0 + R_1 + R_2 = R_{X_1, X_2}(\Delta_1, \Delta_2)$ the *Pangloss Plane*.

Theorem 2.2 is our main theorem for set up (1)-(4). From this theorem follows Proposition 2.3 that parametrizes the region $\mathscr{R}_{GW}(\Delta_1, \Delta_2)$ by a Gaussian RV *W*, and the weak stochastic realization of the joint distribution of (X_1, X_2, W) .

C. Wyner's Lossy Common Information

Viswanatha, Akyol, and Rose [5], and Xu, Liu, and Chen [6], characterized the minimum lossy common message rate on the rate region $\Re_{GW}(\Delta_1, \Delta_2)$, as follows.

Theorem 4 in [6]: Let $C_{GW}(X_1, X_2; \Delta_1, \Delta_2)$ denote the minimum common message rate R_0 on the Gray and Wyner lossy rate region $\mathscr{R}_{GW}(\Delta_1, \Delta_2)$, with sum rate not exceeding the joint rate distortion function $R_{X_1,X_2}(\Delta_1, \Delta_2)$.

Then $C_{GW}(X_1, X_2; \Delta_1, \Delta_2)$ is characterized by

$$C_{GW}(X_1, X_2; \Delta_1, \Delta_2) \stackrel{\triangle}{=} \inf I(X_1, X_2; W)$$
(22)

such that the following identity holds

$$R_{X_1|W}(\Delta_1) + R_{X_2|W}(\Delta_2) + I(X_1, X_2; W) = R_{X_1, X_2}(\Delta_1, \Delta_2)$$
(23)

where the infimum is over all RVs W in \mathbb{W} , which parametrize the source distribution via $\mathbf{P}_{X_1,X_2,W}$, having a $\mathbb{X}_1 \times \mathbb{X}_2$ -marginal source distribution \mathbf{P}_{X_1,X_2} , and induce joint distributions $\mathbf{P}_{W,X_1,X_2,\hat{X}_1,\hat{X}_2}$ which satisfy the constraint.

 $C_{GW}(X_1, X_2; \Delta_1, \Delta_2)$ is also given the interpretation of Wyner's lossy common information, due to its operational meaning [5], [6]. We should mention that from Appendix B in [6] it follows that a necessary condition for the equality

constraint (23) is $R_{X_1,X_2|W}(\Delta_1,\Delta_2) = R_{X_1|W}(\Delta_1) + R_{X_2|W}(\Delta_2)$, and sufficient condition for this equality to hold is the conditional independence condition [6]: $\mathbf{P}_{X_1,X_2|W} = \mathbf{P}_{X_1|W}\mathbf{P}_{X_2|W}$. Hence, a sufficient condition for any rate $(R_0,R_1,R_2) \in \mathscr{R}_{GW}(\Delta_1,\Delta_2)$ to lie on the Pangloss plane, i.e., to satisfy (23) is the conditional independence.

It is shown in [5], [6], that there exists a distortion region $\mathscr{D}_W \subseteq [0,\infty] \times [0,\infty]$ such that $C_{GW}(X_1,X_2;\Delta_1,\Delta_2) = C_W(X_1,X_2)$, i.e., it is independent of the distortions (Δ_1,Δ_2) , i.e. it equals the Wyner's information theoretic characterization of common information defined by (16).

From Theorem 2.2 follows Theorem 2.4 that gives the closed form expression of $C_{GW}(X_1, X_2; \Delta_1, \Delta_2) = C_W(X_1, X_2)$ and identifies the region \mathcal{D}_W , for the multivariate Gaussian RVs (X_1, X_2) with respect to the avarage distortions (1)-(4).

II. MAIN RESULTS

Given the tuple of multivariate Gaussian RVs and distortion functions (1)-(4), the main contributions of the paper are: (1) the theorem and the proof of Wyner's common information (information definition). The existing proof of this result in [12] is incomplete (see discussion below Theorem 2.2).

(2) Paremetrization of rate triples $(R_0, R_1, R_2) \in \mathscr{R}_{GW}(\Delta_1, \Delta_2)$, and Wyner's lossy common information.

Below we state the expression of mutual information as a function of the canonical correlation coefficients, discussed in Gelfand and Yaglom [10].

Theorem 2.1: Consider a tuple of multivariable jointly Gaussian RVs $X_1 : \Omega \to \mathbb{R}^{p_1}, X_2 : \Omega \to \mathbb{R}^{p_2}, (X_1, X_2) \in$ $G(0, Q_{(X_1, X_2)})$. Compute the canonical variable form of the tuple of Gaussian RVs according to Algorithm 2.2 of [3]. This yields the indices $p_{11} = p_{21}, p_{12} = p_{22}, p_{13}, p_{23}$, and $n = p_{11} + p_{12} = p_{21} + p_{22}$ and the diagonal matrix D with canonical correlation coefficients $d_i \in (0,1)$ for i = 1, ..., n(as in [3, Definition 2.2]).

Then mutual information $I(X_1; X_2)$ is given by the formula,

$$I(X_1;X_2) = \begin{cases} 0, & 0 = p_{11} = p_{12}, \\ -\frac{1}{2} \sum_{i=1}^n \ln\left(1 - d_i^2\right), & 0 = p_{11}, p_{12} > 0, \\ \infty, & p_{11} > 0 \end{cases}$$

where d_i are the canonical correlation coefficients.

 $I(X_1;X_2)$ is a generalization of the well-known formula of a tuple of scalar RVs, i.e., $p_1 = p_2 = 1$, $I(X_1;X_2) = -\frac{1}{2}\ln(1-\rho^2)$, where $\rho \stackrel{\triangle}{=} \mathbf{E}\{X_1X_2\} \in [-1,1]$ is the correlation coefficient.

The case $p_{11} = p_{21} > 0$ gives $I(X_1; X_2) = +\infty$; if such components are present they should be removed. Hence, we state the next theorem under the restriction $p_{11} = p_{21} = 0$.

Theorem 2.2: Consider a tuple of multivariable jointly Gaussian RVs $X_1 : \Omega \to \mathbb{R}^{p_1}, X_2 : \Omega \to \mathbb{R}^{p_2}, (X_1, X_2) \in G(0, Q_{(X_1, X_2)})$ and without loss of generality assume S = Block-diag (S_1, S_2) produces a canonical variable form such that $p_{11} = p_{21} = 0$ (see [3, Definition 2.2]).

For any joint distribution $\mathbf{P}_{X_1,X_2,W}$ parametrized by an arbitrary RV $W: \Omega \to \mathbb{R}^k$ with fixed marginal distribution $\mathbf{P}_{X_1,X_2} = G(0, Q_{(X_1,X_2)})$ the following hold.

(a) The mutual information $I(X_1, X_2; W)$ satisfies

$$I(X_1, X_2; W) = I(X_{12}^c, X_{22}^c; W), \quad p_{12} = p_{22} = n$$

$$\geq H(X_{12}^c, X_{22}^c) - H(X_{12}^c|W) - H(X_{22}^c|W),$$
(24)
(25)

$$= \frac{1}{2} \sum_{i=1}^{n} \ln(1 - d_i^2) - \frac{1}{2} \ln(\det([I - D^{1/2}Q_W^{-1}D^{1/2}][I - D^{1/2}Q_WD^{1/2}])) \quad (26)$$

where the lower bound is parametrized by $Q_W \in \mathbf{Q}_W$,

$$\mathbf{Q}_{\mathbf{W}} = \left\{ Q_{W} \in \mathbb{R}^{n \times n} | \ Q_{W} = Q_{W}^{T}, \ 0 < D \le Q_{W} \le D^{-1} \right\}$$
(27)

and such that $\mathbf{P}_{X_{11}^c, X_{22}^c, W}$ is jointly Gaussian.

(b) The lower bound in (25) is achieved if $\mathbf{P}_{X_1,X_2,W}$ is jointly Gaussian and $\mathbf{P}_{X_{12}^c,X_{22}^c|W} = \mathbf{P}_{X_{12}^c|W}\mathbf{P}_{X_{22}^c|W}, W : \Omega \to \mathbb{R}^n$, and a realization of the RVs (X_{12}^c,X_{22}^c) which achieves the lower bound is

$$X_{12}^c = Q_{X_{12}^c, W} Q_W^{-1} W + Z_1, (28)$$

$$Q_{X_{12}^c,W} = D^{1/2}, \ Z_1 \in G(0, (I - D^{1/2}Q_W^{-1}D^{1/2})),$$
 (29)

$$X_{22}^c = Q_{X_{22}^c, W} Q_W^{-1} W + Z_2 \tag{30}$$

$$Q_{X_{22}^c,W} = D^{1/2}Q_W, \quad Z_2 \in G(0, (I - D^{1/2}Q_W D^{1/2})), \quad (31)$$

$$(Z_1, Z_2, W)$$
, are independent. (32)

(c) A lower bound on (26) occurs if $Q_W = Q_{W^*} \in \mathbf{Q}_W$ is diagonal, i.e., $Q_{W^*} = \text{Diag}(Q_{W_1^*}, \dots, Q_{W_n^*}), d_i \leq Q_{W_i^*} \leq d_i^{-1}, \forall i$, and it is achieved by realization (28)-(32), with $Q_W = Q_{W^*}$.

(d) Wyner's information common information is given by

$$C_W(X_1, X_2) = \begin{cases} \frac{1}{2} \sum_{i=1}^n \ln\left(\frac{1+d_i}{1-d_i}\right) \in (0, \infty) & \text{if } n > 0\\ 0 & \text{if } n = 0 \end{cases}$$
(33)

and it is achieved by a Gaussian RV $W = W^* \in G(0, Q_{W^*}), W^* : \Omega \to \mathbb{R}^n, Q_{W^*} = I_n$ an $n \times n$ identity covariance matrix, and the realization of part (b) with $Q_W = I_n$.

The characterization of the subset \mathscr{P}_{min}^{CIG} of the set \mathscr{P}^{CIG} of two RVs (X_1, X_2) in canonical variable form by the set $\mathbf{Q}_{\mathbf{W}}$ is due to Van Putten and Van Schuppen [11].

In [12] the proof of (33) is incomplete because there is no optimization over the set of measures $Q_W \in \mathbf{Q}_W$ achieving the conditional independence. In that reference there is an assumption that three cross-covariances can be simultaneously diagonalized, which is not true in general. This assumption implies that case (d) of the above theorem holds. This assumption is repeated in [13].

From Theorem 2.2 follows directly the proposition below.

Proposition 2.3: Consider the statement of Theorem 2.2, with (X_1, X_2) in canonical variable form. Then $\mathscr{R}_{GW}(\Delta_1, \Delta_2)$ is determined from

$$T(\alpha_1, \alpha_2) = \inf_{\mathbf{Q}_{\mathbf{W}}} \left\{ I(X_1, X_2; W) + \alpha_1 R_{X_1|W}(d_1) + \alpha_2 R_{X_2|W}(d_2) \right\}$$

 $0 \le \alpha_i \le 1, i = 1, 2, \alpha_1 + \alpha_2 \ge 1$, and the infimum occurs at the diagonal $Q_W = Q_{W^*} \in \mathbf{Q}_W$ of Theorem 2.2, part (c). Moreover, $R_{X_i|W}(d_i), i = 1, 2$ is given by

$$R_{X_i|W}(\Delta_i) = \inf_{\sum_{j=1}^n \Delta_{i,j} = \Delta_i} \frac{1}{2} \sum_{j=1}^n \log\left(\frac{(1 - d_j/Q_{W_j}^*)}{\Delta_{i,j}}\right)^+ \quad (34)$$

where $(\cdot)^+ \stackrel{\triangle}{=} \max\{1, \cdot\}$, $\mathbf{E}||X_{i2}^c - \hat{X}_{i2}^c||_{\mathbb{R}^n}^2 = \sum_{j=1}^n \Delta_{i,j} = \Delta_i, i = 1, 2$, and the water-filling equations hold:

$$\Delta_{i,j} = \begin{cases} \lambda, & \lambda < 1 - d_j \\ 1 - d_j, & \lambda \ge 1 - d_j, \end{cases} \Delta_i \in (0, \infty), i = 1, 2.$$
(35)

Proof Follows from Gray and Wyner [4, (4) of page 1703, eqn(42)] and Theorem 2.2. (34) follows from RDF of Gaussian RVs.

Theorem 2.4: Consider the tuple of jointly Gaussian RVs of Theorem 2.2. Then

$$C_{GW}(X_1, X_2; \Delta_1, \Delta_2) = C_W(X_1, X_2)$$
(36)

$$= \frac{1}{2} \sum_{j=1}^{n} \ln\left(\frac{1+d_j}{1-d_j}\right), \quad (\Delta_1, \Delta_2) \in \mathscr{D}_W$$
(37)

$$\mathcal{D}_W \stackrel{\triangle}{=} \left\{ (\Delta_1, \Delta_2) \in [0, \infty] \times [0, \infty] \middle| 0 \le \Delta_1 \le \sum_{j=1}^n (1 - d_j), \\ 0 \le \Delta_2 \le \sum_{j=1}^n (1 - d_j) \right\}, d_j \in (0, 1), j = 1, \dots, n.$$

Formula (37) is a generalization of the analogous formula derived in [4]–[6], for a tuple of jointly Gaussian scalar RVs (X_1, X_2) , zero mean, $\mathbf{E}[X_1^2] = \mathbf{E}[X_2^2] = 1$, $\mathbf{E}[X_1X_2] = \boldsymbol{\rho} \in [0, 1]$.

III. PROOFS OF MAIN THEOREMS

We present in this section additional exposition on the Concepts of Section I-A, and outlines of the proofs of the main theorems (see [3] for additional exposition).

A. Further Discussion on the Three Conecpts

First we state a few facts.

(A1) The parametrization of the family of Gaussian probability distributions \mathscr{P}^{CIG} and \mathscr{P}^{CIG}_{min} require the solution of the weak stochastic realization problem of Gaussian RVs (defined by Problem 2.15 in [3]) given in [14, Theorem 4.2] (see also [3, Theorem 3.8]), and reproduced below.

Theorem 3.1: [14, Theorem 4.2] Consider a tuple (X_1, X_2) of Gaussian RVs in the canonical variable form. Restrict attention to the correlated parts of these RVs, as follows:

$$(X_1, X_2) \in G(0, Q_{(X_1, X_2)}) = \mathbf{P}_0, \quad X_1, X_2 : \Omega \to \mathbb{R}^n,$$
 (38)

$$Q_{(x_1,x_2)} = \begin{pmatrix} I & D \\ D & I \end{pmatrix}, \ p_{11} = p_{21} = 0, p_{13} = p_{23} = 0, \ (39)$$

$$D = \operatorname{Diag}(d_1, \dots, d_n) \in \mathbb{R}^{n \times n}, \ 1 > d_1 \ge \dots \ge d_n > 0.$$
(40)

(a) There exists a probability measure \mathbf{P}_1 , and a triple of Gaussian RVs X_1, X_2 , $W : \Omega \to \mathbb{R}^n$ defined on it, such that (i) $\mathbf{P}_1|_{(X_1, X_2)} = \mathbf{P}_0$ and (ii) X_1 and X_2 are conditional independent given W with W having minimal dimension.

(b) There exist a family of Gaussian measures denoted by $\mathbf{P_{ci}} \subseteq \mathscr{P}_{min}^{CIG}$, that satisfy (i) and (ii) of (a), and moreover this family is parametrized by the matrices and sets:

$$G(0, \mathcal{Q}_{s}(\mathcal{Q}_{W})), \ \mathcal{Q}_{W} \in \mathbf{Q}_{\mathbf{W}}, \tag{41}$$

$$Q_s(Q_W) = \begin{pmatrix} I & D & D^{1/2} \\ D & I & D^{1/2} Q_W \\ D^{1/2} & Q_W D^{1/2} & Q_W \end{pmatrix}, \quad (42)$$

$$\mathbf{Q}_{\mathbf{W}} = \left\{ \mathcal{Q}_{W} \in \mathbb{R}^{n \times n} \middle| \mathcal{Q}_{W} = \mathcal{Q}_{W}^{T}, \ 0 < D \leq \mathcal{Q}_{W} \leq D^{-1} \right\},\tag{43}$$

$$\mathbf{P_{ci}} = \left\{ G(0, \mathcal{Q}_s(\mathcal{Q}_W)) \text{ on } (\mathbb{R}^{3n}, \mathscr{B}(\mathbb{R}^{3n})) \middle| \mathcal{Q}_W \in \mathbf{Q}_W \right\}$$

and $\mathbf{P_{ci}} \subseteq \mathscr{P}_{min}^{CIG}$.

(A2) The weak stochastic realization of a Gaussian measure $G(0,Q_0)$ on the Borel space $(\mathbb{R}^{p_1+p_2}, \mathscr{B}(\mathbb{R}^{p_1+p_2}))$ is then defined and characterized as in Def. 2.17 and Prop. 2.18, Alg. 3.4 of [3].

B. Proofs of Main Theorems

(B) For the calculatation of $C_W(X_1, X_2)$ via Theorem 2.2 and $C_{GW}(X_1, X_2; \Delta_1, \Delta_2)$ via Theorem 2.4 it is sufficient to impose the conditional independence $\mathbf{P}_{X_1, X_2|W} = \mathbf{P}_{X_1|W} \mathbf{P}_{X_2|W}$, due to, (a) the well-known inequality

$$I(X_1, X_2; W) \ge H(X_1, X_2) - H(X_1|W) - H(X_2|W)$$
(44)

which is achieved if $\mathbf{P}_{X_1,X_2|W} = \mathbf{P}_{X_1|W}\mathbf{P}_{X_2|W}$.

(b) A necessary condition for the equality constraint (23) to hold (see Appendix B in [6]) is

$$R_{X_1,X_2|W}(\Delta_1,\Delta_2) = R_{X_1|W}(\Delta_1) + R_{X_2|W}(\Delta_2).$$
(45)

Further, a sufficient condition for (45) to hold is the conditional independence condition [6]: $\mathbf{P}_{X_1,X_2|W} = \mathbf{P}_{X_1|W}\mathbf{P}_{X_2|W}$. a sufficient condition for any rate $(R_0,R_1,R_2) \in \mathscr{R}_{GW}(\Delta_1,\Delta_2)$ to lie on the Pangloss plane is the conditional independence.

(c) For jointly Gaussian RVs (X_1, X_2) with square-error distortion, then by the maximum entropy principle the optimal joint distribution $\mathbf{P}_{X_1, X_2, \hat{X}_1, \hat{X}_2, W}$ of the optimization problem $C_{GW}(X_1, X_2; \Delta_1, \Delta_2)$ is Gaussian.

(d) The characterization of Wyner's information common information $C_W(X_1, X_2)$ for jointly Gaussian multivariate RVs (X_1, X_2) occurs in the set of jointly Gaussian RVs (X_1, X_2, W) such that $\mathbf{P}_{X_1, X_2|W} = \mathbf{P}_{X_1|W} \mathbf{P}_{X_2|W}$ and $C_W(X_1, X_2)$ is invariant with respect to Hotelling's nonsingular basis transformation

(e) For data (1)-(4), any rate triple (R_0, R_1, R_2) that belongs to $\mathscr{R}_{GW}(\Delta_1, \Delta_2)$, characterized by (17)-(21), is equivalently computed by transforming the tuple (X_1, X_2) into their canonical variable form (5)-(14).

Remark 3.2: Theorem 3.1 is a parametrization of the familiy of Gaussian measures $\mathbf{P}_{ci} \subseteq \mathscr{P}_{min}^{CIG}$ by Q_W and Q_{W^*} .

(a) Theorem 3.1 applies to other network problems, i.e., the

Gaussian MACs by incorporating average power constraints. (b) The weak stochastic realization of RVs (X_1, X_2) , in terms of the random variable W is given in Theorem 2.2.

(c) An alternative proof of Proposition 2.3, i.e., that $\mathscr{R}_{GW}(\Delta_1, \Delta_2) = \mathscr{R}^*_{GW}(\Delta_1, \Delta_2)$ is generated from distributions $\mathbf{P_{ci}} \subseteq \mathscr{P}_{min}^{CIG} \subseteq \mathscr{P}$ is given in [15].

Proof of Theorem 2.1: Follows from the non-singular transformations (5), (6), and chain rule of mutual information applied to $I(X_1; X_2) = I(S_1X_1; S_1X_2) = I(X_{11}^c, X_{12}^c, X_{13}^c; X_{21}^c, X_{22}^c, X_{23}^c).$

Proof of Theorem 2.2: An alternative derivation based on inequalities of linear algebra is given in Theorem 3.11 of [3], and is based on [16, Theorem 9.E.6], with reference to Hua LooKeng [17]. Below, we present a simplified derivation.

(a) Equality (24) follows from the non-singular transformations (5), (6); inequality (25) is due to (B).(a); (26) follows by evaluation of entropies; (27) is due to Theorem 3.1. (b) The lower bound is achieved by the maximum entropy principle of Gaussian RVs, and the realization is due to Theorem 3.1. (c) We identify a further lower bound on the second right-hand-side term of (26) that depends on $Q_W \in \mathbf{Q}_W$ (and corresponds to $-H(X_{12}^c|W) - H(X_{22}^c|W)$, by letting $Q_W = Q_{W^*}$. By the chain rule of entropy then

$$H(X_{12}^c|W) = \sum_{j=1}^n H(X_{12,j}^c|W_1, \dots, W_n, X_{12,1}^c, X_{12,2}^c, \dots, X_{12,j-1}^c)$$

$$\leq \sum_{i=1}^n H(X_{12,j}^c|W_j)$$

and the upper bound is achieved if $(X_{12,j}, W_j), j = 1, ..., n$ are jointly independent, hence $Q_W = Q_{W^*}$. Similarly, the upper bound $H(X_{22}^c|W) \leq \sum_{j=1}^n H(X_{22,j}^c|W_j)$ is achieved if $(X_{22,j}, W_j), j = 1, ..., n$ are jointly independent, i.e., $Q_W = Q_{W^*}$. Such joint distribution is induced by the realization of part (b), with $Q_W = Q_{W^*}$. (d) By part (c), and simple algebra we can show that the optimal Q_{W^*} for $C_W(X_1, X_2)$ is $Q_{W^*} = I_n \in \mathbf{Q}_{W^*}$ and then follows (33).

Proof of Theorem 2.4: A direct way to prove the statement is to compute the rate distortion functions $R_{X_i}(\Delta_i), R_{X_i|W}(\Delta_i), i = 1, 2$ and $R_{X_1,X_2}(\Delta_1, \Delta_2)$, using the weak stochastic realization of Theorem 3.1.(b), and then verify that identity (23) holds, i.e., $R_{X_1|W}(\Delta_1) + R_{X_2|W}(\Delta_2) + I(X_1,X_2;W) = R_{X_1,X_2}(\Delta_1,\Delta_2)$ for $(\Delta_1,\Delta_2) \in \mathcal{D}_W$, for the choice $W = W^*$ with $Q_{W^*} = I_n$ given in Theorem 2.2.(b).

IV. CONCLUDING REMARKS

This paper calculates rates on the Gray and Wyner lossy rate region of a tuple of jointly Gaussian RVs, $X_1 : \Omega \to \mathbb{R}^{p_1}, X_2 : \Omega \to \mathbb{R}^{p_2}$ with square-error fidelity at the two decoders, by making use of van Putten's and van Schuppen's [11] parametrization of all jointly Gaussian distributions $\mathbf{P}_{X_1,X_2,W}$, by another Gaussian RV $W : \Omega \to \mathbb{R}^n$, such that $\mathbf{P}_{X_1,X_2|W} = \mathbf{P}_{X_1|W}\mathbf{P}_{X_2|W}$, and their weak stochastic realization. However, much remains to be done to exploit the new approach to other multi-user problems of information theory.

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