# Minimizing the alphabet size in codes with restricted error sets 

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#### Abstract

This paper focuses on error-correcting codes that can handle a predefined set of specific error patterns. The need for such codes arises in many settings of practical interest, including wireless communication and flash memory systems. In many such settings, a smaller field size is achievable than that offered by MDS and other standard codes. We establish a connection between the minimum alphabet size for this generalized setting and the combinatorial properties of a hypergraph that represents the prespecified collection of error patterns. We also show a connection between error and erasure correcting codes in this specialized setting. This allows us to establish bounds on the minimum alphabet size and show an advantage of non-linear codes over linear codes in a generalized setting. We also consider a variation of the problem which allows a small probability of decoding error and relate it to an approximate version of hypergraph coloring.


## I. Introduction

In many practical settings, there is a need to design errorcorrecting codes that can handle specific error patterns. For example, in wireless communications, magnetic recording, flash memory systems, and Dynamic Random-Access Memories (DRAMs) the errors can appear in correlated locations such as bursts, single-row errors, or crisscrosss errors, e.g., [1]-[6]. These settings benefit from customized error correcting codes, that may improve on the best known parameters of standard error correcting codes. For example, the optimal error-correcting capabilities of the classical $(n, k)$ Maximum Distance Separable (MDS) code, such as the Reed-Solomon code, come at the price of a significant alphabet size of $q \geq n-k+1,[7]$ As we show in this paper, in many settings with specific error patters, a much smaller alphabet size is needed.

In this work, we present a general framework for code design that can handle any possible collection of predefined error patterns. Our framework applies to both linear and nonlinear codes. For an error-correcting code of length $n$, we use an $n$-vertex hypergraph $G$ to represent the given collection of error sets. Specifically, nodes of $G$ represent the coordinates (symbols) of the codewords, while the hyperedges of $G$ represent possible locations for errors, i.e., each hyperedge $e$ represents the set of coordinates that can be corrupted in the specific scenario represented by $e$. For each collection of

[^0]error sets represented by $G$, we are interested in finding the minimum alphabet size over which there exists a code that can correct all error sets specified by edges in $G$. In our setting, $(n, k)$-MDS codes can correct error patterns corresponding to the complete $(n-k) / 2$-uniform $n$-vertex hypergraph.

In this work, we relate the minimum alphabet size of error-correcting codes with predefined error patterns to certain variants of hypergraph coloring. Through reductive arguments to erasure codes, and in particular to our prior work [8] in the context of erasure codes with generalized decoding sets, we propose code design for the error setting at hand, and show that non-linear error-correcting codes outperform linear ones. We then turn to study a variation of the problem which allows a small probability of decoding error and relate it to an approximate version of hypergraph coloring.

Our work is structured as follows. In Section [ $\mathbb{I}$ we give some preliminaries and, in particular, we introduce our model for generalized erasure and error patterns. We also review our previous study on erasure codes in the generalized setting of a predefined collection of decoding sets [8]. In Section III, we present bounds on the minimum alphabet size of the corresponding codes through hypergraph coloring. In Section IV] we reduce the error-correcting setting to the erasure setting. In Section $\bar{\square}$ we extend our studies to the problem of error detection. Finally, in Section VI we relax the zeroerror requirement for decoding a correct message and analyze settings which allow small $\varepsilon>0$ probability of decoding error.

## II. Model and Preliminaries

Since our paper makes a connection between erasure and error correction in a generalized setting, we present definitions for both scenarios. We begin by presenting a definition and our prior results for erasure correction scenarios.

## A. Erasure Correction with predefined decoding sets.

We start by studying the design of erasure-codes in a generalized setting in which decoding is required from a collection of predefined decoding sets. In this setting, the decoding sets include the set of coordinates that can be used to decode the message. The setting is represented by a hypergraph $G=([n], E)$, with the set $[n]=\{1, \ldots, n\}$ of nodes representing coordinates and set of hyperedges $E$ representing decoding sets.

We define the $q_{k}$ parameter of a given hypergraph $G=$ $([n], E)$ as the minimum alphabet size of a $(n, k)$ erasure code that enables the receiver to decode the original message from every subset $e \in E$.

Definition 1 (The $q_{k}$ parameter [8]) Let $G=([n], E)$ be a hypergraph on the vertex set $[n]=\{1, \ldots, n\}$. Let $k$ be integer. Let $q_{k}(G)$ denote the smallest size $q$ of an alphabet $F$ for which there exist an encoding function

$$
C: F^{k} \rightarrow F^{n}
$$

and a decoding function

$$
D:(F \cup\{\perp\})^{n} \rightarrow F^{k}
$$

such that for every edge $e \in E$ and every message $m \in F^{k}$ it holds that

$$
D\left(C_{e}(m)\right)=m
$$

Here, $C_{e}(m)$ stands for the word obtained from the codeword $C(m)$ by replacing the symbols in the locations of $[n] \backslash e$ by the erasure symbol $\perp$.

Similarly, let $q_{k, l i n}(G)$ denote the smallest prime power $q$ for which there exist linear encoding and decoding functions defined above when $F$ is a field of size $q$.

In Definition 1 notice that for $G$ that includes edges of size less than $k$ no such $(C, D)$ pair exists (no matter what the size of $F$ is). In this case we define $q_{k}(G)$ and $q_{k, l i n}(G)$ to be $\infty$. Moreover, for every $G$ with edges of size at least $k$, MDS codes satisfy the requirements on $(C, D)$ and thus $q_{k}(G)<\infty$. Specifically, observe that for the complete $n$ vertex $k$-uniform hypergraph, denoted by $\kappa_{n, k}$, the values of $q_{k}\left(\kappa_{n, k}\right)$ and $q_{k, l i n}\left(\kappa_{n, k}\right)$ are equal to the minimum alphabet sizes of general and linear $(n, k)$ MDS codes, respectively. We state below the MDS conjectures for general and for linear codes (see, e.g., [7], [9]-[11]).

Conjecture 1 (MDS Conjecture for general codes) For given integers $k<q \neq 6$, let $n(q, k)$ be the largest integer $n$ such that $q_{k}\left(\kappa_{n, k}\right) \leq q$. Then,

$$
n(q, k) \leq \begin{cases}q+2 & \text { if } 4 \mid q \text { and } k \in\{3, q-1\}  \tag{1}\\ q+1 & \text { otherwise }\end{cases}
$$

Conjecture 2 (MDS Conjecture for linear codes) For given integers $k<q$ where $q$ is a prime power, let $n(q, k)$ be the largest integer $n$ such that $q_{k, l i n}\left(\kappa_{n, k}\right) \leq q$. Then,

$$
n(q, k) \leq \begin{cases}q+2 & \text { if } q \text { is even and } k \in\{3, q-1\}  \tag{2}\\ q+1 & \text { otherwise. }\end{cases}
$$

There are strong relations between the $q$ parameter of hypergraphs and certain colorings.

Definition 2 (Hypergraph strong-coloring) A valid strongcoloring of a hypergraph $G$ is an assignment of colors to its vertices so that the vertices of each edge are assigned to distinct colors. The chromatic number $\chi(G)$ of $G$ is the minimum number of colors that allows a valid strong-coloring of $G$. At times, we refer to $\chi$ simply as the chromatic number of $G$.

Definition 3 (Hypergraph $k$-coloring) $A$ valid $k$-coloring of a hypergraph $G$ is an assignment of colors to its vertices so that the vertices of each edge are assigned to at least $k$ distinct colors. The $k$-chromatic number $\chi_{k}(G)$ of $G$ is the minimum number of colors that allows a valid $k$-coloring of $G$. If $G$ has edges of size less than $k$, we define $\chi_{k}(G)=\infty$.

Note that a $k$-coloring of a $k$-uniform hypergraph is exactly a strong-coloring. Also, note that every hypergraph $G$ for which $q_{k}(G)<\infty$ (i.e., all edges are of size at least $k$ ) satisfies $\chi_{k}(G) \leq \chi(G)$. In particular, for $k$-uniform hypergraphs $G$, $\chi_{k}(G)=\chi(G)$.

Theorem 1 (Connecting $q_{k}(G)$ with $\chi_{k}(G),[\mathbf{8}]$ ) For every hypergraph $G$ for which $q_{k}(G)<\infty$,

$$
q_{k}(G) \leq q_{k}\left(\kappa_{\chi_{k}(G), k}\right) \quad \text { and } \quad q_{k, l i n}(G) \leq q_{k, l i n}\left(\kappa_{\chi_{k}(G), k}\right)
$$

In particular,

$$
q_{k}(G) \leq q_{k, l i n}(G) \leq\left[\chi_{k}(G)-1\right]_{p p}
$$

Here, for an integer $x,[x]_{p p}$ represents the smallest prime power that is greater or equal to $x$.

Theorem 1 formalizes the natural intuition that for simple collections of erasure patterns $G$, i.e., the setting in which $\chi_{k}(G)$ is small, a small alphabet size $q$ suffices for a suitable erasure code. In particular, the theorem states that $q_{k}(G)$ is upper bounded by $q_{k}\left(\kappa_{\chi_{k}(G), k}\right)$, which is the minimum alphabet size of a $\left(\chi_{k}(G), k\right)$ MDS code.

The graph family $G_{q, k}$, defined next, is helpful in analyzing the tightness of the upper bound provided by Theorem 1

Definition 4 (The graph family $G_{q, k}$ ) For integers $q$ and $k$, let $G_{q, k}$ be the $k$-uniform hypergraph whose vertex set consists of all the balanced vectors of length $q^{k}$ over $F=$ $\{0,1, \ldots, q-1\}$, that is, the vectors $u \in F^{q^{k}}$ such that $\left|\left\{i \in\left[q^{k}\right] \mid u_{i}=j\right\}\right|=q^{k-1}$ for every $j \in F$, where $k$ vertices $u^{1}=\left(u_{1}^{1}, \ldots, u_{q^{k}}^{1}\right), \ldots, u^{k}=\left(u_{1}^{k}, \ldots, u_{q^{k}}^{k}\right)$ form an edge if the collection of $k$-tuples $\left\{\left(u_{i}^{1}, u_{i}^{2}, \ldots, u_{i}^{k}\right)\right\}_{i \in\left[q^{k}\right]}$ is equal to $[q]^{k}$.

The following lemma identified hypergraphs $G$ for which the gap between $q_{k}(G)$ and $\chi_{k}(G)$ is maximal.

Lemma 1 (The extremal nature of $G_{q, k},[\mathbf{8}]$ ) For integers $q$ and $k$,

1) $q_{k}\left(G_{q, k}\right) \leq q$, and
2) $\chi_{k}(G) \leq \chi_{k}\left(G_{q, k}\right)$ for every graph $G$ with $q_{k}(G)=q$.

Extending results in [8], below we present (rater loose) bounds on $\chi\left(G_{q, k}\right)$.

Proposition 1 (Bounds of $\chi_{k}\left(G_{k, q}\right)$ ) For every prime power $q$ and $k \geq 2$,

$$
\frac{q^{k}-1}{q-1} \leq \chi_{k}\left(G_{q, k}\right) \leq\binom{ q^{k-1}+1}{q^{k-2}+1}
$$

Proof: We first study the collection of vertices in $G_{q, k}$ corresponding to normalized linear functions $F^{k} \rightarrow F$ for field $F$ of size $q$. A normalized linear function is one in which the leading nonzero coefficient equals 1 . Such functions, when considered in vector form $\left(u_{1}, \ldots u_{q^{k}}\right) \in F^{q^{k}}$ are balanced and thus correspond to vertices of $G_{q, k}$. Moreover, it is not hard to verify that any two vertices of $G_{q, k}$ corresponding to distinct normalized linear functions are included in an edge of $G_{q, k}$ (i.e., there exist $k-2$ additional vertices of $G_{q, k}$ corresponding to normalized linear functions that complete a linearly
independent collection of functions). Thus, any $k$-coloring of $G_{q, k}$ must color all vertices corresponding to normalized linear functions with distinct colors. The number of normalized linear functions over $F$ corresponding to vertices of $G_{q, k}$ is $\sum_{i=1}^{k} q^{k-i}=\left(q^{k}-1\right) /(q-1)$. Therefore $\chi\left(G_{q, k}\right) \geq \frac{q^{k}-1}{q-1}$.

On the other hand, we now show that $\chi\left(G_{q, k}\right) \leq\binom{ q^{k-1}+1}{q^{k-2}+1}$. For any vector $u$ in $G_{q, k}$, consider the first $q^{k-1}+1$ entries of $u$. By the pigeonhole principal, $u_{i_{1}}=u_{i_{2}}=\ldots=u_{i^{k-2}{ }^{k+1}}$ for some collection of entries indexed by $i_{1}<i_{2}<\ldots<$ $i_{q^{k-2}+1} \in\left[q^{k-1}+1\right]$. Now, for any $q^{k-2}+1$ distinct indices $i_{1}<i_{2}<\ldots<i_{q^{k-2}+1} \in\left[q^{k-1}+1\right]$ let $A_{i_{1}, i_{2}, \ldots, i_{q^{k-2}+1}}$ be the set of all vertices $u \in F^{q^{k}}$ of $G_{q, k}$ that satisfy $u_{i_{1}}=u_{i_{2}}=$ $\ldots=u_{i_{q^{k-2}+1}}$. Every set $A_{i_{1}, i_{2}, \ldots, i_{q^{k-2}+1}}$ forms an independent set in $G_{q, k}^{q^{k-2}}$, i.e., a set that does not include any two vertices from an edge of $G_{q, k}$. This follows, since for every two distinct vertices $u, v \in A_{i_{1}, i_{2}, \ldots, i_{q^{k-2}+1}}$ we have $u_{i_{1}}=u_{i_{2}}=\ldots=$ $u_{i^{k-2}+1}$ and $v_{i_{1}}=v_{i_{2}}=\ldots=v_{i_{q^{k-2}+1}}$, which is too large of an overlap to allow the balanced nature of vertices included in edges of $G_{q, k}$. Specifically, for vertices $u$ and $v$ that appear in an edge of $G_{q, k}$, it must be for any $j_{1}$ and $j_{2}$ in $F$ that $\mid\{i \in$ $\left.\left.\left[q^{k}\right] \mid u_{i}=j_{1}, v_{i}=j_{2}\right]\right\} \mid=q^{k-2}$. Such independent sets were referred to as canonical in [8]. As the $\binom{q^{k-1}+1}{q^{k-2}+1}$ independent sets $A_{i_{1}, i_{2}, \ldots, i_{q^{k-2}+1}}$ of $G_{q, k}$ with $i_{1}, i_{2}, \ldots, i_{q^{k-2}+1} \in\left[q^{k-1}+1\right]$ cover the entire vertex set of $G_{q, k}$, coloring each one with a distinct color implies the required upper bound on $\chi_{k}$.

Lemma 1 and Proposition 1 imply a gap between $q_{k}\left(G_{q, k}\right)$ and $\chi_{k}\left(G_{q, k}\right)$ which can be extended to one between $q_{k, l i n}$ and the $k$-chromatic number of the subgraph of $G_{q, k}$ induced by vertices that correspond to normalized linear functions.

Proposition 2 (Gap between $q_{k, l i n}(G)$ and $\left.\chi_{k}(G),[\mathbf{8}]\right)$
For every $k \geq 3$ and every prime power $q$, there exists a $k$-uniform hypergraph $G$ with $q_{k, l i n}(G) \leq q$ and yet $\chi_{k}(G) \geq \frac{q^{k}-1}{q-1}$.

We finally state a modest known gap between $q_{k, l i n}$ and $q_{k}$. Identifying graphs that exhibit a larger gap than that presented below is a problem left open in this work.

Proposition 3 (Gap between $q_{k, l i n}$ and $q_{k}$, [12]) For $q=3$ and $k=2$ it holds that

$$
q_{k, l i n}\left(G_{q, k}\right)=\left[\chi_{k}\left(G_{q, k}\right)-1\right]_{p p}=5>3 \geq q_{k}\left(G_{q, k}\right)
$$

## B. Error Correction with predefined error sets.

In what follows, we extend our discussion beyond erasures to the context of errors. As we will see, several of our results on the $q$-parameter corresponding to erasures extend naturally to the $p$-parameter (defined below) corresponding to codes with restricted error sets. Similarly, to the erasure setting, we represent the collection of error sets by using a hypergraph $G=([n], E)$, in which the set of vertices $[n]$ represents coordinates of a codeword. Each edge $e \in E$ of $G$ represents an error set, i.e., the set of the coordinates that can be altered. Note that this is different from the notation used in Definition 1
for the erasure case in which edges $e$ represented decoding sets (i.e., sets of uncorrupted symbols).

Definition 5 (The $p_{k}$ parameter) Let $G=([n], E)$ be a hypergraph on the vertex set $[n]=\{1, \ldots, n\}$. Let $k$ be an integer. Let $p_{k}(G)$ denote the smallest size $p$ of an alphabet $F$ for which there exist an encoding function

$$
C: F^{k} \rightarrow F^{n}
$$

and a decoding function

$$
D: F^{n} \rightarrow F^{k}
$$

such that for every edge $e \in E$, every message $m \in F^{k}$, and every error vector $v=\left(v_{1}, \ldots, v_{n}\right) \in F^{n}$,

$$
D\left(C(m) \diamond_{e} v\right)=m
$$

Here, for $C(m)=c_{1}, \ldots, c_{n}$, the term $C(m) \diamond_{e} v$ refers to the vector $y=y_{1}, \ldots, y_{n}$ for which for $i \in[n], y_{i}=v_{i}$ if $i \in e$, and otherwise $y_{i}=c_{i}$ (i.e., we overwrite $C(m)$ with values of $v$ in the coordinates $i \in e$ ).

Similarly, let $p_{k, l i n}(G)$ denote the smallest prime power $p$ for which there exist linear encoding and decoding functions as above when $F$ is a field of size $p$.

In Definition 55 the pair $(C, D)$ corresponds to a code that is resilient to errors on locations corresponding to an edge $e \in E$. That is, the edge set $E$ represents the possible error patterns (i.e., sets of potentially corrupted symbols).

Similar to Definition 1 in Definition 5, if $G$ has edges of size greater than $\left\lfloor\frac{n-k}{2}\right\rfloor$, no such codes $(C, D)$ exist, and we define $p_{k}(G)=p_{k, l i n}(G)=\infty$.

As with erasures, for the complete hypergraph $\mathcal{K}_{n,\left\lfloor\frac{n-k}{2}\right\rfloor}$, the values of $p_{k}\left(\kappa_{n,\left\lfloor\frac{n-k}{2}\right\rfloor}\right)$ and $p_{k, l i n}\left(\kappa_{n,\left\lfloor\frac{n-k}{2}\right\rfloor}\right)$ are equal to the minimum alphabet sizes of general and linear $(n, k)$ MDS codes, respectively. That is, $p_{k}\left(\kappa_{n,\left\lfloor\frac{n-k}{2}\right\rfloor}\right)=q_{k}\left(\kappa_{n, k}\right)$ and $p_{k, l i n}\left(\kappa_{n,\left\lfloor\frac{n-k}{2}\right\rfloor}\right)=q_{k, l i n}\left(\kappa_{n, k}\right)$.

Note that Definitions 1 and 5 assume zero-error decoding. We relax this requirement in Section VI

## III. Bounds on the Alphabet Size

Proposition 4 (Analog of Theorem 11) Let $k$ be an integer. For every hypergraph $G=([n], E)$ for which $p_{k}(G)<\infty$ it holds that

$$
p_{k}(G) \leq p_{k}\left(\kappa_{\chi,\left\lfloor\frac{x-k}{2}\right\rfloor}\right)
$$

where $\chi=\chi(\bar{G})$ and $\bar{G}=(V, \bar{E})$ is the hypergraph with vertex set $V=[n]$ and edges $\bar{E}=\{V \backslash e \mid e \in E\}$.

Proof: To ease our notation, we assume that $n-k$ and $\chi-k$ are even (minor modifications in notation are needed otherwise). Let $G$ be as above and let $\chi=\chi(\bar{G})$. Denoting $p=p_{k}\left(\kappa_{\chi,(\chi-k) / 2}\right)$, it follows that there exist a $(\chi, k) \operatorname{MDS}$ code $C$ over an alphabet $F$ of size $p$. To prove that $p_{k}(G) \leq p$, we define a coding scheme for $G$ over the alphabet $F$ that includes the following two steps. First, fix a valid coloring $g:[n] \rightarrow[\chi]$ of $\bar{G}$. Second, consider the encoding function $\widetilde{C}: F^{k} \rightarrow F^{n}$ that given a message $m \in F^{k}$ outputs the vector in $F^{n}$ whose $i$ 'th entry $\widetilde{C}_{i}(m)$ is $C_{g(i)}(m)$, i.e., $\widetilde{C}_{i}(m)$ is the coordinate in the codeword $C(m)$ which corresponds
to the color of the $i$ 'th vertex. Here, and throughout, we use the notation $C_{i}(m)$ to denote the $i$ 'th entry in the codeword $C(m)$.

The decoder $\widetilde{D}: F^{n} \rightarrow F^{k}$ for $G$ is now defined using the following procedure. Consider an error vector $v \in F^{n}$, edge $e_{0} \in E$, and the corresponding received word $y=\widetilde{C}(m) \diamond_{e_{0}} v$. For each edge $\bar{e}$ in $\bar{E}$, the decoder $\widetilde{D}$ considers $y_{\bar{e}}$ consisting of the entries of $y$ restricted to the indices in $\bar{e}$, and detects whether $y_{\bar{e}}$ has been corrupted, i.e., whether $\widetilde{C}_{\bar{e}}(m)=y_{\bar{e}}$. As for at least one such edge $\bar{e}_{0}$ it holds that $\widetilde{C}_{\bar{e}_{0}}(m)=y_{\bar{e}_{0}}$ (e.g. for $\bar{e}_{0}=[n] \backslash e_{0}$ ), the decoder $\widetilde{D}$ can use $y_{\bar{e}_{0}}$ to decode $m$. We are left to show, given $\bar{e} \in \bar{E}$, how $\widetilde{D}$ can detect whether $\widetilde{C}_{\bar{e}}(m)=y_{\bar{e}}$, and if so decode $m$.

To detect whether a given $\bar{e}$ in $\bar{E}$ satisfies $\widetilde{C}_{\bar{e}}(m)=y_{\bar{e}}$ we note, by the definition of $\widetilde{C}$ and the fact that all vertices in $\bar{e}$ have distinct colors under the coloring $g$, that the entries in $\widetilde{C}_{\bar{e}}(m)$ correspond to at least $(n+k) / 2$ distinct entries $C(m)$. The latter, in turn, implies that $\widetilde{C}_{\bar{e}}(m)$ is itself a $(|\bar{e}|, k)$ MDS code. As such, $\widetilde{C}_{\bar{e}}(m)$ can detect up to $|\bar{e}|-k \geq \frac{n-k}{2}$ errors and correct up to $(|\bar{e}|-k) / 2 \geq \frac{n-k}{4}$ errors. We conclude, as all error sets $e$ are of size at most $(n-k) / 2$, that given $\bar{e}$ in $\bar{E}$, the decoder $\widetilde{D}$ can detect whether or not $y_{\bar{e}}$ has been corrupted, and if not, recover $m$ as required.

Proposition 4 is not tight, meaning that $p_{k}(G)$ might be smaller than $p_{k}\left(\kappa_{\chi,(\chi-k) / 2}\right)$. For $k=2$ take for example $G=([6], E)$ to be the 6 -cycle, i.e., the graph on 6 vertices in which its edges $E=\{(i, i+1) \mid i=0,1, \ldots, 5\}$ (with addition $\bmod 6)$. Then $p_{2}(G)=2$, since the binary encoding $C: F^{2} \rightarrow F^{6}$ in which for a message $m=(x, y) \in F^{2}$ equals $C(x, y)=(x, y, x, y, x, y)$ allows majority decoding for any 2 errors along an edge in $G$. However, $\chi=\chi(\bar{G})=6$, since every pair of vertices in $\bar{G}$ is included in some edge in $\bar{E}$, and by [13] it holds that $p_{2}\left(\kappa_{\left.\chi, \left\lvert\, \frac{\chi-k}{2}\right.\right]}\right)=p_{2}\left(\kappa_{6,2}\right)=5$. In the next section, we improve on Proposition 4 by connecting the $p_{k}$ and $q_{k}$ parameters.

## IV. Connecting Error and Erasure Correcting Codes

For parameters $n$ and $k$, we say that encoder $C: F^{k} \rightarrow F^{n}$ is good for a given hypergraph $G=([n], E)$ with respect to erasures (res., errors) if there exists a decoder $D$ satisfying Definition 1 (res., Definition 5). The following proposition is proven from basic principles.

Proposition 5 (From errors to erasures) Let $n$ and $k$ be parameters. Consider a hypergraph $G^{\mathrm{err}}=\left([n], E^{\mathrm{err}}\right)$ corresponding to errors. Let $G^{\text {era }}=\left([n], E^{\mathrm{era}}\right)$ be the hypergraph (corresponding to erasures) for which

$$
E^{\mathrm{era}}=\left\{[n] \backslash\left(e_{1}^{\mathrm{err}} \cup e_{2}^{\mathrm{err}}\right) \mid e_{1}^{\mathrm{err}}, e_{2}^{\mathrm{err}} \in E^{\mathrm{err}}\right\}
$$

Let $C: F^{k} \rightarrow F^{n}$ be any encoder. Then, $C$ is good for $G^{\mathrm{err}}$ if and only if $C$ is good for $G^{\mathrm{era}}$.
Proof: First assume that $C$ is good for $G^{\text {err }}$. We show that for every edge $e=e^{\text {era }} \in E^{\text {era }}$, one can decode $m$ from $C_{e}(m)$. Assume in contradiction that there are two messages $m_{1} \neq m_{2}$ such that $C_{e}\left(m_{1}\right)=C_{e}\left(m_{2}\right)$. Recall that $e=[n] \backslash\left(e_{1} \cup e_{2}\right)$ for $e_{1}=e_{1}^{\mathrm{err}} \in E^{\mathrm{err}}$ and
$e_{2}=e_{2}^{\mathrm{err}} \in E^{\mathrm{err}}$. Consider the word $y=\left(y_{1}, \ldots, y_{n}\right) \in F^{n}$ such that for $i \in e=[n] \backslash\left(e_{1} \cup e_{2}\right): y_{i}=C_{i}\left(m_{1}\right)=C_{i}\left(m_{2}\right)$, for $i \in e_{1} \backslash e_{2}: y_{i}=C_{i}\left(m_{2}\right)$, and for $i \in e_{2}: y_{i}=C_{i}\left(m_{1}\right)$. It is not hard to verify that there exist vectors $v_{1}$ and $v_{2}$ such that $y=C\left(m_{1}\right) \diamond_{e_{1}} v_{1}=C\left(m_{2}\right) \diamond_{e_{2}} v_{2}$. Namely, $y$ could be obtained from the codeword $C\left(m_{1}\right)$ with error vector $v_{1}$ corresponding to $e_{1}$ or from the codeword $C\left(m_{2}\right)$ with error vector $v_{2}$ corresponding to $e_{2}$, contradicting the existence of a decoder $D$ according to Definition 5

For the other direction, if code $C$ is not good for $G^{\text {err }}$ then there exist two messages, $m_{1}$ and $m_{2}$, two error vectors $v_{1}$ and $v_{2}$, and two edges $e_{1}$ and $e_{2}$ in $E^{\text {err }}$ such that $C\left(m_{1}\right) \diamond_{e_{1}}$ $v_{1}=C\left(m_{2}\right) \diamond_{e_{2}} v_{2}$. Otherwise, it is not hard to verify the existence of a natural decoder $D$ according to Definition 5 . Let $e=[n] \backslash\left(e_{1} \cup e_{2}\right) \in E^{\text {era }}$. The equality $C\left(m_{1}\right) \diamond_{e_{1}} v_{1}=$ $C\left(m_{2}\right) \diamond_{e_{2}} v_{2}$ now implies that $C_{e}\left(m_{1}\right)=C_{e}\left(m_{2}\right)$, which in turn implies that $C$ is not good for $G^{\text {era }}$.

The proposition above has an operational perspective. Namely, one can design an error-correcting code $C$ and decoder $D$ for a given graph $G^{\text {err }}$, by designing an erasurecode for the graph $G^{\text {era }}$. The latter can be done, e.g., using Theorem 1 to obtain the following corollary.

Corollary 2 Let $k$ be an integer. For every hypergraph $G^{\mathrm{err}}=([n], E)$ for which $p_{k}\left(G^{\mathrm{err}}\right)<\infty$ it holds that
$p_{k}\left(G^{\mathrm{err}}\right) \leq q_{k}\left(G^{\mathrm{era}}\right) \leq q_{k}\left(\kappa_{\chi_{k}\left(G^{\mathrm{era}}\right), k}\right) \leq\left[\chi_{k}\left(G^{\mathrm{era}}\right)-1\right]_{p p}$, which, in turn, implies that

$$
p_{k}\left(G^{\mathrm{err}}\right) \leq p_{k}\left(\kappa_{\chi,\left\lfloor\frac{\chi-k}{2}\right\rfloor}\right)
$$

where $\chi=\chi_{k}\left(G^{\text {era }}\right)$.
We now extend the connections implied by Proposition 5 to capture the $p_{k}$ and $q_{k}$ parameters.

Theorem 3 (Connecting $p_{k}$ with $q_{k}$ ) Let $n, k$ be parameters such that $n-k \geq k$. Let $G_{0}^{\text {era }}=\left([n], E_{0}^{\mathrm{era}}\right)$ be a hypergraph corresponding to erasures such that $q_{k}\left(G_{0}^{\text {era }}\right)<\infty$. Then, for $N=2 n-k$ there exists a hypergraph $G^{\text {err }}$ on $N$ vertices such that $p_{k}\left(G^{\mathrm{err}}\right)=q_{k}\left(G_{0}^{\text {era }}\right)$ and $p_{k, l i n}\left(G^{\mathrm{err}}\right)=$ $q_{k, l i n}\left(G_{0}^{\text {era }}\right)$.
Proof: Let $G_{0}^{\text {era }}=\left([n], E_{0}^{\text {era }}\right)$ be as above. We define two graphs according to $G_{0}^{\text {era }}$. First consider the graph $G^{\text {err }}=$ $\left([n] \cup U, E^{\mathrm{err}}\right)$ corresponding to errors for which $U$ is a vertex set of size $n-k$ and

$$
E^{\mathrm{err}}=\{U\} \cup\left\{[n] \backslash e^{\mathrm{era}} \mid e^{\mathrm{era}} \in E_{0}^{\mathrm{era}}\right\}
$$

Here, we use the fact that edges in $E_{0}^{\text {era }}$ are subsets of $[n]$. Namely, the vertex set $[n] \cup U$ of $G^{\text {err }}$ is of size $N=2 n-k$ and each edge in $E^{\mathrm{err}}$ is of size at most $\frac{N-k}{2}=n-k$. We refer to the edges in $\left\{[n] \backslash e^{\text {era }} \mid e^{\text {era }} \in E_{0}^{\text {era }}\right\} \subset E^{\text {err }}$ as ordinary edges, and to the edge $U \in E^{\mathrm{err}}$ as the special edge.

Let $G^{\text {era }}$ be the graph corresponding to erasures defined by $G^{\mathrm{err}}$ as in Proposition 5 Namely, $G^{\text {era }}=\left([n] \cup U, E^{\text {era }}\right)$ where

$$
E^{\mathrm{era}}=\left\{([n] \cup U) \backslash\left(e_{1}^{\mathrm{err}} \cup e_{2}^{\mathrm{err}}\right) \mid e_{1}^{\mathrm{err}}, e_{2}^{\mathrm{err}} \in E^{\mathrm{err}}\right\}
$$

Taking a closer look at the edge set $E^{\mathrm{era}}$, if an edge $e^{\text {era }}$ in $E^{\mathrm{era}}$ is defined by two ordinary edges of $E^{\mathrm{err}}$, then it is not hard to verify that $U \subseteq e^{e r a}$. If an edge $e^{e r a}$ in $E^{\text {era }}$ is defined by the special edge $U$ and an ordinary edge $e \in E^{\text {err }}$, then $e^{\text {era }}=[n] \backslash e$. As the ordinary edge $e \in E^{\mathrm{err}}$, by definition, equals $[n] \backslash e_{0}^{\text {era }}$ for an edge $e_{0}^{\text {era }} \in E_{0}^{\text {era }}$ we conclude that $e^{\mathrm{era}}=e_{0}^{\mathrm{era}}$. Finally, if an edge $e$ in $E^{\mathrm{era}}$ is defined solely by $U$ (i.e., we set $e_{1}=e_{2}=U$ ), then $e=[n]$. All in all, we conclude that the edge set $E^{\text {era }}$ equals the edges $E_{0}^{\text {era }} \cup\{[n]\}$ and an additional set of edges $e^{\text {era }}$ for which $U \subseteq e^{\text {era }}$.
We now show that $p_{k}\left(G^{\text {err }}\right)=q_{k}\left(G_{0}^{\text {era }}\right)$. We start by studying codes for $G_{0}^{\text {era }}$ and $G^{\text {era }}$. For any code $C_{0}: F^{k} \rightarrow$ $F^{n}$ for $G_{0}^{\text {era }}$, define the code $C: F^{k} \rightarrow F^{n+(n-k)}$ for $G^{\text {era }}$ in which for every message $m$ it holds that $C(m)=C_{0}(m)$ on the first $[n]$ entries, that $C(m)=m$ on entries $n+1, \ldots, n+k$ and that $C(m)$ equals the symbol $a \in F$ for the remaining entries $n+k+1, \ldots, 2 n-k$. Here, we use the fact that $n-k \geq k$. Similarly, for any code $C: F^{k} \rightarrow F^{n+n-k}$ for $G^{\text {era }}$, let the code $C_{0}: F^{k} \rightarrow F^{n}$ for $G_{0}^{\text {era }}$ be the restriction of $C$ to the first $n$ entries. It is now not hard to verify that $C_{0}$ is good for $G_{0}^{\text {era }}$ if and only if $C$ is good for $G^{\text {era }}$. More specifically, let $C_{0}$ be a code that is good for $G_{0}^{\text {era }}$, and let $D_{0}$ be the corresponding decoder. For any message $m$ and edge $e=e_{0}^{\text {era }}$ it holds that $D_{0}\left(\left(C_{0}\right)_{e}(m)\right)=m$. To show that $C$ is good for $G^{\text {era }}$ we define the decoder $D$, that for $e^{\text {era }} \in E^{\text {era }}$ either runs $D_{0}$ on the first $n$ entries of $C$ if $e^{\text {era }} \subseteq[n]$, or decodes using the identity mapping from $U$ if $U \subseteq e^{\text {era }}$. For the opposite direction, let $C$ be good for $G^{\text {era }}$, and let $D$ be the corresponding decoder. To show that $C_{0}$ is good for $G_{0}^{\text {era }}$ we define the decoder $D_{0}$ as the restriction of $D$ that takes into account only the first $n$ entries of $C$. Correctness follows as $E_{0}^{\mathrm{era}} \subseteq E^{\mathrm{era}}$ and as $C_{0}$ is a restriction of $C$ to the first $n$ entries.

To show that $p_{k}\left(G^{\mathrm{err}}\right)=q_{k}\left(G_{0}^{\text {era }}\right)$, let $N=2 n-k$ and let $C: F^{k} \rightarrow F^{N}$ be any encoder. By Proposition 5, $C$ is good for $G^{e r r}$ if and only if $C$ is good for $G^{\text {era }}$. By the discussion above, $C$ is good for $G^{\text {era }}$ if and only if the corresponding $C_{0}$ is good for $G_{0}^{\mathrm{era}}$. Thus, $C_{0}$ is good for $G_{0}^{\mathrm{era}}$ if and only if $C$ is good for $G^{\mathrm{err}}$. Optimizing over $|F|$, we conclude $p_{k}\left(G^{\mathrm{err}}\right)=$ $q_{k}\left(G_{0}^{\text {era }}\right)$. As the reductions described above between $C$ and $C_{0}$ preserves linearity, we also conclude that $p_{k, l i n}\left(G^{\mathrm{err}}\right)=$ $q_{k, l i n}\left(G_{0}^{\text {era }}\right)$.

By Theorem 3 the gap between the $q_{k}$ parameter and the $q_{k, \text { lin }}$ parameter for erasure codes stated in Proposition $3 \mathrm{im}-$ plies a gap between the $p_{k}$ parameter and the $p_{k, l i n}$ parameter for error correcting codes. We summarize this results in the following corollary.

## Corollary 4 (Non-linear codes outperform linear codes)

For $k=2$, there exists a hypergraph $G$ with $p_{k, l i n}(G)=5$ and yet $p_{k}(G)=3$.

## V. ERROR DETECTION

Similar to the case of errors and erasures, one can define analogs of Definitions 1 and 5 for the case of error detection. Namely, for a given hypergraph $G=([n], E)$ the $r_{k}$ parameter defined below equals the minimum size alphabet of an $(n, k)$
error detection code that can detect error patters represented by $E$.

Definition 6 (The $r_{k}$ parameter) Let $G=([n], E)$ be a hypergraph on the vertex set $[n]=\{1, \ldots, n\}$, and let $k$ be an integer. Let $r_{k}(G)$ denote the smallest size $r$ of an alphabet $F$ for which there exist an encoding function

$$
C: F^{k} \rightarrow F^{n}
$$

and a decoding function

$$
D: F^{n} \rightarrow\{\text { error, no-error }\}
$$

such that for every edge $e \in E$, every message $m \in F^{k}$, and every error vector $v=\left(v_{1}, \ldots, v_{n}\right) \in F^{n}$,
$D\left(C(m) \diamond_{e} v\right)=$ "error" if and only if $C(m) \neq C(m) \diamond_{e} v$, ( $\diamond_{e}$ is defined in Definition 5]).

Similar to Definition 11 in Definition $6 r_{k}(G)$ is defined if and only if all edges of $G$ are of size at most $n-k$, otherwise we define $r_{k}(G)=\infty$. Also, similar to Proposition 55 the following proposition is proven from basic principles (its proof is sketched here for completeness).

## Proposition 6 (Detecting errors vs. correcting erasures)

Let $n$ and $k$ be parameters such that $n-k \geq k$. For $a$ hypergraph $G=([n], E)$, let $\bar{G}=([n], \bar{E})$ be the hypergraph for which $\bar{E}=\{[n] \backslash e \mid e \in E\}$. Then, $r_{k}(G)=q_{k}(\bar{G})$.

Proof: Assume that $\bar{C}$ is a good erasure code for $\bar{G}$. The same code can be used for detection on G. Namely, given a received word $y$, to check if $y$ is corrupted in locations corresponding to $e \in E$, decode to $m$ using $y_{\bar{e}}$ (via the erasure decoding) and compare $\bar{C}(m)$ to $y$. For the other direction, assume that $C$ is a good detection code for $G$. Use the same code $C$ for erasures. To decode from $C_{\bar{e}}(m)$, construct the collection $Y$ of size $|F|^{n-|\bar{e}|}$ of words $y \in F^{n}$ that equal $C_{\bar{e}}(m)$ on the locations of $\bar{e}$ and otherwise equal a (distinct) word in $F^{n-|\bar{e}|}$. As $C$ is a detection code for errors with support $e=[n] \backslash \bar{e}$, we can detect the unique $y \in Y$ that is a codeword, and accordingly decode $m$.

## VI. AVERAGE ERROR $\varepsilon$

In what follows, we generalize the $q_{k}, p_{k}$, and $r_{k}$ parameters to include a decoding error. In our prior work [8], for $k=2$ in the context of erasures, we considered decoding error when averaged over the message set $F^{k}$. We here consider a looser notion of error that is also averaged over edges in the edge set $E$ of the hypergraph at hand. As shown below, allowing a slight error in decoding will in turn allow the construction of codes with small alphabet sizes (independent of the blocklength $n$ ).

Definition 7 (The $q_{\varepsilon, k}, p_{\varepsilon, k}$, and $r_{\varepsilon, k}$ parameters) Let $k$ be an integer. Let $G=([n], E)$ be a hypergraph on the vertex set $[n]$ and let $\varepsilon>0$. Let $q_{\varepsilon, k}(G)$ denote the smallest size $q$ of an alphabet $F$ for which there exist an encoding function $C: F^{k} \rightarrow F^{n}$ and a decoding function $D:(F \cup\{\perp\})^{n} \rightarrow F^{k}$ such that

$$
\underset{e, m}{\operatorname{Pr}}\left[D\left(C_{e}(m)\right)=m\right] \geq 1-\varepsilon
$$

where $m$ is uniformly chosen from $F^{k}$, and $e$ is uniformly chosen from $E$. One may define $p_{\varepsilon, k}(G)$ and $r_{\varepsilon, k}(G)$ in an analogous manner.

We will need the following approximate version of coloring.
Definition 8 (Hypergraph $(1-\varepsilon)$ - $k$-coloring) $A$ valid ( $1-$ $\varepsilon)$ - $k$-coloring of a hypergraph $G=(V, E)$ is an assignment of colors to its vertices $V$ so that for at least $(1-\varepsilon)|E|$ edges $e \in E$, the vertices of $e$ are assigned to at least $k$ colors. The $(1-\varepsilon)$ - $k$-chromatic number $\chi_{\varepsilon, k}(G)$ of $G$ is the minimum number of colors that allows a valid $(1-\varepsilon)$ - $k$-coloring of $G$.

Theorem 5 Let $G=(V, E)$ be a hypergraph, and $\varepsilon>0$ a parameter. Then $q_{\varepsilon, k}(G) \leq\left[\chi_{\varepsilon, k}(G)-1\right]_{p p}$. In particular, let $n$ and $k$ be any integers, $q_{\varepsilon, k}\left(\kappa_{n, k}\right) \leq O\left(k^{2} / \varepsilon\right)$.

Proof: The proof that $q_{\varepsilon, k}(G) \leq\left[\chi_{\varepsilon, k}(G)-1\right]_{p p}$ is almost identical to the proof of Theorem 1 (presented in [8]) and is obtained by replacing $q_{k}$ and $\chi_{k}$ by $q_{\varepsilon, k}$ and $\chi_{\varepsilon, k}$ respectively. The second part of the theorem follows by showing that $\kappa_{n, k}$ can be $(1-\varepsilon)-k$ colored using $k^{2} / \varepsilon$ colors. Consider partitioning $[n]$ into $k^{2} / \varepsilon$ subsets, each of size $\varepsilon n / k^{2}$. Assign the same color to all the vertices in the same subset, and distinct colors to vertices in distinct subsets. We now show that this is a $(1-\varepsilon)-k$ coloring. The fraction of edges that are assigned to at least $k$ colors is

$$
\frac{\binom{k^{2} / \varepsilon}{k} \cdot\left(\varepsilon n / k^{2}\right)^{k}}{\binom{n}{k}}
$$

Now, for integers $a$ and $b,\binom{a}{b}=\frac{\prod_{j=0}^{b-1}(a-j)}{b!}$, and $a^{b} \geq$ $\prod_{j=0}^{b-1}(a-j) \geq(a-b)^{b} \geq\left(1-b^{2} / a\right) a^{b}$, thus

$$
\frac{\left(1-b^{2} / a\right) a^{b}}{b!} \leq\binom{ a}{b} \leq \frac{a^{b}}{b!}
$$

Therefore $\frac{\binom{k^{2} / \varepsilon}{k} \cdot\left(\varepsilon n / k^{2}\right)^{k}}{\binom{n}{k}} \geq 1-\varepsilon$.
Notice that implications corresponding to those in Theorem 5 on parameters $p_{\varepsilon, k}$ and $r_{\varepsilon, k}$ can be derived using Theorem 3 and Proposition 6 respectively.

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    ${ }^{1}$ The minimum alphabet size of an $(n, k)$ MDS code is unknown, see Conjectures 1 and 2 in the paper.

