

Synthesizing Correlated Randomness using Algebraic Structured Codes

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Abstract

In this problem, Alice and Bob, are provided X_1^n and X_2^n that are IID $p_{X_1X_2}$. Alice and Bob can communicate to Charles over (noiseless) links of rate R_1 and R_2 , respectively. Their goal is to enable Charles generate samples Y^n such that the triple (X_1^n, X_2^n, Y^n) has a PMF that is close, in total variation, to $\prod p_{X_1X_2Y}$. In addition, the three parties may possess shared common randomness at rate C . We address the problem of characterizing the set of rate triples (R_1, R_2, C) for which the above goal can be accomplished. We build on our recent findings and propose a new coding scheme based on coset codes. We analyze its information-theoretic performance and derive a new inner bound. We identify examples for which the derived inner bound is analytically proven to contain rate triples that are not achievable via any known unstructured code based coding techniques. Our findings build on a variant of soft-covering which generalizes its applicability to the algebraic structured code ensembles. This adds to the advancement of the use structured codes in network information theory.

I. INTRODUCTION

The task of generating correlated randomness at different terminals in a network finds its applications in several communication and computing paradigms. This task is also fundamental to several cryptographic protocols. In this article, we provide a new information-theoretic coding framework for generating such correlated randomness in network scenarios.

We consider the scenario which was originally studied by authors in [1], as depicted in Fig 1. Three distributed parties, say Alice, Bob and Charles, have to generate samples that are independent and identically distributed (IID) with a target probability mass function (PMF) $p_{X_1X_2Y}$. Alice and Bob are provided with samples that are IID according to $p_{X_1X_2}$ - the marginal of the target PMF $p_{X_1X_2Y}$. They

have access to unlimited private randomness and share noiseless communication links of rates R_1, R_2 with Charles. In addition, the three parties share common randomness at rate C . The authors in [1] provided a set of sufficient conditions, i.e., an achievable rate region for such a scenario. However, can this rate-region be improved? This article answers the above question in the affirmative.

It is well established that traditional coding techniques using unstructured codes do not achieve optimality for the several multi-terminal scenarios. For instance, the work by Körner-Marton [2] demonstrated this sub-optimality for a classical distributed lossless compression problem with symmetric binary sources using random linear codes. We harness analogous gains for the problem of generating correlated randomness at distributed parties. Specifically, we propose a coding scheme based on coset codes, analyze its information-theoretic performance and thereby derive a new inner bound (see Thm. 1). We identify an example for which the derived inner bound is analytically proven to contain rate triples that are not achievable in the earlier known results [1]. While the derived inner bound does not subsume the one characterized in [], one can adopt the technique in [3, Sec. VII] - also demonstrated in a related context [4] - to derive an inner bound that subsumes the inner bounds derived in [1] and Thm. 1.

The problem of generating correlated randomness can be traced back to Wyner [5], whose work discovered the important technical tool, called the *soft covering*. This tool has found its application in diverse fields including cryptography and quantum information theory. The work in [1] further refined this tool by introducing a joint-typicality based application. As we illustrate in the sequel, this work adds another dimension to our current understanding of soft covering, what we term as the *change of measure soft covering*.

A renewed interest in soft covering led Cuff [6], [7] to consider a point-to-point (PTP) version of the scenario depicted in Fig. 1, wherein Bob (or X_2) is absent. A side-information based scenario was subsequently studied in [8] and a converse provided in [1]. In [1] we studied the above scenario using unstructured coding techniques. A similar sequence of problems were also studied in the quantum setting [9]–[11].

While all of the above works leverage the unstructured IID random codes, it has been proven that algebraic structured codes provide gains in network communication involving distributed encoders [4], [12]–[17]. Motivated by this, we consider the distributed correlation synthesis problem depicted in Fig. 1 and present a new achievable rate-region using structured coding techniques. We highlight two main challenges in this endeavour. The first challenge is to be able to achieve rates corresponding to non-uniform distributions. In particular, codewords within a random linear code has uniform empirical distributions. This requires us to enlarge our codes to be able to identify codeword with the desired single-letter distribution. We address this challenge by using a random shifts of cosets of a linear code as our

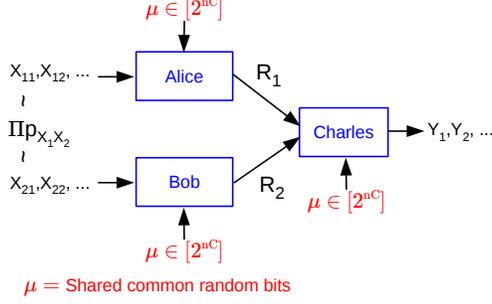


Fig. 1. Source Coding for Synthesizing Correlated Randomness

code, henceforth referred to as Unionized Coset Codes (UCCs) [16]. The second challenge concerns the statistical dependence among codewords of a coset code. In contrast to IID codes, the codewords of a UCC are only pairwise independent [18]. This prevents us from using the Chernoff concentration bound. We therefore develop novel techniques for our information theoretic study.

II. PRELIMINARIES AND PROBLEM STATEMENT

We supplement standard information theory notation with the following. For a PMF p_X , we let $p_X^n = \prod_{i=1}^n p_X$. For an integer $n \geq 1$, $[n] \triangleq \{1, \dots, n\}$. The total variation between PMFs p_X and q_X defined over \mathcal{X} is denoted $\|p_X - q_X\|_1 = \frac{1}{2} \sum_{x \in \mathcal{X}} |p_X(x) - q_X(x)|$. \mathbb{F}_p is used to denote a finite field of size p with addition \oplus .

Building on this, we address the network scenario (Fig. 1) for which we state the problem below. In the following, we let $\underline{X} = (X_1, X_2)$, $\underline{x}^n = (x_1^n, x_2^n)$.

Definition 1. Given a PMF $p_{X_1 X_2 Y}$ on $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}$, a rate triple (R_1, R_2, C) is achievable, if $\forall \epsilon > 0$ and sufficiently large n , there exists 2^{nC} randomized encoder pairs $E_j^{(\mu)} : \mathcal{X}_j^n \rightarrow [\Theta_j] : j \in [2], \mu \in [2^{nC}]$, and a corresponding collection of 2^{nC} randomized decoders $D^{(\mu)} : [\Theta_1] \times [\Theta_2] \rightarrow \mathcal{Y}^n$ for $\mu \in [2^{nC}]$ such that $\left| p_{\underline{X}^n Y^n}^n - p_{\underline{X}^n Y^n} \right|_1 \leq \epsilon$, $\frac{1}{n} \log_2 \Theta_j \leq R_j + \epsilon : j \in [2]$, where

$$p_{\underline{X}^n Y^n}(\underline{x}^n, y^n) = \sum_{\mu \in [2^{nC}]} 2^{-nC} \sum_{\substack{(m_1, m_2) \in \\ [\Theta_1] \times [\Theta_2]}} p_{\underline{X}^n Y^n}^n(\underline{x}^n, y^n)$$

$$p_{M_1 | X_1^n}^{(\mu)}(m_1 | x_1^n) p_{M_2 | X_2^n}^{(\mu)}(m_2 | x_2^n) p_{Y^n | M_1, M_2}^{(\mu)}(y^n | m_1, m_2)$$

$p_{M_j | X_j^n}^{(\mu)} : j \in [2], p_{Y^n | M_1, M_2}^{(\mu)}$ are the PMFs induced by the two randomized encoders and decoder respectively, corresponding to common randomness message μ . We let $\mathcal{R}_d(p_{\underline{X}^n Y^n})$ denote the set of achievable rate triples.

Theorem 1 provides a new characterization of $\mathcal{R}_d(p_{\underline{X}Y})$ based on coset codes, for the above described problem statement. This characterization provides a new inner bound to the achievable rate-region. An essential aspect of our work is the identification of a PMF $p_{X_1X_2Y}$ for which the coding scheme described in [1], [19] is strictly sub-optimal.

III. DISTRIBUTED SOFT COVERING USING ALGEBRAIC STRUCTURED RANDOM CODES

A. Change of Measure Soft Covering

Before presenting the main result of the paper, we develop the necessary tools and provide a lemma which is crucial for the upcoming results. This lemma extends the cloud mixing result of [7] with a mismatched codebook generation process. The lemma is as follows.

Lemma 1. *Consider a PMF p_{XY} on $\mathcal{X} \times \mathcal{Y}$, and let R be a finite non-negative integer. Additionally, assume that there exists some set $\bar{\mathcal{X}}$ containing the set \mathcal{X} , with $p_{XY}(x, y) = 0$ for all $x \in \bar{\mathcal{X}} \setminus \mathcal{X}$. Suppose q_X is any PMF on the set $\bar{\mathcal{X}}$ such that the PMF p_X is absolutely continuous with respect to the q_X . Let a random code $\mathbb{C} \triangleq \{X^n(m) : m \in [2^{nR}]\}$ be defined as a collection of codewords chosen pairwise independently from the set $\bar{\mathcal{X}}$ according to the PMF q_X^n . Then we have for $R \geq H_q(X) - H_p(Y|X) = I_p(X; Y) - H_p(X) + H_q(X)$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{C}} \left[\sum_{y^n \in \mathcal{Y}^n} \left| p_Y^n(y^n) - \frac{1}{M} \sum_{m=1}^{2^{nR}} \frac{p_X^n(X^n(m))}{q_X^n(X^n(m))} p_{Y|X}^n(y^n|X^n(m)) \right| \right] = 0$$

Proof. The proof follows similar analysis as the proof of [20, Lemma 19] as hence is omitted. □

B. Main Result

Our main result is the characterization of $\mathcal{R}_s(p_{\underline{X}Y})$ which is the inner bound to $\mathcal{R}_d(p_{\underline{X}Y})$. In the following, we let $\underline{X} = (X_1, X_2)$, $\underline{W} = (W_1, W_2)$, $\underline{x} = (x_1, x_2)$ and $\underline{w} = (w_1, w_2)$.

Theorem 1. *Given a PMF $p_{X_1X_2Y}$, let $\mathcal{P}(p_{X_1X_2Y})$ denote the collection of all PMFs $p_{QW_1W_2\underline{X}Y}$ defined on $\mathcal{Q} \times \mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{X} \times \mathcal{Y}$ such that (i) $p_{\underline{X}Y}(\underline{x}, y) = \sum_{(q, \underline{w}) \in \mathcal{Q} \times \mathcal{W}} p_{QW\underline{X}Y}(q, \underline{w}, \underline{x}, y)$ for all $(\underline{x}, y) \in \mathcal{X} \times \mathcal{Y}$, (ii) $W_1 - QX_1 - QX_2 - W_2$ and $\underline{X} - Q\underline{W} - Y$ are Markov chains, (iii) $|\mathcal{W}_1| \leq |\mathcal{X}_1|$, $|\mathcal{W}_2| \leq |\mathcal{X}_2|$. Further, let $\beta(p_{QW\underline{X}Y})$ denote the set of rates and common randomness triple (R_1, R_2, C) that satisfy*

$$R_1 \geq I(X_1; W_1|W_2, Q) + I(W_1 \oplus W_2; W_2|Q)$$

$$R_2 \geq I(X_2; W_2|W_1, Q) + I(W_1 \oplus W_2; W_1|Q)$$

$$R_1 + C \geq I(\underline{X}; W_1|W_2, Q) + I(Y; W_1|\underline{X}, Q) + I(W_1 \oplus W_2; W_2|Q)$$

$$R_2 + C \geq I(\underline{X}; W_2 | W_1, Q) + I(Y; W_2 | \underline{X}, Q) + I(W_1 \oplus W_2; W_1 | Q)$$

$$R_1 + R_2 + C \geq I(\underline{X}; W_1 | W_2, Q) + I(\underline{X}; W_2 | W_1, Q) + I(W_1 \oplus W_2; W_1 | Q) + I(W_1 \oplus W_2; W_2 | Q), \quad (1)$$

where the above information theoretic terms are evaluated with respect to the PMF $p_{QW_1W_2\underline{X}Y}$. Let

$$\mathcal{R}_s(p_{\underline{X}Y}) \triangleq \text{Closure} \left(\bigcup_{p_{QW_1W_2\underline{X}Y} \in \mathcal{P}(p_{X_1X_2Y})} \beta(p_{QW_1W_2\underline{X}Y}) \right) \quad (2)$$

We have

$$\mathcal{R}_s(p_{\underline{X}Y}) \subseteq \mathcal{R}_d(p_{\underline{X}Y}).$$

In other words, the rate triple $(R_1, R_2, C) \in \left(\bigcup_{p_{QW_1W_2\underline{X}Y} \in \mathcal{P}(p_{X_1X_2Y})} \beta(p_{QW_1W_2\underline{X}Y}) \right)$ is achievable.

Note that the rate-region obtained in Theorem 2 of [19] contains the constraint $R_1 + R_2 + C \geq I(X_1X_2Y; W_1W_2 | Q)$. Hence when $2H(W_1 \oplus W_2 | Q) < H(W_1, W_2 | Q)$, the above theorem gives a lower sum rate constraint. As a result, the rate-region above contains points that are not contained within the rate-region provided in [19]. To illustrate this fact further, consider the following example.

Example 1. Let X_1 and X_2 be a pair of binary symmetric correlated sources with $P(X_2 = 1 | X_1 = 0) = p$, for some $p \in (0, 0.5)$. Let $Y = X_1 \oplus X_2 \oplus Q$, where $P(Q = 1) = q$, for some $q \in (0, 0.5)$. Consider $q = p = 0.1$ for a numerical evaluation. Let us first consider the inner bound $\mathcal{R}_u(p_{\underline{X}, Y})$ to the rate region $\mathcal{R}(p_{\underline{X}, Y})$ given in [1], developed using unstructured code ensemble. Due to symmetry in the example, it turns out that the search over the auxiliary random variables for minimization reduces to a single-parameter minimization which can be computed through derivative techniques. The computation details are not provided for the sake of brevity. In particular, the minimum value of $R_1 + R_2 + C$ can be computed to be 1.3965. Next let us consider the new inner bound $\mathcal{R}_s(p_{\underline{X}, Y})$ developed using structured code ensemble (Theorem 1). The minimum value of $R_1 + R_2 + C$ can be computed to be 0.9596.

The results can also be verified for the special case of $q = 0$ which we provide in the following. Using the arguments given in proof of Proposition 1 of [2], one can show that

$$\begin{aligned} \mathcal{R}_u(p_{\underline{X}, Y}) &= \{(R_1, R_2, C) : R_1 \geq h_b(p), R_2 \geq h_b(p), \\ &R_1 + R_2 \geq 1 + h_b(p), C \geq 0\}. \end{aligned}$$

Next let us consider the new inner bound $\mathcal{R}_s(p_{\underline{X}, Y})$ developed using structured code ensemble (Theorem 1). By choosing $W_1 = X_1$ and $W_2 = X_2$, we see that the following triple of rates is achievable:

$$\{(R_1, R_2, C) : R_1 \geq h_b(p), R_2 \geq h_b(p), C \geq 0\}.$$

In fact, one can show that this is optimal using the side information argument. If X_2 is sent losslessly, then from the converse argument in the side information case, we see that $R_1 \geq H(X_2|X_1) = h_b(p)$.

IV. PROOF OF DISTRIBUTED SOFT COVERING USING ALGEBRAIC STRUCTURED RANDOM CODES

The coding strategy used here is based on Unionized Coset Codes, defined in Definition (2). The structure in these codes provides a method to exploit the structure present in the stochastic processing applied by decoder, i.e., $P_{Y|W_1+W_2}$. Using this technique, we aim to strictly reduce the rate constraints compared to the ones obtained in Theorem 1 of [1].

Let $\mu \in [2^{nC}]$ denote the common randomness shared amidst all terminals. The first encoder uses a part of the entire common randomness available to it, say C_1 bits out of the C bits, which is denoted by $\mu_1 \in [2^{nC_1}]$. Similarly, let $\mu_2 \in [2^{nC_2}]$ denote the common randomness used by the second encoder. Our goal is to prove the existence of PMFs $p_{M_1|X_1}^{(\mu_1)}(m_1|x_1^n) : x_1^n \in \mathcal{X}_1^n, m_1 \in [\Theta_1], \mu_1 \in [2^{nC_1}]$, $p_{M_2|X_2}^{(\mu_2)}(m_2|x_2^n) : x_2^n \in \mathcal{X}_2^n, m_2 \in [\Theta_2], \mu_2 \in [2^{nC_2}]$, $p_{Y^n|M_1, M_2}(y^n|m_1, m_2) : y^n \in \mathcal{Y}^n, (m_1, m_2) \in [\Theta_1] \times [\Theta_2]$ such that

$$\mathcal{Q} \triangleq \frac{1}{2} \sum_{\underline{x}^n, y^n} \left| p_{\underline{X}Y}^n(\underline{x}^n, y^n) - \sum_{\mu \in [2^{nC}]} \sum_{\substack{m_1 \in [\Theta_1], \\ m_2 \in [\Theta_2]}} \frac{p_{\underline{X}}^n(\underline{x}^n)}{2^{nC}} p_{M_1|X_1}^{(\mu_1)}(m_1|x_1^n) p_{M_2|X_2}^{(\mu_2)}(m_2|x_2^n) p_{Y^n|M}^{(\mu)}(y^n|m) \right| \leq \varepsilon,$$

$$\frac{\log \Theta_j}{n} \leq R_j + \epsilon : j \in [2], \quad (3)$$

for sufficiently large n . Fix a block length $n > 0$, a positive integer N and a finite field \mathbb{F}_p . Further, let W_1 and W_2 be random variables defined on the alphabets \mathcal{W}_1 and \mathcal{W}_2 , respectively, where $\mathcal{W}_1 = \mathcal{W}_2 = \mathbb{F}_p$, and let $Z \triangleq W_1 \oplus W_2$. In building the code, we use the Unionized Coset Codes (UCCs) [16] defined as below. These codes involve two layers of codes (i) a coarse code and (ii) a fine code. The coarse code is a coset of the linear code and the fine code is the union of several cosets of the linear code.

For a fixed $k \times n$ matrix $G \in \mathbb{F}_p^{k \times n}$ with $k \leq n$, and a $1 \times n$ vector $B \in \mathbb{F}_p^n$, define the coset code as

$$\mathbb{C}(G, B) \triangleq \{x^n : x^n = a^k G + B, \text{ for some } a^k \in \mathbb{F}_p^k\}.$$

In other words, $\mathbb{C}(G, B)$ is a shift of the row space of the matrix G . The row space of G is a linear code. If the rank of G is k , then there are p^k codewords in the coset code.

Definition 2. An (n, k, l, p) UCC is a pair (G, h) consisting of a $k \times n$ matrix $G \in \mathbb{F}_p^{k \times n}$, and a mapping $h : \mathbb{F}_p^l \rightarrow \mathbb{F}_p^n$. In the context of UCC, define the composite code as $\mathbb{C} = \bigcup_{i \in \mathbb{F}_p^l} \mathbb{C}(G, h(i))$.

For every $\mu \triangleq (\mu_1, \mu_2)$, consider two UCCs $(G, h_1^{(\mu_1)})$ and $(G, h_2^{(\mu_2)})$, each with parameters (n, k, l_1, p) and (n, k, l_2, p) , respectively. Note that, for every $\mu \in [N]$, the generator matrix G remains the same.

For each (μ_1, μ_2) , the generator matrix G along with the function $h_1^{\mu_1}$ and $h_2^{\mu_2}$ generates p^{k+l_1} and p^{k+l_2} codewords, respectively. Each of these codewords are characterized by a triple (a_i, m_i, μ_i) , where $a_i \in \mathbb{F}_p^k$ and $m_i \in \mathbb{F}_p^{l_i}$ corresponds to the coarse code and the fine code indices, respectively, for $i \in [2]$. Let $\mathbf{w}_1(a_1, m_1, \mu_1)$ and $\mathbf{w}_2(a_2, m_2, \mu_2)$ denote the codewords associated with Alice and Bob, generated using the above procedure, respectively, where $\mathbf{w}_1(a_1, m_1, \mu_1) \triangleq a_1 G + h_1^{(\mu_1)}(i)$, and $\mathbf{w}_2(a_2, m_2, \mu_2) \triangleq a_2 G + h_2^{(\mu_2)}(j)$.

Consider the collections $c_1 = (c_1^{(\mu_1)} : 1 \leq \mu_1 \leq 2^{nC_1})$ where $c_1^{(\mu_1)} = (\mathbf{w}_1(l_1, \mu_1) : 1 \leq l_1 \leq 2^{n\tilde{R}_1})$ and $c_2 = (c_2^{(\mu_2)} : 1 \leq \mu_2 \leq 2^{nC_2})$ where $c_2^{(\mu_2)} = (\mathbf{w}_2(l_2, \mu_2) : 1 \leq l_2 \leq 2^{n\tilde{R}_2})$. For this collection, we let

$$\begin{aligned} E_{L_1|X_1^n}^{(\mu_1)}(a_1, m_1|x_1^n) &\triangleq \sum_{w_1^n \in T_\delta(W_1|x_1^n)} p^n \frac{p_{W_1|X_1^n}^n(w_1^n|x_1^n)}{2^{nS_1}(1+\eta)} \mathbb{1}_{\{\mathbf{w}_1(a_1, m_1, \mu_1) = w_1^n\}}, \\ E_{L_2|X_2^n}^{(\mu_2)}(a_2, m_2|x_2^n) &\triangleq \sum_{w_2^n \in T_\delta(W_2|x_2^n)} p^n \frac{p_{W_2|X_2^n}^n(w_2^n|x_2^n)}{2^{nS_2}(1+\eta)} \mathbb{1}_{\{\mathbf{w}_2(a_2, m_2, \mu_2) = w_2^n\}}. \end{aligned} \quad (4)$$

The definition of $E_{L_1|X_1^n}^{(\mu_1)}$ and $E_{L_2|X_2^n}^{(\mu_2)}$ can be thought of as encoding rules that do not exploit the additional rebate obtained by using binning techniques, specifically in a distributed setup.

A. Binning of Random Encoders

We next proceed to binning the above constructed collection of random encoders. Since, UCC is already a union of several cosets, we associate a bin to each coset, and place all the codewords of a coset in the same bin. For each $i \in \mathbb{F}_p^{l_1}$ and $j \in \mathbb{F}_p^{l_2}$, let $\mathcal{B}_1^{(\mu_1)}(i) \triangleq \mathbb{C}(G, h_1^{(\mu_1)}(i))$ and $\mathcal{B}_2^{(\mu_2)}(j) \triangleq \mathbb{C}(G, h_2^{(\mu_2)}(j))$ denote the i^{th} and the j^{th} bins, respectively. Formally, we define the following PMFs.

$$p_{M_1|X_1^n}^{(\mu_1)}(m_1|x_1^n) = \begin{cases} \mathbb{1}_{\{m_1=0\}} & \text{if } s_i^{(\mu_1)}(x_1^n) > 1, \\ 1 - s_i^{(\mu_1)}(x_1^n) & \text{if } m_1 = 0 \text{ and } s_i^{(\mu_1)}(x_1^n) \in [0, 1], \\ \sum_{a_i \in \mathbb{F}_p^k} E_{L_i|X_i^n}^{(\mu_i)}(a_i, m_i|x_i^n) & \text{if } m_i \neq 0 \text{ and } s_i^{(\mu_i)}(x_i^n) \in [0, 1], \end{cases}$$

for all $x_i^n \in T_\delta(X_i)$, $s_i^{(\mu_i)}(x_i^n)$ defined as $s_i^{(\mu_i)}(x_i^n) \triangleq \sum_{a_i \in \mathbb{F}_p^k} \sum_{m_i \in \mathbb{F}_p^{l_i}} E_{L_i|X_i^n}^{(\mu_i)}(a_i, m_i|x_i^n)$ and $i \in [2]$. For $x_1^n \notin T_\delta(X_1)$, we let $p_{M_1|X_1^n}^{(\mu_1)}(m_1|x_1^n) = \mathbb{1}_{\{m_1=0\}}$.

With this definition note that, $\sum_{m_1=0}^{2^{nR_1}} p_{M_1|X_1^n}^{(\mu_1)}(m_1|x_1^n) = 1$ for all $\mu_1 \in [2^{nC_1}]$ and $x_1^n \in \mathcal{X}_1^n$ and similarly, $\sum_{m_2=0}^{2^{nR_2}} p_{M_2|X_2^n}^{(\mu_2)}(m_2|x_2^n) = 1$ for all $\mu_2 \in [2^{nC_2}]$ and $x_2^n \in \mathcal{X}_2^n$.

Also, note that the effect of introducing binning (by defining the above PMFs) is in reducing the communication rates from (S_1, S_2) to (R_1, R_2) , where $R_i = \frac{l_i}{n} \log p, i \in \{1, 2\}$. Now, we move on to describing the decoder.

B. Decoder mapping

We create a decoder that takes as an input a pair of bin numbers and produces a sequence $W^n \in \mathbb{F}_p^n$. More precisely, we define a mapping $f^{(\mu)}$ for $\mu \triangleq (\mu_1, \mu_2)$, acting on the messages (m_1, m_2) as follows. On observing μ and the classical indices $(m_1, m_2) \in \mathbb{F}_p^{l_1} \times \mathbb{F}_p^{l_2}$ communicated by the encoder, the decoder constructs $D_{i,j}^{(\mu)} \triangleq \{\tilde{a} \in \mathbb{F}_p^k : \tilde{a}G + h_1^{(\mu_1)}(i) + h_2^{(\mu_2)}(j) \in \mathcal{T}_{\hat{\delta}}^{(n)}(Z)\}$, and $f^{(\mu)}(m_1, m_2)$

$$\triangleq \begin{cases} \tilde{a}G + h_1^{(\mu_1)}(i) + h_2^{(\mu_2)}(j) & \text{if } D_{i,j}^{(\mu_1, \mu_2)} \equiv \{\tilde{a}\} \\ w_0^n & \text{otherwise,} \end{cases} \quad (5)$$

where $\hat{\delta} = p\delta$ and w_0^n is an additional sequence added to \mathbb{F}_p^n . Further, $f^{(\mu)}(m_1, m_2) = w_0^n$ for $i = 0$ or $j = 0$. The decoder then performs a stochastic processing of the output and chooses y^n according to PMF $p_{Y^n|Z}^n(y^n | f^{(\mu)}(m_1, m_2))$. This implies the PMF $p_{Y^n|M_1M_2}^{(\mu)}(\cdot | \cdot)$ is given by

$$p_{Y^n|M_1M_2}^{(\mu)}(\cdot | m_1, m_2) = p_{Y^n|Z}^n(y^n | f^{(\mu)}(m_1, m_2)). \quad (6)$$

We now begin our analysis of the total variation term given in (3).

C. Analysis of Total Variation

Our goal is to prove the existence of a collections c_1, c_2 for which (3) holds. We do this via random coding. Specifically, we prove that $\mathbb{E}[K] \leq \epsilon$, where the expectation is over the ensemble of codebooks. The PMF induced on the ensemble of codebooks is as specified below. The codewords of the random codebook $C_i^{(\mu_i)} = (W_i(a_i, m_i, \mu_i) : a_i \in \mathbb{F}_p^k, m_i \in \mathbb{F}_p^{l_i})$ for each $\mu_i \in [2^{nC_i}]$ are only pairwise independent [16] and distributed with PMF $\mathbb{P}(W_i(a_i, m_i, \mu_i) = w_i^n) = \frac{1}{p^n}$ for each $i \in [2]$.

Step 1: Error caused by not covering

We begin by splitting K into two terms using the triangle inequality as $K \leq S + \tilde{S}$, where

$$S \triangleq \sum_{\underline{x}^n, \underline{y}^n} \left| p_{\underline{X}^n \underline{Y}^n}^n(\underline{x}^n, \underline{y}^n) - \sum_{\substack{\mu_1, \mu_2 \\ m_1 > 0, \\ m_2 > 0}} \frac{p_{\underline{X}^n}^n(\underline{x}^n) p_{M_1|X_1^n}^{(\mu_1)}(m_1 | x_1^n)}{2^{n(C_1+C_2)}} p_{M_2|X_2^n}^{(\mu_2)}(m_2 | x_2^n) p_{Y^n|\underline{M}}^{(\mu)}(y^n | \underline{m}) \right|,$$

$$\tilde{S} \triangleq \sum_{\underline{x}^n, \underline{y}^n} \left| \sum_{\mu_1, \mu_2} \sum_{m_1=0 \cup m_2=0} \frac{p_{\underline{X}^n}^n(\underline{x}^n)}{2^{n(C_1+C_2)}} p_{M_1|X_1^n}^{(\mu_1)}(m_1 | x_1^n) p_{M_2|X_2^n}^{(\mu_2)}(m_2 | x_2^n) p_{Y^n|\underline{M}}^{(\mu)}(y^n | \underline{m}) \right|.$$

Note that \tilde{S} captures the error induced by not covering $p_{\underline{X}^n \underline{Y}^n}^n$. For the term corresponding to \tilde{S} , we prove the following result by developing the following lemma below followed by a proposition.

Lemma 2. *For the above defined notations, for $i \in \{1, 2\}$, if $S_i \geq I(X_i; W_i) + \delta_{c_i}$, then the following holds true*

$$\frac{1}{2^{nC_i}} \sum_{\mu_i} \sum_{x_i^n} p_{X_i}^n(x_i^n) \mathbb{P} \left(\left[\sum_{a_i \in \mathbb{F}_p^k} \sum_{m_i \in \mathbb{F}_p^{l_i}} E_{L_i|X_i^n}^{(\mu_i)}(a_i, m_i | x_i^n) \right] > 1 \right) \leq \epsilon_{c_i} \quad (7)$$

Proof. The proof is provided in Appendix A-A. \square

Proposition 1. *There exist functions $\epsilon_{\tilde{S}}(\delta)$, and $\delta_{\tilde{S}}(\delta)$, such that for all sufficiently small δ and sufficiently large n , we have $\mathbb{E}[\tilde{S}] \leq \epsilon_{\tilde{S}}(\delta)$, if $S_1 > I(X_1; W_1) - H(W_1) + \log p + \delta_{\tilde{S}}$ and $S_2 > I(X_2; W_2) - H(W_2) + \log p + \delta_{\tilde{S}}$, where $\epsilon_{\tilde{S}}, \delta_{\tilde{S}} \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof is provided in Appendix B-A. \square

Now we move on to removing from S the error that is induced due to binning.

Step 2: Error caused by binning

Note that S can be simplified using the definitions of $P_{M_1|X_1^n}^{(\mu_1)}(\cdot|\cdot)$, $P_{M_2|X_2^n}^{(\mu_2)}(\cdot|\cdot)$, and $p_{Y^n|M}^{(\mu)}(y^n|m)$ as

$$S \triangleq \sum_{\underline{x}^n, y^n} \left| p_{XY}^n(\underline{x}^n, y^n) - \sum_{\substack{\mu_1, \mu_2 \\ m_1 > 0 \\ m_2 > 0}} \sum_{\substack{a_1 \in \mathbb{F}_p^k \\ a_2 \in \mathbb{F}_p^k}} \sum_{w_1, w_2 \in \mathbb{F}_p^n} \frac{p_X^n(\underline{x}^n)}{2^{n(C_1+C_2)}} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) E_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) \mathbb{1}_{\{\mathbf{w}_1(a_1, m_1, \mu_1) = w_1^n\}} \mathbb{1}_{\{\mathbf{w}_2(a_2, m_2, \mu_2) = w_2^n\}} p_{Y|Z}^n(y^n | f^{(\mu)}(m_1, m_2)) \right|,$$

where $E_{W_i^n|X_i^n}^{(\mu_i)}(w_i^n|x_i^n)$ is defined as

$$E_{W_i^n|X_i^n}^{(\mu_i)}(w_i^n|x_i^n) \triangleq \frac{p^n}{2^{nS_i}(1+\eta)} p_{W_i|X_i}^n(w_i^n|x_i^n) \mathbb{1}_{\{w_i^n \in T_\delta(W_i|x_i^n)\}} \mathbb{1}_{\{s_i^{(\mu_i)}(x_i^n) \leq 1\}}, \quad \text{for } i \in \{1, 2\}.$$

Further by defining $\gamma_{w_1^n}^{(\mu_1)}$ and $\zeta_{w_2^n}^{(\mu_2)}$ as

$$\begin{aligned} \gamma_{w_1^n}^{(\mu_1)} &\triangleq |\{\mathbf{w}_1(a_1, m_1, \mu_1) = w_1^n\}| = \sum_{m_1 > 0} \sum_{a_1 \in \mathbb{F}_p^k} \mathbb{1}_{\{\mathbf{w}_1(a_1, m_1, \mu_1) = w_1^n\}} \quad \text{and} \\ \zeta_{w_2^n}^{(\mu_2)} &\triangleq |\{\mathbf{w}_2(a_2, m_2, \mu_2) = w_2^n\}| = \sum_{m_2 > 0} \sum_{a_2 \in \mathbb{F}_p^k} \mathbb{1}_{\{\mathbf{w}_2(a_2, m_2, \mu_2) = w_2^n\}}. \end{aligned} \quad (8)$$

we bound S using triangle inequality as $S \leq S_1 + S_2$, where

$$\begin{aligned} S_1 &\triangleq \sum_{\underline{x}^n, y^n} \left| p_{XY}^n(\underline{x}^n, y^n) - \sum_{\mu_1, \mu_2} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} \frac{p_X^n(\underline{x}^n)}{2^{n(C_1+C_2)}} \gamma_{w_1^n}^{(\mu_1)} \zeta_{w_2^n}^{(\mu_2)} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) E_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) p_{Y|Z}^n(y^n | w_1^n \oplus w_2^n) \right|, \\ S_2 &\triangleq \sum_{\underline{x}^n, y^n} \sum_{\mu_1, \mu_2} \sum_{\substack{m_1 > 0 \\ m_2 > 0}} \sum_{\substack{a_1 \in \mathbb{F}_p^k \\ a_2 \in \mathbb{F}_p^k}} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} \frac{p_X^n(\underline{x}^n)}{2^{n(C_1+C_2)}} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) E_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) \mathbb{1}_{\{\mathbf{w}_1(a_1, m_1, \mu_1) = w_1^n\}} \mathbb{1}_{\{\mathbf{w}_2(a_2, m_2, \mu_2) = w_2^n\}} \left| p_{Y|Z}^n(y^n | w_1^n \oplus w_2^n) - p_{Y|Z}^n(y^n | f^{(\mu)}(m_1, m_2)) \right|. \end{aligned}$$

To bound the term corresponding to S_2 , we provide the following proposition.

Proposition 2 (Mutual Packing). *There exist $\epsilon_{S_2}(\delta)$, such that for all sufficiently small δ and sufficiently large n , we have $\mathbb{E}[S_2] \leq \epsilon_{S_2}(\delta)$, if $S_1 - R_1 < \log p - H(Z)$, or equivalently, $S_2 - R_2 < \log p - H(Z)$, where $\epsilon_{S_2} \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof is provided in Appendix B-B. □

Now, we move on to analyzing the term corresponding to S_1 .

Step 3: Term concerning Alice's encoding

In this step, we separately analyze the action of the two encoders in approximating the product distribution $p_{\underline{X}Y}^n(\cdot)$. For that, we split S_1 as $S_1 \leq Q_1 + Q_2$, where

$$Q_1 \triangleq \sum_{\underline{x}^n, y^n} \left| p_{\underline{X}Y}^n(\underline{x}^n, y^n) - \frac{1}{2^{nC_1}} \sum_{\mu_1} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) \gamma_{w_1^n}^{(\mu_1)} E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \right. \\ \left. p_{W_2^n | X_2^n}^n(w_2^n | x_2^n) p_{Y|Z}^n(y^n | w_1^n + w_2^n) \right|,$$

$$Q_2 \triangleq \sum_{\underline{x}^n, y^n} \left| \frac{1}{2^{nC_1}} \sum_{\mu_1} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) \gamma_{w_1^n}^{(\mu_1)} E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \right. \\ \left. \left(p_{W_2^n | X_2^n}^n(w_2^n | x_2^n) - \zeta_{w_2^n}^{(\mu_2)} E_{W_2^n | X_2^n}^{(\mu_2)}(w_2^n | x_2^n) \right) p_{Y|Z}^n(y^n | w_1^n + w_2^n) \right|.$$

With this partition, the terms within the trace norm of Q_1 differ only in the action of Alice's encoder. And similarly, the terms within the norm of Q_2 differ only in the action of Bob's encoder. Showing that these two terms are small forms a major portion of the achievability proof.

Analysis of Q_1 : To prove Q_1 is small, we characterize the rate constraints which ensure that an upper bound to Q_1 can be made to vanish in an expected sense. In addition, this upper bound becomes useful in obtaining a single-letter characterization for the rate needed to make the term corresponding to Q_2 vanish. For this, we define J as

$$J \triangleq \sum_{\underline{x}^n, w_2^n, y^n} \left| p_{\underline{X}W_2Y}^n(\underline{x}^n, y^n) - \frac{1}{2^{nC_1}} \sum_{\mu_1} \sum_{w_1^n} p_{\underline{X}}^n(\underline{x}^n) \gamma_{w_1^n}^{(\mu_1)} E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \right. \\ \left. p_{W_2^n | X_2^n}^n(w_2^n | x_2^n) p_{Y|Z}^n(y^n | w_1^n + w_2^n) \right|.$$

By again using triangle inequality we obtain $J \leq J_1 + J_2$, where

$$J_1 \triangleq \sum_{\underline{x}^n, w_2^n, y^n} \left| p_{\underline{X}W_2Y}^n(\underline{x}^n, y^n) - \frac{1}{2^{nC_1}} \sum_{\mu_1} \sum_{w_1^n} p_{\underline{X}}^n(\underline{x}^n) \gamma_{w_1^n}^{(\mu_1)} \bar{E}_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \right. \\ \left. p_{W_2^n | X_2^n}^n(w_2^n | x_2^n) p_{Y|Z}^n(y^n | w_1^n + w_2^n) \right|$$

$$J_2 \triangleq \sum_{\underline{x}^n, w_2^n, y^n} \left| \frac{1}{2^{nC_1}} \sum_{\mu_1} \sum_{w_1^n} p_{\underline{X}}^n(\underline{x}^n) \gamma_{w_1^n}^{(\mu_1)} \left(\bar{E}_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) - E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \right) \right. \\ \left. p_{W_2^n | X_2^n}^n(w_2^n | x_2^n) p_{Y^n | Z}^n(y^n | w_1^n + w_2^n) \right|$$

where $\bar{E}_{W_1^n | X_1^n}^{(\mu_1)}(\cdot | \cdot)$ is defined as

$$E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \triangleq \frac{p^n}{2^{nS_1}(1+\eta)} p_{W_1 | X_1}^n(w_1^n | x_1^n) \mathbb{1}_{\{w_1^n \in T_\delta(W_1 | x_1^n)\}}. \quad (9)$$

To prove the term corresponding to J_1 is small, consider the following proposition.

Proposition 3. *There exist $\epsilon_{J_1}(\delta), \delta_{J_1}(\delta)$ such that for all sufficiently small δ and sufficiently large n , we have $\mathbb{E}[J_1] \leq \epsilon_{J_1}$ if $S_1 + C_1 \geq I(W_1; X_1 X_2 Z W_2) + \log p - H(W_1) + \delta_{J_1}$, where $\epsilon_{J_1}, \delta_{J_1} \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof is provided in Appendix B-C. □

Now, consider the term corresponding to J_2 . This can be simplified as

$$J_2 = \frac{1}{2^{nC_1}} \sum_{x_1^n} \sum_{\mu_1} p_{\underline{X}}^n(x^n) \left(\sum_{w_1^n} \gamma_{w_1^n}^{(\mu_1)} \bar{E}_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \right) \mathbb{1}_{\{s_1^{(\mu_1)}(x_1^n) > 1\}} \\ \leq \frac{1}{2^{nC_1}} \sum_{x_1^n} \sum_{\mu_1} p_{\underline{X}}^n(x^n) \mathbb{E} \left[\sum_{w_1^n} \gamma_{w_1^n}^{(\mu_1)} \bar{E}_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \right] \mathbb{1}_{\{s_1^{(\mu_1)}(x_1^n) > 1\}} \\ + \frac{1}{2^{nC_1}} \sum_{x_1^n} \sum_{\mu_1} p_{\underline{X}}^n(x^n) \left| \sum_{w_1^n} \gamma_{w_1^n}^{(\mu_1)} \bar{E}_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) - \mathbb{E} \left[\sum_{w_1^n} \gamma_{w_1^n}^{(\mu_1)} \bar{E}_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \right] \right| \\ \leq \frac{1}{2^{nC_1}} \sum_{x_1^n} \sum_{\mu_1} p_{\underline{X}}^n(x^n) \mathbb{E} \left[\sum_{w_1^n} \gamma_{w_1^n}^{(\mu_1)} \bar{E}_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \right] \mathbb{1}_{\{s_1^{(\mu_1)}(x_1^n) > 1\}} \\ + \frac{1}{2^{nC_1}} \sum_{x_1^n} \sum_{\mu_1} p_{\underline{X}}^n(x^n) \left| \sum_{w_1^n} \gamma_{w_1^n}^{(\mu_1)} \bar{E}_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) - \mathbb{E} \left[\sum_{w_1^n} \gamma_{w_1^n}^{(\mu_1)} \bar{E}_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \right] \right| \\ \leq H_0 + H_1 + \epsilon'',$$

for

$$H_0 \triangleq \frac{1}{2^{nC_1}} \sum_{x_1^n} \sum_{\mu_1} p_{\underline{X}}^n(x^n) \mathbb{1}_{\{s_1^{(\mu_1)}(x_1^n) > 1\}} \\ H_1 \triangleq \frac{1}{2^{nC_1}} \sum_{x_1^n} \sum_{\mu_1} \left| p_{\underline{X}}^n(x^n) - \frac{p^n}{2^{nS_1}(1+\eta)} \sum_{w_1^n} \sum_{m_1 > 0} \sum_{a_1 \in \mathbb{F}_q^k} p_{X_1 W_1}^n(x_1^n, w_1^n) \mathbb{1}_{\{w_1^n(a_1, m_1, \mu_1) = w_1^n\}} \right|$$

where we use $\mathbb{E} \left[\sum_{w_1^n} \gamma_{w_1^n}^{(\mu_1)} \bar{E}_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \right] \leq 1$ in defining H_0 and H_1 is obtained by adding the sequences $w_1^n \notin \mathcal{T}_\delta^{(n)}(W_1)$ within the summation. Now, we can provide an upper bound on H_0 and

H_1 using the Lemmas 2 and 1, respectively, as $\mathbb{E}[H_0 + H_1] \leq \epsilon_H$ if $S_1 \geq I(X_1; W_1) + \delta_H$. Therefore, since $Q_1 \leq J$, hence Q_1 , can be made arbitrarily small for sufficiently large n , if $S_1 + C_1 > I(W_1; X_1 X_2 Y W_2) - H(W_1) + \log p + \delta_J$. Now we move on to bounding Q_2 .

Step 4: Analysis of Bob's encoding

Step 3 ensured that the random variables $X_1 X_2 Y W_2$ are close to a product PMF in total variation. In this step, we approximate the PMF of random variables $X_1 X_2 Y$ using the Bob's encoding rule and bound the theorem corresponding to Q_2 . We proceed with the following proposition.

Proposition 4. *There exist functions $\epsilon_{Q_2}(\delta)$ and $\delta_{Q_2}(\delta)$, such that for all sufficiently small δ and sufficiently large n , we have $\mathbb{E}[Q_2] \leq \epsilon_{Q_2}$, if $S_1 + C_1 \geq I(W_1; X_1 X_2 Y W_2) - H(W_1) + \log p + \delta_{Q_2}$ and $S_2 + C_2 \geq I(W_2; X_1 X_2 Y) - H(W_2) + \log p + \delta_{Q_2}$, where $\epsilon_{Q_2}, \delta_{Q_2} \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof is provided in Appendix B-D. □

Hence, in bounding the terms corresponding to Q_1 and Q_2 , we have obtained the following constraints:

$$\begin{aligned} S_1 + C_1 &\geq I(W_1; X_1 X_2 Y W_2) - H(W_1) + \log p, \\ S_2 + C_2 &\geq I(W_2; X_1 X_2 Y) - H(W_2) + \log p. \end{aligned} \tag{10}$$

By doing an exact symmetric analysis, but by replacing the first encoder by a product distribution instead of the second encoder in S_1 , we obtain the following constraints

$$\begin{aligned} S_1 + C_1 &\geq I(W_1; X_1 X_2 Y) - H(W_1) + \log p, \\ S_2 + C_2 &\geq I(W_2; X_1 X_2 Y W_1) - H(W_2) + \log p. \end{aligned} \tag{11}$$

By time sharing between the above rates (10) and (11), one can obtain the following rate constraints

$$\begin{aligned} S_1 + C_1 &\geq I(W_1; X_1 X_2 Y) - H(W_1) + \log p, \\ S_2 + C_2 &\geq I(W_2; X_1 X_2 Y) - H(W_2) + \log p, \\ S_1 + S_2 + C_1 + C_2 &\geq I(W_1 W_2; X_1 X_2 Y) - H(W_1, W_2) + 2 \log p. \end{aligned}$$

D. Rate Constraints

To sum-up, we showed that the (3) holds for sufficiently large n and with probability sufficiently close to 1, if the following bounds holds while incorporating the time sharing random variable Q taking values over the finite set \mathcal{Q} ¹:

$$S_1 \geq I(X_1; W_1 | Q) - H(W_1 | Q) + \log p,$$

¹Since Q , the time sharing random variable is employed in the standard way we omit its discussion here.

$$\begin{aligned}
S_2 &\geq I(X_2; W_2|Q) - H(W_2|Q) + \log p, \\
S_1 + C_1 &\geq I(X_1 X_2 Y; W_1|Q) - H(W_1|Q) + \log p, \\
S_2 + C_2 &\geq I(X_1 X_2 Y; W_2|Q) - H(W_2|Q) + \log p, \\
S_1 + S_2 + C_1 + C_2 &\geq I(W_1 W_2; X_1 X_2 Y|Q) - H(W_1, W_2|Q) + 2 \log p, \\
S_1 - R_1 &= S_2 - R_2 \leq \log p - H(W_1 \oplus W_2|Q), \\
0 &\leq R_1 \leq S_1, \quad 0 \leq R_2 \leq S_2, \\
C_1 + C_2 &\leq C, \quad C \geq 0
\end{aligned} \tag{12}$$

Lastly, we complete the proof of the theorem using the following lemma.

Lemma 3. *Let \mathcal{R}_1 denote the set of all (R_1, R_2, C) for which there exists (S_1, S_2) such that the septuple $(R_1, R_2, C, S_1, S_2, C_1, C_2)$ satisfies the inequalities in (12). Let, \mathcal{R}_2 denote the set of all triples (R_1, R_2, C) that satisfies the inequalities in (1) given in the statement of the theorem. Then, $\mathcal{R}_1 = \mathcal{R}_2$.*

Proof. This follows from Fourier-Motzkin elimination [21]. \square

APPENDIX A

PROOF OF LEMMAS

A. Proof of Lemma 2

Let K denote the left hand side of (7). Further, for the purpose of this proof, we skip the subscript i . Bounding the a-typical sequences of x^n from the summation gives $K = K_1 + \epsilon_X$, where

$$K_1 \triangleq \frac{1}{2^{nC}} \sum_{\mu} \sum_{x^n \in \mathcal{T}_{\delta}^{(n)}(X)} p_X^n(x^n) \mathbb{P} \left(\left[\sum_{a \in \mathbb{F}_p^k} \sum_{m \in \mathbb{F}_p^l} E_{L|X^n}^{(\mu)}(a, m|x^n) \right] > 1 \right),$$

and $\epsilon_X \triangleq \sum_{x^n \notin \mathcal{T}_{\delta}^{(n)}(X)} p_X^n(x^n)$. Note that $\epsilon_X(\delta) \searrow 0$ as $\delta \searrow 0$. With that, it remains to show the K_1 can be made arbitrarily small in expected sense. Toward that, define

$$Z_{x^n}^{(\mu)}(a, m) \triangleq \frac{p^n}{(1 + \eta)} \sum_{w^n} p_{XW}^n(x^n, w^n) \mathbb{1}_{\{w^n(a, m, \mu) = w^n\}} \mathbb{1}_{\{w^n \in \mathcal{T}_{\delta}^{(n)}(W|x^n)\}}, \quad Z_{x^n}^{(\mu)} \triangleq \frac{1}{2^{nS}} \sum_{m > 0} \sum_{a \in \mathbb{F}_q^k} Z_{x^n}^{(\mu)}(a, m)$$

Observe the following upper and lower bounds on $\mathbb{E}[Z_{x^n}^{(\mu)}]$.

$$\mathbb{E}[Z_{x^n}^{(\mu)}] = \frac{p^n}{(1 + \eta)} \sum_{w^n \in \mathcal{T}_{\delta}^{(n)}(W|x^n)} p_{XW}^n(x^n, w^n) \frac{1}{p^n} \leq \frac{p_X^n(x^n)}{(1 + \eta)} \tag{13}$$

$$\mathbb{E}[Z_{x^n}^{(\mu)}] = \frac{1}{(1 + \eta)} \sum_{w^n \in \mathcal{T}_{\delta}^{(n)}(W|x^n)} p_{XW}^n(x^n, w^n) \geq \frac{p_X^n(x^n) 2^{n\delta_w}}{(1 + \eta)}, \tag{14}$$

where the inequalities above uses the typicality arguments and $\delta_w(\delta) \searrow 0$ as $\delta \searrow 0$. Using these bounds, we perform the following simplification.

$$\mathbb{P}\left(\left[\sum_{a \in \mathbb{F}_p^k} \sum_{m \in \mathbb{F}_p^l} E_{L|X^n}^{(\mu)}(a, m|x^n)\right] > 1\right) = \mathbb{P}\left(Z_{x^n}^{(\mu)} > p_X^n(x^n)\right) \leq \mathbb{P}\left(Z_{x^n}^{(\mu)} > (1 + \eta)\mathbb{E}[Z_{x^n}^{(\mu)}]\right) \quad (15)$$

where the inequality above uses the inequality from (13). Further, we have

$$\mathbb{P}\left(\left|Z_{x^n}^{(\mu)} - \mathbb{E}[Z_{x^n}^{(\mu)}]\right| > \eta\mathbb{E}[Z_{x^n}^{(\mu)}]\right) \leq \frac{\mathbb{E}\left|Z_{x^n}^{(\mu)} - \mathbb{E}[Z_{x^n}^{(\mu)}]\right|}{\eta\mathbb{E}[Z_{x^n}^{(\mu)}]} \leq \frac{(1 + \eta)\sqrt{\text{Var}\left(Z_{x^n}^{(\mu)}\right)}}{\eta 2^{-n\delta_w} p_X^n(x^n)} \quad (16)$$

where the first inequality follows from Markov Inequality, and the second uses (i) the Jensen's inequality for square-root function and (ii) the bound from (14). Combining the inequalities (15) and (16) using union bound, we obtain

$$\mathbb{P}\left(\left[\sum_{a \in \mathbb{F}_p^k} \sum_{m \in \mathbb{F}_p^l} E_{L|X^n}^{(\mu)}(a, m|x^n)\right] > 1\right) \leq \frac{\sqrt{\text{Var}\left(Z_{x^n}^{(\mu)}\right)}}{\eta 2^{-n\delta_w} p_X^n(x^n)} \leq \frac{(1 + \eta)\sqrt{2^{-n(S+H(X|W)+H(W)-\delta')}}}{\eta 2^{-n\delta_w} p_X^n(x^n)} \quad (17)$$

where the last inequality follows by simplifying $\text{Var}\left(Z_{x^n}^{(\mu)}\right)$ similar to the one in [20, Lemma 19] to obtain

$$\text{Var}\left(Z_{x^n}^{(\mu)}\right) \leq 2^{-n(S+H(X|W)+H(W)-\delta')} \frac{1}{(1 + \eta)^2}.$$

Substituting the simplification of (17) in K_1 completes the proof.

APPENDIX B

PROOF OF PROPOSITIONS

A. Proof of Proposition 1

We bound \tilde{S} as $\tilde{S} \leq \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3$, where

$$\tilde{S}_1 \triangleq \sum_{\underline{x}^n, y^n} \sum_{\mu_1, \mu_2} \sum_{m_2 > 0} \frac{p_X^n(\underline{x}^n)}{2^{n(C_1+C_2)}} p_{M_1|X_1^n}^{(\mu_1)}(0|x_1^n) p_{M_2|X_2^n}^{(\mu_2)}(m_2|x_2^n) p_{Y^n|M_1 M_2}^{(\mu)}(y^n|0, m_2)$$

$$\tilde{S}_2 \triangleq \sum_{\underline{x}^n, y^n} \sum_{\mu_1, \mu_2} \sum_{m_2 > 0} \frac{p_X^n(\underline{x}^n)}{2^{n(C_1+C_2)}} p_{M_1|X_1^n}^{(\mu_1)}(m_1|x_1^n) p_{M_2|X_2^n}^{(\mu_2)}(0|x_2^n) p_{Y^n|M_1 M_2}^{(\mu)}(y^n|m_1, 0)$$

$$\tilde{S}_3 \triangleq \sum_{\underline{x}^n, y^n} \sum_{\mu_1, \mu_2} \sum_{m_2 > 0} \frac{p_X^n(\underline{x}^n)}{2^{n(C_1+C_2)}} p_{M_1|X_1^n}^{(\mu_1)}(0|x_1^n) p_{M_2|X_2^n}^{(\mu_2)}(0|x_2^n) p_{Y^n|M_1 M_2}^{(\mu)}(y^n|0, 0)$$

Analysis of \tilde{S}_1 : Consider the following simplification with regards to \tilde{S}_1 .

$$\tilde{S}_1 = \sum_{\underline{x}^n} \sum_{\mu_1, \mu_2} \frac{p_X^n(\underline{x}^n) p_{M_1|X_1^n}^{(\mu_1)}(0|x_1^n)}{2^{n(C_1+C_2)}} \left(\sum_{m_2 > 0} p_{M_2|X_2^n}^{(\mu_2)}(m_2|x_2^n) \left(\sum_{y^n} p_{Y^n|M_1 M_2}^{(\mu)}(y^n|0, m_2) \right) \right)$$

$$\begin{aligned}
&= \sum_{x_1^n} \sum_{\mu_1} \frac{p_{X_1}^n(x_1^n) p_{M_1|X_1^n}^{(\mu_1)}(0|x_1^n)}{2^{nC_1}} \\
&\leq \sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \sum_{\mu_1} \frac{p_{X_1}^n(x_1^n) p_{M_1|X_1^n}^{(\mu_1)}(0|x_1^n)}{2^{nC_1}} \left(\mathbb{1}_{\{s_1^{(\mu_1)}(x_1^n) \leq 1\}} + \mathbb{1}_{\{s_1^{(\mu_1)}(x_1^n) > 1\}} \right) + \epsilon_{X_1} \\
&= \tilde{S}_{11} + \tilde{S}_{12} + \epsilon_{X_1}
\end{aligned}$$

where we define $\epsilon_{X_1}(\delta) \triangleq 1 - \sum_{x^n \in \mathcal{T}_\delta^{(n)}(X_1)} p_{X_1}^n(x_1^n)$ and

$$\begin{aligned}
\tilde{S}_{11} &\triangleq \sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \sum_{\mu_1} \frac{p_{X_1}^n(x_1^n)}{2^{nC_1}} (1 - s_1^{(\mu_1)}(x_1^n)) \mathbb{1}_{\{s_1^{(\mu_1)}(x_1^n) \leq 1\}}, \\
\tilde{S}_{12} &\triangleq \sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \sum_{\mu_1} \frac{p_{X_1}^n(x_1^n)}{2^{nC_1}} \mathbb{1}_{\{s_1^{(\mu_1)}(x_1^n) > 1\}}.
\end{aligned}$$

Now, we bound each of the above terms. For the term corresponding to \tilde{S}_{11} , consider the following.

$$\begin{aligned}
\tilde{S}_{11} &\stackrel{(a)}{\leq} \sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \sum_{\mu_1} \frac{p_{X_1}^n(x_1^n)}{2^{nC_1}} \left| 1 - \sum_{a_1 \in \mathbb{F}_p^k} \sum_{m_1 \in \mathbb{F}_p^{l_1}} E_{L_1|X_1^n}^{(\mu_1)}(a_1, m_1|x_1^n) \right| \\
&\stackrel{(b)}{=} \sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \sum_{\mu_1} \frac{1}{2^{nC_1}} \left| p_{X_1}^n(x_1^n) - \frac{p^n}{2^{nS_1}} \sum_{a_1 \in \mathbb{F}_p^k} \sum_{m_1 \in \mathbb{F}_p^{l_1}} \sum_{w_1^n \in \mathcal{T}_\delta(W_1|x_1^n)} \frac{p_{W_1 X_1}^n(w_1^n, x_1^n)}{(1+\eta)} \mathbb{1}_{\{w_1(a_1, m_1, \mu_1) = w_1^n\}} \right| \\
&\stackrel{(c)}{\leq} \sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \sum_{\mu_1} \frac{1}{2^{nC_1}} \left| p_{X_1}^n(x_1^n) - \frac{p^n}{2^{nS_1}} \sum_{a_1 \in \mathbb{F}_p^k} \sum_{m_1 \in \mathbb{F}_p^{l_1}} \sum_{w_1^n} \frac{p_{W_1 X_1}^n(w_1^n, x_1^n)}{(1+\eta)} \mathbb{1}_{\{w_1(a_1, m_1, \mu_1) = w_1^n\}} \right| \\
&\quad + \sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \sum_{\mu_1} \frac{p^n}{2^{n(S_1+C_1)}} \sum_{a_1 \in \mathbb{F}_p^k} \sum_{m_1 \in \mathbb{F}_p^{l_1}} \sum_{w_1^n \notin \mathcal{T}_\delta(W_1|x_1^n)} \frac{p_{W_1 X_1}^n(w_1^n, x_1^n)}{(1+\eta)} \mathbb{1}_{\{w_1(a_1, m_1, \mu_1) = w_1^n\}} \quad (18)
\end{aligned}$$

where (a) follows by bounding the indicator by 1 and using the definition of $s_1^{(\mu_1)}(\cdot)$, (b) uses the definition of $E_{L_1|X_1^n}^{(\mu_1)}(a_1, m_1|x_1^n)$ as defined in Definition (4), the inequality (c) follows from triangle inequality.

Taking expectation on (18) over the first encoders codebook generation, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{C}_1}[\tilde{S}_{11}] &\leq \frac{1}{2^{nC_1}} \sum_{\mu_1} \mathbb{E}_{\mathbb{C}_1} \left[\sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \left| p_{X_1}^n(x_1^n) - \frac{p^n}{2^{nS_1}} \sum_{a_1 \in \mathbb{F}_p^k} \sum_{m_1 \in \mathbb{F}_p^{l_1}} \sum_{w_1^n} \frac{p_{W_1 X_1}^n(w_1^n, x_1^n)}{(1+\eta)} \mathbb{1}_{\{w_1(a_1, m_1, \mu_1) = w_1^n\}} \right| \right] \\
&\quad + \sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \sum_{\mu_1} \frac{p^n}{2^{n(S_1+C_1)}} \sum_{a_1 \in \mathbb{F}_p^k} \sum_{m_1 \in \mathbb{F}_p^{l_1}} \sum_{w_1^n \notin \mathcal{T}_\delta(W_1|x_1^n)} \frac{p_{W_1 X_1}^n(w_1^n, x_1^n)}{(1+\eta)} \frac{1}{p^n} \quad (19)
\end{aligned}$$

For the first term in (19), we use Lemma (1) and obtain $\mathbb{E}[\tilde{S}_{11}] \leq \epsilon_{\tilde{S}_{11}}$ if $S_1 \leq I(X_1; W_1) - H(W_1) + \log p + \delta_{\tilde{S}_{11}}$. As for the second term we can use typicality arguments and bound it as

$$\sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \sum_{\mu_1} \frac{p^n}{2^{n(S_1+C_1)}} \sum_{a_1 \in \mathbb{F}_p^k} \sum_{m_1 \in \mathbb{F}_p^{l_1}} \sum_{w_1^n \notin \mathcal{T}_\delta(W_1|x_1^n)} \frac{p_{W_1 X_1}^n(w_1^n, x_1^n)}{(1+\eta)} \frac{1}{p^n}$$

$$\leq \sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \sum_{w_1^n \notin T_\delta(W_1|x_1^n)} \frac{p_{W_1 X_1}^n(w_1^n, x_1^n)}{(1 + \eta)} \leq \epsilon' \quad (20)$$

where $\epsilon' \triangleq \sum_{x_1^n \in \mathcal{T}_\delta^{(n)}(X_1)} \sum_{w_1^n \notin T_\delta(W_1|x_1^n)} p_{W_1 X_1}^n(w_1^n, x_1^n)$

Finally, we have the term corresponding to \tilde{S}_{12} . For this, we use Lemma 2 and hence obtain $\mathbb{E}[\tilde{S}_{12}] \leq \epsilon_{\tilde{S}_{12}}$ if $S_2 \leq I(X_1; W_1) - H(W_1) + \log p + \delta_{\tilde{S}_{12}}$.

Analysis of \tilde{S}_2 : Due to the symmetry in \tilde{S}_1 and \tilde{S}_2 , the analysis of \tilde{S}_2 follows very similar arguments at that of \tilde{S}_1 and therefore we obtain $\mathbb{E}[\tilde{S}_2] \leq \epsilon_{\tilde{S}_2}$ if $S_2 \geq I(X_2; W_2) - H(W_2) + \log p + \delta_{\tilde{S}_2}$.

Analysis of \tilde{S}_3 : Follows by merging the above analysis of \tilde{S}_1 and \tilde{S}_2 .

B. Proof of Proposition 2

Recalling S_2 , we have

$$\begin{aligned} S_2 &= \sum_{\underline{x}^n} \sum_{\mu_1, \mu_2} \sum_{\substack{m_1 > 0, \\ m_2 > 0}} \sum_{\substack{a_1 \in \mathbb{F}_p^k \\ a_2 \in \mathbb{F}_p^k}} \sum_{\substack{w_1^n, w_2^n \in \mathbb{F}_p^n}} \frac{p_{\underline{X}}^n(\underline{x}^n)}{2^{n(C_1+C_2)}} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) E_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) \\ &\quad \left| \mathbb{1}_{\{\mathbf{w}_1(a_1, m_1, \mu_1) = w_1^n\}} \mathbb{1}_{\{\mathbf{w}_2(a_2, m_2, \mu_2) = w_2^n\}} \sum_{y^n} \left| p_{Y^n|Z}^n(y^n|w_1^n \oplus w_2^n) - p_{Y^n|Z}^n(y^n|f^{(\mu)}(m_1, m_2)) \right| \right| \\ &\leq 2 \sum_{\underline{x}^n} \sum_{\mu_1, \mu_2} \sum_{\substack{m_1 > 0, \\ m_2 > 0}} \sum_{\substack{a_1 \in \mathbb{F}_p^k \\ a_2 \in \mathbb{F}_p^k}} \sum_{\substack{w_1^n, w_2^n \in \mathbb{F}_p^n}} \frac{p_{\underline{X}}^n(\underline{x}^n)}{2^{n(C_1+C_2)}} \frac{p^n p^n}{2^{n(S_1+S_2)}(1+\eta)^2} p_{W_1|X_1}^n(w_1^n|x_1^n) p_{W_2|X_2}^n(w_2^n|x_2^n) \\ &\quad \mathbb{1}_{\{w_1^n \in T_\delta(W_1|x_1^n)\}} \mathbb{1}_{\{w_2^n \in T_\delta(W_2|x_2^n)\}} \mathbb{1}_{\{\mathbf{w}_1(a_1, m_1, \mu_1) = w_1^n\}} \mathbb{1}_{\{\mathbf{w}_2(a_2, m_2, \mu_2) = w_2^n\}} \mathbb{1}_{\{w_1^n \oplus w_2^n, m_1, m_2\}} \end{aligned}$$

where we define $\mathbb{1}_{\{w_1^n \oplus w_2^n, m_1, m_2\}}$ as

$$\mathbb{1}_{\{w^n, m_1, m_2\}} \triangleq \mathbb{1} \left\{ \exists (\tilde{w}^n, \tilde{a}) : \tilde{w}^n G + h_1^{(\mu_1)}(m_1) + h_2^{(\mu_2)}(m_2), \tilde{w}^n \in \mathcal{T}_\delta^{(n)}(W_1 \oplus W_2), \tilde{w}^n \neq w^n \right\}.$$

Using this we obtain,

$$\begin{aligned} \mathbb{E}[S_2] &\leq 2 \sum_{\underline{x}^n} \sum_{\mu_1, \mu_2} \sum_{\substack{m_1 > 0, \\ m_2 > 0}} \sum_{\substack{a_1 \in \mathbb{F}_p^k \\ a_2 \in \mathbb{F}_p^k}} \sum_{\substack{w_1^n, w_2^n \in \mathbb{F}_p^n}} \frac{p_{\underline{X}}^n(\underline{x}^n)}{2^{n(C_1+C_2)}} \frac{p^n p^n}{2^{n(S_1+S_2)}(1+\eta)^2} p_{W_1|X_1}^n(w_1^n|x_1^n) p_{W_2|X_2}^n(w_2^n|x_2^n) \\ &\quad \mathbb{1}_{\{w_1^n \in T_\delta(W_1|x_1^n)\}} \mathbb{1}_{\{w_2^n \in T_\delta(W_2|x_2^n)\}} \mathbb{1}_{\{\mathbf{w}_1(a_1, m_1, \mu_1) = w_1^n\}} \mathbb{1}_{\{\mathbf{w}_2(a_2, m_2, \mu_2) = w_2^n\}} \mathbb{1}_{\{w_1^n \oplus w_2^n, m_1, m_2\}} \end{aligned}$$

Note that, we have

$$\begin{aligned} &\mathbb{E} \left[\mathbb{1}_{\{w_1^n \oplus w_2^n, m_1, m_2\}} \mathbb{1}_{\{\mathbf{w}_1(a_1, m_1, \mu_1) = w_1^n\}} \mathbb{1}_{\{\mathbf{w}_2(a_2, m_2, \mu_2) = w_2^n\}} \right] \\ &\leq \sum_{\tilde{a} \neq a} \sum_{\substack{\tilde{w}^n \in \mathcal{T}_\delta^{(n)}(W_1 \oplus W_2) \\ \tilde{w}^n \neq w_1^n \oplus w_2^n}} \frac{1}{p^n} \frac{1}{p^n} \frac{1}{p^n} \leq 2^{n(H(W_1 \oplus W_2) + \delta_z)} p^{3n-k}, \quad (21) \end{aligned}$$

where $\delta_z(\delta) \searrow 0$ as $\delta \searrow 0$. This gives

$$\begin{aligned} \mathbb{E}[S_2] &\leq 2 \sum_{\underline{x}^n} \sum_{\substack{m_1 > 0, \\ m_2 > 0}} \sum_{\substack{a_1 \in \mathbb{F}_p^k \\ a_2 \in \mathbb{F}_p^k}} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) \frac{2^{n(H(W_1 \oplus W_2) + \delta_z)} p^{n-k}}{2^{n(S_1 + S_2)} (1 + \eta)^2} p_{W_1|X_1}^n(w_1^n | x_1^n) p_{W_2|X_2}^n(w_2^n | x_2^n) \\ &\leq 2^n [(S_1 - R_1) - (\log p H(W_1 \oplus W_2) - \delta_{S_2})], \end{aligned}$$

where $\delta_{S_2}(\delta) \searrow 0$ as $\delta \searrow 0$.

C. Proof of Proposition 3

Substituting the definition of $\bar{E}_{W_1^n|X_1^n}^{(\mu_1)}(\cdot|\cdot)$ and $\gamma_{w_1^n}^{(\mu_1)}$ in J_1 , we obtain

$$\begin{aligned} J_1 &= \sum_{\underline{x}^n, w_2^n, y^n} \left| p_{\underline{X}W_2Y}^n(\underline{x}^n, y^n) - \frac{1}{2^{nC_1}} \sum_{\mu_1} \sum_{w_1^n} \sum_{m_1 > 0} \sum_{a_1 \in \mathbb{F}_q^k} p_{\underline{X}}^n(\underline{x}^n) \frac{p^n}{2^{nS_1} (1 + \eta)} p_{W_1|X_1}^n(w_1^n | x_1^n) \right. \\ &\quad \left. \mathbb{1}_{\{w_1^n \in T_\delta(W_1|x_1^n)\}} \mathbb{1}_{\{w_1(a_1, m_1, \mu_1) = w_1^n\}} p_{W_2|X_2}^n(w_2^n | x_2^n) p_{Y|Z}^n(y^n | w_1^n + w_2^n) \right| \\ &= \sum_{\underline{x}^n, w_2^n, y^n} \left| p_{\underline{X}W_2Y}^n(\underline{x}^n, y^n) \right. \\ &\quad \left. - \frac{1}{(1 + \eta)} \frac{p^n}{2^{n(S_1 + C_1)}} \sum_{\mu_1} \sum_{w_1^n \in T_\delta(W_1|x_1^n)} \sum_{m_1 > 0} \sum_{a_1 \in \mathbb{F}_q^k} p_{\underline{X}WY}^n(\underline{x}^n, \underline{w}^n, y^n) \mathbb{1}_{\{w_1(a_1, m_1, \mu_1) = w_1^n\}} \right| \\ &\leq J_{11} + J_{12}, \end{aligned}$$

where

$$\begin{aligned} J_{11} &\triangleq \sum_{\underline{x}^n, w_2^n, y^n} \left| p_{\underline{X}W_2Y}^n(\underline{x}^n, y^n) - \frac{1}{(1 + \eta)} \frac{p^n}{2^{n(S_1 + C_1)}} \sum_{\mu_1} \sum_{w_1^n} \sum_{\substack{m_1 > 0 \\ a_1 \in \mathbb{F}_q^k}} p_{\underline{X}WY}^n(\underline{x}^n, \underline{w}^n, y^n) \mathbb{1}_{\{w_1(a_1, m_1, \mu_1) = w_1^n\}} \right| \\ J_{12} &\triangleq \sum_{\underline{x}^n, w_2^n, y^n} \left| \frac{1}{(1 + \eta)} \frac{p^n}{2^{n(S_1 + C_1)}} \sum_{\mu_1} \sum_{w_1^n \notin T_\delta^{(n)}(W_1)} \sum_{m_1 > 0} \sum_{a_1 \in \mathbb{F}_q^k} p_{\underline{X}WY}^n(\underline{x}^n, \underline{w}^n, y^n) \mathbb{1}_{\{w_1(a_1, m_1, \mu_1) = w_1^n\}} \right| \end{aligned}$$

As for the term J_{11} , we use Lemma 1 and obtain the following bound on $\mathbb{E}[J_{11}]$ as, $\mathbb{E}[J_{11}] \leq \epsilon_{J_{11}}$ if $S_1 + C_1 \geq I(W_1; X_1 X_2 Z W_2) + \log p - H(W_1) + \delta_{J_{11}}$.

For the term J_{12} , applying expectation gives

$$\mathbb{E}[J_{12}] \leq \frac{1}{(1 + \eta)} \sum_{\underline{x}^n, w_2^n, y^n} \sum_{w_1^n \notin T_\delta^{(n)}(W_1)} p_{\underline{X}WY}^n(\underline{x}^n, \underline{w}^n, y^n) \leq \epsilon'.$$

where $\epsilon'(\delta) \searrow 0$ as $\delta \searrow 0$.

D. Proof of Proposition 4

We begin by applying triangle inequality on Q_2 to obtain $Q_2 \leq F_1 + F_2$, where

$$F_1 \triangleq \sum_{\underline{x}^n, y^n} \left| \frac{1}{2^{n(C_1+C_2)}} \sum_{\mu_1, \mu_2} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) \gamma_{w_1^n}^{(\mu_1)} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) \right. \\ \left. \left(p_{W_2|X_2}^n(w_2^n|x_2^n) - \zeta_{w_2^n}^{(\mu_2)} \bar{E}_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) \right) p_{Y|Z}^n(y^n|w_1^n \oplus w_2^n) \right|, \\ F_2 \triangleq \sum_{\underline{x}^n, y^n} \left| \frac{1}{2^{n(C_1+C_2)}} \sum_{\mu_1, \mu_2} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) \gamma_{w_1^n}^{(\mu_1)} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) \right. \\ \left. \zeta_{w_2^n}^{(\mu_2)} \left(\bar{E}_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) - E_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) \right) p_{Y|Z}^n(y^n|w_1^n \oplus w_2^n) \right|,$$

where $\bar{E}_{W_2^n|X_2^n}^{(\mu_2)}(\cdot|\cdot)$ is defined as

$$\bar{E}_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) \triangleq \frac{p^n}{2^{nS_2}(1+\eta)} p_{W_2|X_2}^n(w_2^n|x_2^n) \mathbb{1}_{\{w_2^n \in T_\delta(W_2|x_2^n)\}}$$

Considering the term corresponding to F_1 , we bound it using triangle inequality applied by adding and subtracting the following terms within its modulus:

$$(i) \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) p_{W_1|X_1}^n(w_1^n|x_1^n) p_{W_2|X_2}^n(w_2^n|x_2^n) p_{Y|Z}^n(y^n|w_1^n \oplus w_2^n) \\ (ii) \frac{1}{2^{nC_2}} \sum_{\mu_2} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) p_{W_1|X_1}^n(w_1^n|x_1^n) \frac{p^n \zeta_{w_2^n}^{(\mu_2)}}{2^{nS_2}(1+\eta)} p_{W_2|X_2}^n(w_2^n|x_2^n) p_{Y|Z}^n(y^n|w_1^n \oplus w_2^n) \\ (iii) \frac{1}{2^{n(C_1+C_2)}} \sum_{\mu_1, \mu_2} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) \gamma_{w_1^n}^{(\mu_1)} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) \frac{p^n \zeta_{w_2^n}^{(\mu_2)}}{2^{nS_2}(1+\eta)} p_{W_2|X_2}^n(w_2^n|x_2^n) p_{Y|Z}^n(y^n|w_1^n \oplus w_2^n)$$

This gives the following bound $F_1 \leq F_{11} + F_{12} + F_{13} + F_{14}$, where

$$F_{11} \triangleq \sum_{\underline{x}^n, y^n} \left| \frac{1}{2^{nC_1}} \sum_{\mu_1} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) \left(\gamma_{w_1^n}^{(\mu_1)} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) - p_{W_1^n|X_1^n}^n(w_1^n|x_1^n) \right) p_{Y|Z}^n(y^n|w_1^n \oplus w_2^n) \right| \\ F_{12} \triangleq \sum_{\underline{x}^n, y^n} \left| \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) p_{W_1|X_1}^n(w_1^n|x_1^n) \left(p_{W_2|X_2}^n(w_2^n|x_2^n) \right. \right. \\ \left. \left. - \frac{p^n \zeta_{w_2^n}^{(\mu_2)}}{2^{nS_2}(1+\eta)} p_{W_2|X_2}^n(w_2^n|x_2^n) \right) p_{Y|Z}^n(y^n|w_1^n \oplus w_2^n) \right| \\ F_{13} \triangleq \sum_{\underline{x}^n, y^n} \left| \frac{1}{2^{nC_2}} \sum_{\mu_2} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) \left(p_{W_1|X_1}^n(w_1^n|x_1^n) \right. \right. \\ \left. \left. - \frac{1}{2^{nC_1}} \sum_{\mu_1} \gamma_{w_1^n}^{(\mu_1)} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) \right) \frac{p^n \zeta_{w_2^n}^{(\mu_2)}}{2^{nS_2}(1+\eta)} p_{W_2|X_2}^n(w_2^n|x_2^n) p_{Y|Z}^n(y^n|w_1^n \oplus w_2^n) \right|$$

$$F_{14} \triangleq \sum_{\underline{x}^n, y^n} \left| \frac{1}{2^{n(C_1+C_2)}} \sum_{\mu_1, \mu_2} \sum_{w_1^n \in \mathbb{F}_p^n} \sum_{w_2^n \notin \mathcal{T}_\delta^{(n)}(W_2)} p_{\underline{X}}^n(\underline{x}^n) \gamma_{w_1^n}^{(\mu_1)} \bar{E}_{W_1^n|X_1^n}(w_1^n|x_1^n) \frac{p^n \zeta_{w_2^n}^{(\mu_2)}}{2^{nS_2}(1+\eta)} \right. \\ \left. p_{W_2|X_2}^n(w_2^n|x_2^n) p_{Y|Z}^n(y^n|w_1^n \oplus w_2^n) \right|$$

We start by analyzing F_{11} . Note that F_{11} is exactly similar to the term Q_1 and hence using the same rate constraints as Q_1 , this term can be bounded. Next consider the term corresponding to F_{12} . Substituting the definition of $\zeta_{w_2^n}^{(\mu_2)}$ gives

$$F_{12} = \sum_{\underline{x}^n, y^n} \left| p_{\underline{X}Y}^n(\underline{x}^n, y^n) - \frac{p^n}{2^{nS_2}(1+\eta)} \sum_{w_2^n \in \mathbb{F}_p^n} \sum_{\substack{m_2 > 0 \\ a_2 \in \mathbb{F}_q^k}} \mathbb{1}_{\{w_2(a_2, m_2, \mu_2) = w_2^n\}} p_{\underline{X}W_2Y}^n(\underline{x}^n, w_2^n, y^n) \right|$$

Lemma 1 gives us functions $\epsilon_{F_{12}}(\delta), \delta_{F_{12}}(\delta)$ such that if

$$S_1 \geq I(W_2; X_1 X_2 Y) - H(W_2) + \log p + \delta_{F_{12}},$$

then $\mathbb{E}[F_{12}] \leq \epsilon_{F_{12}}$, where $\epsilon_{F_{12}}(\delta), \delta_{F_{12}}(\delta) \searrow 0$ as $\delta \searrow 0$.

Now, we move on to considering in the term corresponding to F_{13} . Taking expectation with respect to $G, h_1^{(\mu_1)}$ and $h_2^{(\mu_2)}$ gives

$$\mathbb{E}[F_{13}] = \mathbb{E}_{G, h_1} \left[\sum_{\underline{x}^n, y^n} \left| \frac{1}{2^{nC_2}} \sum_{\mu_2} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) \left(p_{W_1|X_1}^n(w_1^n|x_1^n) \right. \right. \right. \\ \left. \left. - \frac{1}{2^{nC_1}} \sum_{\mu_1} \gamma_{w_1^n}^{(\mu_1)} E_{W_1^n|X_1^n}(w_1^n|x_1^n) \right) \mathbb{E}_{h_2|G} \left[\frac{p^n \zeta_{w_2^n}^{(\mu_2)}}{2^{nS_2}(1+\eta)} \right] p_{W_2|X_2}^n(w_2^n|x_2^n) p_{Y|Z}^n(y^n|w_1^n \oplus w_2^n) \right] \\ = \frac{1}{(1+\eta)} \mathbb{E}_{G, h_1} \left[\sum_{\underline{x}^n, y^n} \left| \frac{1}{2^{nC_2}} \sum_{\mu_2} \sum_{w_1^n, w_2^n \in \mathbb{F}_p^n} p_{\underline{X}}^n(\underline{x}^n) \left(p_{W_1|X_1}^n(w_1^n|x_1^n) \right. \right. \right. \\ \left. \left. - \frac{1}{2^{nC_1}} \sum_{\mu_1} \gamma_{w_1^n}^{(\mu_1)} E_{W_1^n|X_1^n}(w_1^n|x_1^n) \right) p_{W_2|X_2}^n(w_2^n|x_2^n) p_{Y|Z}^n(y^n|w_1^n \oplus w_2^n) \right] \\ = \mathbb{E} \left[\frac{J}{(1+\eta)} \right],$$

where the above equalities follows from the fact that $h_1^{(\mu_1)}$ and $h_2^{(\mu_2)}$ were generated independently and from using the definition of J as stated earlier. Therefore, using the same analysis and rate constraints as J , we can bound the term F_{13} . Finally, we remain with the term F_{14} . Applying expectation on F_{14} gives

$$\mathbb{E}[F_{14}] \leq \mathbb{E}_{G, h_2} \left[\sum_{\underline{x}^n} \left| \sum_{\substack{\mu_1, \mu_2 \\ w_1^n \in \mathbb{F}_p^n \\ w_2^n \in \mathcal{T}_\delta^{(n)}(W_2)}} p_{\underline{X}}^n(\underline{x}^n) \frac{E_{h_1} \left[\gamma_{w_1^n}^{(\mu_1)} \right]}{2^{n(C_1+C_2)}} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) \frac{p^n \zeta_{w_2^n}^{(\mu_2)}}{2^{nS_2}(1+\eta)} p_{W_2|X_2}^n(w_2^n|x_2^n) \right| \right]$$

$$\begin{aligned}
&\leq \sum_{\underline{x}^n} \frac{1}{2^{nC_2}} \sum_{\mu_2} \sum_{\substack{w_1^n \in \mathcal{T}_\delta^{(n)}(W_1) \\ w_2^n \notin \mathcal{T}_\delta^{(n)}(W_2)}} p_{\underline{X}}^n(\underline{x}^n) \frac{p_{W_1|X_1}^n(w_1^n|x_1^n)}{(1+\eta)} \mathbb{E} \left[\frac{p^n \zeta_{w_2^n}^{(\mu_2)}}{2^{nS_2}(1+\eta)} \right] p_{W_2|X_2}^n(w_2^n|x_2^n) \\
&\leq \frac{1}{(1+\eta)^2} \sum_{\underline{x}^n} \sum_{w_2^n \notin \mathcal{T}_\delta^{(n)}(W_2)} p_{\underline{X}}^n(\underline{x}^n) p_{W_2|X_2}^n(w_2^n|x_2^n) \leq \epsilon'_w,
\end{aligned}$$

where $\epsilon'_w(\delta) \searrow 0$ as $\delta \searrow 0$. This completes the analysis for the term corresponding to F_1 . Finally we remain with the analysis of the term F_2 . Simplifying F_2 gives

$$\begin{aligned}
F_2 &\leq \frac{1}{2^{n(C_1+C_2)}} \sum_{\mu_1, \mu_2} \sum_{\underline{x}^n} p_{\underline{X}}^n(\underline{x}^n) \left(\sum_{w_1^n \in \mathbb{F}_p^n} \gamma_{w_1^n}^{(\mu_1)} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) \right) \\
&\quad \left| \sum_{w_2^n \in \mathbb{F}_p^n} \zeta_{w_2^n}^{(\mu_2)} \left(\bar{E}_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) - E_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) \right) \right|, \\
&= \frac{1}{2^{nC_2}} \sum_{\mu_1, \mu_2} \sum_{\underline{x}^n} p_{\underline{X}}^n(\underline{x}^n) \left(\sum_{m_1 > 0} p_{M_1|X_1^n}(m_1|x_1^n) \right) \\
&\quad \left| \sum_{w_2^n \in \mathbb{F}_p^n} \zeta_{w_2^n}^{(\mu_2)} \left(\bar{E}_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) - E_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) \right) \right|, \\
&\leq \frac{1}{2^{nC_2}} \sum_{\mu_1, \mu_2} \sum_{\underline{x}^n} p_{\underline{X}}^n(\underline{x}^n) \left| \sum_{w_2^n \in \mathbb{F}_p^n} \zeta_{w_2^n}^{(\mu_2)} \left(\bar{E}_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) - E_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) \right) \right|, \\
&= \tilde{S}_2,
\end{aligned}$$

where the last inequality above follows by using $\left(\sum_{m_1 > 0} p_{M_1|X_1^n}(m_1|x_1^n) \right) \leq 1$ and the last equality follows by recalling the definition of \tilde{S}_2 . Therefore, using the constraints obtained in the analysis of \tilde{S}_2 , we complete the proof of the proposition.

REFERENCES

- [1] T. Anwar Atif, A. Padakandla, and S. Sandeep Pradhan, “Source coding for synthesizing correlated randomness,” in *2020 IEEE International Symposium on Information Theory (ISIT)*, 2020, pp. 1570–1575.
- [2] J. Korner and K. Marton, “How to encode the modulo-two sum of binary sources (corresp.),” *IEEE Transactions on Information Theory*, vol. 25, no. 2, pp. 219–221, 1979.
- [3] R. Ahlswede and T. Han, “On source coding with side information via a multiple-access channel and related problems in multi-user information theory,” *IEEE Transactions on Information Theory*, vol. 29, no. 3, pp. 396–412, 1983.
- [4] A. Padakandla, A. G. Sahebi, and S. S. Pradhan, “An achievable rate region for the three-user interference channel based on coset codes,” *IEEE Transactions on Information Theory*, vol. 62, no. 3, pp. 1250–1279, 2016.
- [5] A. Wyner, “Recent results in the shannon theory,” *Information Theory, IEEE Transactions on*, vol. 20, no. 1, pp. 2 – 10, Jan 1974.
- [6] P. W. Cuff, “Communication in networks for coordinating behavior,” Ph.D. dissertation, Stanford, CA, USA, 2009.
- [7] P. Cuff, “Distributed channel synthesis,” *IEEE Transactions on Information Theory*, vol. 59, no. 11, pp. 7071–7096, Nov 2013.
- [8] M. H. Yassaee, A. Gohari, and M. R. Aref, “Channel simulation via interactive communications,” *IEEE Transactions on Information Theory*, vol. 61, no. 6, pp. 2964–2982, 2015.
- [9] A. Winter, ““Extrinsic” and “Intrinsic” data in quantum measurements: Asymptotic convex decomposition of positive operator valued measures,” *Communications in mathematical physics*, vol. 244, no. 1, pp. 157–185, 2004.
- [10] M. M. Wilde, P. Hayden, F. Buscemi, and M.-H. Hsieh, “The information-theoretic costs of simulating quantum measurements,” *Journal of Physics A: Mathematical and Theoretical*, vol. 45, no. 45, p. 453001, 2012.
- [11] M. Heidari, T. A. Atif, and S. Sandeep Pradhan, “Faithful simulation of distributed quantum measurements with applications in distributed rate-distortion theory,” in *2019 IEEE International Symposium on Information Theory (ISIT)*, July 2019, pp. 1162–1166.
- [12] D. Krithivasan and S. S. Pradhan, “Distributed source coding using abelian group codes: A new achievable rate-distortion region,” *IEEE Transactions on Information Theory*, vol. 57, no. 3, pp. 1495–1519, 2011.
- [13] B. Nazer and M. Gastpar, “Computation over multiple-access channels,” *IEEE Trans. on Info. Th.*, vol. 53, no. 10, pp. 3498 –3516, oct. 2007.
- [14] T. Philosof and R. Zamir, “On the loss of single-letter characterization: The dirty multiple access channel,” *IEEE Trans. on Info. Th.*, vol. 55, pp. 2442–2454, June 2009.
- [15] A. Jafarian and S. Vishwanath, “Achievable rates for k-user Gaussian interference channels,” submitted to *IEEE Trans. of Information theory* 2011, available at <http://arxiv.org/abs/1109.5336>.
- [16] S. S. Pradhan, A. Padakandla, and F. Shirani, *An Algebraic and Probabilistic Framework for Network Information Theory*. Foundations and Trends in Communications and Information Theory, 2021, vol. 18, no. 2.
- [17] A. Padakandla and S. S. Pradhan, “Computing sum of sources over an arbitrary multiple access channel,” in *2013 IEEE International Symposium on Information Theory*. IEEE, 2013, pp. 2144–2148.
- [18] R. G. Gallager, *Information Theory and Reliable Communication*. New York: John Wiley & Sons, 1968.
- [19] T. A. Atif, A. Padakandla, and S. S. Pradhan, “Source coding for synthesizing correlated randomness,” *arXiv preprint arXiv:2004.03651*, 2020.
- [20] P. W. Cuff, *Communication in networks for coordinating behavior*. Stanford University, 2009.
- [21] G. M. Ziegler, *Lectures on polytopes*. Springer Science & Business Media, 2012, vol. 152.