Polynomial Approximations of Conditional Expectations in Scalar Gaussian Channels

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Abstract

We consider a channel Y = X + N where X is a random variable satisfying $\mathbb{E}[|X|] < \infty$ and N is an independent standard normal random variable. We show that the minimum mean-square error estimator of X from Y, which is given by the conditional expectation $\mathbb{E}[X \mid Y]$, is a polynomial in Y if and only if it is linear or constant; these two cases correspond to X being Gaussian or a constant, respectively. We also prove that the higher-order derivatives of $y \mapsto \mathbb{E}[X \mid Y = y]$ are expressible as multivariate polynomials in the functions $y \mapsto \mathbb{E}\left[(X - \mathbb{E}[X \mid Y])^k \mid Y = y\right]$ for $k \in \mathbb{N}$. These expressions yield bounds on the 2-norm of the derivatives of the conditional expectation. These bounds imply that, if X has a compactly-supported density that is even and decreasing on the positive half-line, then the error in approximating the conditional expectation $\mathbb{E}[X \mid Y]$ by polynomials in Y of degree at most n decays faster than any polynomial in n.

1 Introduction

We investigate the extent to which polynomials can approximate conditional expectations in the scalar Gaussian channel. For

$$Y = X + N,\tag{1}$$

where X has finite variance and $N \sim \mathcal{N}(0, 1)$ is independent of X, the conditional expectation $\mathbb{E}[X \mid Y]$ is the minimum mean-square error (MMSE) estimator:

$$\min_{Z} \mathbb{E}\left[|X - Z|^2\right] = \mathbb{E}\left[|X - \mathbb{E}[X \mid Y]|^2\right],\tag{2}$$

where the minimization is taken over all $\sigma(Y)$ -measurable random variables Z. It is well-known that $\mathbb{E}[X \mid Y]$ is linear (i.e., a first degree polynomial in Y) if and only if X is Gaussian (see, e.g., [1]). We take this a step further and examine when $\mathbb{E}[X \mid Y]$ is close to being a polynomial. Specifically, we focus on two questions:

- (Q1) For which distributions of X is a polynomial estimator optimal (in the mean-square sense) for reconstructing X from Y?
- (Q2) When the MMSE estimator $\mathbb{E}[X \mid Y]$ is not a polynomial, how well can it be approximated by a polynomial?

In the course of answering $(\mathbf{Q2})$, we answer another fundamental question:

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(Q3) How can the higher-order derivatives of $\mathbb{E}[X \mid Y = y]$ in y be expressed and bounded?

We provide a full answer for (Q1) in Theorem 1, where we show that the MMSE estimator is a polynomial if and only if X is Gaussian or constant. In other words, the only way $\mathbb{E}[X \mid Y]$ can be a polynomial is if it is linear in Y or is a constant.

For the second question, if X has a probability density function (PDF) or a probability mass function (PMF) p_X that is compactly-supported, even, and decreasing over $[0, \infty) \cap \text{supp}(p_X)$, then we show in Theorem 3 that for all positive integers n and k satisfying $n \ge \max(k-1, 1)$ we have that

$$\inf_{q \in \mathscr{P}_n} \|\mathbb{E}[X \mid Y] - q(Y)\|_2 = O_{X,k}\left(\frac{1}{n^{k/2}}\right).$$
(3)

Here, \mathscr{P}_n denotes the set of all polynomials with real coefficients of degree at most n, the implicit constant in (3) can depend on both X and k, and $\|\cdot\|_2$ refers to the P_Y -weighted 2-norm, i.e., $\|f(Y)\|_2^2 = \mathbb{E}[f(Y)^2].$

The result in (3) hinges on our answer to (Q3) in virtue of it giving a uniform upper bound on the derivatives of the conditional expectation (see Theorem 2): there are absolute constants $\{\eta_k\}_{k\geq 1}$ such that

$$\sup_{\mathbb{E}[|X|] < \infty} \left\| \frac{d^k}{dy^k} \mathbb{E}[X \mid Y = y] \right\|_2 \le \eta_k.$$
(4)

The bound in (4) is a corollary of our answer to the other half of (Q3), where we express the derivatives of the conditional expectation in the form (see Proposition 1)

$$\frac{d^{r-1}}{dy^{r-1}} \mathbb{E}[X \mid Y=y] = \sum_{\substack{2\lambda_2 + \dots + r\lambda_r = r\\\lambda_2, \dots, \lambda_r \in \mathbb{N}}} e_{\lambda_2, \dots, \lambda_r} \prod_{i=2}^r \mathbb{E}\left[\left(X - \mathbb{E}[X \mid Y]\right)^i \mid Y=y \right]^{\lambda_i}$$
(5)

for some explicit integers $e_{\lambda_2,\dots,\lambda_r}$ that we define in the sequel. Setting r = 2 in (5) recovers the first derivative [2]

$$\frac{d}{dy} \mathbb{E} \left[X \mid Y = y \right] = \operatorname{Var} \left[X \mid Y = y \right].$$
(6)

These results complement our previous work in [3], where we show that if X has a moment generating function (MGF), then there are constants $\{c_{n,j}\}_{n \in \mathbb{N}, j \in [n]}$ such that

$$\mathbb{E}[X \mid Y] = \lim_{n \to \infty} \sum_{j \in [n]} c_{n,j} Y^j \tag{7}$$

holds in the mean-square sense. In fact, we may choose

 $(c_{n,0},\cdots,c_{n,n}) = \mathbb{E}\left[(X, XY,\cdots,XY^n) \right] \boldsymbol{M}_{Y,n}^{-1}$ (8)

where the Hankel matrix of moments of Y is denoted by

$$\boldsymbol{M}_{Y,n} := \left(\mathbb{E}\left[Y^{i+j} \right] \right)_{(i,j) \in [n]^2}.$$

$$\tag{9}$$

Denoting $\mathbf{Y}^{(n)} = (1, Y, \cdots, Y^n)^T$, the polynomial

$$E_n[X \mid Y] := \mathbb{E}\left[(X, XY, \cdots, XY^n) \right] \boldsymbol{M}_{Y,n}^{-1} \boldsymbol{Y}^{(n)}$$
(10)

is the orthogonal projection of $\mathbb{E}[X \mid Y]$ onto the subspace $\mathscr{P}_n(Y) := \{p(Y) \mid p \in \mathscr{P}_n\}$. This projection characterization, in turn, makes $E_n[X \mid Y]$ the best polynomial approximation (in the

weighted L^2 -norm sense) of the conditional expectation $\mathbb{E}[X \mid Y]$. Specifically, $E_n[X \mid Y]$ uniquely solves the approximation problem

$$E_n[X \mid Y] = \underset{q(Y) \in \mathscr{P}_n(Y)}{\operatorname{argmin}} \|q(Y) - \mathbb{E}[X \mid Y]\|_2.$$
(11)

For (3), we apply solutions to the Bernstein approximation problem (see [4] for a comprehensive survey). The original Bernstein approximation problem extends Weierstrass approximation to polynomial approximation in $L^{\infty}(\mathbb{R},\mu)$ for a measure μ that is absolutely continuous with respect to the Lebesgue measure. The work by Ditzian and Totik [5]—which introduces moduli of smoothness—shows that tools used to solve the Bernstein approximation problem can also be useful for polynomial approximation in $L^{p}(\mathbb{R},\mu)$ for all $p \geq 1$. We apply their results for the case p = 2.

MMSE estimation in Gaussian channels plays a central role in several information-theoretic applications (e.g., [1, 6-9]). The MMSE dimension [10] is a measure of nonlinearity of the MMSE estimator. The first-order derivative of the conditional expectation in Gaussian channels has been treated in [2]. In particular, formula (6) is generalized in [2] to the multivariate case. To the best of our knowledge, no generalization such as (5) to the higher-order derivatives exists in the literature.

The bound in (3) induces a bound on the gap between the MSE achieved by polynomial estimators and the MMSE. Indeed, the loss from replacing the MMSE estimator $\mathbb{E}[X \mid Y]$ with its best polynomial approximation $E_n[X \mid Y]$ is

$$\Delta_{n,X} := \|X - E_n[X \mid Y]\|_2^2 - \|X - \mathbb{E}[X \mid Y]\|_2^2,$$
(12)

which satisfies

$$\Delta_{n,X} \le 2\|X - E_n[X \mid Y]\|_2 \|E_n[X \mid Y] - \mathbb{E}[X \mid Y]\|_2.$$
(13)

Hence, (3) yields the bounds $\Delta_{n,X} = O_{X,\ell}(n^{-\ell})$ for every fixed $\ell > 0$. We note that utilizing higherorder polynomials as proxies of the MMSE has appeared, e.g., in approaches to denoising [11].

Formulas for the conditional expectation that do not require computation of conditional distributions are desirable in practice. For example, the Tweedie formula for the conditional expectation $\mathbb{E}[X \mid Y = y] = y + p'_Y(y)/p_Y(y)$ helped develop the empirical Bayes method [12]. Similarly, the formula for the higher-order derivatives (5) might shed light on practical applications. For instance, one may obtain a uniform bound $|(d^k/dy^k)\mathbb{E}[X \mid Y = y]| \leq M^k k!$ if, e.g., X is bounded. This implies that the conditional expectation is real analytic. In particular, knowledge of the moments $\mathbb{E}[X^{\ell} \mid Y = 0]$ (for $\ell \in \mathbb{N}$) suffices to obtain $\mathbb{E}[X \mid Y = y]$ on the neighborhood $y \in (-1/M, 1/M)$ via Taylor's expansion and the derivative expressions (5). Further, the value of the conditional expectation $\mathbb{E}[X \mid Y = y]$ over an interval $y \in (\alpha, \beta)$ is retrievable by its evaluations at only $[M(\beta - \alpha)/2] + 1$ points.

1.1 Notation

The probability measure induced by a random variable (RV) X is denoted by P_X . If X is continuous (resp. discrete), then its PDF (resp. PMF) is denoted by p_X . We use the notation $\|\cdot\|_q$ for norms of RVs, i.e., for $q \ge 1$ we have $\|X\|_q^q = \mathbb{E}[|X|^q]$. We say that a RV X is *n*-times integrable if it satisfies $\|X\|_n < \infty$, and it is integrable if $\|X\|_1 < \infty$. The norm of the Banach space $L^q(\mathbb{R})$ (for $q \ge 1$) is denoted by $\|\cdot\|_{L^q(\mathbb{R})}$.

The characteristic function of a RV Z is denoted by $\varphi_Z(t) := \mathbb{E}\left[e^{itZ}\right]$. We let \mathscr{P}_n denote the set of polynomials of degree at most n with real coefficients. For $n \in \mathbb{N}$, we set $[n] := \{0, 1, \dots, n\}$ and denote the set of all finite-length tuples of non-negative integers by \mathbb{N}^* .

For every integer $r \ge 2$, let Π_r be the set of unordered integer partitions $r = r_1 + \cdots + r_k$ of r into integers $r_j \ge 2$. We encode Π_r via a list of the multiplicities of the parts as

$$\Pi_r := \{ (\lambda_2, \cdots, \lambda_\ell) \in \mathbb{N}^* ; \ 2\lambda_2 + \cdots + \ell \lambda_\ell = r \}.$$
(14)

In (14), $\ell \geq 2$ is free, and trailing zeros are ignored (i.e., $\lambda_{\ell} > 0$). For a partition $(\lambda_2, \dots, \lambda_{\ell}) = \lambda \in \Pi_r$ having $m = \lambda_2 + \dots + \lambda_{\ell}$ parts, we denote¹

$$c_{\lambda} := \frac{1}{m} \binom{m}{\lambda_2, \cdots, \lambda_{\ell}} \binom{r}{\underbrace{2, \cdots, 2}}; \cdots; \underbrace{\ell, \cdots, \ell}_{\lambda_{\ell}}$$
(15)

and

$$e_{\boldsymbol{\lambda}} := (-1)^{m-1} c_{\boldsymbol{\lambda}}.$$
(16)

We set² $C_r := \sum_{\lambda \in \Pi_r} c_{\lambda}$. Let $\binom{r}{m}$ denote the Stirling numbers of the second kind (i.e., the number of unordered set-partitions of an *r*-element set into *m* nonempty subsets). The integer C_r can be expressed as

$$C_r = \sum_{k=1}^r (k-1)! \sum_{j=0}^k (-1)^j \binom{r}{j} \binom{r-j}{k-j}.$$
(17)

The first few values of C_r (for $2 \le r \le 7$) are given by 1, 1, 4, 11, 56, 267, and as $r \to \infty$ we have the asymptotic $C_r \sim (r-1)!/\alpha^r$ for some constant $\alpha \approx 1.146$ (see [13]). The crude bound $C_r < r^r$ can also be seen by a combinatorial argument. For completeness, equation (17) is derived in Appendix A.

1.2 Assumptions

We assume only that X is integrable and $N \sim \mathcal{N}(0, 1)$ is independent of X to prove that the conditional expectation $\mathbb{E}[X \mid X + N]$ cannot be a polynomial of degree exceeding 1 (Theorem 1) and derive the formula for the higher-order derivatives of the conditional expectation (Proposition 1) along with the ensuing bounds on the norms of the derivatives (Theorem 2). For the Bernstein approximation theorem we prove for $\mathbb{E}[X \mid X + N]$ (Theorem 3), we impose the additional assumption that X is either continuous or discrete with a PDF or a PMF belonging to the set we define next.

Definition 1. Let \mathscr{D} denote the set of compactly-supported even PDFs or PMFs p that are non-increasing over $[0, \infty) \cap \operatorname{supp}(p)$.

2 Polynomial Conditional Expectation

We start by showing that the only way $\mathbb{E}[X \mid Y]$ can be a polynomial, for integrable X and Y = X + N a Gaussian perturbation, is if X is Gaussian or constant. The proof is carried in two steps. First, we show that a degree-*m* non-constant polynomial $\mathbb{E}[X \mid Y]$ requires $p_Y = e^{-h}$ for some polynomial *h* with deg h = m + 1. The second step is showing that, because $p_Y = e^{-h}$ is a convolution of the Gaussian kernel, m = 1.

The following lemma will be useful for the proof of Theorem 1.

¹The integer c_{λ} counts the number of cyclically-invariant ordered set-partitions of an *r*-element set into $m = \lambda_2 + \cdots + \lambda_{\ell}$ subsets where, for each $k \in \{2, \cdots, \ell\}$, exactly λ_k parts have size k.

²The integer C_r counts the total number of cyclically-invariant ordered set-partitions of an *r*-element set into subsets of sizes at least 2.

Lemma 1. For a RV X and a polynomial p, if p(X) is integrable then so is $X^{\text{deg}(p)}$.

Proof. See Appendix **B**.

This lemma will allow us to conclude the finiteness of all moments of X directly from the hypotheses that $\mathbb{E}[X \mid Y]$ is a polynomial of degree exceeding 1 and $||X||_1 < \infty$, because we have the inequalities $||\mathbb{E}[X \mid Y]||_k \leq ||X||_k$ for every $k \geq 1$.

Theorem 1. For Y = X + N where X is an integrable RV and $N \sim \mathcal{N}(0, 1)$ independent of X, the conditional expectation $\mathbb{E}[X \mid Y]$ cannot be a polynomial in Y with degree greater than 1. Therefore, the MMSE estimator in a Gaussian channel with finite-variance input is a polynomial if and only if the input is Gaussian or constant.

Proof. Suppose, for the sake of contradiction, that

$$\mathbb{E}[X \mid Y] = q(Y) \tag{18}$$

for some polynomial with real coefficients q of degree $m := \deg q > 1$. The contradiction we derive will be that the probability measure defined by

$$Q(B) := \frac{1}{a} \int_{B} e^{-x^{2}/2} dP_{X}(x)$$
(19)

for every Borel subset $B \subset \mathbb{R}$, where $a = \mathbb{E}\left[e^{-X^2/2}\right]$ is the normalization constant, would necessarily have a cumulant generating function that is a polynomial of degree m + 1 > 2. Let R be a RV distributed according to Q. We note that the polynomial q is uniquely determined by (18) because Y is continuous, for if q(Y) = g(Y) for a polynomial g then the support of Y must be a subset of the roots of p - q.

The proof strategy is to compute the PDF p_Y in two ways. One way is to compute p_Y as a convolution

$$p_Y(y) = \frac{1}{\sqrt{2\pi}} \mathbb{E}\left[e^{-(X-y)^2/2}\right].$$
 (20)

This equation shows by Lebesgue's dominated convergence that p_Y is continuous. The second way to compute p_Y is via the inverse Fourier transform of φ_Y . We consider the Fourier transform that takes an integrable function φ to $\widehat{\varphi}(y) := \int_{\mathbb{R}} \varphi(t) e^{-iyt} dt$, so the inverse Fourier transform takes an integrable function p to $(2\pi)^{-1} \int_{\mathbb{R}} p(y) e^{ity} dy$. Now, $\varphi_Y = \varphi_X \varphi_N$ is integrable; indeed, $|\varphi_X| \leq 1$ and $\varphi_N(t) = e^{-t^2/2}$. Also, being a characteristic function, φ_Y is continuous too. Therefore, by the Fourier inversion theorem, since $\varphi_Y/(2\pi)$ is the inverse Fourier transform of p_Y , we obtain that $p_Y = \widehat{\varphi_Y}/(2\pi)$. Equating this latter equation with (20), then multiplying both sides by $\sqrt{2\pi}e^{y^2/2}/a$, that $R \sim Q$ (see (19)) implies

$$\mathbb{E}\left[e^{Ry}\right] = \frac{1}{a\sqrt{2\pi}}e^{y^2/2}\widehat{\varphi_Y}(y).$$
(21)

Equation (21) holds for every $y \in \mathbb{R}$. The rest of the proof derives a contradiction by showing that $\widehat{\varphi}_Y = e^G$ for some polynomial G of degree m + 1 > 2.

Integrability of X implies integrability of $\mathbb{E}[X \mid Y]$, so for every $t \in \mathbb{R}$

$$\mathbb{E}\left[e^{itY}(X - \mathbb{E}[X \mid Y])\right] = 0.$$
(22)

Substituting X = Y - N and $\mathbb{E}[X \mid Y] = q(Y)$ into (22),

$$\mathbb{E}\left[e^{itY}(Y-N-q(Y))\right] = 0.$$
(23)

Because the RVs $e^{itY}(Y - q(Y))$ and $e^{itY}N$ are integrable, we may split the expectation to obtain

$$\mathbb{E}\left[e^{itY}(Y-q(Y))\right] - \mathbb{E}\left[e^{itY}N\right] = 0.$$
(24)

We rewrite equation (24) in terms of the characteristic functions of Y and N.

Since q(Y) is integrable, Lemma 1 implies that Y is *m*-times integrable. In particular, we have $\mathbb{E}\left[|(X+z)^m|\right] < \infty$ for some $z \in \mathbb{R}$. By Lemma 1 again, X is *m*-times integrable. Hence, for each $k \in [m]$ and $Z \in \{X, N, Y\}$, that $\mathbb{E}\left[|Z|^k\right] < \infty$ implies that the *k*-th derivative $\varphi_Z^{(k)}$ exists everywhere and

$$(-i)^k \varphi_Z^{(k)}(t) = \mathbb{E}\left[e^{itZ} Z^k\right].$$
(25)

For the term $\mathbb{E}\left[e^{itY}N\right]$ in (24), plugging in Y = X + N, we infer from (25) that

$$\mathbb{E}\left[e^{itY}N\right] = \varphi_X(t)\mathbb{E}\left[e^{itN}N\right] = -i\varphi_X(t)\varphi'_N(t).$$
(26)

But $\varphi_N(t) = e^{-t^2/2}$, so $\varphi'_N(t) = -t\varphi_N(t)$, hence (26) yields

$$\mathbb{E}\left[e^{itY}N\right] = it\varphi_X(t)\varphi_N(t) = it\varphi_Y(t).$$
(27)

Let α_k for $k \in [m]$ be real constants such that $q(u) = \sum_{k \in [m]} \alpha_k u^k$ identically over \mathbb{R} , so $\alpha_m \neq 0$. For the first term in (24), utilizing (25) repeatedly we obtain

$$\mathbb{E}\left[e^{itY}(Y-q(Y))\right] = -i\sum_{k\in[m]} c_k \varphi_Y^{(k)}(t)$$
(28)

where we define the constants

$$c_k := (-i)^{k+1} \alpha_k + \delta_{1,k} = \begin{cases} (-i)^{k+1} \alpha_k & \text{if } k \in [m] \setminus \{1\}, \\ 1 - \alpha_1 & \text{if } k = 1. \end{cases}$$
(29)

Plugging (27) and (28) in (24), we get the differential equation

$$t\varphi_Y(t) + \sum_{k \in [m]} c_k \varphi_Y^{(k)}(t) = 0.$$
(30)

We will transform the differential equation (30) into a linear differential equation in the Fourier transform of φ_Y . For this, we need first to show that for each $k \in [m]$ the derivative $\varphi_Y^{(k)}$ is integrable so that its Fourier transform is well-defined.

Now, repeated differentiation of $\varphi_Y(t) = \varphi_X(t)e^{-t^2/2}$ shows that for each $k \in [m]$ there is a polynomial r_k in k+2 variables such that

$$\varphi_Y^{(k)}(t) = r_k \left(t, \varphi_X(t), \varphi_X'(t), \cdots, \varphi_X^{(k)}(t) \right) e^{-t^2/2}.$$
(31)

Indeed, we start with $r_0(t, u) = u$ because $\varphi_Y(t) = \varphi_X(t)e^{-t^2/2}$. Now, suppose (31) holds for some $k \in [m-1]$. The derivative (with respect to t) of the r_k term is

$$\frac{d}{dt}r_k\left(t,\varphi_X(t),\cdots,\varphi_X^{(k)}(t)\right) = s_k\left(t,\varphi_X(t),\cdots,\varphi_X^{(k+1)}(t)\right)$$
(32)

for some polynomial s_k in k+3 variables. Therefore, differentiating (31), we get

$$\varphi_Y^{(k+1)}(t) = r_{k+1} \left(t, \varphi_X(t), \varphi'_X(t), \cdots, \varphi_X^{(k+1)}(t) \right) e^{-t^2/2}$$
(33)

where

$$r_{k+1}(t, u_0, \cdots, u_{k+1}) := s_k(t, u_0, \cdots, u_{k+1}) - t \cdot r_k(t, u_0, \cdots, u_k)$$
(34)

is a polynomial in k + 3 variables. Therefore (31) holds for all $k \in [m]$. Now, for each $j \in [m]$, we have by (25) the uniform bound $|\varphi_X^{(j)}(t)| \leq \mathbb{E}[|X|^j]$. Therefore, for each $k \in [m]$, letting v_k be the same polynomial as r_k but with the coefficients replaced with their absolute values, the triangle inequality applied to (31) yields the bound $|\varphi_Y^{(k)}(t)| \leq \eta_k(t)e^{-t^2/2}$ where $\eta_k(t) := v_k(|t|, 1, \mathbb{E}[|X|], \cdots, \mathbb{E}[|X|^k])$ is a (positive) polynomial in |t|. Since $\int_{\mathbb{R}} \eta_k(t)e^{-t^2/2} dt < \infty$, we obtain that $\varphi_Y^{(k)}$ is integrable for each $k \in [m]$.

Taking the Fourier transform in the differential equation (30) we infer

$$i\widehat{\varphi_Y}'(y) + \widehat{\varphi_Y}(y) \sum_{k \in [m]} c_k(iy)^k = 0.$$
(35)

We rewrite this equation in terms of the α_k (see (29)) as

$$\widehat{\varphi_Y}'(y) - \widehat{\varphi_Y}(y) \sum_{k \in [m]} (\alpha_k - \delta_{1,k}) y^k = 0.$$
(36)

Equation (35) necessarily implies

$$\widehat{\varphi_Y}(y) = D \exp\left(\sum_{k \in [m]} \frac{\alpha_k - \delta_{1,k}}{k+1} y^{k+1}\right)$$
(37)

for some constant D. Since $p_Y = \widehat{\varphi_Y}/(2\pi)$, we necessarily have D > 0. Therefore, we obtain the desired form for $\widehat{\varphi_Y}$, namely, $\widehat{\varphi_Y} = e^G$ where $G \in \mathscr{P}_{m+1} \setminus \mathscr{P}_m$ is given by³

$$G(y) := \sum_{k \in [m]} \frac{\alpha_k - \delta_{1,k}}{k+1} y^{k+1} + \log(D).$$
(38)

Plugging in this formula for $\widehat{\varphi_Y}$ in (21), we obtain that the cumulant-generating function of the RV R is the degree-(m + 1) polynomial $G(y) + y^2/2 - \log(a\sqrt{2\pi})$, contradicting Marcinkiewicz's theorem that a cumulant-generating function has degree at most 2 if it were a polynomial (see, e.g., [14, Theorem 2.5.3]). This concludes the proof by contradiction that $\mathbb{E}[X \mid Y]$ cannot be a polynomial of degree at least 2.

For the second statement in the theorem, we consider the remaining two cases that $\mathbb{E}[X | Y]$ is a linear expression in Y or is a constant. If $\mathbb{E}[X | Y]$ is constant, then differentiating and taking the expectation in (6) yields that $||X - \mathbb{E}[X | Y]||_2 = 0$, i.e., $X = \mathbb{E}[X | Y]$ is constant. Finally, under the assumption that X has finite variance, $\mathbb{E}[X | Y]$ is linear if and only if X is Gaussian (see, e.g., [1]). We note that if one requires only that X be integrable, then one may deduce directly from the differential equation (30) that a linear $\mathbb{E}[X | Y]$ implies a Gaussian X in this case too, and, for completeness, we end with a proof of this fact.

Assume that $\mathbb{E}[X \mid Y] = \alpha_1 Y + \alpha_0$ is linear (so $\alpha_1 \neq 0$). The differential equation (30) becomes

$$(t - i\alpha_0)\varphi_Y(t) + (1 - \alpha_1)\varphi'_Y(t) = 0.$$
(39)

³It can also be shown that we necessarily have $\alpha_m < 0$ and m is odd, but these points are most since we eventually have a contradiction.

From (39), we see that $\alpha_1 \neq 1$, because φ_Y is nonzero on an open neighborhood around the origin (since $\varphi_Y(0) = 1$ and φ_Y is continuous). Therefore,

$$\varphi_Y(t) = C e^{\frac{1}{\alpha_1 - 1} \left(\frac{1}{2}t^2 - i\alpha_0 t\right)},\tag{40}$$

for some constant C. Taking t = 0 in (40), we see that C = 1. Therefore, the characteristic function of Y is equal to the characteristic function of a $\mathcal{N}\left(\frac{\alpha_0}{1-\alpha_1}, \frac{1}{1-\alpha_1}\right)$ random variable (by taking $t \to \infty$ in (40), we get $\alpha_1 < 1$). In fact, since $\varphi_Y = \varphi_X \cdot \varphi_N$, we obtain

$$\varphi_X(t) = e^{-\frac{1}{2} \cdot \frac{\alpha_1}{1 - \alpha_1} \cdot t^2 + it \cdot \frac{\alpha_0}{1 - \alpha_1}}.$$
(41)

Taking $t \to \infty$, we see that $\alpha_1/(1 - \alpha_1) > 0$, i.e., $\alpha_1 \in (0, 1)$ (note that $\alpha_1 \neq 0$ by the assumption that $\mathbb{E}[X \mid Y]$ is linear). Therefore, uniqueness of characteristic functions implies that X is Gaussian too.

3 Conditional Expectation Derivatives

We develop formulas for the higher-order derivatives of the conditional expectation, and establish upper bounds. The bounds in Theorem 2 on the norm of the derivatives of the conditional expectation will be crucial in Section 4 for establishing a Bernstein approximation theorem that shows how well polynomials can approximate the conditional expectation in the mean-square sense.

Theorem 2. Fix an integrable RV X and an independent $N \sim \mathcal{N}(0,1)$, and set Y = X + N. Let $r \geq 2$ be an integer, let C_r be as defined in (17), and denote $q_r := \lfloor (\sqrt{8r+9}-3)/2 \rfloor$ and $\gamma_r := (2rq_r)!^{1/(4q_r)}$. We have the bound

$$\left\| \frac{d^{r-1}}{dy^{r-1}} \mathbb{E}[X \mid Y = y] \right\|_{2} \le 2^{r} C_{r} \min\left(\gamma_{r}, \|X\|_{2rq_{r}}^{r}\right).$$
(42)

For $2 \le r \le 7$, we obtain the first few values of q_r as 1, 1, 1, 2, 2, 2, and we have $q_r \sim \sqrt{2r}$ as $r \to \infty$ (see Remark 1 for a way to reduce q_r). To prove Theorem 2, we first express the derivatives of $y \mapsto \mathbb{E}[X \mid Y = y]$ as polynomials in the moments of the RV $X_y - \mathbb{E}[X_y]$, where X_y denotes the RV obtained from X by conditioning on $\{Y = y\}$.

Proposition 1. Fix an integrable RV X and an independent $N \sim \mathcal{N}(0,1)$, and let Y = X + N. For each $(y,k) \in \mathbb{R} \times \mathbb{N}$, denote $f(y) := \mathbb{E}[X \mid Y = y]$ and

$$g_k(y) := \mathbb{E}\left[\left(X - \mathbb{E}[X \mid Y] \right)^k \mid Y = y \right].$$
(43)

For $(\lambda_2, \dots, \lambda_\ell) = \lambda \in \mathbb{N}^*$, denote $g^{\lambda} := \prod_{i=2}^{\ell} g_i^{\lambda_i}$, with the understanding that $g_i^0 = 1$. Then, for every integer $r \geq 2$, we have that

$$f^{(r-1)} = \sum_{\boldsymbol{\lambda} \in \Pi_r} e_{\boldsymbol{\lambda}} \boldsymbol{g}^{\boldsymbol{\lambda}},\tag{44}$$

where the integers e_{λ} are as defined in (15)-(16).

Proof. See Appendix C.

Now we are ready to prove Theorem 2.

Proof of Theorem 2. We use the notation of Proposition 1. Fix $(\lambda_2, \dots, \lambda_\ell) = \lambda \in \Pi_r$. By the generalization of Hölder's inequality stating $\|\psi_1 \cdots \psi_k\|_1 \leq \prod_{i=1}^k \|\psi_i\|_k$, we have that

$$\left\| \boldsymbol{g}^{\boldsymbol{\lambda}}(Y) \right\|_{2}^{2} = \left\| \prod_{\lambda_{i} \neq 0} g_{i}^{2\lambda_{i}}(Y) \right\|_{1} \leq \prod_{\lambda_{i} \neq 0} \left\| g_{i}^{2\lambda_{i}}(Y) \right\|_{s}$$
(45)

where s is the number of nonzero entries in λ . By Jensen's inequality for conditional expectation, for each i such that $\lambda_i \neq 0$ we have that

$$\left\|g_i^{2\lambda_i}(Y)\right\|_s \le \|X - \mathbb{E}[X \mid Y]\|_{2i\lambda_i s}^{2i\lambda_i}.$$
(46)

Now, $r = \sum_{i=2}^{\ell} i\lambda_i \ge \sum_{i=2}^{s+1} i = \frac{(s+1)(s+2)}{2} - 1$, so we have that $s^2 + 3s - 2r \le 0$, i.e., $s \le q_r$. Further, $i\lambda_i \le r$ for each *i*. Hence, monotonicity of norms and inequalities (45) and (46) imply the uniform (in λ) bound

$$\left\| \boldsymbol{g}^{\boldsymbol{\lambda}}(Y) \right\|_{2} \leq \| X - \mathbb{E}[X \mid Y] \|_{2rq_{r}}^{r}.$$

$$\tag{47}$$

Observe that $||X - \mathbb{E}[X | Y]||_k \le 2 \min((k!)^{1/(2k)}, ||X||_k)$ (see [1]), so applying the triangle inequality in (44) we obtain

$$\left\| f^{(r-1)}(Y) \right\|_{2} \leq \sum_{\boldsymbol{\lambda} \in \Pi_{r}} c_{\boldsymbol{\lambda}} \left\| \boldsymbol{g}^{\boldsymbol{\lambda}}(Y) \right\|_{2} \leq 2^{r} C_{r} \min\left(\gamma_{r}, \|X\|_{2rq_{r}}^{r}\right),$$
(48)

where $\gamma_r = (2rq_r)!^{1/(4q_r)}$, as desired.

Remark 1. A closer analysis reveals that $i\lambda_i s$ in (46) cannot exceed $\beta_r := t_r^2(t_r + 1/2)$ for $t_r := (\sqrt{6r+7}-1)/3$. For $r \to \infty$, we have $rq_r/\beta_r \sim 3^{3/2}/2 \approx 2.6$. The reduction when, e.g., r = 7, is from $rq_r = 14$ to $\beta_r = 10$.

4 A Bernstein Approximation Theorem

We show that, if $p \in \mathscr{D}$ (see Definition 1) and $X \sim p$, then the approximation error given by $||E_n[X \mid Y] - \mathbb{E}[X \mid Y]||_2$ decays faster than any polynomial in n.

Theorem 3. Fix $p \in \mathscr{D}$, let $X \sim p$, suppose $N \sim \mathcal{N}(0,1)$ is independent of X, and set Y = X + N. There exists a sequence $\{D(p,k)\}_{k\in\mathbb{N}}$ of constants such that for all integers $n \geq \max(k-1,1)$ we have

$$\|E_n[X \mid Y] - \mathbb{E}[X \mid Y]\|_2 \le \frac{D(p,k)}{n^{k/2}}.$$
(49)

The proof relies on results on the Bernstein approximation problem in weighted L^p spaces. In particular, we consider the Freud case, where the weight is of the form e^{-Q} for Q of polynomial growth, e.g., a Gaussian weight.

Definition 2 (Freud Weights). A function $W : \mathbb{R} \to (0, \infty)$ is called a *Freud Weight*, and we write $W \in \mathscr{F}$, if it is of the form $W = e^{-Q}$ for $Q : \mathbb{R} \to \mathbb{R}$ satisfying:

- (1) Q is even,
- (2) Q is differentiable, and Q'(y) > 0 for y > 0,

- (3) $y \mapsto yQ'(y)$ is strictly increasing over $(0, \infty)$,
- (4) $yQ'(y) \to 0$ as $y \to 0^+$, and
- (5) there exist $\lambda, a, b, c > 1$ such that for every y > c

$$a \le \frac{Q'(\lambda y)}{Q'(y)} \le b.$$
(50)

The convolution of a weight in \mathscr{D} with the Gaussian weight $\varphi(x) := e^{-x^2/2}/\sqrt{2\pi}$ is a Freud weight. This can be shown by noting that with $p_Y = e^{-Q}$ we have $Q'(y) = \mathbb{E}[N \mid Y = y]$.

Theorem 4. If $p \in \mathscr{D}$ and $X \sim p$, then the probability density function of X + N, for $N \sim \mathcal{N}(0, 1)$ independent of X, is a Freud weight.

Proof. See Appendix D.

To be able to state the theorem we borrow from the Bernstein approximation literature, we need first to define the Mhaskar–Rakhmanov–Saff number.

Definition 3. If $Q : \mathbb{R} \to \mathbb{R}$ satisfies conditions (2)–(4) in Definition 2, and if $yQ'(y) \to \infty$ as $y \to \infty$, then the *n*-th Mhaskar–Rakhmanov–Saff number $a_n(Q)$ of Q is defined as the unique positive root a_n of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1 - t^2}} \, dt.$$
(51)

Remark 2. The condition $yQ'(y) \to \infty$ as $y \to \infty$ in Definition 3 is satisfied if e^{-Q} is a Freud weight. Indeed, in view of properties (2)–(3) in Definition 2, the quantity $\ell := \lim_{y\to\infty} yQ'(y)$ is well-defined and it belongs to $(0,\infty]$. If $\ell \neq \infty$, then because $\lim_{y\to\infty} \lambda yQ'(\lambda y) = \ell$ too, property (5) would imply that $a \leq 1/\lambda \leq b$ contradicting that $\lambda, a > 1$. Therefore, $\ell = \infty$.

For example, the weight $W(y) = e^{-y^2}$, for which $Q(y) = y^2$, has $a_n(Q) = \sqrt{n}$ because $\int_0^1 t^2 / \sqrt{1-t^2} dt = \frac{\pi}{4}$. If $X \sim p \in \mathcal{D}$, say $\operatorname{supp}(p) \subset [-M, M]$, and $p_Y = e^{-Q}$ (where $N \sim \mathcal{N}(0, 1)$ is independent of X, and Y = X + N), then (see Appendix E)

$$a_n(Q) \le \left(2M + \sqrt{2}\right)\sqrt{n}.\tag{52}$$

We apply the following Bernstein approximation theorem [4, Corollary 3.6] to prove Theorem 3.

Theorem 5. Fix $W \in \mathscr{F}$, and let u be an r-times continuously differentiable function such that $u^{(r)}$ is absolutely continuous. Let $a_n = a_n(Q)$ where $W = e^{-Q}$, and fix $1 \le s \le \infty$. Then, for some constant D(W, r, s) and every $n \ge \max(r - 1, 1)$

$$\inf_{q\in\mathscr{P}_n} \|(q-u)W\|_{L^s(\mathbb{R})} \le D(W,r,s) \left(\frac{a_n}{n}\right)^r \|u^{(r)}W\|_{L^s(\mathbb{R})}.$$
(53)

Proof of Theorem 3. Fix $k \in \mathbb{N}$ and $n \geq \max(k-1,1)$. We apply Theorem 5 for the function $u(y) = \mathbb{E}[X \mid Y = y]$, the weight $W = \sqrt{p_Y}$, and for s = 2. By our choice of weight, $\|hW\|_{L^2(\mathbb{R})} = \|h(Y)\|_2$ for any Borel $h : \mathbb{R} \to \mathbb{R}$. Recall from (11) that $E_n[X \mid Y]$ minimizes $\|q(Y) - \mathbb{E}[X \mid Y]\|_2$ over $q(Y) \in \mathscr{P}_n(Y)$. By (52), we have the bound $a_n = O_p(\sqrt{n})$. Furthermore, by Theorem 2, $\|(d^k/dy^k)\mathbb{E}[X \mid Y]\|_2 = O_k(1)$. Note that $W \in \mathscr{F}$, because $W^2 = p_Y \in \mathscr{F}$ by Theorem 4. Therefore, by Theorem 5, we obtain a constant D(p, k) such that

$$||E_n[X | Y] - \mathbb{E}[X | Y]||_2 \le \frac{D(p,k)}{n^{k/2}},$$
(54)

as desired.

Appendix A A Derivation of Equation (17)

Using the notation of [15], we have that

$$C_r = \sum_{k=1}^r (k-1)! \binom{r}{k}_{\geq 2}$$
(55)

where ${r \atop k}_{\geq 2}$ denotes the number of partitions of an *r*-element set into *k* subsets each of which contains at least 2 elements (note that there are (k-1)! cyclically-invariant arrangements of *k* parts). The exponential generating function of the sequence $r \mapsto {r \atop k}_{\geq 2}$ is $(e^x - 1 - x)^k / k!$. Now, we may write

$$(e^{x} - 1 - x)^{k} = \sum_{a+b \le k} \binom{k}{a, b} (-1)^{k-a} x^{b} \sum_{t \in \mathbb{N}} \frac{(ax)^{t}}{t!}.$$
(56)

Therefore, the coefficient of x^r in $(e^x - 1 - x)^k/k!$ is

$$\frac{1}{r!} \begin{Bmatrix} r \\ k \end{Bmatrix}_{\geq 2} = \sum_{a+b \leq k} \frac{(-1)^{k-a} a^{r-b}}{a! b! (k-a-b)! (r-b)!}$$
(57)

$$= \frac{1}{r!} \sum_{b=0}^{k} \binom{r}{b} \sum_{a=0}^{k-b} (-1)^{k-a} \frac{a^{r-b}}{a!(k-a-b)!}$$
(58)

$$= \frac{1}{r!} \sum_{b=0}^{k} {\binom{r}{b}} {\binom{r-b}{k-b}} (-1)^{b},$$
(59)

which when combined with (55) gives (17).

Appendix B Proof of Lemma 1

Assume that $\mathbb{E}\left[|X|^{\deg(p)}\right] = \infty$ (so $\deg(p) \ge 1$), and we will show that $\mathbb{E}\left[|p(X)|\right] = \infty$ too. Let $k \in [\deg(p) - 1]$ be the largest integer for which $\mathbb{E}\left[|X|^k\right] < \infty$, and write $p(u) = u^{k+1}q(u) + r(u)$ for a nonzero polynomial q and a remainder $r \in \mathscr{P}_k$. By monotonicity of norms, $\mathbb{E}\left[|X|^j\right] < \infty$ for every $j \in [k]$. Hence, r(X) is integrable. Therefore, it suffices to prove that $X^{k+1}q(X)$ is non-integrable, which we show next.

Consider the set $\mathcal{D} = \{u \in \mathbb{R} ; |q(u)| < |a|\}$ where $a \neq 0$ is the leading coefficient of q. If q is constant, then \mathcal{D} is empty, whereas if deg $q \geq 1$ then $|q(u)| \to \infty$ as $|u| \to \infty$ implies that \mathcal{D} is bounded; in either case, there is an $M \in \mathbb{R}$ such that $\mathcal{D} \subset [-M, M]$. Now, writing $1 = 1_{\mathcal{D}} + 1_{\mathcal{D}^c}$, we obtain

$$\mathbb{E}\left[|X|^{k+1}|q(X)|\right] \ge |a| \mathbb{E}\left[|X|^{k+1} \mathbb{1}_{\mathcal{D}^c}(X)\right].$$
(60)

But we also have that

$$\infty = \mathbb{E}\left[|X|^{k+1}\right] \le M^{k+1} + \mathbb{E}\left[|X|^{k+1} \mathbf{1}_{\mathcal{D}^c}(X)\right],\tag{61}$$

so $\mathbb{E}\left[|X|^{k+1}\mathbf{1}_{\mathcal{D}^c}(X)\right] = \infty$. Therefore, inequality (60) yields that $\mathbb{E}\left[|X^{k+1}q(X)|\right] = \infty$, concluding the proof.

Appendix C Proof of Proposition 1

Recall that the conditional expectation can be expressed as

$$\mathbb{E}[Z \mid Y = y] = \frac{\mathbb{E}\left[Ze^{-(X-y)^2/2}\right]}{\mathbb{E}\left[e^{-(X-y)^2/2}\right]}$$
(62)

for any RV Z for which $Ze^{-(X-y)^2/2}$ is integrable. This formula applies for both Z = X and $Z = (X - \mathbb{E}[X | Y = y])^k$, where $(y, k) \in \mathbb{R} \times \mathbb{N}$, because they are polynomials in X and the map $x \mapsto q(x)e^{-(x-y)^2/2}$ is bounded for any polynomial q.

Differentiating (62) for Z = X and rearranging terms, we obtain

$$\frac{d}{dy} \mathbb{E}[X \mid Y = y] = \frac{\mathbb{E}\left[(X - \mathbb{E}[X \mid Y = y])^2 e^{-(X - y)^2/2} \right]}{\mathbb{E}\left[e^{-(X - y)^2/2} \right]},$$
(63)

i.e., $f' = g_2$. Note that $g_0 \equiv 1$ and $g_1 \equiv 0$. Differentiating g_r for $r \geq 1$, we obtain that

$$g'_r = g_{r+1} - rg_2g_{r-1}. (64)$$

We apply successive differentiation to $f' = g_2$ and recover patterns by utilizing (64) at each step.

From $f' = g_2$ and (64), we infer the first few derivatives

$$f^{(2)} = g_3, \ f^{(3)} = g_4 - 3g_2^2, \ f^{(4)} = g_5 - 10g_2g_3.$$
 (65)

We see a homogeneity in (65), namely, $f^{(r-1)}$ is an integer linear combination of terms of the form $g_{i_1}^{\alpha_1} \cdots g_{i_\ell}^{\alpha_\ell}$ with $i_1\alpha_1 + \cdots + i_\ell\alpha_\ell = r$. This homogeneity can be shown to hold for a general r by induction, which we show next. For most of the remainder of the proof, we forget the numerical values of the $f^{(k)}$ and the $g_r^{(k)}$ and only treat them as symbols satisfying $f' = g_2$ and $g'_r = g_{r+1} - rg_2g_{r-1}$ that respect rules of differentiation and which commute.

We call $\sum_{j=1}^{\ell} i_j \alpha_j$ the weighted degree of any nonzero integer multiple of $g_{i_1}^{\alpha_1} \cdots g_{i_{\ell}}^{\alpha_{\ell}}$. This is a well-defined degree because it is invariant to the way the product is arranged. We also say that a sum is of weighted degree r if each summand is of weighted degree r. To prove the claim of homogeneity, i.e., that $f^{(r-1)}$ is of weighted degree r, we differentiate and apply the relation in (64) to a generic term $g_{i_1}^{\alpha_1} \cdots g_{i_{\ell}}^{\alpha_{\ell}}$ whose weighted degree is r. We have the derivative

$$\left(g_{i_1}^{\alpha_1}\cdots g_{i_\ell}^{\alpha_\ell}\right)' = \left(g_{i_1}^{\alpha_1}\right)'\cdots g_{i_\ell}^{\alpha_\ell} + \cdots + g_{i_1}^{\alpha_1}\cdots \left(g_{i_\ell}^{\alpha_\ell}\right)'.$$
(66)

From (64), for integers $i, \alpha \geq 1$,

$$(g_i^{\alpha})' = \alpha g_i^{\alpha - 1} g_{i+1} - \alpha i g_2 g_{i-1} g_i^{\alpha - 1}.$$
(67)

Therefore, the derivative of g_i^{α} has weighted degree $i\alpha + 1$. In other words, differentiation increased the weighted degree of g_i^{α} by 1. From (66), then, we see that the weighted degree of $\left(g_{i_1}^{\alpha_1} \cdots g_{i_\ell}^{\alpha_\ell}\right)'$ is r + 1. Since $f' = g_2$ is of weighted degree 2, induction and linearity of differentiation yield that $f^{(r-1)}$ is of weighted degree r for each $r \geq 2$.

Now, we fix the way we are writing products of the g_i . We ignore explicitly writing g_0 and g_1 , collect identical terms into an exponent, and write lower indices first. One way to keep this notation is via integer partitions. Consider the "homogeneous" sets

$$G_r := \left\{ \sum_{\boldsymbol{\lambda} \in \Pi_r} \beta_{\boldsymbol{\lambda}} \boldsymbol{g}^{\boldsymbol{\lambda}} \; ; \; \beta_{\boldsymbol{\lambda}} \in \mathbb{Z} \text{ for each } \boldsymbol{\lambda} \in \Pi_r \right\}.$$
(68)

The homogeneity property for the derivatives of f can be written as $f^{(r-1)} \in G_r$ for each $r \ge 2$.

Next, we investigate the exact integer coefficients h_{λ} in the expression of the derivatives of f in terms of the g^{λ} . Homogeneity of the derivatives of f says that we may write each $f^{(r-1)}$, $r \geq 2$, as an integer linear combination of $\{g^{\lambda}\}_{\lambda \in \Pi_r}$. One way to obtain such a representation is via repeated differentiation of $f' = g_2$, applying the relation (67), and discarding any term that is a multiple of g_1 . Applying these steps, we arrive at representations

$$f^{(r-1)} = \sum_{\boldsymbol{\lambda} \in \Pi_r} h_{\boldsymbol{\lambda}} \boldsymbol{g}^{\boldsymbol{\lambda}}, \quad c_{\boldsymbol{\lambda}} \in \mathbb{Z}.$$
 (69)

The terms g^{ν} that appear upon differentiating a term g^{λ} can be described as follows. For $(\lambda_2, \dots, \lambda_{\ell}) = \lambda \in \Pi_r$, we call λ_2 the leading term of λ . Consider for a tuple $\lambda \in \Pi_r$ the following two sets of tuples $\tau_+(\lambda), \tau_-(\lambda) \subset \Pi_{r+1}$:

- The set $\tau_+(\lambda)$ consists of all tuples obtainable from λ via replacing a pair $(\lambda_i, \lambda_{i+1})$ with $(\lambda_i 1, \lambda_{i+1} + 1)$ (so, necessarily $\lambda_i \ge 1$) while keeping all other entries unchanged;
- The set $\tau_{-}(\boldsymbol{\lambda})$ consists of all tuples obtainable from $\boldsymbol{\lambda}$ via replacing a pair $(\lambda_{i-1}, \lambda_i)$, for which $i \geq 3$, with the pair $(\lambda_{i-1} + 1, \lambda_i 1)$ (so, necessarily $\lambda_i \geq 1$) and additionally increasing the leading term by 1 while keeping all other terms unchanged.

For example, if $\boldsymbol{\lambda} = (0, 5, 0, 1) \in \Pi_{20}$ then

$$\tau_{+}(\boldsymbol{\lambda}) = \{(0,4,1,1), (0,5,0,0,1)\} \subset \Pi_{21}$$
(70)

and

$$\tau_{-}(\boldsymbol{\lambda}) = \{(2,4,0,1), (1,5,1)\} \subset \Pi_{21}.$$
(71)

The relation (67) yields, in view of

$$\left(g_2^{\lambda_2}\cdots g_\ell^{\lambda_\ell}\right)' = \left(g_2^{\lambda_2}\right)'\cdots g_\ell^{\lambda_\ell} + \cdots + g_2^{\lambda_2}\cdots \left(g_\ell^{\lambda_\ell}\right)',\tag{72}$$

that

$$\left(\boldsymbol{g}^{\boldsymbol{\lambda}}\right)' = \sum_{\boldsymbol{\nu}\in\tau_{+}(\boldsymbol{\lambda})} a_{\boldsymbol{\lambda},\boldsymbol{\nu}} \boldsymbol{g}^{\boldsymbol{\nu}} - \sum_{\boldsymbol{\nu}\in\tau_{-}(\boldsymbol{\lambda})} b_{\boldsymbol{\lambda},\boldsymbol{\nu}} \boldsymbol{g}^{\boldsymbol{\nu}}$$
(73)

for some positive integers $a_{\boldsymbol{\lambda},\boldsymbol{\nu}}$ and $b_{\boldsymbol{\lambda},\boldsymbol{\nu}}$, which we describe next. Finding $a_{\boldsymbol{\lambda},\boldsymbol{\nu}}$ and $b_{\boldsymbol{\lambda},\boldsymbol{\nu}}$ can be straightforwardly done from (67) in view of (72). If $\boldsymbol{\nu} \in \tau_+(\boldsymbol{\lambda})$, say

$$(\nu_i, \nu_{i+1}) = (\lambda_i - 1, \lambda_{i+1} + 1), \tag{74}$$

then $a_{\boldsymbol{\lambda},\boldsymbol{\nu}} = \lambda_i$. If $\boldsymbol{\nu} \in \tau_-(\boldsymbol{\lambda})$, say

$$(\nu_{i-1}, \nu_i) = (\lambda_{i-1} + 1, \lambda_i - 1), \tag{75}$$

then $b_{\boldsymbol{\lambda},\boldsymbol{\nu}} = i\lambda_i$. In our example of $\boldsymbol{\lambda} = (0, 5, 0, 1)$, we get

$$a_{(0,5,0,1),(0,4,1,1)} = 5 \tag{76}$$

$$a_{(0,5,0,1),(0,5,0,0,1)} = 1, (77)$$

whereas

$$b_{(0,5,0,1),(2,4,0,1)} = 15 \tag{78}$$

$$b_{(0,5,0,1),(1,5,1)} = 5. (79)$$

Note that the two sets $\tau_{+}(\lambda)$ and $\tau_{-}(\lambda)$ are disjoint because, e.g., the sum of entries of a tuple in $\tau_{+}(\lambda)$ is the same as that for λ , whereas the sum of entries of a tuple in $\tau_{-}(\lambda)$ is one more than that for λ .

We next describe how to use what we have shown thus far to deduce a recurrence relation for the h_{λ} . Let θ be a process inverting τ , i.e., define for $\boldsymbol{\nu} \in \Pi_{r+1}$ the two sets

$$\theta_{+}(\boldsymbol{\nu}) := \{ \boldsymbol{\lambda} \in \Pi_{r} ; \; \boldsymbol{\nu} \in \tau_{+}(\boldsymbol{\lambda}) \}$$
(80)

and

$$\theta_{-}(\boldsymbol{\nu}) := \{ \boldsymbol{\lambda} \in \Pi_r ; \; \boldsymbol{\nu} \in \tau_{-}(\boldsymbol{\lambda}) \}$$
(81)

The two sets $\theta_+(\boldsymbol{\nu})$ and $\theta_-(\boldsymbol{\nu})$ are disjoint because the two sets $\tau_+(\boldsymbol{\lambda})$ and $\tau_-(\boldsymbol{\lambda})$ are disjoint for each fixed $\boldsymbol{\lambda}$. Recall our process for defining $h_{\boldsymbol{\lambda}}$: we start with $f' = g_2$, so $h_{(1)} = 1$; we successively differentiate $f' = g_2$; after each differentiation, we use (67) and (72) (recall that we have the understanding $g_i^0 = 1$); we discard any ensuing multiple of g_1 ; after r-2 differentiations, we get an equation $f^{(r-1)} = \sum_{\boldsymbol{\lambda} \in \Pi_r} h_{\boldsymbol{\lambda}} \boldsymbol{g}^{\boldsymbol{\lambda}}$, which we take to be the definition of the $h_{\boldsymbol{\lambda}}$. The point here is that it could be that $f^{(r-1)}$ is representable as an integer linear combination of the $\boldsymbol{g}^{\boldsymbol{\lambda}}$ in more than one way, which can only be verified after the numerical values for the g_i are taken into account, but we are not doing that: our approach treats the g_i as symbols following the laid out rules. Now, we look at one of the steps of this procedure, starting at differentiating $f^{(r-1)} = \sum_{\boldsymbol{\lambda} \in \Pi_r} h_{\boldsymbol{\lambda}} \boldsymbol{g}^{\boldsymbol{\lambda}}$, so $f^{(r)} = \sum_{\boldsymbol{\lambda} \in \Pi_r} h_{\boldsymbol{\lambda}} (\boldsymbol{g}^{\boldsymbol{\lambda}})'$. Replacing $(\boldsymbol{g}^{\boldsymbol{\lambda}})'$ via (73),

$$f^{(r)} = \sum_{\boldsymbol{\lambda} \in \Pi_r} \left(\sum_{\boldsymbol{\nu} \in \tau_+(\boldsymbol{\lambda})} a_{\boldsymbol{\lambda}, \boldsymbol{\nu}} \boldsymbol{g}^{\boldsymbol{\nu}} - \sum_{\boldsymbol{\nu} \in \tau_-(\boldsymbol{\lambda})} b_{\boldsymbol{\lambda}, \boldsymbol{\nu}} \boldsymbol{g}^{\boldsymbol{\nu}} \right).$$
(82)

Exchanging the order of summations (for which we use θ),

$$f^{(r)} = \sum_{\boldsymbol{\nu} \in \Pi_{r+1}} \left(\sum_{\boldsymbol{\lambda} \in \theta_+(\boldsymbol{\nu})} h_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}, \boldsymbol{\nu}} - \sum_{\boldsymbol{\lambda} \in \theta_-(\boldsymbol{\nu})} h_{\boldsymbol{\lambda}} b_{\boldsymbol{\lambda}, \boldsymbol{\nu}} \right) \boldsymbol{g}^{\boldsymbol{\nu}}.$$
(83)

Therefore, by definition of the h_{λ} , we have the recurrence: for each $\nu \in \Pi_{r+1}$

$$h_{\boldsymbol{\nu}} = \sum_{\boldsymbol{\lambda} \in \theta_{+}(\boldsymbol{\nu})} h_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}, \boldsymbol{\nu}} - \sum_{\boldsymbol{\lambda} \in \theta_{-}(\boldsymbol{\nu})} h_{\boldsymbol{\lambda}} b_{\boldsymbol{\lambda}, \boldsymbol{\nu}}, \quad h_{(1)} = 1.$$
(84)

One instance of this recurrence is, e.g.,

$$h_{(2,1)} = 3h_{(3)} - 4h_{(1,0,1)} - 6h_{(0,2)}.$$
(85)

Now, we show that the recurrence in (84) also generates e_{λ} as defined in (16). For $(\lambda_2, \dots, \lambda_{\ell}) = \lambda \in \Pi_r$, denote $\sigma(\lambda) = \lambda_2 + \dots + \lambda_{\ell}$. If $\boldsymbol{\nu} \in \tau_+(\lambda)$ then $\sigma(\boldsymbol{\nu}) = \sigma(\lambda)$, and if $\boldsymbol{\nu} \in \tau_-(\lambda)$ then $\sigma(\boldsymbol{\nu}) = \sigma(\lambda) + 1$. Therefore, $\lambda \in \theta_+(\boldsymbol{\nu})$ implies $\sigma(\boldsymbol{\nu}) = \sigma(\lambda)$, and $\lambda \in \theta_-(\boldsymbol{\nu})$ implies $\sigma(\boldsymbol{\nu}) = \sigma(\lambda) + 1$. Multiplying (84) by $(-1)^{\sigma(\boldsymbol{\nu})-1}$ yields the equivalent recurrence

$$t_{\boldsymbol{\nu}} = \sum_{\boldsymbol{\lambda} \in \theta_{+}(\boldsymbol{\nu})} t_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}, \boldsymbol{\nu}} + \sum_{\boldsymbol{\lambda} \in \theta_{-}(\boldsymbol{\nu})} t_{\boldsymbol{\lambda}} b_{\boldsymbol{\lambda}, \boldsymbol{\nu}}, \quad t_{(1)} = 1,$$
(86)

where $t_{\lambda} := (-1)^{\sigma(\lambda)-1}h_{\lambda}$. We show that $c_{\lambda} = (-1)^{\sigma(\lambda)-1}e_{\lambda}$ (see (16)) satisfies this recurrence, which is equivalent to e_{λ} satisfying the recurrence (84). Clearly, $c_{(1)} = 1$, so consider c_{ν} for $\nu \in \Pi_r$ with $r \geq 3$. Consider labelled elements s_1, s_2, \cdots , and let $S_k = \{s_1, \cdots, s_k\}$ for each $k \geq 2$. For any $\lambda \in \Pi_k$, let \mathcal{C}_{λ} be the set of arrangements of cyclically-invariant set-partitions of S_k according to λ , so $|\mathcal{C}_{\lambda}| = c_{\lambda}$. Now, fix $\boldsymbol{\nu} \in \Pi_{r+1}$, and we will build $\mathcal{C}_{\boldsymbol{\nu}}$ from the \mathcal{C}_{λ} where λ ranges over $\theta_+(\boldsymbol{\nu}) \cup \theta_-(\boldsymbol{\nu})$. Consider first $\lambda \in \theta_+(\boldsymbol{\nu})$, where a partition in $\mathcal{C}_{\boldsymbol{\nu}}$ is constructed from a partition in \mathcal{C}_{λ} by appending s_{r+1} to one of the parts of the latter partition. Note that adding s_{r+1} to two distinct partitions of S_r cannot produce the same partition of S_{r+1} ; indeed, just removing s_{r+1} shows that that is impossible. Now, let i be the unique index such that $(\nu_i, \nu_{i+1}) = (\lambda_i - 1, \lambda_{i+1} + 1)$. Then, a partition $\mathcal{P} \in \mathcal{C}_{\boldsymbol{\nu}}$ of S_{r+1} is induced by a partition $\mathcal{P}' \in \mathcal{C}_{\lambda}$ of S_r if and only if s_{r+1} is added to a part in \mathcal{P}' of size i, of which there are exactly $\lambda_i = a_{\lambda, \nu}$. Therefore, we get a contribution of $\sum_{\lambda \in \theta_+(\boldsymbol{\nu})} c_{\lambda} a_{\lambda, \nu}$ towards $c_{\boldsymbol{\nu}}$, which is the first part in (86).

For the second part, $\sum_{\lambda \in \theta_{-}(\nu)} c_{\lambda} b_{\lambda,\nu}$, we consider the remaining ways of generating a partition in \mathcal{C}_{ν} from a partition according to some $\lambda \in \theta_{-}(\nu)$. In this case, s_{r+1} is not appended to an existing part, but it is used to create a new part of size 2. Thus, we need to also move an element s_{j} , $1 \leq j \leq r$, from a part of size at least 3 to be combined with s_{r+1} to create a new part of size 2. It is also clear in this case that such a procedure applied to two distinct partitions in \mathcal{C}_{λ} cannot produce the same partition in \mathcal{C}_{ν} . Let *i* be the unique index for which $(\nu_{i-1}, \nu_i) = (\lambda_{i-1} + 1, \lambda_i - 1)$. There are λ_i parts to choose from, and *i* elements to choose from once a part is chosen, so there are a total of $i\lambda_i = b_{\lambda,\nu}$ ways to generate a partition in \mathcal{C}_{ν} from a partition in \mathcal{C}_{λ} . This gives the second sum in (86), and we conclude that

$$c_{\boldsymbol{\nu}} = \sum_{\boldsymbol{\lambda} \in \theta_{+}(\boldsymbol{\nu})} c_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}, \boldsymbol{\nu}} + \sum_{\boldsymbol{\lambda} \in \theta_{-}(\boldsymbol{\nu})} c_{\boldsymbol{\lambda}} b_{\boldsymbol{\lambda}, \boldsymbol{\nu}}.$$
(87)

Therefore, the c_{λ} and the t_{λ} satisfy the same recurrence, which takes the form: for $\nu \in \Pi_{r+1}$ there are integers $\{d_{\lambda,\nu}\}_{\lambda \in \Pi_r}$ such that

$$u_{\boldsymbol{\nu}} = \sum_{\boldsymbol{\lambda} \in \Pi_r} d_{\boldsymbol{\lambda}, \boldsymbol{\nu}} u_{\boldsymbol{\lambda}}$$
(88)

with the initial condition $u_{(1)} = 1$. Then, we can induct on r to conclude that the c_{λ} and the t_{λ} are the same sequence. Since $\Pi_2 = \{(1)\}$, we see that $c_{\lambda} = t_{\lambda}$ for every $\lambda \in \Pi_2$. Suppose $r \ge 2$ is such that $c_{\lambda} = t_{\lambda}$ for every $\lambda \in \Pi_r$. Hence, for every $\nu \in \Pi_{r+1}$, we have that

$$\sum_{\boldsymbol{\lambda}\in\Pi_r} d_{\boldsymbol{\lambda},\boldsymbol{\nu}} c_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\lambda}\in\Pi_r} d_{\boldsymbol{\lambda},\boldsymbol{\nu}} t_{\boldsymbol{\lambda}}.$$
(89)

Since both sequences c_{λ} and t_{λ} satisfy the recurrence (88), we obtain from (89) that $c_{\nu} = t_{\nu}$ for every $\nu \in \Pi_{r+1}$. Therefore, we obtain by induction that $c_{\lambda} = t_{\lambda}$ for every $\lambda \in \Pi_r$ for every r, as desired.

Appendix D Proof of Theorem 4

Fix $p \in \mathscr{D}$, suppose $X \sim p$, and write Y = X + N and $p_Y = e^{-Q}$. First, we note that Q'(y) is equal to $\mathbb{E}[N \mid Y = y]$.

Lemma 2. Fix a random variable X and let Y = X + N where $N \sim \mathcal{N}(0, 1)$ is independent of X. Writing $p_Y = e^{-Q}$, we have that $Q'(y) = \mathbb{E}[N \mid Y = y]$. *Proof.* We have that $p_Y(y) = \mathbb{E}[e^{-(y-X)^2/2}]/\sqrt{2\pi}$. Differentiating this equation, we obtain that $p'_Y(y) = \mathbb{E}[(X-y)e^{-(y-X)^2/2}]/\sqrt{2\pi}$, where the exchange of differentiation and integration is warranted since $t \mapsto te^{-t^2/2}$ is bounded. Now, $Q = -\log p_Y$, so $Q' = -p'_Y/p_Y$, i.e.,

$$Q'(y) = y - \frac{\mathbb{E}[Xe^{-(y-X)^2/2}]}{\mathbb{E}[e^{-(y-X)^2/2}]} = y - \mathbb{E}[X \mid Y = y].$$
(90)

The proof is completed by substituting X = Y - N.

In view of Lemma 2, that p is even and non-increasing over $[0, \infty) \cap \operatorname{supp}(p)$ imply that Q satisfies conditions (1)–(4) of Definition 2. It remains to show that property (5) holds. To this end, we show that if $\operatorname{supp}(p) \subset [-M, M]$ and $\lambda = M + 2$, then for every y > M + 4 we have that

$$1 < \frac{M^2 + 5M + 8}{2(M+2)} \le \frac{Q'(\lambda y)}{Q'(y)} \le \frac{M^2 + 7M + 8}{4}.$$
(91)

First, since $Q'(y) = y - \mathbb{E}[X \mid Y = y]$ (see (90)), we have the bounds $y - M \leq Q'(y) \leq y + M$ for every $y \in \mathbb{R}$. Therefore, y > M and $\lambda > 1$ imply that

$$\frac{\lambda y - M}{y + M} \le \frac{Q'(\lambda y)}{Q'(y)} \le \frac{\lambda y + M}{y - M}.$$
(92)

Further, since y > M + 4 and $\lambda = M + 2$, we have

$$\frac{M^2 + 5M + 8}{2(M+2)} < \lambda - \frac{M(M+3)}{y+M} = \frac{\lambda y - M}{y+M}$$
(93)

and

$$\frac{\lambda y + M}{y - M} = \lambda + \frac{M(M+3)}{y - M} \le \frac{M^2 + 7M + 8}{4}.$$
(94)

The fact that $1 < \frac{M^2 + 5M + 8}{2(M+2)}$ follows since the discriminant of $M^2 + 3M + 4$ is -7 < 0. Therefore, p_Y is a Freud weight.

Appendix E Proof of Inequality (52)

By Lemma 2,

$$Q'(y) = \mathbb{E}[N \mid Y = y] = y - \mathbb{E}[X \mid Y = y].$$
(95)

Therefore $X \leq M$ implies that, for any constant $z \geq 0$, we have

$$\int_{0}^{1} \frac{ztQ'(zt)}{\sqrt{1-t^2}} dt = \frac{\pi}{4}z^2 - z \int_{0}^{1} \frac{t}{\sqrt{1-t^2}} \frac{\mathbb{E}\left[Xe^{-(X-zt)^2/2}\right]}{\mathbb{E}\left[e^{-(X-zt)^2/2}\right]} dt$$
(96)

$$\geq \frac{\pi}{4}z^2 - Mz. \tag{97}$$

We have $\pi z^2/4 - Mz > n$ for $z = (2M + \sqrt{2})\sqrt{n}$. Since $y \mapsto yQ'(y)$ is strictly increasing over $(0, \infty)$ (condition (3) of Definition 2), we conclude that $a_n(Q) \leq (2M + \sqrt{2})\sqrt{n}$.

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