# Polynomial Approximations of Conditional Expectations in Scalar Gaussian Channels 

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#### Abstract

We consider a channel $Y=X+N$ where $X$ is a random variable satisfying $\mathbb{E}[|X|]<\infty$ and $N$ is an independent standard normal random variable. We show that the minimum mean-square error estimator of $X$ from $Y$, which is given by the conditional expectation $\mathbb{E}[X \mid Y]$, is a polynomial in $Y$ if and only if it is linear or constant; these two cases correspond to $X$ being Gaussian or a constant, respectively. We also prove that the higher-order derivatives of $y \mapsto \mathbb{E}[X \mid Y=y]$ are expressible as multivariate polynomials in the functions $y \mapsto \mathbb{E}\left[(X-\mathbb{E}[X \mid Y])^{k} \mid Y=y\right]$ for $k \in \mathbb{N}$. These expressions yield bounds on the 2 -norm of the derivatives of the conditional expectation. These bounds imply that, if $X$ has a compactly-supported density that is even and decreasing on the positive half-line, then the error in approximating the conditional expectation $\mathbb{E}[X \mid Y]$ by polynomials in $Y$ of degree at most $n$ decays faster than any polynomial in $n$.


## 1 Introduction

We investigate the extent to which polynomials can approximate conditional expectations in the scalar Gaussian channel. For

$$
\begin{equation*}
Y=X+N \tag{1}
\end{equation*}
$$

where $X$ has finite variance and $N \sim \mathcal{N}(0,1)$ is independent of $X$, the conditional expectation $\mathbb{E}[X \mid Y]$ is the minimum mean-square error (MMSE) estimator:

$$
\begin{equation*}
\min _{Z} \mathbb{E}\left[|X-Z|^{2}\right]=\mathbb{E}\left[|X-\mathbb{E}[X \mid Y]|^{2}\right], \tag{2}
\end{equation*}
$$

where the minimization is taken over all $\sigma(Y)$-measurable random variables $Z$. It is well-known that $\mathbb{E}[X \mid Y]$ is linear (i.e., a first degree polynomial in $Y$ ) if and only if $X$ is Gaussian (see, e.g., [1]). We take this a step further and examine when $\mathbb{E}[X \mid Y]$ is close to being a polynomial. Specifically, we focus on two questions:
(Q1) For which distributions of $X$ is a polynomial estimator optimal (in the mean-square sense) for reconstructing $X$ from $Y$ ?
(Q2) When the MMSE estimator $\mathbb{E}[X \mid Y]$ is not a polynomial, how well can it be approximated by a polynomial?

In the course of answering (Q2), we answer another fundamental question:

[^0](Q3) How can the higher-order derivatives of $\mathbb{E}[X \mid Y=y]$ in $y$ be expressed and bounded?
We provide a full answer for (Q1) in Theorem 1, where we show that the MMSE estimator is a polynomial if and only if $X$ is Gaussian or constant. In other words, the only way $\mathbb{E}[X \mid Y]$ can be a polynomial is if it is linear in $Y$ or is a constant.

For the second question, if $X$ has a probability density function (PDF) or a probability mass function (PMF) $p_{X}$ that is compactly-supported, even, and decreasing over $[0, \infty) \cap \operatorname{supp}\left(p_{X}\right)$, then we show in Theorem 3 that for all positive integers $n$ and $k$ satisfying $n \geq \max (k-1,1)$ we have that

$$
\begin{equation*}
\inf _{q \in \mathscr{P}_{n}}\|\mathbb{E}[X \mid Y]-q(Y)\|_{2}=O_{X, k}\left(\frac{1}{n^{k / 2}}\right) . \tag{3}
\end{equation*}
$$

Here, $\mathscr{P}_{n}$ denotes the set of all polynomials with real coefficients of degree at most $n$, the implicit constant in (3) can depend on both $X$ and $k$, and $\|\cdot\|_{2}$ refers to the $P_{Y}$-weighted 2-norm, i.e., $\|f(Y)\|_{2}^{2}=\mathbb{E}\left[f(Y)^{2}\right]$.

The result in (3) hinges on our answer to (Q3) in virtue of it giving a uniform upper bound on the derivatives of the conditional expectation (see Theorem 2): there are absolute constants $\left\{\eta_{k}\right\}_{k \geq 1}$ such that

$$
\begin{equation*}
\sup _{\mathbb{E}[|X|]<\infty}\left\|\frac{d^{k}}{d y^{k}} \mathbb{E}[X \mid Y=y]\right\|_{2} \leq \eta_{k} \tag{4}
\end{equation*}
$$

The bound in (4) is a corollary of our answer to the other half of (Q3), where we express the derivatives of the conditional expectation in the form (see Proposition 1)

$$
\begin{equation*}
\frac{d^{r-1}}{d y^{r-1}} \mathbb{E}[X \mid Y=y]=\sum_{\substack{2 \lambda_{2}+\cdots+r \lambda_{r}=r \\ \lambda_{2}, \cdots, \lambda_{r} \in \mathbb{N}}} e_{\lambda_{2}, \cdots, \lambda_{r}} \prod_{i=2}^{r} \mathbb{E}\left[(X-\mathbb{E}[X \mid Y])^{i} \mid Y=y\right]^{\lambda_{i}} \tag{5}
\end{equation*}
$$

for some explicit integers $e_{\lambda_{2}, \cdots, \lambda_{r}}$ that we define in the sequel. Setting $r=2$ in (5) recovers the first derivative [2]

$$
\begin{equation*}
\frac{d}{d y} \mathbb{E}[X \mid Y=y]=\operatorname{Var}[X \mid Y=y] . \tag{6}
\end{equation*}
$$

These results complement our previous work in [3], where we show that if $X$ has a moment generating function (MGF), then there are constants $\left\{c_{n, j}\right\}_{n \in \mathbb{N}, j \in[n]}$ such that

$$
\begin{equation*}
\mathbb{E}[X \mid Y]=\lim _{n \rightarrow \infty} \sum_{j \in[n]} c_{n, j} Y^{j} \tag{7}
\end{equation*}
$$

holds in the mean-square sense. In fact, we may choose

$$
\begin{equation*}
\left(c_{n, 0}, \cdots, c_{n, n}\right)=\mathbb{E}\left[\left(X, X Y, \cdots, X Y^{n}\right)\right] M_{Y, n}^{-1} \tag{8}
\end{equation*}
$$

where the Hankel matrix of moments of $Y$ is denoted by

$$
\begin{equation*}
\boldsymbol{M}_{Y, n}:=\left(\mathbb{E}\left[Y^{i+j}\right]\right)_{(i, j) \in[n]^{2}} . \tag{9}
\end{equation*}
$$

Denoting $\boldsymbol{Y}^{(n)}=\left(1, Y, \cdots, Y^{n}\right)^{T}$, the polynomial

$$
\begin{equation*}
E_{n}[X \mid Y]:=\mathbb{E}\left[\left(X, X Y, \cdots, X Y^{n}\right)\right] \boldsymbol{M}_{Y, n}^{-1} \boldsymbol{Y}^{(n)} \tag{10}
\end{equation*}
$$

is the orthogonal projection of $\mathbb{E}[X \mid Y]$ onto the subspace $\mathscr{P}_{n}(Y):=\left\{p(Y) \mid p \in \mathscr{P}_{n}\right\}$. This projection characterization, in turn, makes $E_{n}[X \mid Y]$ the best polynomial approximation (in the
weighted $L^{2}$-norm sense) of the conditional expectation $\mathbb{E}[X \mid Y]$. Specifically, $E_{n}[X \mid Y]$ uniquely solves the approximation problem

$$
\begin{equation*}
E_{n}[X \mid Y]=\underset{q(Y) \in \mathscr{P}_{n}(Y)}{\operatorname{argmin}}\|q(Y)-\mathbb{E}[X \mid Y]\|_{2} . \tag{11}
\end{equation*}
$$

For (3), we apply solutions to the Bernstein approximation problem (see [4] for a comprehensive survey). The original Bernstein approximation problem extends Weierstrass approximation to polynomial approximation in $L^{\infty}(\mathbb{R}, \mu)$ for a measure $\mu$ that is absolutely continuous with respect to the Lebesgue measure. The work by Ditzian and Totik [5]-which introduces moduli of smoothness - shows that tools used to solve the Bernstein approximation problem can also be useful for polynomial approximation in $L^{p}(\mathbb{R}, \mu)$ for all $p \geq 1$. We apply their results for the case $p=2$.

MMSE estimation in Gaussian channels plays a central role in several information-theoretic applications (e.g., [1, 6-9]). The MMSE dimension [10] is a measure of nonlinearity of the MMSE estimator. The first-order derivative of the conditional expectation in Gaussian channels has been treated in [2]. In particular, formula (6) is generalized in [2] to the multivariate case. To the best of our knowledge, no generalization such as (5) to the higher-order derivatives exists in the literature.

The bound in (3) induces a bound on the gap between the MSE achieved by polynomial estimators and the MMSE. Indeed, the loss from replacing the MMSE estimator $\mathbb{E}[X \mid Y]$ with its best polynomial approximation $E_{n}[X \mid Y]$ is

$$
\begin{equation*}
\Delta_{n, X}:=\left\|X-E_{n}[X \mid Y]\right\|_{2}^{2}-\|X-\mathbb{E}[X \mid Y]\|_{2}^{2} \tag{12}
\end{equation*}
$$

which satisfies

$$
\begin{equation*}
\Delta_{n, X} \leq 2\left\|X-E_{n}[X \mid Y]\right\|_{2}\left\|E_{n}[X \mid Y]-\mathbb{E}[X \mid Y]\right\|_{2} \tag{13}
\end{equation*}
$$

Hence, (3) yields the bounds $\Delta_{n, X}=O_{X, \ell}\left(n^{-\ell}\right)$ for every fixed $\ell>0$. We note that utilizing higherorder polynomials as proxies of the MMSE has appeared, e.g., in approaches to denoising [11].

Formulas for the conditional expectation that do not require computation of conditional distributions are desirable in practice. For example, the Tweedie formula for the conditional expectation $\mathbb{E}[X \mid Y=y]=y+p_{Y}^{\prime}(y) / p_{Y}(y)$ helped develop the empirical Bayes method [12]. Similarly, the formula for the higher-order derivatives (5) might shed light on practical applications. For instance, one may obtain a uniform bound $\left|\left(d^{k} / d y^{k}\right) \mathbb{E}[X \mid Y=y]\right| \leq M^{k} k$ ! if, e.g., $X$ is bounded. This implies that the conditional expectation is real analytic. In particular, knowledge of the moments $\mathbb{E}\left[X^{\ell} \mid Y=0\right]$ (for $\ell \in \mathbb{N}$ ) suffices to obtain $\mathbb{E}[X \mid Y=y]$ on the neighborhood $y \in(-1 / M, 1 / M)$ via Taylor's expansion and the derivative expressions (5). Further, the value of the conditional expectation $\mathbb{E}[X \mid Y=y]$ over an interval $y \in(\alpha, \beta)$ is retrievable by its evaluations at only $\lceil M(\beta-\alpha) / 2\rceil+1$ points.

### 1.1 Notation

The probability measure induced by a random variable (RV) $X$ is denoted by $P_{X}$. If $X$ is continuous (resp. discrete), then its PDF (resp. PMF) is denoted by $p_{X}$. We use the notation $\|\cdot\|_{q}$ for norms of RVs, i.e., for $q \geq 1$ we have $\|X\|_{q}^{q}=\mathbb{E}\left[|X|^{q}\right]$. We say that a RV $X$ is $n$-times integrable if it satisfies $\|X\|_{n}<\infty$, and it is integrable if $\|X\|_{1}<\infty$. The norm of the Banach space $L^{q}(\mathbb{R})$ (for $q \geq 1)$ is denoted by $\|\cdot\|_{L^{q}(\mathbb{R})}$.

The characteristic function of a RV $Z$ is denoted by $\varphi_{Z}(t):=\mathbb{E}\left[e^{i t Z}\right]$. We let $\mathscr{P}_{n}$ denote the set of polynomials of degree at most $n$ with real coefficients. For $n \in \mathbb{N}$, we set $[n]:=\{0,1, \cdots, n\}$ and denote the set of all finite-length tuples of non-negative integers by $\mathbb{N}^{*}$.

For every integer $r \geq 2$, let $\Pi_{r}$ be the set of unordered integer partitions $r=r_{1}+\cdots+r_{k}$ of $r$ into integers $r_{j} \geq 2$. We encode $\Pi_{r}$ via a list of the multiplicities of the parts as

$$
\begin{equation*}
\Pi_{r}:=\left\{\left(\lambda_{2}, \cdots, \lambda_{\ell}\right) \in \mathbb{N}^{*} ; 2 \lambda_{2}+\cdots+\ell \lambda_{\ell}=r\right\} \tag{14}
\end{equation*}
$$

In (14), $\ell \geq 2$ is free, and trailing zeros are ignored (i.e., $\lambda_{\ell}>0$ ). For a partition $\left(\lambda_{2}, \cdots, \lambda_{\ell}\right)=$ $\boldsymbol{\lambda} \in \Pi_{r}$ having $m=\lambda_{2}+\cdots+\lambda_{\ell}$ parts, we denote ${ }^{1}$

$$
\begin{equation*}
c_{\boldsymbol{\lambda}}:=\frac{1}{m}\binom{m}{\lambda_{2}, \cdots, \lambda_{\ell}}(\underbrace{2, \cdots, 2}_{\lambda_{2}} ; \cdots ; \underbrace{\ell, \cdots, \ell}_{\lambda_{\ell}}) \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
e_{\boldsymbol{\lambda}}:=(-1)^{m-1} c_{\boldsymbol{\lambda}} \tag{16}
\end{equation*}
$$

We $\operatorname{set}^{2} C_{r}:=\sum_{\boldsymbol{\lambda} \in \Pi_{r}} c_{\boldsymbol{\lambda}}$. Let $\left\{\begin{array}{c}r \\ m\end{array}\right\}$ denote the Stirling numbers of the second kind (i.e., the number of unordered set-partitions of an $r$-element set into $m$ nonempty subsets). The integer $C_{r}$ can be expressed as

$$
C_{r}=\sum_{k=1}^{r}(k-1)!\sum_{j=0}^{k}(-1)^{j}\binom{r}{j}\left\{\begin{array}{l}
r-j  \tag{17}\\
k-j
\end{array}\right\}
$$

The first few values of $C_{r}$ (for $2 \leq r \leq 7$ ) are given by $1,1,4,11,56,267$, and as $r \rightarrow \infty$ we have the asymptotic $C_{r} \sim(r-1)!/ \alpha^{r}$ for some constant $\alpha \approx 1.146$ (see [13]). The crude bound $C_{r}<r^{r}$ can also be seen by a combinatorial argument. For completeness, equation (17) is derived in Appendix A.

### 1.2 Assumptions

We assume only that $X$ is integrable and $N \sim \mathcal{N}(0,1)$ is independent of $X$ to prove that the conditional expectation $\mathbb{E}[X \mid X+N]$ cannot be a polynomial of degree exceeding 1 (Theorem 1 ) and derive the formula for the higher-order derivatives of the conditional expectation (Proposition 1) along with the ensuing bounds on the norms of the derivatives (Theorem 2). For the Bernstein approximation theorem we prove for $\mathbb{E}[X \mid X+N]$ (Theorem 3), we impose the additional assumption that $X$ is either continuous or discrete with a PDF or a PMF belonging to the set we define next.

Definition 1. Let $\mathscr{D}$ denote the set of compactly-supported even PDFs or PMFs $p$ that are nonincreasing over $[0, \infty) \cap \operatorname{supp}(p)$.

## 2 Polynomial Conditional Expectation

We start by showing that the only way $\mathbb{E}[X \mid Y]$ can be a polynomial, for integrable $X$ and $Y=X+N$ a Gaussian perturbation, is if $X$ is Gaussian or constant. The proof is carried in two steps. First, we show that a degree-m non-constant polynomial $\mathbb{E}[X \mid Y]$ requires $p_{Y}=e^{-h}$ for some polynomial $h$ with $\operatorname{deg} h=m+1$. The second step is showing that, because $p_{Y}=e^{-h}$ is a convolution of the Gaussian kernel, $m=1$.

The following lemma will be useful for the proof of Theorem 1.

[^1]Lemma 1. For a $R V X$ and a polynomial $p$, if $p(X)$ is integrable then so is $X^{\operatorname{deg}(p)}$.
Proof. See Appendix B.
This lemma will allow us to conclude the finiteness of all moments of $X$ directly from the hypotheses that $\mathbb{E}[X \mid Y]$ is a polynomial of degree exceeding 1 and $\|X\|_{1}<\infty$, because we have the inequalities $\|\mathbb{E}[X \mid Y]\|_{k} \leq\|X\|_{k}$ for every $k \geq 1$.

Theorem 1. For $Y=X+N$ where $X$ is an integrable $R V$ and $N \sim \mathcal{N}(0,1)$ independent of $X$, the conditional expectation $\mathbb{E}[X \mid Y]$ cannot be a polynomial in $Y$ with degree greater than 1. Therefore, the MMSE estimator in a Gaussian channel with finite-variance input is a polynomial if and only if the input is Gaussian or constant.

Proof. Suppose, for the sake of contradiction, that

$$
\begin{equation*}
\mathbb{E}[X \mid Y]=q(Y) \tag{18}
\end{equation*}
$$

for some polynomial with real coefficients $q$ of degree $m:=\operatorname{deg} q>1$. The contradiction we derive will be that the probability measure defined by

$$
\begin{equation*}
Q(B):=\frac{1}{a} \int_{B} e^{-x^{2} / 2} d P_{X}(x) \tag{19}
\end{equation*}
$$

for every Borel subset $B \subset \mathbb{R}$, where $a=\mathbb{E}\left[e^{-X^{2} / 2}\right]$ is the normalization constant, would necessarily have a cumulant generating function that is a polynomial of degree $m+1>2$. Let $R$ be a RV distributed according to $Q$. We note that the polynomial $q$ is uniquely determined by (18) because $Y$ is continuous, for if $q(Y)=g(Y)$ for a polynomial $g$ then the support of $Y$ must be a subset of the roots of $p-g$.

The proof strategy is to compute the PDF $p_{Y}$ in two ways. One way is to compute $p_{Y}$ as a convolution

$$
\begin{equation*}
p_{Y}(y)=\frac{1}{\sqrt{2 \pi}} \mathbb{E}\left[e^{-(X-y)^{2} / 2}\right] . \tag{20}
\end{equation*}
$$

This equation shows by Lebesgue's dominated convergence that $p_{Y}$ is continuous. The second way to compute $p_{Y}$ is via the inverse Fourier transform of $\varphi_{Y}$. We consider the Fourier transform that takes an integrable function $\varphi$ to $\widehat{\varphi}(y):=\int_{\mathbb{R}} \varphi(t) e^{-i y t} d t$, so the inverse Fourier transform takes an integrable function $p$ to $(2 \pi)^{-1} \int_{\mathbb{R}} p(y) e^{i t y} d y$. Now, $\varphi_{Y}=\varphi_{X} \varphi_{N}$ is integrable; indeed, $\left|\varphi_{X}\right| \leq 1$ and $\varphi_{N}(t)=e^{-t^{2} / 2}$. Also, being a characteristic function, $\varphi_{Y}$ is continuous too. Therefore, by the Fourier inversion theorem, since $\varphi_{Y} /(2 \pi)$ is the inverse Fourier transform of $p_{Y}$, we obtain that $p_{Y}=\widehat{\varphi_{Y}} /(2 \pi)$. Equating this latter equation with (20), then multiplying both sides by $\sqrt{2 \pi} e^{y^{2} / 2} / a$, that $R \sim Q$ (see (19)) implies

$$
\begin{equation*}
\mathbb{E}\left[e^{R y}\right]=\frac{1}{a \sqrt{2 \pi}} e^{y^{2} / 2} \widehat{\varphi_{Y}}(y) \tag{21}
\end{equation*}
$$

Equation (21) holds for every $y \in \mathbb{R}$. The rest of the proof derives a contradiction by showing that $\widehat{\varphi_{Y}}=e^{G}$ for some polynomial $G$ of degree $m+1>2$.

Integrability of $X$ implies integrability of $\mathbb{E}[X \mid Y]$, so for every $t \in \mathbb{R}$

$$
\begin{equation*}
\mathbb{E}\left[e^{i t Y}(X-\mathbb{E}[X \mid Y])\right]=0 \tag{22}
\end{equation*}
$$

Substituting $X=Y-N$ and $\mathbb{E}[X \mid Y]=q(Y)$ into (22),

$$
\begin{equation*}
\mathbb{E}\left[e^{i t Y}(Y-N-q(Y))\right]=0 \tag{23}
\end{equation*}
$$

Because the RVs $e^{i t Y}(Y-q(Y))$ and $e^{i t Y} N$ are integrable, we may split the expectation to obtain

$$
\begin{equation*}
\mathbb{E}\left[e^{i t Y}(Y-q(Y))\right]-\mathbb{E}\left[e^{i t Y} N\right]=0 . \tag{24}
\end{equation*}
$$

We rewrite equation (24) in terms of the characteristic functions of $Y$ and $N$.
Since $q(Y)$ is integrable, Lemma 1 implies that $Y$ is $m$-times integrable. In particular, we have $\mathbb{E}\left[\left|(X+z)^{m}\right|\right]<\infty$ for some $z \in \mathbb{R}$. By Lemma 1 again, $X$ is $m$-times integrable. Hence, for each $k \in[m]$ and $Z \in\{X, N, Y\}$, that $\mathbb{E}\left[|Z|^{k}\right]<\infty$ implies that the $k$-th derivative $\varphi_{Z}^{(k)}$ exists everywhere and

$$
\begin{equation*}
(-i)^{k} \varphi_{Z}^{(k)}(t)=\mathbb{E}\left[e^{i t Z} Z^{k}\right] . \tag{25}
\end{equation*}
$$

For the term $\mathbb{E}\left[e^{i t Y} N\right]$ in (24), plugging in $Y=X+N$, we infer from (25) that

$$
\begin{equation*}
\mathbb{E}\left[e^{i t Y} N\right]=\varphi_{X}(t) \mathbb{E}\left[e^{i t N} N\right]=-i \varphi_{X}(t) \varphi_{N}^{\prime}(t) \tag{26}
\end{equation*}
$$

But $\varphi_{N}(t)=e^{-t^{2} / 2}$, so $\varphi_{N}^{\prime}(t)=-t \varphi_{N}(t)$, hence (26) yields

$$
\begin{equation*}
\mathbb{E}\left[e^{i t Y} N\right]=i t \varphi_{X}(t) \varphi_{N}(t)=i t \varphi_{Y}(t) . \tag{27}
\end{equation*}
$$

Let $\alpha_{k}$ for $k \in[m]$ be real constants such that $q(u)=\sum_{k \in[m]} \alpha_{k} u^{k}$ identically over $\mathbb{R}$, so $\alpha_{m} \neq 0$. For the first term in (24), utilizing (25) repeatedly we obtain

$$
\begin{equation*}
\mathbb{E}\left[e^{i t Y}(Y-q(Y))\right]=-i \sum_{k \in[m]} c_{k} \varphi_{Y}^{(k)}(t) \tag{28}
\end{equation*}
$$

where we define the constants

$$
c_{k}:=(-i)^{k+1} \alpha_{k}+\delta_{1, k}=\left\{\begin{array}{cl}
(-i)^{k+1} \alpha_{k} & \text { if } k \in[m] \backslash\{1\},  \tag{29}\\
1-\alpha_{1} & \text { if } k=1 .
\end{array}\right.
$$

Plugging (27) and (28) in (24), we get the differential equation

$$
\begin{equation*}
t \varphi_{Y}(t)+\sum_{k \in[m]} c_{k} \varphi_{Y}^{(k)}(t)=0 \tag{30}
\end{equation*}
$$

We will transform the differential equation (30) into a linear differential equation in the Fourier transform of $\varphi_{Y}$. For this, we need first to show that for each $k \in[m]$ the derivative $\varphi_{Y}^{(k)}$ is integrable so that its Fourier transform is well-defined.

Now, repeated differentiation of $\varphi_{Y}(t)=\varphi_{X}(t) e^{-t^{2} / 2}$ shows that for each $k \in[m]$ there is a polynomial $r_{k}$ in $k+2$ variables such that

$$
\begin{equation*}
\varphi_{Y}^{(k)}(t)=r_{k}\left(t, \varphi_{X}(t), \varphi_{X}^{\prime}(t), \cdots, \varphi_{X}^{(k)}(t)\right) e^{-t^{2} / 2} \tag{31}
\end{equation*}
$$

Indeed, we start with $r_{0}(t, u)=u$ because $\varphi_{Y}(t)=\varphi_{X}(t) e^{-t^{2} / 2}$. Now, suppose (31) holds for some $k \in[m-1]$. The derivative (with respect to $t$ ) of the $r_{k}$ term is

$$
\begin{equation*}
\frac{d}{d t} r_{k}\left(t, \varphi_{X}(t), \cdots, \varphi_{X}^{(k)}(t)\right)=s_{k}\left(t, \varphi_{X}(t), \cdots, \varphi_{X}^{(k+1)}(t)\right) \tag{32}
\end{equation*}
$$

for some polynomial $s_{k}$ in $k+3$ variables. Therefore, differentiating (31), we get

$$
\begin{equation*}
\varphi_{Y}^{(k+1)}(t)=r_{k+1}\left(t, \varphi_{X}(t), \varphi_{X}^{\prime}(t), \cdots, \varphi_{X}^{(k+1)}(t)\right) e^{-t^{2} / 2} \tag{33}
\end{equation*}
$$

where

$$
\begin{equation*}
r_{k+1}\left(t, u_{0}, \cdots, u_{k+1}\right):=s_{k}\left(t, u_{0}, \cdots, u_{k+1}\right)-t \cdot r_{k}\left(t, u_{0}, \cdots, u_{k}\right) \tag{34}
\end{equation*}
$$

is a polynomial in $k+3$ variables. Therefore (31) holds for all $k \in[m]$. Now, for each $j \in[m]$, we have by (25) the uniform bound $\left|\varphi_{X}^{(j)}(t)\right| \leq \mathbb{E}\left[|X|^{j}\right]$. Therefore, for each $k \in[m]$, letting $v_{k}$ be the same polynomial as $r_{k}$ but with the coefficients replaced with their absolute values, the triangle inequality applied to (31) yields the bound $\left|\varphi_{Y}^{(k)}(t)\right| \leq \eta_{k}(t) e^{-t^{2} / 2}$ where $\eta_{k}(t):=$ $v_{k}\left(|t|, 1, \mathbb{E}[|X|], \cdots, \mathbb{E}\left[|X|^{k}\right]\right)$ is a (positive) polynomial in $|t|$. Since $\int_{\mathbb{R}} \eta_{k}(t) e^{-t^{2} / 2} d t<\infty$, we obtain that $\varphi_{Y}^{(k)}$ is integrable for each $k \in[m]$.

Taking the Fourier transform in the differential equation (30) we infer

$$
\begin{equation*}
i{\widehat{\varphi_{Y}}}^{\prime}(y)+\widehat{\varphi_{Y}}(y) \sum_{k \in[m]} c_{k}(i y)^{k}=0 . \tag{35}
\end{equation*}
$$

We rewrite this equation in terms of the $\alpha_{k}$ (see (29)) as

$$
\begin{equation*}
{\widehat{\varphi_{Y}}}^{\prime}(y)-\widehat{\varphi_{Y}}(y) \sum_{k \in[m]}\left(\alpha_{k}-\delta_{1, k}\right) y^{k}=0 . \tag{36}
\end{equation*}
$$

Equation (35) necessarily implies

$$
\begin{equation*}
\widehat{\varphi_{Y}}(y)=D \exp \left(\sum_{k \in[m]} \frac{\alpha_{k}-\delta_{1, k}}{k+1} y^{k+1}\right) \tag{37}
\end{equation*}
$$

for some constant $D$. Since $p_{Y}=\widehat{\varphi_{Y}} /(2 \pi)$, we necessarily have $D>0$. Therefore, we obtain the desired form for $\widehat{\varphi_{Y}}$, namely, $\widehat{\varphi_{Y}}=e^{G}$ where $G \in \mathscr{P}_{m+1} \backslash \mathscr{P}_{m}$ is given by ${ }^{3}$

$$
\begin{equation*}
G(y):=\sum_{k \in[m]} \frac{\alpha_{k}-\delta_{1, k}}{k+1} y^{k+1}+\log (D) . \tag{38}
\end{equation*}
$$

Plugging in this formula for $\widehat{\varphi_{Y}}$ in (21), we obtain that the cumulant-generating function of the RV $R$ is the degree- $(m+1)$ polynomial $G(y)+y^{2} / 2-\log (a \sqrt{2 \pi})$, contradicting Marcinkiewicz's theorem that a cumulant-generating function has degree at most 2 if it were a polynomial (see, e.g., [14, Theorem 2.5.3]). This concludes the proof by contradiction that $\mathbb{E}[X \mid Y]$ cannot be a polynomial of degree at least 2 .

For the second statement in the theorem, we consider the remaining two cases that $\mathbb{E}[X \mid Y]$ is a linear expression in $Y$ or is a constant. If $\mathbb{E}[X \mid Y]$ is constant, then differentiating and taking the expectation in (6) yields that $\|X-\mathbb{E}[X \mid Y]\|_{2}=0$, i.e., $X=\mathbb{E}[X \mid Y]$ is constant. Finally, under the assumption that $X$ has finite variance, $\mathbb{E}[X \mid Y]$ is linear if and only if $X$ is Gaussian (see, e.g., [1]). We note that if one requires only that $X$ be integrable, then one may deduce directly from the differential equation (30) that a linear $\mathbb{E}[X \mid Y]$ implies a Gaussian $X$ in this case too, and, for completeness, we end with a proof of this fact.

Assume that $\mathbb{E}[X \mid Y]=\alpha_{1} Y+\alpha_{0}$ is linear (so $\alpha_{1} \neq 0$ ). The differential equation (30) becomes

$$
\begin{equation*}
\left(t-i \alpha_{0}\right) \varphi_{Y}(t)+\left(1-\alpha_{1}\right) \varphi_{Y}^{\prime}(t)=0 \tag{39}
\end{equation*}
$$

[^2]From (39), we see that $\alpha_{1} \neq 1$, because $\varphi_{Y}$ is nonzero on an open neighborhood around the origin (since $\varphi_{Y}(0)=1$ and $\varphi_{Y}$ is continuous). Therefore,

$$
\begin{equation*}
\varphi_{Y}(t)=C e^{\frac{1}{\alpha_{1}-1}\left(\frac{1}{2} t^{2}-i \alpha_{0} t\right)}, \tag{40}
\end{equation*}
$$

for some constant $C$. Taking $t=0$ in (40), we see that $C=1$. Therefore, the characteristic function of $Y$ is equal to the characteristic function of a $\mathcal{N}\left(\frac{\alpha_{0}}{1-\alpha_{1}}, \frac{1}{1-\alpha_{1}}\right)$ random variable (by taking $t \rightarrow \infty$ in (40), we get $\alpha_{1}<1$ ). In fact, since $\varphi_{Y}=\varphi_{X} \cdot \varphi_{N}$, we obtain

$$
\begin{equation*}
\varphi_{X}(t)=e^{-\frac{1}{2} \cdot \frac{\alpha_{1}}{1-\alpha_{1}} \cdot t^{2}+i t \cdot \frac{\alpha_{0}}{1-\alpha_{1}}} \tag{41}
\end{equation*}
$$

Taking $t \rightarrow \infty$, we see that $\alpha_{1} /\left(1-\alpha_{1}\right)>0$, i.e., $\alpha_{1} \in(0,1)$ (note that $\alpha_{1} \neq 0$ by the assumption that $\mathbb{E}[X \mid Y]$ is linear). Therefore, uniqueness of characteristic functions implies that $X$ is Gaussian too.

## 3 Conditional Expectation Derivatives

We develop formulas for the higher-order derivatives of the conditional expectation, and establish upper bounds. The bounds in Theorem 2 on the norm of the derivatives of the conditional expectation will be crucial in Section 4 for establishing a Bernstein approximation theorem that shows how well polynomials can approximate the conditional expectation in the mean-square sense.

Theorem 2. Fix an integrable $R V X$ and an independent $N \sim \mathcal{N}(0,1)$, and set $Y=X+N$. Let $r \geq 2$ be an integer, let $C_{r}$ be as defined in (17), and denote $q_{r}:=\lfloor(\sqrt{8 r+9}-3) / 2\rfloor$ and $\gamma_{r}:=\left(2 r q_{r}\right)!^{1 /\left(4 q_{r}\right)}$. We have the bound

$$
\begin{equation*}
\left\|\frac{d^{r-1}}{d y^{r-1}} \mathbb{E}[X \mid Y=y]\right\|_{2} \leq 2^{r} C_{r} \min \left(\gamma_{r},\|X\|_{2 r q_{r}}^{r}\right) \tag{42}
\end{equation*}
$$

For $2 \leq r \leq 7$, we obtain the first few values of $q_{r}$ as $1,1,1,2,2,2$, and we have $q_{r} \sim \sqrt{2 r}$ as $r \rightarrow \infty$ (see Remark 1 for a way to reduce $q_{r}$ ). To prove Theorem 2, we first express the derivatives of $y \mapsto \mathbb{E}[X \mid Y=y]$ as polynomials in the moments of the $\mathrm{RV} X_{y}-\mathbb{E}\left[X_{y}\right]$, where $X_{y}$ denotes the RV obtained from $X$ by conditioning on $\{Y=y\}$.

Proposition 1. Fix an integrable $R V X$ and an independent $N \sim \mathcal{N}(0,1)$, and let $Y=X+N$. For each $(y, k) \in \mathbb{R} \times \mathbb{N}$, denote $f(y):=\mathbb{E}[X \mid Y=y]$ and

$$
\begin{equation*}
g_{k}(y):=\mathbb{E}\left[(X-\mathbb{E}[X \mid Y])^{k} \mid Y=y\right] \tag{43}
\end{equation*}
$$

For $\left(\lambda_{2}, \cdots, \lambda_{\ell}\right)=\boldsymbol{\lambda} \in \mathbb{N}^{*}$, denote $\boldsymbol{g}^{\boldsymbol{\lambda}}:=\prod_{i=2}^{\ell} g_{i}^{\lambda_{i}}$, with the understanding that $g_{i}^{0}=1$. Then, for every integer $r \geq 2$, we have that

$$
\begin{equation*}
f^{(r-1)}=\sum_{\lambda \in \Pi_{r}} e_{\lambda} g^{\boldsymbol{\lambda}}, \tag{44}
\end{equation*}
$$

where the integers $e_{\boldsymbol{\lambda}}$ are as defined in (15)-(16).
Proof. See Appendix C.
Now we are ready to prove Theorem 2.

Proof of Theorem 2. We use the notation of Proposition 1. Fix $\left(\lambda_{2}, \cdots, \lambda_{\ell}\right)=\boldsymbol{\lambda} \in \Pi_{r}$. By the generalization of Hölder's inequality stating $\left\|\psi_{1} \cdots \psi_{k}\right\|_{1} \leq \prod_{i=1}^{k}\left\|\psi_{i}\right\|_{k}$, we have that

$$
\begin{equation*}
\left\|\boldsymbol{g}^{\boldsymbol{\lambda}}(Y)\right\|_{2}^{2}=\left\|\prod_{\lambda_{i} \neq 0} g_{i}^{2 \lambda_{i}}(Y)\right\|_{1} \leq \prod_{\lambda_{i} \neq 0}\left\|g_{i}^{2 \lambda_{i}}(Y)\right\|_{s} \tag{45}
\end{equation*}
$$

where $s$ is the number of nonzero entries in $\boldsymbol{\lambda}$. By Jensen's inequality for conditional expectation, for each $i$ such that $\lambda_{i} \neq 0$ we have that

$$
\begin{equation*}
\left\|g_{i}^{2 \lambda_{i}}(Y)\right\|_{s} \leq\|X-\mathbb{E}[X \mid Y]\|_{2 i \lambda_{i} s^{2}}^{2 i \lambda_{i}} . \tag{46}
\end{equation*}
$$

Now, $r=\sum_{i=2}^{\ell} i \lambda_{i} \geq \sum_{i=2}^{s+1} i=\frac{(s+1)(s+2)}{2}-1$, so we have that $s^{2}+3 s-2 r \leq 0$, i.e., $s \leq q_{r}$. Further, $i \lambda_{i} \leq r$ for each $i$. Hence, monotonicity of norms and inequalities (45) and (46) imply the uniform (in $\boldsymbol{\lambda}$ ) bound

$$
\begin{equation*}
\left\|\boldsymbol{g}^{\boldsymbol{\lambda}}(Y)\right\|_{2} \leq\|X-\mathbb{E}[X \mid Y]\|_{2 r q_{r}}^{r} \tag{47}
\end{equation*}
$$

Observe that $\|X-\mathbb{E}[X \mid Y]\|_{k} \leq 2 \min \left((k!)^{1 /(2 k)},\|X\|_{k}\right)$ (see [1]), so applying the triangle inequality in (44) we obtain

$$
\begin{equation*}
\left\|f^{(r-1)}(Y)\right\|_{2} \leq \sum_{\lambda \in \Pi_{r}} c_{\boldsymbol{\lambda}}\left\|\boldsymbol{g}^{\boldsymbol{\lambda}}(Y)\right\|_{2} \leq 2^{r} C_{r} \min \left(\gamma_{r},\|X\|_{2 r q_{r}}^{r}\right), \tag{48}
\end{equation*}
$$

where $\gamma_{r}=\left(2 r q_{r}\right)!^{1 /\left(4 q_{r}\right)}$, as desired.
Remark 1. A closer analysis reveals that $i \lambda_{i} s$ in (46) cannot exceed $\beta_{r}:=t_{r}^{2}\left(t_{r}+1 / 2\right)$ for $t_{r}:=$ $(\sqrt{6 r+7}-1) / 3$. For $r \rightarrow \infty$, we have $r q_{r} / \beta_{r} \sim 3^{3 / 2} / 2 \approx 2.6$. The reduction when, e.g., $r=7$, is from $r q_{r}=14$ to $\beta_{r}=10$.

## 4 A Bernstein Approximation Theorem

We show that, if $p \in \mathscr{D}$ (see Definition 1) and $X \sim p$, then the approximation error given by $\left\|E_{n}[X \mid Y]-\mathbb{E}[X \mid Y]\right\|_{2}$ decays faster than any polynomial in $n$.

Theorem 3. Fix $p \in \mathscr{D}$, let $X \sim p$, suppose $N \sim \mathcal{N}(0,1)$ is independent of $X$, and set $Y=X+N$. There exists a sequence $\{D(p, k)\}_{k \in \mathbb{N}}$ of constants such that for all integers $n \geq \max (k-1,1)$ we have

$$
\begin{equation*}
\left\|E_{n}[X \mid Y]-\mathbb{E}[X \mid Y]\right\|_{2} \leq \frac{D(p, k)}{n^{k / 2}} \tag{49}
\end{equation*}
$$

The proof relies on results on the Bernstein approximation problem in weighted $L^{p}$ spaces. In particular, we consider the Freud case, where the weight is of the form $e^{-Q}$ for $Q$ of polynomial growth, e.g., a Gaussian weight.

Definition 2 (Freud Weights). A function $W: \mathbb{R} \rightarrow(0, \infty)$ is called a Freud Weight, and we write $W \in \mathscr{F}$, if it is of the form $W=e^{-Q}$ for $Q: \mathbb{R} \rightarrow \mathbb{R}$ satisfying:
(1) $Q$ is even,
(2) $Q$ is differentiable, and $Q^{\prime}(y)>0$ for $y>0$,
(3) $y \mapsto y Q^{\prime}(y)$ is strictly increasing over $(0, \infty)$,
(4) $y Q^{\prime}(y) \rightarrow 0$ as $y \rightarrow 0^{+}$, and
(5) there exist $\lambda, a, b, c>1$ such that for every $y>c$

$$
\begin{equation*}
a \leq \frac{Q^{\prime}(\lambda y)}{Q^{\prime}(y)} \leq b \tag{50}
\end{equation*}
$$

The convolution of a weight in $\mathscr{D}$ with the Gaussian weight $\varphi(x):=e^{-x^{2} / 2} / \sqrt{2 \pi}$ is a Freud weight. This can be shown by noting that with $p_{Y}=e^{-Q}$ we have $Q^{\prime}(y)=\mathbb{E}[N \mid Y=y]$.
Theorem 4. If $p \in \mathscr{D}$ and $X \sim p$, then the probability density function of $X+N$, for $N \sim \mathcal{N}(0,1)$ independent of $X$, is a Freud weight.

Proof. See Appendix D.
To be able to state the theorem we borrow from the Bernstein approximation literature, we need first to define the Mhaskar-Rakhmanov-Saff number.
Definition 3. If $Q: \mathbb{R} \rightarrow \mathbb{R}$ satisfies conditions (2)-(4) in Definition 2, and if $y Q^{\prime}(y) \rightarrow \infty$ as $y \rightarrow \infty$, then the $n$-th Mhaskar-Rakhmanov-Saff number $a_{n}(Q)$ of $Q$ is defined as the unique positive root $a_{n}$ of the equation

$$
\begin{equation*}
n=\frac{2}{\pi} \int_{0}^{1} \frac{a_{n} t Q^{\prime}\left(a_{n} t\right)}{\sqrt{1-t^{2}}} d t \tag{51}
\end{equation*}
$$

Remark 2. The condition $y Q^{\prime}(y) \rightarrow \infty$ as $y \rightarrow \infty$ in Definition 3 is satisfied if $e^{-Q}$ is a Freud weight. Indeed, in view of properties (2)-(3) in Definition 2, the quantity $\ell:=\lim _{y \rightarrow \infty} y Q^{\prime}(y)$ is well-defined and it belongs to $(0, \infty]$. If $\ell \neq \infty$, then because $\lim _{y \rightarrow \infty} \lambda y Q^{\prime}(\lambda y)=\ell$ too, property (5) would imply that $a \leq 1 / \lambda \leq b$ contradicting that $\lambda, a>1$. Therefore, $\ell=\infty$.

For example, the weight $W(y)=e^{-y^{2}}$, for which $Q(y)=y^{2}$, has $a_{n}(Q)=\sqrt{n}$ because $\int_{0}^{1} t^{2} / \sqrt{1-t^{2}} d t=\frac{\pi}{4}$. If $X \sim p \in \mathscr{D}$, say $\operatorname{supp}(p) \subset[-M, M]$, and $p_{Y}=e^{-Q}($ where $N \sim \mathcal{N}(0,1)$ is independent of $X$, and $Y=X+N$ ), then (see Appendix E)

$$
\begin{equation*}
a_{n}(Q) \leq(2 M+\sqrt{2}) \sqrt{n} . \tag{52}
\end{equation*}
$$

We apply the following Bernstein approximation theorem [4, Corollary 3.6] to prove Theorem 3.
Theorem 5. Fix $W \in \mathscr{F}$, and let $u$ be an r-times continuously differentiable function such that $u^{(r)}$ is absolutely continuous. Let $a_{n}=a_{n}(Q)$ where $W=e^{-Q}$, and fix $1 \leq s \leq \infty$. Then, for some constant $D(W, r, s)$ and every $n \geq \max (r-1,1)$

$$
\begin{equation*}
\inf _{q \in \mathscr{P}_{n}}\|(q-u) W\|_{L^{s}(\mathbb{R})} \leq D(W, r, s)\left(\frac{a_{n}}{n}\right)^{r}\left\|u^{(r)} W\right\|_{L^{s}(\mathbb{R})} \tag{53}
\end{equation*}
$$

Proof of Theorem 3. Fix $k \in \mathbb{N}$ and $n \geq \max (k-1,1)$. We apply Theorem 5 for the function $u(y)=\mathbb{E}[X \mid Y=y]$, the weight $W=\sqrt{p_{Y}}$, and for $s=2$. By our choice of weight, $\|h W\|_{L^{2}(\mathbb{R})}=$ $\|h(Y)\|_{2}$ for any Borel $h: \mathbb{R} \rightarrow \mathbb{R}$. Recall from (11) that $E_{n}[X \mid Y]$ minimizes $\| q(Y)-\mathbb{E}\left[X \mid Y \|_{2}\right.$ over $q(Y) \in \mathscr{P}_{n}(Y)$. By (52), we have the bound $a_{n}=O_{p}(\sqrt{n})$. Furthermore, by Theorem 2, $\left\|\left(d^{k} / d y^{k}\right) \mathbb{E}[X \mid Y]\right\|_{2}=O_{k}(1)$. Note that $W \in \mathscr{F}$, because $W^{2}=p_{Y} \in \mathscr{F}$ by Theorem 4. Therefore, by Theorem 5 , we obtain a constant $D(p, k)$ such that

$$
\begin{equation*}
\left\|E_{n}[X \mid Y]-\mathbb{E}[X \mid Y]\right\|_{2} \leq \frac{D(p, k)}{n^{k / 2}} \tag{54}
\end{equation*}
$$

as desired.

## Appendix A A Derivation of Equation (17)

Using the notation of [15], we have that

$$
C_{r}=\sum_{k=1}^{r}(k-1)!\left\{\begin{array}{l}
r  \tag{55}\\
k
\end{array}\right\}_{\geq 2}
$$

where $\left\{\begin{array}{l}r \\ k\end{array}\right\}_{\geq 2}$ denotes the number of partitions of an $r$-element set into $k$ subsets each of which contains at least 2 elements (note that there are ( $k-1$ )! cyclically-invariant arrangements of $k$ parts). The exponential generating function of the sequence $r \mapsto\left\{\begin{array}{l}r \\ k\end{array}\right\}_{\geq 2}$ is $\left(e^{x}-1-x\right)^{k} / k$ !. Now, we may write

$$
\begin{equation*}
\left(e^{x}-1-x\right)^{k}=\sum_{a+b \leq k}\binom{k}{a, b}(-1)^{k-a} x^{b} \sum_{t \in \mathbb{N}} \frac{(a x)^{t}}{t!} . \tag{56}
\end{equation*}
$$

Therefore, the coefficient of $x^{r}$ in $\left(e^{x}-1-x\right)^{k} / k$ ! is

$$
\begin{align*}
\frac{1}{r!}\left\{\begin{array}{l}
r \\
k
\end{array}\right\}_{\geq 2} & =\sum_{a+b \leq k} \frac{(-1)^{k-a} a^{r-b}}{a!b!(k-a-b)!(r-b)!}  \tag{57}\\
& =\frac{1}{r!} \sum_{b=0}^{k}\binom{r}{b} \sum_{a=0}^{k-b}(-1)^{k-a} \frac{a^{r-b}}{a!(k-a-b)!}  \tag{58}\\
& =\frac{1}{r!} \sum_{b=0}^{k}\binom{r}{b}\left\{\begin{array}{l}
r-b \\
k-b
\end{array}\right\}(-1)^{b}, \tag{59}
\end{align*}
$$

which when combined with (55) gives (17).

## Appendix B Proof of Lemma 1

Assume that $\mathbb{E}\left[|X|^{\operatorname{deg}(p)}\right]=\infty($ so $\operatorname{deg}(p) \geq 1)$, and we will show that $\mathbb{E}[|p(X)|]=\infty$ too. Let $k \in[\operatorname{deg}(p)-1]$ be the largest integer for which $\mathbb{E}\left[|X|^{k}\right]<\infty$, and write $p(u)=u^{k+1} q(u)+r(u)$ for a nonzero polynomial $q$ and a remainder $r \in \mathscr{P}_{k}$. By monotonicity of norms, $\mathbb{E}\left[|X|^{j}\right]<\infty$ for every $j \in[k]$. Hence, $r(X)$ is integrable. Therefore, it suffices to prove that $X^{k+1} q(X)$ is non-integrable, which we show next.

Consider the set $\mathcal{D}=\{u \in \mathbb{R} ;|q(u)|<|a|\}$ where $a \neq 0$ is the leading coefficient of $q$. If $q$ is constant, then $\mathcal{D}$ is empty, whereas if $\operatorname{deg} q \geq 1$ then $|q(u)| \rightarrow \infty$ as $|u| \rightarrow \infty$ implies that $\mathcal{D}$ is bounded; in either case, there is an $M \in \mathbb{R}$ such that $\mathcal{D} \subset[-M, M]$. Now, writing $1=1_{\mathcal{D}}+1_{\mathcal{D}^{c}}$, we obtain

$$
\begin{equation*}
\mathbb{E}\left[|X|^{k+1}|q(X)|\right] \geq|a| \mathbb{E}\left[|X|^{k+1} 1_{\mathcal{D}^{c}}(X)\right] \tag{60}
\end{equation*}
$$

But we also have that

$$
\begin{equation*}
\infty=\mathbb{E}\left[|X|^{k+1}\right] \leq M^{k+1}+\mathbb{E}\left[|X|^{k+1} 1_{\mathcal{D}^{c}}(X)\right], \tag{61}
\end{equation*}
$$

so $\mathbb{E}\left[|X|^{k+1} 1_{\mathcal{D}^{c}}(X)\right]=\infty$. Therefore, inequality (60) yields that $\mathbb{E}\left[\left|X^{k+1} q(X)\right|\right]=\infty$, concluding the proof.

## Appendix C Proof of Proposition 1

Recall that the conditional expectation can be expressed as

$$
\begin{equation*}
\mathbb{E}[Z \mid Y=y]=\frac{\mathbb{E}\left[Z e^{-(X-y)^{2} / 2}\right]}{\mathbb{E}\left[e^{-(X-y)^{2} / 2}\right]} \tag{62}
\end{equation*}
$$

for any RV $Z$ for which $Z e^{-(X-y)^{2} / 2}$ is integrable. This formula applies for both $Z=X$ and $Z=(X-\mathbb{E}[X \mid Y=y])^{k}$, where $(y, k) \in \mathbb{R} \times \mathbb{N}$, because they are polynomials in $X$ and the map $x \mapsto q(x) e^{-(x-y)^{2} / 2}$ is bounded for any polynomial $q$.

Differentiating (62) for $Z=X$ and rearranging terms, we obtain

$$
\begin{equation*}
\frac{d}{d y} \mathbb{E}[X \mid Y=y]=\frac{\mathbb{E}\left[(X-\mathbb{E}[X \mid Y=y])^{2} e^{-(X-y)^{2} / 2}\right]}{\mathbb{E}\left[e^{-(X-y)^{2} / 2}\right]} \tag{63}
\end{equation*}
$$

i.e., $f^{\prime}=g_{2}$. Note that $g_{0} \equiv 1$ and $g_{1} \equiv 0$. Differentiating $g_{r}$ for $r \geq 1$, we obtain that

$$
\begin{equation*}
g_{r}^{\prime}=g_{r+1}-r g_{2} g_{r-1} . \tag{64}
\end{equation*}
$$

We apply successive differentiation to $f^{\prime}=g_{2}$ and recover patterns by utilizing (64) at each step.
From $f^{\prime}=g_{2}$ and (64), we infer the first few derivatives

$$
\begin{equation*}
f^{(2)}=g_{3}, f^{(3)}=g_{4}-3 g_{2}^{2}, f^{(4)}=g_{5}-10 g_{2} g_{3} \tag{65}
\end{equation*}
$$

We see a homogeneity in (65), namely, $f^{(r-1)}$ is an integer linear combination of terms of the form $g_{i_{1}}^{\alpha_{1}} \cdots g_{i_{\ell}}^{\alpha_{\ell}}$ with $i_{1} \alpha_{1}+\cdots+i_{\ell} \alpha_{\ell}=r$. This homogeneity can be shown to hold for a general $r$ by induction, which we show next. For most of the remainder of the proof, we forget the numerical values of the $f^{(k)}$ and the $g_{r}^{(k)}$ and only treat them as symbols satisfying $f^{\prime}=g_{2}$ and $g_{r}^{\prime}=g_{r+1}-r g_{2} g_{r-1}$ that respect rules of differentiation and which commute.

We call $\sum_{j=1}^{\ell} i_{j} \alpha_{j}$ the weighted degree of any nonzero integer multiple of $g_{i_{1}}^{\alpha_{1}} \cdots g_{i_{\ell}}^{\alpha_{\ell}}$. This is a well-defined degree because it is invariant to the way the product is arranged. We also say that a sum is of weighted degree $r$ if each summand is of weighted degree $r$. To prove the claim of homogeneity, i.e., that $f^{(r-1)}$ is of weighted degree $r$, we differentiate and apply the relation in (64) to a generic term $g_{i_{1}}^{\alpha_{1}} \cdots g_{i_{\ell}}^{\alpha_{\ell}}$ whose weighted degree is $r$. We have the derivative

$$
\begin{equation*}
\left(g_{i_{1}}^{\alpha_{1}} \cdots g_{i_{\ell}}^{\alpha_{\ell}}\right)^{\prime}=\left(g_{i_{1}}^{\alpha_{1}}\right)^{\prime} \cdots g_{i_{\ell}}^{\alpha_{\ell}}+\cdots+g_{i_{1}}^{\alpha_{1}} \cdots\left(g_{i_{\ell}}^{\alpha_{\ell}}\right)^{\prime} . \tag{66}
\end{equation*}
$$

From (64), for integers $i, \alpha \geq 1$,

$$
\begin{equation*}
\left(g_{i}^{\alpha}\right)^{\prime}=\alpha g_{i}^{\alpha-1} g_{i+1}-\alpha i g_{2} g_{i-1} g_{i}^{\alpha-1} \tag{67}
\end{equation*}
$$

Therefore, the derivative of $g_{i}^{\alpha}$ has weighted degree $i \alpha+1$. In other words, differentiation increased the weighted degree of $g_{i}^{\alpha}$ by 1 . From (66), then, we see that the weighted degree of $\left(g_{i_{1}}^{\alpha_{1}} \cdots g_{i_{\ell}}^{\alpha_{\ell}}\right)^{\prime}$ is $r+1$. Since $f^{\prime}=g_{2}$ is of weighted degree 2 , induction and linearity of differentiation yield that $f^{(r-1)}$ is of weighted degree $r$ for each $r \geq 2$.

Now, we fix the way we are writing products of the $g_{i}$. We ignore explicitly writing $g_{0}$ and $g_{1}$, collect identical terms into an exponent, and write lower indices first. One way to keep this notation is via integer partitions. Consider the "homogeneous" sets

$$
\begin{equation*}
G_{r}:=\left\{\sum_{\lambda \in \Pi_{r}} \beta_{\lambda} g^{\lambda} ; \beta_{\boldsymbol{\lambda}} \in \mathbb{Z} \text { for each } \boldsymbol{\lambda} \in \Pi_{r}\right\} . \tag{68}
\end{equation*}
$$

The homogeneity property for the derivatives of $f$ can be written as $f^{(r-1)} \in G_{r}$ for each $r \geq 2$.
Next, we investigate the exact integer coefficients $h_{\lambda}$ in the expression of the derivatives of $f$ in terms of the $\boldsymbol{g}^{\boldsymbol{\lambda}}$. Homogeneity of the derivatives of $f$ says that we may write each $f^{(r-1)}, r \geq 2$, as an integer linear combination of $\left\{\boldsymbol{g}^{\boldsymbol{\lambda}}\right\}_{\boldsymbol{\lambda} \in \Pi_{r}}$. One way to obtain such a representation is via repeated differentiation of $f^{\prime}=g_{2}$, applying the relation (67), and discarding any term that is a multiple of $g_{1}$. Applying these steps, we arrive at representations

$$
\begin{equation*}
f^{(r-1)}=\sum_{\boldsymbol{\lambda} \in \Pi_{r}} h_{\boldsymbol{\lambda}} g^{\boldsymbol{\lambda}}, \quad c_{\boldsymbol{\lambda}} \in \mathbb{Z} . \tag{69}
\end{equation*}
$$

The terms $\boldsymbol{g}^{\boldsymbol{\nu}}$ that appear upon differentiating a term $\boldsymbol{g}^{\boldsymbol{\lambda}}$ can be described as follows. For $\left(\lambda_{2}, \cdots, \lambda_{\ell}\right)=\boldsymbol{\lambda} \in \Pi_{r}$, we call $\lambda_{2}$ the leading term of $\boldsymbol{\lambda}$. Consider for a tuple $\boldsymbol{\lambda} \in \Pi_{r}$ the following two sets of tuples $\tau_{+}(\boldsymbol{\lambda}), \tau_{-}(\boldsymbol{\lambda}) \subset \Pi_{r+1}$ :

- The set $\tau_{+}(\boldsymbol{\lambda})$ consists of all tuples obtainable from $\boldsymbol{\lambda}$ via replacing a pair $\left(\lambda_{i}, \lambda_{i+1}\right)$ with $\left(\lambda_{i}-1, \lambda_{i+1}+1\right)$ (so, necessarily $\lambda_{i} \geq 1$ ) while keeping all other entries unchanged;
- The set $\tau_{-}(\boldsymbol{\lambda})$ consists of all tuples obtainable from $\boldsymbol{\lambda}$ via replacing a pair $\left(\lambda_{i-1}, \lambda_{i}\right)$, for which $i \geq 3$, with the pair $\left(\lambda_{i-1}+1, \lambda_{i}-1\right)$ (so, necessarily $\lambda_{i} \geq 1$ ) and additionally increasing the leading term by 1 while keeping all other terms unchanged.

For example, if $\boldsymbol{\lambda}=(0,5,0,1) \in \Pi_{20}$ then

$$
\begin{equation*}
\tau_{+}(\boldsymbol{\lambda})=\{(0,4,1,1),(0,5,0,0,1)\} \subset \Pi_{21} \tag{70}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau_{-}(\boldsymbol{\lambda})=\{(2,4,0,1),(1,5,1)\} \subset \Pi_{21} . \tag{71}
\end{equation*}
$$

The relation (67) yields, in view of

$$
\begin{equation*}
\left(g_{2}^{\lambda_{2}} \cdots g_{\ell}^{\lambda_{\ell}}\right)^{\prime}=\left(g_{2}^{\lambda_{2}}\right)^{\prime} \cdots g_{\ell}^{\lambda_{\ell}}+\cdots+g_{2}^{\lambda_{2}} \cdots\left(g_{\ell}^{\lambda_{\ell}}\right)^{\prime}, \tag{72}
\end{equation*}
$$

that

$$
\begin{equation*}
\left(g^{\boldsymbol{\lambda}}\right)^{\prime}=\sum_{\nu \in \tau_{+}(\boldsymbol{\lambda})} a_{\lambda, \nu} g^{\nu}-\sum_{\nu \in \tau_{-}(\boldsymbol{\lambda})} b_{\lambda, \nu} g^{\nu} \tag{73}
\end{equation*}
$$

for some positive integers $a_{\lambda, \nu}$ and $b_{\lambda, \nu}$, which we describe next. Finding $a_{\boldsymbol{\lambda}, \boldsymbol{\nu}}$ and $b_{\boldsymbol{\lambda}, \boldsymbol{\nu}}$ can be straightforwardly done from (67) in view of (72). If $\boldsymbol{\nu} \in \tau_{+}(\boldsymbol{\lambda})$, say

$$
\begin{equation*}
\left(\nu_{i}, \nu_{i+1}\right)=\left(\lambda_{i}-1, \lambda_{i+1}+1\right), \tag{74}
\end{equation*}
$$

then $a_{\boldsymbol{\lambda}, \boldsymbol{\nu}}=\lambda_{i}$. If $\boldsymbol{\nu} \in \tau_{-}(\boldsymbol{\lambda})$, say

$$
\begin{equation*}
\left(\nu_{i-1}, \nu_{i}\right)=\left(\lambda_{i-1}+1, \lambda_{i}-1\right), \tag{75}
\end{equation*}
$$

then $b_{\lambda, \boldsymbol{\nu}}=i \lambda_{i}$. In our example of $\boldsymbol{\lambda}=(0,5,0,1)$, we get

$$
\begin{align*}
a_{(0,5,0,1),(0,4,1,1)} & =5  \tag{76}\\
a_{(0,5,0,1),(0,5,0,0,1)} & =1, \tag{77}
\end{align*}
$$

whereas

$$
\begin{align*}
b_{(0,5,0,1),(2,4,0,1)} & =15  \tag{78}\\
b_{(0,5,0,1),(1,5,1)} & =5 . \tag{79}
\end{align*}
$$

Note that the two sets $\tau_{+}(\boldsymbol{\lambda})$ and $\tau_{-}(\boldsymbol{\lambda})$ are disjoint because, e.g., the sum of entries of a tuple in $\tau_{+}(\boldsymbol{\lambda})$ is the same as that for $\boldsymbol{\lambda}$, whereas the sum of entries of a tuple in $\tau_{-}(\boldsymbol{\lambda})$ is one more than that for $\boldsymbol{\lambda}$.

We next describe how to use what we have shown thus far to deduce a recurrence relation for the $h_{\boldsymbol{\lambda}}$. Let $\theta$ be a process inverting $\tau$, i.e., define for $\boldsymbol{\nu} \in \Pi_{r+1}$ the two sets

$$
\begin{equation*}
\theta_{+}(\boldsymbol{\nu}):=\left\{\boldsymbol{\lambda} \in \Pi_{r} ; \boldsymbol{\nu} \in \tau_{+}(\boldsymbol{\lambda})\right\} \tag{80}
\end{equation*}
$$

and

$$
\begin{equation*}
\theta_{-}(\boldsymbol{\nu}):=\left\{\boldsymbol{\lambda} \in \Pi_{r} ; \boldsymbol{\nu} \in \tau_{-}(\boldsymbol{\lambda})\right\} \tag{81}
\end{equation*}
$$

The two sets $\theta_{+}(\boldsymbol{\nu})$ and $\theta_{-}(\boldsymbol{\nu})$ are disjoint because the two sets $\tau_{+}(\boldsymbol{\lambda})$ and $\tau_{-}(\boldsymbol{\lambda})$ are disjoint for each fixed $\boldsymbol{\lambda}$. Recall our process for defining $h_{\boldsymbol{\lambda}}$ : we start with $f^{\prime}=g_{2}$, so $h_{(1)}=1$; we successively differentiate $f^{\prime}=g_{2}$; after each differentiation, we use (67) and (72) (recall that we have the understanding $g_{i}^{0}=1$ ); we discard any ensuing multiple of $g_{1}$; after $r-2$ differentiations, we get an equation $f^{(r-1)}=\sum_{\boldsymbol{\lambda} \in \Pi_{r}} h_{\boldsymbol{\lambda}} \boldsymbol{g}^{\boldsymbol{\lambda}}$, which we take to be the definition of the $h_{\boldsymbol{\lambda}}$. The point here is that it could be that $f^{(r-1)}$ is representable as an integer linear combination of the $\boldsymbol{g}^{\boldsymbol{\lambda}}$ in more than one way, which can only be verified after the numerical values for the $g_{i}$ are taken into account, but we are not doing that: our approach treats the $g_{i}$ as symbols following the laid out rules. Now, we look at one of the steps of this procedure, starting at differentiating $f^{(r-1)}=\sum_{\boldsymbol{\lambda} \in \Pi_{r}} h_{\boldsymbol{\lambda}} \boldsymbol{g}^{\boldsymbol{\lambda}}$, so $f^{(r)}=\sum_{\boldsymbol{\lambda} \in \Pi_{r}} h_{\boldsymbol{\lambda}}\left(\boldsymbol{g}^{\boldsymbol{\lambda}}\right)^{\prime}$. Replacing $\left(\boldsymbol{g}^{\boldsymbol{\lambda}}\right)^{\prime}$ via (73),

$$
\begin{equation*}
f^{(r)}=\sum_{\lambda \in \Pi_{r}}\left(\sum_{\boldsymbol{\nu} \in \tau_{+}(\boldsymbol{\lambda})} a_{\lambda, \nu} g^{\nu}-\sum_{\boldsymbol{\nu} \in \tau_{-}(\boldsymbol{\lambda})} b_{\lambda, \nu} g^{\nu}\right) \tag{82}
\end{equation*}
$$

Exchanging the order of summations (for which we use $\theta$ ),

$$
\begin{equation*}
f^{(r)}=\sum_{\boldsymbol{\nu} \in \Pi_{r+1}}\left(\sum_{\boldsymbol{\lambda} \in \theta_{+}(\boldsymbol{\nu})} h_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}, \boldsymbol{\nu}}-\sum_{\boldsymbol{\lambda} \in \theta_{-}(\boldsymbol{\nu})} h_{\boldsymbol{\lambda}} b_{\boldsymbol{\lambda}, \boldsymbol{\nu}}\right) \boldsymbol{g}^{\boldsymbol{\nu}} . \tag{83}
\end{equation*}
$$

Therefore, by definition of the $h_{\boldsymbol{\lambda}}$, we have the recurrence: for each $\boldsymbol{\nu} \in \Pi_{r+1}$

$$
\begin{equation*}
h_{\boldsymbol{\nu}}=\sum_{\boldsymbol{\lambda} \in \theta_{+}(\boldsymbol{\nu})} h_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}, \boldsymbol{\nu}}-\sum_{\lambda \in \theta_{-}(\boldsymbol{\nu})} h_{\boldsymbol{\lambda}} b_{\boldsymbol{\lambda}, \boldsymbol{\nu}}, \quad h_{(1)}=1 . \tag{84}
\end{equation*}
$$

One instance of this recurrence is, e.g.,

$$
\begin{equation*}
h_{(2,1)}=3 h_{(3)}-4 h_{(1,0,1)}-6 h_{(0,2)} . \tag{85}
\end{equation*}
$$

Now, we show that the recurrence in (84) also generates $e_{\boldsymbol{\lambda}}$ as defined in (16). For $\left(\lambda_{2}, \cdots, \lambda_{\ell}\right)=$ $\boldsymbol{\lambda} \in \Pi_{r}$, denote $\sigma(\boldsymbol{\lambda})=\lambda_{2}+\cdots+\lambda_{\ell}$. If $\boldsymbol{\nu} \in \tau_{+}(\boldsymbol{\lambda})$ then $\sigma(\boldsymbol{\nu})=\sigma(\boldsymbol{\lambda})$, and if $\boldsymbol{\nu} \in \tau_{-}(\boldsymbol{\lambda})$ then $\sigma(\boldsymbol{\nu})=\sigma(\boldsymbol{\lambda})+1$. Therefore, $\boldsymbol{\lambda} \in \theta_{+}(\boldsymbol{\nu})$ implies $\sigma(\boldsymbol{\nu})=\sigma(\boldsymbol{\lambda})$, and $\boldsymbol{\lambda} \in \theta_{-}(\boldsymbol{\nu})$ implies $\sigma(\boldsymbol{\nu})=\sigma(\boldsymbol{\lambda})+1$. Multiplying (84) by $(-1)^{\sigma(\boldsymbol{\nu})-1}$ yields the equivalent recurrence

$$
\begin{equation*}
t_{\boldsymbol{\nu}}=\sum_{\lambda \in \theta_{+}(\boldsymbol{\nu})} t_{\lambda} a_{\lambda, \boldsymbol{\nu}}+\sum_{\lambda \in \theta_{-}(\boldsymbol{\nu})} t_{\boldsymbol{\lambda}} b_{\lambda, \boldsymbol{\nu}}, \quad t_{(1)}=1, \tag{86}
\end{equation*}
$$

where $t_{\boldsymbol{\lambda}}:=(-1)^{\sigma(\boldsymbol{\lambda})-1} h_{\boldsymbol{\lambda}}$. We show that $c_{\boldsymbol{\lambda}}=(-1)^{\sigma(\boldsymbol{\lambda})-1} e_{\boldsymbol{\lambda}}$ (see (16)) satisfies this recurrence, which is equivalent to $e_{\boldsymbol{\lambda}}$ satisfying the recurrence (84). Clearly, $c_{(1)}=1$, so consider $c_{\boldsymbol{\nu}}$ for $\boldsymbol{\nu} \in \Pi_{r}$ with $r \geq 3$.

Consider labelled elements $s_{1}, s_{2}, \cdots$, and let $S_{k}=\left\{s_{1}, \cdots, s_{k}\right\}$ for each $k \geq 2$. For any $\boldsymbol{\lambda} \in \Pi_{k}$, let $\mathcal{C}_{\boldsymbol{\lambda}}$ be the set of arrangements of cyclically-invariant set-partitions of $S_{k}$ according to $\boldsymbol{\lambda}$, so $\left|\mathcal{C}_{\boldsymbol{\lambda}}\right|=c_{\boldsymbol{\lambda}}$. Now, fix $\boldsymbol{\nu} \in \Pi_{r+1}$, and we will build $\mathcal{C}_{\boldsymbol{\nu}}$ from the $\mathcal{C}_{\boldsymbol{\lambda}}$ where $\boldsymbol{\lambda}$ ranges over $\theta_{+}(\boldsymbol{\nu}) \cup \theta_{-}(\boldsymbol{\nu})$. Consider first $\boldsymbol{\lambda} \in \theta_{+}(\boldsymbol{\nu})$, where a partition in $\mathcal{C}_{\boldsymbol{\nu}}$ is constructed from a partition in $\mathcal{C}_{\boldsymbol{\lambda}}$ by appending $s_{r+1}$ to one of the parts of the latter partition. Note that adding $s_{r+1}$ to two distinct partitions of $S_{r}$ cannot produce the same partition of $S_{r+1}$; indeed, just removing $s_{r+1}$ shows that that is impossible. Now, let $i$ be the unique index such that $\left(\nu_{i}, \nu_{i+1}\right)=\left(\lambda_{i}-1, \lambda_{i+1}+1\right)$. Then, a partition $\mathcal{P} \in \mathcal{C}_{\boldsymbol{\nu}}$ of $S_{r+1}$ is induced by a partition $\mathcal{P}^{\prime} \in \mathcal{C}_{\boldsymbol{\lambda}}$ of $S_{r}$ if and only if $s_{r+1}$ is added to a part in $\mathcal{P}^{\prime}$ of size $i$, of which there are exactly $\lambda_{i}=a_{\boldsymbol{\lambda}, \boldsymbol{\nu}}$. Therefore, we get a contribution of $\sum_{\boldsymbol{\lambda} \in \theta_{+}(\boldsymbol{\nu})} c_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}, \boldsymbol{\nu}}$ towards $c_{\boldsymbol{\nu}}$, which is the first part in (86).

For the second part, $\sum_{\boldsymbol{\lambda} \in \theta_{-}(\boldsymbol{\nu})} c_{\boldsymbol{\lambda}} b_{\boldsymbol{\lambda}, \boldsymbol{\nu}}$, we consider the remaining ways of generating a partition in $\mathcal{C}_{\boldsymbol{\nu}}$ from a partition according to some $\boldsymbol{\lambda} \in \theta_{-}(\boldsymbol{\nu})$. In this case, $s_{r+1}$ is not appended to an existing part, but it is used to create a new part of size 2 . Thus, we need to also move an element $s_{j}, 1 \leq j \leq r$, from a part of size at least 3 to be combined with $s_{r+1}$ to create a new part of size 2. It is also clear in this case that such a procedure applied to two distinct partitions in $\mathcal{C}_{\boldsymbol{\lambda}}$ cannot produce the same partition in $\mathcal{C}_{\boldsymbol{\nu}}$. Let $i$ be the unique index for which $\left(\nu_{i-1}, \nu_{i}\right)=\left(\lambda_{i-1}+1, \lambda_{i}-1\right)$. There are $\lambda_{i}$ parts to choose from, and $i$ elements to choose from once a part is chosen, so there are a total of $i \lambda_{i}=b_{\boldsymbol{\lambda}, \boldsymbol{\nu}}$ ways to generate a partition in $\mathcal{C}_{\boldsymbol{\nu}}$ from a partition in $\mathcal{C}_{\boldsymbol{\lambda}}$. This gives the second sum in (86), and we conclude that

$$
\begin{equation*}
c_{\boldsymbol{\nu}}=\sum_{\boldsymbol{\lambda} \in \theta_{+}(\boldsymbol{\nu})} c_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}, \boldsymbol{\nu}}+\sum_{\boldsymbol{\lambda} \in \theta_{-}(\boldsymbol{\nu})} c_{\boldsymbol{\lambda}} b_{\boldsymbol{\lambda}, \boldsymbol{\nu}} . \tag{87}
\end{equation*}
$$

Therefore, the $c_{\boldsymbol{\lambda}}$ and the $t_{\boldsymbol{\lambda}}$ satisfy the same recurrence, which takes the form: for $\boldsymbol{\nu} \in \Pi_{r+1}$ there are integers $\left\{d_{\lambda, \nu}\right\}_{\boldsymbol{\lambda} \in \Pi_{r}}$ such that

$$
\begin{equation*}
u_{\boldsymbol{\nu}}=\sum_{\lambda \in \Pi_{r}} d_{\lambda, \nu} u_{\boldsymbol{\lambda}} \tag{88}
\end{equation*}
$$

with the initial condition $u_{(1)}=1$. Then, we can induct on $r$ to conclude that the $c_{\boldsymbol{\lambda}}$ and the $t_{\boldsymbol{\lambda}}$ are the same sequence. Since $\Pi_{2}=\{(1)\}$, we see that $c_{\boldsymbol{\lambda}}=t_{\boldsymbol{\lambda}}$ for every $\boldsymbol{\lambda} \in \Pi_{2}$. Suppose $r \geq 2$ is such that $c_{\boldsymbol{\lambda}}=t_{\boldsymbol{\lambda}}$ for every $\boldsymbol{\lambda} \in \Pi_{r}$. Hence, for every $\boldsymbol{\nu} \in \Pi_{r+1}$, we have that

$$
\begin{equation*}
\sum_{\lambda \in \Pi_{r}} d_{\lambda, \nu} c_{\lambda}=\sum_{\lambda \in \Pi_{r}} d_{\lambda, \nu} t_{\boldsymbol{\lambda}} . \tag{89}
\end{equation*}
$$

Since both sequences $c_{\boldsymbol{\lambda}}$ and $t_{\boldsymbol{\lambda}}$ satisfy the recurrence (88), we obtain from (89) that $c_{\boldsymbol{\nu}}=t_{\boldsymbol{\nu}}$ for every $\boldsymbol{\nu} \in \Pi_{r+1}$. Therefore, we obtain by induction that $c_{\boldsymbol{\lambda}}=t_{\boldsymbol{\lambda}}$ for every $\boldsymbol{\lambda} \in \Pi_{r}$ for every $r$, as desired.

## Appendix D Proof of Theorem 4

Fix $p \in \mathscr{D}$, suppose $X \sim p$, and write $Y=X+N$ and $p_{Y}=e^{-Q}$. First, we note that $Q^{\prime}(y)$ is equal to $\mathbb{E}[N \mid Y=y]$.

Lemma 2. Fix a random variable $X$ and let $Y=X+N$ where $N \sim \mathcal{N}(0,1)$ is independent of $X$. Writing $p_{Y}=e^{-Q}$, we have that $Q^{\prime}(y)=\mathbb{E}[N \mid Y=y]$.

Proof. We have that $p_{Y}(y)=\mathbb{E}\left[e^{-(y-X)^{2} / 2}\right] / \sqrt{2 \pi}$. Differentiating this equation, we obtain that $p_{Y}^{\prime}(y)=\mathbb{E}\left[(X-y) e^{-(y-X)^{2} / 2}\right] / \sqrt{2 \pi}$, where the exchange of differentiation and integration is warranted since $t \mapsto t e^{-t^{2} / 2}$ is bounded. Now, $Q=-\log p_{Y}$, so $Q^{\prime}=-p_{Y}^{\prime} / p_{Y}$, i.e.,

$$
\begin{equation*}
Q^{\prime}(y)=y-\frac{\mathbb{E}\left[X e^{-(y-X)^{2} / 2}\right]}{\mathbb{E}\left[e^{-(y-X)^{2} / 2}\right]}=y-\mathbb{E}[X \mid Y=y] . \tag{90}
\end{equation*}
$$

The proof is completed by substituting $X=Y-N$.
In view of Lemma 2, that $p$ is even and non-increasing over $[0, \infty) \cap \operatorname{supp}(p)$ imply that $Q$ satisfies conditions (1)-(4) of Definition 2. It remains to show that property (5) holds. To this end, we show that if $\operatorname{supp}(p) \subset[-M, M]$ and $\lambda=M+2$, then for every $y>M+4$ we have that

$$
\begin{equation*}
1<\frac{M^{2}+5 M+8}{2(M+2)} \leq \frac{Q^{\prime}(\lambda y)}{Q^{\prime}(y)} \leq \frac{M^{2}+7 M+8}{4} . \tag{91}
\end{equation*}
$$

First, since $Q^{\prime}(y)=y-\mathbb{E}[X \mid Y=y]$ (see (90)), we have the bounds $y-M \leq Q^{\prime}(y) \leq y+M$ for every $y \in \mathbb{R}$. Therefore, $y>M$ and $\lambda>1$ imply that

$$
\begin{equation*}
\frac{\lambda y-M}{y+M} \leq \frac{Q^{\prime}(\lambda y)}{Q^{\prime}(y)} \leq \frac{\lambda y+M}{y-M} . \tag{92}
\end{equation*}
$$

Further, since $y>M+4$ and $\lambda=M+2$, we have

$$
\begin{equation*}
\frac{M^{2}+5 M+8}{2(M+2)}<\lambda-\frac{M(M+3)}{y+M}=\frac{\lambda y-M}{y+M} \tag{93}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\lambda y+M}{y-M}=\lambda+\frac{M(M+3)}{y-M} \leq \frac{M^{2}+7 M+8}{4} . \tag{94}
\end{equation*}
$$

The fact that $1<\frac{M^{2}+5 M+8}{2(M+2)}$ follows since the discriminant of $M^{2}+3 M+4$ is $-7<0$. Therefore, $p_{Y}$ is a Freud weight.

## Appendix E Proof of Inequality (52)

By Lemma 2,

$$
\begin{equation*}
Q^{\prime}(y)=\mathbb{E}[N \mid Y=y]=y-\mathbb{E}[X \mid Y=y] . \tag{95}
\end{equation*}
$$

Therefore $X \leq M$ implies that, for any constant $z \geq 0$, we have

$$
\begin{align*}
\int_{0}^{1} \frac{z t Q^{\prime}(z t)}{\sqrt{1-t^{2}}} d t & =\frac{\pi}{4} z^{2}-z \int_{0}^{1} \frac{t}{\sqrt{1-t^{2}}} \frac{\mathbb{E}\left[X e^{-(X-z t)^{2} / 2}\right]}{\mathbb{E}\left[e^{-(X-z t)^{2} / 2}\right]} d t  \tag{96}\\
& \geq \frac{\pi}{4} z^{2}-M z \tag{97}
\end{align*}
$$

We have $\pi z^{2} / 4-M z>n$ for $z=(2 M+\sqrt{2}) \sqrt{n}$. Since $y \mapsto y Q^{\prime}(y)$ is strictly increasing over $(0, \infty)$ (condition (3) of Definition 2), we conclude that $a_{n}(Q) \leq(2 M+\sqrt{2}) \sqrt{n}$.

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[^1]:    ${ }^{1}$ The integer $c_{\boldsymbol{\lambda}}$ counts the number of cyclically-invariant ordered set-partitions of an $r$-element set into $m=$ $\lambda_{2}+\cdots+\lambda_{\ell}$ subsets where, for each $k \in\{2, \cdots, \ell\}$, exactly $\lambda_{k}$ parts have size $k$.
    ${ }^{2}$ The integer $C_{r}$ counts the total number of cyclically-invariant ordered set-partitions of an $r$-element set into subsets of sizes at least 2 .

[^2]:    ${ }^{3}$ It can also be shown that we necessarily have $\alpha_{m}<0$ and $m$ is odd, but these points are moot since we eventually have a contradiction.

