Linear Coding for AWGN channels with Noisy Output Feedback via Dynamic Programming

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Abstract

The optimal coding scheme for communicating a Gaussian message over an Additive White Gaussian noise (AWGN) channel with AWGN output feedback, with a limited number of transmissions, is unknown. Even if we restrict the scope of the coding scheme to linear schemes, still, deriving the optimal coding scheme is a challenging task. The state-of-the-art linear scheme for channels with noisy feedback is by Chance and Love, where the coefficients of the linear scheme are numerically optimized based on unique observations [2]. In this paper, we introduce a new class of linear coding schemes, which we call sequential linear schemes, for this channel by introducing a linear state process at the transmitter. We then derive the optimal scheme outperforms the state-of-the-art linear scheme in [2] for noisy feedback and coincides with the Shalkwijk-Kailath scheme for noiseless feedback. This problem is an instance of decentralized control without any common information and to the best of our knowledge the first such scenario where we can derive analytical solutions using a DP. Finally, we show that in communicating message *bits* instead of a Gaussian message, a learning-based approach further improves the reliability of sequential linear schemes.

I. INTRODUCTION

The study of channels with *output feedback* was initiated by Shannon [3], where he shows that the feedback of the output does not increase the capacity of point to point AWGN channels and any discrete memoryless channels. Despite such negative results, feedback is shown to improve the reliability in the *finite blocklength* regime [4]–[7]. For channels with *noiseless* output feedback, Horstein studied binary symmetric channel (BSC) channel and presented a scheme that achieves capacity [4]. Schalkwijk and Kailath in [5], [6] studied AWGN channel and proposed an optimal linear coding scheme that achieves capacity and a doubly exponential error exponent [5]. Both Horstein and Schalkwijk-Kailath (SK) schemes were later generalized by Shayevitz and Feder by proposing a posterior matching scheme [7] for an arbitrary discrete memoryless channel where the encoder transmits the generalized inverse of the capacity-achieving input Cummulative Distribution Function (CDF) at every transmission.

For channels with *noisy* output feedback, on the other hand, far less is known. The celebrated SK scheme does not readily generalize to noisy feedback channels [7]. Chance and Love proposed a linear scheme that significantly outperforms the SK scheme for AWGN channels with AWGN noisy feedback [2]. They also introduced a concatenated coding scheme where their linear scheme is used as an inner code and the forward error-correcting code is used as an outer code. Nevertheless, whether the Chance and Love (CL) scheme is optimal has remained unknown over the last decade. We make progress on this long-standing open problem; we derive a linear coding scheme that outperforms the CL scheme for channels with noisy output feedback.

While we focus on AWGN channels with AWGN noisy output feedback, we note that there are various other models for channels with feedback. In [8], Martins and Weissman consider a channel where the feedback is affected by quantization noise or an additive bounded noise and provide a scheme that performs close to capacity. In [9], Burnashev and Yamamoto consider a BSC as both forward as well as the feedback

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channel and show that for some positive crossover probability of the feedback channel, the achievable error exponent outperforms that of the no-feedback case. This result was further refined in [10] and extended by Xiang and Kim to show that the error exponent of channels with feedback is strictly higher compared to channels with no feedback if the noise variance of the feedback channel is sufficiently small [11]. The error exponents for channels with noisy feedback for different power constraint assumptions on the feedback channel are derived by Kim and Weissman [12]. For channels with *active* feedback, where the receiver is allowed to actively *encode* its outputs, Ben-Yishai and Shayevitz introduce the Modulo-SK scheme which is shown to improve the reliability upon the SK scheme by posing the problem as a joint source-channel coding with side information and solving it using an interactive modulo-lattice solution [13].

In general, communication with *noiseless* feedback can be posed as a multi-user decentralized stochastic control problem [14] with common information at the transmitter and the receiver. In general, there is a conceptual framework to solve such problems within the framework of dynamic programming [15]. However, when there is no common information among the decision-makers, there is no such framework available, mainly because in such problems each player needs to put a belief on other players' private information, and a belief on their beliefs ad infinitum. This is referred to as the infinite regress of higher-order beliefs in both control and economics literature. Point-to-point channel with noisy feedback is one such decentralized control problem without any common information and has thus lacked any mathematical framework to study this problem. However, recently, in [16], Vasal presented a sequential decomposition methodology to decompose a general discrete memoryless point-to-point channel with noisy feedback by providing a notion of the state of this channel based on an auxiliary controller at the transmitter.

In this paper, we consider an AWGN channel with AWGN noisy output feedback and extend the notion of the state mentioned above to propose a DP algorithm that solves for the coefficients of a linear encoding scheme in closed-form. More specifically we introduce a class of linear schemes, called *sequential linear* schemes, and derive the optimal solution within this class for AWGN channels with noisy and noiseless feedback. For channels with noiseless feedback, we recover the SK scheme as the optimal scheme in our framework as a special case. Note that SK scheme was inspired by Robbin's scheme and was used in AWGN channel based on human intuition and there was no analytical framework to *derive* the celebrated SK scheme. For AWGN channels with noisy feedback, surprisingly, we show that our sequential linear coding scheme strictly outperforms the CL scheme. Our main contributions are as follows:

- We introduce a family of sequential linear schemes that are naturally equipped with a recursive optimal decoder, i.e., Kalman linear Minimum Mean-squared error (MMSE) filter, for AWGN channels with noisy output feedback (Section III). We derive the closed-form solution for the optimal sequential linear code via dynamic programming under a constant peak power constraint. To do so, we introduce a novel Markov decision process (MDP) framework that uses variances in the estimation at the transmitter and receiver as states (Section IV).
- We characterize the MMSE that the optimal schemes achieve as a function of the number of transmissions in a closed-form solution. We observe that the MMSE approximately drops exponentially for noiseless feedback while the drop in variance is polynomial for noisy feedback settings (Section V).
- We extend the results to a total power constraint. We provide a dynamic program to compute a linear scheme with the optimal power allocations for each of the transmissions for both noiseless and noisy feedback settings. We show that our scheme outperforms the state-of-the-art scheme for noisy feedback channels (Section VI).
- We evaluate the performance of our proposed sequential linear coding schemes for communicating message bits over AWGN channels with noisy output feedback. We conclude that for codes with finite constellation, the sequential linear coding scheme derived using the DP is sub-optimal. We develop a learning-based approach that improves the performance of sequential linear schemes for such cases (Section VII).

II. PROBLEM SETUP AND PRIOR WORK

We consider AWGN channels with noisy output feedback, where the transmitter aims to communicate a message $W \sim \mathcal{N}(0, \sigma_w^2)$ to the receiver over T rounds of communications. As depicted in Fig. 1, the forward communication channel is modeled as an AWGN channel,

$$y_t = x_t + n_t,$$

where x_t denotes the transmitted symbol, and $N_t \sim \mathcal{N}(0, \sigma_f^2)$ denotes the additive Gaussian noise for $t = 0, 1, \dots, T$. Each forward transmission is followed by an output feedback, where the transmitter receives

$$\tilde{y}_t = y_t + \tilde{n}_t,$$

where $\tilde{N}_t \sim \mathcal{N}(0, \sigma_b^2)$ denotes the additive Gaussian noise in the feedback channel.



Fig. 1: AWGN channel with noisy output feedback

The encoding process is inherently causal. For every transmission except the initial transmission, the output feedback $\tilde{y}^t = \{\tilde{y}_0 \cdots \tilde{y}_t\}$ is used to frame the next transmission symbol as

$$x_t = \phi_t^{\text{enc}}\left(w, \tilde{y}^{t-1}\right)$$

for $t \in [1, T]$, where the transmission power is constrained either in the peak power or total power. For the peak power constraint, we let $\mathsf{E}[X_t^2] \leq P \ \forall \ t \in [0, T]$. For the total power constraint, we let $\mathsf{E}[\sum_{t=0}^T X_t^2] \leq (T+1)P$. The decoding process, on the other hand, does not need to be causal; after T+1 rounds of transmissions, the decoder generates an estimated message \hat{w}_T based on the entire received sequence $y^T = (y_0, \dots, y_T)$ as

$$\hat{w}_T = \psi^{\text{dec}}(y^T).$$

Hence, designing a coding scheme for channels that utilizes output feedback involves jointly designing multiple encoding functions $\{\phi_t^{\text{enc}}(\cdot)\}$ for $t \in [1,T]$ and $\psi^{\text{dec}}(\cdot)$ that minimize the MMSE defined as $\mathsf{E}\left[\left(W - \hat{W}_T\right)^2\right]$.

Designing a coding scheme for channels with feedback is challenging due to (a) the necessity to optimize the encoder and decoder jointly and (b) the high dimensional encoding space. Thus, literature has focused on linear coding schemes, for which the transmitted symbol is a linear function of the message and the output feedback, formally defined in Definition 1 and in [2]. Linear schemes allow theoretical analysis and efficient implementation; they are naturally equipped with an optimal decoder, i.e., the linear MMSE estimator, assuming that the message w is Gaussian, which eases the design of linear schemes.

Definition 1 (Linear schemes). A sequence of schemes is called linear if the encoding function can be expressed as

$$\phi_t^{\text{enc}}\left(w,\tilde{y}^{t-1}\right) = a_t w + \sum_{j=0}^{t-1} b_{t,j}\tilde{y}_j$$

for some $a_t, b_{t,0}, \cdots, b_{t,t-1} \in \mathbb{R}$ for every $t \in [0, T]$.

Nevertheless, there has been limited success in deriving the optimal linear schemes for AWGN channels with output feedback. For AWGN channels with noiseless feedback, the celebrated SK scheme is shown to be optimal among all linear schemes [17]. For AWGN channels with noisy output feedback, on the other hand, the optimal linear coding scheme is still unknown [2]. In the following, we review the known results on the linear coding for AWGN channels with output feedback. We begin by reviewing the Schalkwijk and Kailath [18], which is the optimal linear coding scheme for channels with noiseless output feedback [18] (Section II-A). We then review the best known linear coding scheme for channels with *noisy* feedback by Chance and Love [2] (Section II-B).

A. Prior work on noiseless feedback: Optimal linear scheme by Schalkwijk and Kailath

SK scheme introduced by Schalkwijk and Kailath in [18] is a linear coding scheme for channels with *noiseless* feedback (i.e., $\tilde{y}_t = y_t$) that is shown to be optimal [19]. In the SK scheme, as illustrated in Algorithm 1, the encoder transmits its raw message w in the first transmission. Afterwards, the encoder computes the error between w and the receiver's MMSE estimate $\hat{w}_t := \mathsf{E}[W|y^t]$ and transmits the error (with a scaling to satisfy the power constraint), i.e., $x_{t+1} \propto (w - \hat{w}_t)$.

Algorithm 1: Schalkwijk-Kailath scheme

Input: w Output: Final Estimate $\hat{w}_T = \mathsf{E}[W|y_0, \cdots, y_T]$ Encoder: Initialize $u_0 = w$ and $\gamma_0 = \sqrt{P/\mathsf{E}[W^2]}$ $x_0 = \gamma_0 u_0$ Decoder: $y_0 = x_0 + n_0$ for $t = 0, \cdots, T$ do $\left| \begin{array}{c} \underline{\mathsf{Encoder}};\\ u_{t+1} = u_t - \mathsf{E}[U_t|y_t]\\ x_{t+1} = \gamma_{t+1}u_{t+1}, \text{ where } \gamma_{t+1} = \sqrt{P/\mathsf{E}[U_{t+1}^2]}\\ \underline{\mathsf{Decoder}};\\ y_{t+1} = x_{t+1} + n_{t+1}\\ \mathbf{end}\\ \underline{\mathsf{Decoder}}; \ \hat{w}_T = \sum_{t=0}^T \mathsf{E}[U_t|y_t]. \end{array} \right|$

Extending the SK scheme to channels with noisy output feedback, however, is not straightforward since the encoder is not aware of the received values y^t and thus cannot compute the receiver's MMSE estimate $\hat{w}_t = \mathsf{E}[W|y^t]$.

B. Prior work on noisy feedback: linear coding scheme by Chance and Love

For AWGN channels with output feedback, Chance and Love in [2] introduce a *concatenated* coding scheme, which consists of a linear code as the inner code concatenated with an error-correcting code as the outer code. In the following, we describe the methodology proposed in [2] to optimize the linear encoding scheme for noisy feedback. The authors begin with the most general assumption of the linear feedback scheme given as

$$\mathbf{x} = \mathbf{F} \left(\mathbf{n} + \tilde{\mathbf{n}} \right) + \mathbf{g} w_{z}$$

where w is the message, $\mathbf{g} \in \mathbb{R}^T$ is a unit-norm vector, \mathbf{F} is a $T \times T$ lower triangular encoding matrix, $\tilde{\mathbf{n}} + \mathbf{n}$ is the combined $T \times 1$ noise vector, and x is the final $T \times 1$ vector to be transmitted. At the

$$\hat{w}_T = \mathbf{q}^T \mathbf{y}.$$

In order to find the optimal set of parameters \mathbf{F} , \mathbf{g} , and \mathbf{q} , one could represent the received Signal to Noise Ratio (SNR) in terms of these parameters and then find \mathbf{F} , \mathbf{g} , and \mathbf{q} that maximizes the received SNR. This optimization, however, is intractable. To mitigate this challenge, Chance and Love first show that for a given \mathbf{F} and \mathbf{g} , the optimal vector \mathbf{q} can be obtained in a closed form, which leaves two variables \mathbf{F} and \mathbf{g} to optimize. However, simultaneous optimization of the remaining two variables is still intractable; they propose an iterative approach wherein one variable is optimized, keeping the other fixed and vice versa.

In [2], they provide a concatenated code with an inner and an outer code. The inner code is the linear coding scheme using the optimized variables **F**, **g** and **q**. Nevertheless, the CL scheme we refer to in this paper does not include the concatenation (as the concatenation can be applied to any inner coding scheme). The CL scheme is shown to significantly outperform the SK scheme for AWGN channels with noisy output feedback and coincides with the SK scheme for AWGN channels with noiseless output feedback, which shows that a tailored coding scheme for channels with noisy output feedback does provide an additional reliability gain. They showcase an improvement of nearly 10 dB of received SNR compared to the SK scheme for certain settings [2].

Two very interesting questions, following Chance and Love's work, are (a) whether the CL scheme is optimal within the class of linear schemes and (b) whether one could derive a closed-form linear code. The CL scheme includes an iterative update of the matrices F and q and thus does not provide a closed-form expression. In the rest of the paper, we provide an answer to both questions. We derive a linear coding scheme with a closed-form expression that strictly generalizes the CL scheme, i.e., the proposed scheme is equal to or strictly more reliable than the CL scheme. As we elaborate on in the following section, our scheme is inspired by the CL scheme, which has a sequential structure.

III. SEQUENTIAL LINEAR SCHEMES

In this section, we introduce a new family of linear schemes, namely sequential linear schemes defined in Definition 2, and depicted in Fig. 2. We prove that the family of these sequential schemes, although a strict subset of linear schemes, strictly generalizes the SK and CL schemes (Remarks 2 and 3). We then show that for the class of sequential linear schemes, a recursive Kalman filter can be used as an optimal estimator at the receiver (Section III-A), which is an essential precursor toward deriving the optimal sequential linear schemes.

Definition 2 (Sequential linear schemes). A sequence of schemes is called *sequentially linear* if the encoder maintains a state $u_t \in \mathbb{R}$ which is updated based on the most recent feedback \tilde{y}_t in a linear manner as

$$u_{t+1} = u_t + c_t \tilde{y}_t \tag{1}$$

for some $c_t \in \mathbb{R}$ and transmits a scaled version of u_{t+1} as

$$x_{t+1} = \gamma_{t+1} u_{t+1}, \tag{2}$$

to satisfy the power constraint $\mathsf{E}[X_t^2] \leq P$, for $t \in [0,T]$ under the peak power constraint or to satisfy the power constraint $\sum_{t=0}^{t=T} \mathsf{E}[X_t^2] \leq (T+1)P$ under the total power constraint.

For the convenience of notation, we let $\phi_t(\cdot)$ denote the sequential linear encoding operation at time t parameterized by (γ_{t+1}, c_t) , i.e.,

$$x_{t+1} = \phi_t (u_t, \tilde{y}_t) = \gamma_{t+1} (u_t + c_t \tilde{y}_t).$$
(3)



Fig. 2: Sequential linear encoder for channels with noisy output feedback: The encoder state is updated as $u_{t+1} = b_t u_t + c_t \tilde{y}_t$ as in (1) and a scaled version of u_t is sent as $x_t = \gamma_t u_t$

Remark 1 (Not all linear schemes are sequential). As we restrict the state u_t to be a scalar, sequential linear schemes are a *strict* subset of linear schemes. In other words, not all linear schemes are sequential. On the other hand, if we allow u_t to be a vector of length T, i.e., $u_t \in \mathbb{R}^T$, where T denotes the length of total transmissions, then sequential linear schemes include the entire class of linear schemes.

While the restriction on the dimension of u_t makes the sequential linear schemes a strict subset of linear schemes, in the following, we show that both the SK scheme and the CL scheme belong to the family of sequential linear schemes.

Remark 2 (SK scheme is sequentially linear). SK scheme falls into the family of sequential linear schemes. Let u_t denote the estimation error, i.e., $u_t = w - \hat{w}_t$. Then it follows that $u_{t+1} = u_t + c_t \tilde{y}_t$, where $c_t = -\mathsf{E}[U_t \tilde{Y}_t]/\mathsf{E}[\tilde{Y}_t^2]$ and $x_{t+1} = \gamma_{t+1}u_{t+1}$, where γ_{t+1} is chosen to satisfy the power constraint $\mathsf{E}[X_{t+1}^2] \leq P$ for every $t \in [0, T]$. (See Appendix IX-A for detailed proof.)

Remark 3 (CL scheme is sequentially linear). CL scheme starts with the general form of linear schemes, but the conjectured optimal linear schemes fall into the family of sequential linear encoding. (See Appendix IX-B for detailed proof.)

A. Linear MMSE Estimation

In this section, we show that (a) sequential linear schemes are naturally equipped with an efficient and recursive MMSE estimator, namely, the Kalman filter, and (b) the MMSE can be represented as a function of the parameters of the encoding scheme. In other words, for any choice of $\{c_t, \gamma_t\}_{t=1}^T$ in the sequential linear encoding scheme, we can represent the MMSE as a function of $\{c_t, \gamma_t\}_{t=1}^T$. Kalman filter provides a recursive MMSE estimation of a state variable when the state variable follows a state-space equation, and the observation variable can be written as a sum of the observation variable and the noise.

In order to derive the recursive Kalman estimator for sequential linear schemes, we begin with the observation that the encoder's state u_t satisfies the linear state space equation in (1) and y_t is a scaled version of u_t plus the Gaussian noise. Under these conditions, the Kalman filter allows one to recursively update the estimation of the *state* u_t given a series of observations y^t . Nevertheless, this alone is insufficient to estimate the *message* w given y^t . In order to derive a recursive estimation of w given y^t , we let $p_t = \begin{bmatrix} w & u_t \end{bmatrix}^T$ denote the pair of the message w and the encoder state u_t as shown below.

Lemma 1 (Kalman Filter as Decoder). Let us define the Kalman state variable $p_t = \begin{bmatrix} w & u_t \end{bmatrix}^T$, where w is the message and u_t is the encoder state defined in the sequential linear encoding scheme at the transmitter from Fig. 2. Then, p_t and y_t satisfy the following

$$p_{t+1} = A_t p_t + G_t z_t, \quad y_t = C_t p_t + n_t,$$
(4)

$$A_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \gamma_t c_t \end{bmatrix}, \qquad G_t = \begin{bmatrix} 0 & 0 \\ c_t & c_t \end{bmatrix}, \qquad C_t = \begin{bmatrix} 0 & \gamma_t \end{bmatrix},$$

and the vectors $z_t = \begin{bmatrix} n_t & \tilde{n}_t \end{bmatrix}$ and forward Gaussian noise n_t represent the noises in the state update and the observations, respectively.



Fig. 3: Sequential linear update of the state p_t where w is the message, $z_t = [n_t \tilde{n}_t]$ with n_t and \tilde{n}_t being the forward and the feedback noise respectively

Proof. Lemma 1 follows immediately from the definition of the sequential linear schemes in Definition 2. \Box

Based on Lemma 1, we establish the following theorem.

Theorem 1 (MMSE estimation via Kalman filter). Let $p_t = \begin{bmatrix} w & u_t \end{bmatrix}^T$ denote the pair of the message and the t^{th} encoder state. Let $\hat{p}_t := \mathsf{E}[P_t|y^t]$ denote its estimate and $\Sigma_t := \mathsf{E}[(P_t - \hat{P}_t)(P_t - \hat{P}_t)^T]$ denote the covariance of the corresponding error, given the observation vector $y^t = (y_0, \dots, y_t)$. Then the estimate \hat{p}_t and its error covariance matrix Σ_t can be recursively computed via Kalman filter as

$$\hat{p}_{t+1} = g_1(\hat{p}_t, \gamma_t, \gamma_{t+1}, c_t, y_t), \qquad \Sigma_{t+1} = g_2(\Sigma_t, \gamma_t, \gamma_{t+1}, c_t),$$
(5)

where $g_1(\cdot)$ and $g_2(\cdot)$ are defined in (7) and (8), respectively, and γ_t and c_t denote the encoder parameters defined in Definition 2.

Proof. As the state variable p_t and the observation variable y_t follow the canonical form of equations in (4) as shown in Lemma 1, the Kalman filter provides a recursive form of the MMSE estimate and the corresponding MMSE error covariance matrix.

Let $\hat{p}_t := \mathsf{E}[P_t|y^t]$ denote the MMSE estimate of p_t given y^t and Σ_t denote the corresponding error covariance matrix defined as

$$\Sigma_t := \mathbb{E}\left[\left(P_t - \hat{P}_t\right)\left(P_t - \hat{P}_t\right)^T\right] = \begin{bmatrix}\epsilon_{w,t}^2 & \epsilon_{uw,t}^2\\\epsilon_{uw,t}^2 & \epsilon_{u,t}^2\end{bmatrix},\tag{6}$$

where $\epsilon_{w,t}^2 := \mathbb{E}\left[\left(W - \hat{W}_t\right)^2\right]$ denotes the Mean Squared Error (MSE) in estimating the message, $\epsilon_{u,t}^2 := \mathbb{E}\left[\left(U_t - \hat{U}_t\right)^2\right]$ denotes the MMSE in estimating the encoder's most recent state, and $\epsilon_{uw,t}^2 := \mathbb{E}\left[\left(W - \hat{W}_t\right)\left(U_t - \hat{U}_t\right)\right]$ denotes the covariance of the two errors. Using this canonical form and the standard definitions for Kalman filter solutions we can express the MMSE estimate of \hat{p}_t in a recursive form as

$$\hat{p}_{t+1} = A_t \hat{p}_t + L_{t+1} \left(y_{t+1} - C_{t+1} A_t \hat{p}_t \right), \tag{7}$$

where

$$L_{t+1} = \frac{\left(A_t \Sigma_t A_t^T + G_t Q G_t^T\right) C_{t+1}^T}{C_{t+1} \left(A_t \Sigma_t A_t^T + G_t Q G_t^T\right) C_{t+1}^T + \sigma_f^2} \quad \text{and} \quad Q = \mathsf{E}[Z_t Z_t^T] = \begin{bmatrix} \sigma_f^2 & 0\\ 0 & \sigma_b^2 \end{bmatrix}.$$

General MDP



Fig. 4: MDP Model and the analogy to the AWGN channel with feedback

Parameter	Symbol	Value
State	s_t	$\left\{\Sigma_t, \sigma_{u,t}^2\right\}$
Action	a_t	c_t
Cost	r_t	$\begin{array}{ccc} 0 & t \neq T \\ \epsilon_{w,T}^2 & t = T \end{array}$
Value function	V_t	$\sum_{t=0}^{T} r_t$
Transition Function	$ au_t$	$g_{4}\left(\cdot\right),g_{5}\left(\cdot\right)$

TABLE I: Table showing the elements of the proposed MDP

Similarly, Σ_t follows a recursive relation given as

$$\Sigma_{t+1} = \left(I - L_{t+1}C_{t+1}\right) \left(A_t \Sigma_t A_t^T + G_t Q G_t^T\right),\tag{8}$$

where \hat{p}_{-1} and Σ_{-1} are initialized as $\hat{p}_{-1} = \mathbf{0}$ and $\Sigma_{-1} = \begin{bmatrix} \sigma_w^2 & \sigma_w^2 \\ \sigma_w^2 & \sigma_w^2 \end{bmatrix}$.

The Kalman filter provides a recursive solution for the MMSE and the final MSE. In particular, $\epsilon_{w,t}^2$, the top left element of the matrix Σ_t , denotes the MSE in estimating w given the observed sequence y^t and is defined as $\epsilon_{w,t}^2 = \mathsf{E}\left[\left(W - \hat{W}_t\right)^2\right]$. Therefore, $\epsilon_{w,t+1}^2$ can be recursively computed based on Σ_t and parameters γ_t and c_t . In the next section, we propose an MDP which utilizes this recursive property in Theorem 1 to find the optimal parameters for the encoding scheme that minimize the final MSE, $\epsilon_{w,T}^2$.

IV. OPTIMAL SEQUENTIAL LINEAR SCHEMES VIA DYNAMIC PROGRAMMING

In the previous section, we showed that the MMSE can be updated sequentially as a function of the encoder parameters $\{c_t, \gamma_t\}_{t=1}^T$. In this section, we provide a closed-form optimal solution of $\{c_t, \gamma_t\}_{t=1}^T$ that minimizes the MMSE at any time t, for a peak power constraint. We do so by formulating an MDP, for which the parameters of the encoding scheme are modeled as an action and the corresponding MMSE is modeled as a cost (Section IV-B), and computing the optimal action policy via dynamic programming. (Section IV-C). We begin with an overview of the general MDP framework.

A. General MDP Framework

An MDP, in general, is described through a Markov process characterized by the tuple $\{s_t, a_t, \tau_t, r_t\}$. An agent with the state s_t takes an action a_t at a cost r_t and in the process transitions to the future state s_{t+1} governed by the function τ_t as $s_{t+1} = \tau_t(s_t, a_t)$. We define the optimal policy π_t as the mapping from the state space to the action space that minimizes the expected sum of cost from any state to the final state. We define the value function $V_t(s_t)$ to be the expected sum of cost if the agent follows the optimal policy π_t from s_t till the final state s_T . The optimal policy π_t is computed using the Bellman equation as

$$\pi_{t}(s_{t}) = \underset{a_{t}}{\operatorname{argmin}} \left(r_{t}(s_{t}, a_{t}) + V_{t+1}(\tau_{t}(s_{t}, a_{t})) \right),$$

$$V_{t}(s_{t}) = r_{t}(s_{t}, \pi_{t}(s_{t})) + V_{t+1}(\tau_{t}(s_{t}, \pi_{t}(s_{t}))).$$
(9)

B. Proposed MDP framework

In the following, we formulate an MDP with the encoding scheme being the policy and the MSE being the cost, which has to be minimized to get the optimal encoding scheme, as summarized in Table I.

State and Action We define the state s_t as the tuple of the error covariance matrix and the variance of the encoder's state, i.e., $s_t = \{\Sigma_t, \sigma_{u,t}^2\}$, and the action a_t as the encoder parameter, i.e., $a_t = c_t$. Using Theorem 1, we can show the following.

Theorem 2. Let state s_t be the pair of the error covariance matrix and and the variance of the encoder's state, $s_t := \{\Sigma_t, \sigma_{u,t}^2\}$, and the control action a_t be the parameter c_t , we can define the MDP as

$$s_{t+1} = \tau_t(s_t, a_t)$$

where the transition function $\tau_t(\cdot)$ is obtained from (10) and (5).

Proof. From (3), we have, $u_{t+1} = u_t + c_t \tilde{y}_t$, therefore, $\sigma_{u,t+1}^2 = \mathsf{E}\left[U_{t+1}^2\right]$ can be expressed in terms of $\sigma_{u,t}^2$ and the encoder parameters as

$$\sigma_{u,t+1}^2 = (1 + \gamma_t c_t)^2 \sigma_{u,t}^2 + c_t^2 \left(\sigma_f^2 + \sigma_b^2\right).$$
(10)

Under the constant peak power constraint, we have $\gamma_t = \sqrt{\frac{P}{\sigma_{u,t}^2}}$, from which it follows that

$$\sigma_{u,t+1}^2 = \left(1 + \sqrt{\frac{P}{\sigma_{u,t}^2}}c_t\right)^2 \sigma_{u,t}^2 + c_t^2 \left(\sigma_f^2 + \sigma_b^2\right) =: g_4(\sigma_{u,t}^2, c_t).$$
(11)

In Theorem 1, we derive the recursive function $\Sigma_{t+1} = g_2(\Sigma_t, \gamma_t, \gamma_{t+1}, c_t)$. Combining $\gamma_t = \sqrt{\frac{P}{\sigma_{u,t}^2}}$, $\gamma_{t+1} = \sqrt{\frac{P}{\sigma_{u,t+1}^2}}$, and $\sigma_{u,t+1}^2 = g_4(\sigma_{u,t}^2, c_t)$, and substituting the values for γ_t and γ_{t+1} , we can represent Σ_{t+1} as a function of $(\Sigma_t, \sigma_{u,t}, c_t)$. We let $g_5(\cdot)$ denote such a function, i.e.,

$$\Sigma_{t+1} = g_5\left(\Sigma_t, \sigma_{u,t}^2, c_t\right). \tag{12}$$

Given that $s_t = \{\Sigma_t, \sigma_{u,t}^2\}$, and ϕ_t includes $c_t = a_t$, we conclude $s_{t+1} = \tau_t(s_t, a_t)$ from (11) and (12).

We now provide an intuition behind choosing $\{\Sigma_t, \sigma_{u,t}^2\}$ as the state. A naturally proposed MDP would be to let Σ_t denote the state and the encoding function ϕ_t denote the action, where ϕ_t is parameterized by $\{\gamma_{t+1}, c_t\}$. However, γ_{t+1} is not an independent variable given the peak power constraint and can be obtained from the variance of u_t at the encoder, $\sigma_{u,t}^2$.

Therefore, we define our state s_t to be the collection of variances $\{\Sigma_t, \sigma_{u,t}^2\}$ and the parameter c_t becomes the control action a_t . This allows us to define the recursion function τ_t that relates the current state $s_t = \{\Sigma_t, \sigma_{u,t}^2\}$ and control actions $a_t = c_t$ to the future state $s_{t+1} = \{\Sigma_{t+1}, \sigma_{u,t+1}^2\}$, as we show in Theorem 2. Consequently, the problem of finding the optimal encoding scheme $\tilde{\phi}_t$ is now reduced to obtaining the optimal action \tilde{a}_t at any given state s_t .

Cost and Value function We define the cost r_t in a way that the objective function that is minimized at each step of the iteration in (9) is the final MSE. Specifically, we define the cost r_t as

$$r_t = \begin{cases} 0 & t \neq T\\ \epsilon_{w,T}^2 & t = T. \end{cases}$$
(13)

Such a definition for cost r_t ensures the value function, which is the optimized sum of the cost, is always $r_T = \epsilon_{w,T}^2$. This kind of definition for cost function is common in problems where only the final cost is considered.

We define the value function as $V_t(s_t) = \epsilon_{w,T}^2(s_t, \tilde{a}_{t:T-1})$, given any state $s_t, t = 0 : T - 1$ other than that the terminal state, which denotes the the final MMSE if we start at state s_t and undertake the optimal actions $\tilde{a}_{t:T-1}$. Then the following is obtained using the Bellman equation in (9) by substituting the reward function as defined above.

$$V_t(s_t) = \epsilon_{w,T}^2(s_t, \tilde{a}_{t:T})$$
(14a)

$$=\epsilon_{w,T}^{2}\left(s_{t},\tilde{a}_{t},\tilde{a}_{t+1:T}\right)$$
(14b)

$$=\epsilon_{wT}^{2}\left(s_{t+1},\tilde{a}_{t+1:T}\right) \tag{14c}$$

$$= V_{t+1}(s_{t+1}). (14d)$$

In summary, the key idea is to treat the T-step encoding process as an MDP with the action a_t representing the free parameter c_t of the encoder function. The proposed MDP and the analogy to the communication network are showcased in Fig. 4 and are summarized in table I. In the next section, we provide an algorithm for solving this MDP.

C. Solution of the MDP via Dynamic Programming

We presented an MDP and defined the state $s_t := \{\Sigma_t, \sigma_{u,t}^2\}$, the control action $a_t = c_t$, and the set of update equations that describe the transition $s_{t+1} = \tau_t (s_t, a_t)$ in Theorem 2. To solve the MDP and find the optimal control action, we use the *dynamic programming* which returns the analytical expressions for the T + 1 optimal actions $\{a_t\}_{t=0}^T$ and the value function $V_t(s_t)$.

Algorithm 2: Proposed Dynamic Program				
Output: Optimal action $\tilde{a}_{0:T-1}$, Value function $V_{0:T}$				
Initialization : At $t = T$, $\forall s_T$, $V_T(s_T) = \epsilon_{w,T}^2(s_T, \tilde{a}_T)$				
for $t = T - 1, \cdots, 0$ do				
$\tilde{a}_{t} = \operatorname{argmin} V_{t+1} \left(\tau_{t} \left(s_{t}, a_{t} \right) \right)$				
$V_{t}(s_{t}) = V_{t+1}(\tau_{t}(s_{t}, \tilde{a}_{t}))$				
end				

As depicted in Algorithm 2, we backward recursively evaluate the optimal actions from time t = T till t = 0, storing the value function at each instant. The algorithm initializes the value function V_T at t = T for all s_T as $\epsilon_{w,T}^2$. The value is part of the definition of Σ_T given in (6) within the definition of state s_T . The subsequent steps use the formulation proved in (14d) to obtain the value function at each of the previous states s_t from t = T - 1 till 0 while optimizing over the actions to obtain the optimal action \tilde{a}_t .

We solve the dynamic program above and obtain the closed-form expressions for the value function $V_t(s_t)$ and the optimal action \tilde{a}_t for every $t \in [0, T]$ as shown in the following.

Lemma 2 (Solution to the MDP). Let P, σ_f^2 and σ_b^2 be the power constraint, forward and feedback channel variance respectively for the AWGN channel with feedback. Let $V_t(s_t)$ be the value function for state s_t at any time t for the analogous MDP, then the $V_t(s_t)$ and the optimal control action \tilde{a}_t is given as

$$V_t(s_t) = \frac{-\epsilon_{uw,t}^4 + K_{T-t}\epsilon_{w,t}^2 \sigma_{u,t}^2 + \epsilon_{w,t}^2 \epsilon_{u,t}^2}{K_{T-t}\sigma_{u,t}^2 + \epsilon_{u,t}^2},$$
(15)

while the optimal action \tilde{a}_t is given as

$$\tilde{a}_t = -\frac{K_n \sqrt{S} \left(\frac{\sigma_{u,t}}{\sigma_f}\right)}{K_n \eta_0 + \beta},\tag{16}$$

where n = T - t is the number of remaining transmissions and

$$K_n = \frac{\eta_1 K_{n-1}^2 + \eta_2 K_{n-1}}{\eta_3 K_{n-1}^2 + \eta_4 K_{n-1} + \eta_2} =: f(K_{n-1})$$
(17)

$$= f^{n-1}(K_1), (18)$$

with $K_1 = \frac{\eta_1}{\eta_3}$, $\eta_0 = (1+S)(1+\beta)$, $\eta_1 = 1+\beta+S\beta$, $\eta_2 = \beta$, $\eta_3 = S(1+\beta)(1+S)$, $\eta_4 = 1+\beta+S+2S\beta$, with $S = \frac{P}{\sigma_t^2}$ and $\beta = \frac{\sigma_b^2}{\sigma_t^2}$.

Proof. See Appendix IX-C for the proof.

In the next section, we obtain the sequential linear encoding scheme from the solution of the dynamic program for both noiseless and noisy feedback cases. We also derive the final MSE obtained using these schemes from the value function computed from the dynamic program.

V. OPTIMAL SEQUENTIAL LINEAR SCHEMES

In Section IV, we described an MDP analogous to a communication system with feedback and proposed a DP algorithm to solve it. The solution obtained was a set of T + 1 value functions and corresponding optimal control actions. This section uses these solutions to derive closed-form expressions for the optimal encoding scheme and the MMSE and analyze its performance. In addition, we compare our proposed encoding scheme with the SK scheme for channels with noisy output feedback and show that our scheme cannot be obtained as a trivial generalization from the SK scheme. Furthermore, we obtain approximate expressions for MSE for large T and analyze their asymptotic performances for both noiseless and noisy feedback regimes.

A. Optimal sequential linear schemes

The optimal control actions obtained in Lemma 2 and given by (16) are used to derive the linear encoding scheme in a closed-form, as defined in (3). We constructed the MDP such that the action vector was the required coefficient c_t of the encoding scheme. We use this fact to derive the encoding scheme in the following theorem.

Theorem 3 (Closed-form solution for the optimal sequential linear encoding). Let P, σ_f^2 and σ_b^2 be the peak power constraint, forward and feedback channel variances respectively for AWGN channels with noisy output feedback. Then the optimal sequential linear encoder, denoted as $\tilde{\phi}_t(\cdot)$, for transmitting a message is given as

$$x_{t+1} = \gamma_{t+1} u_{t+1}, \tag{19}$$

where the state of the encoder u_{t+1} is updated as

$$u_{t+1} = \left(u_t - \frac{K_n \sqrt{S} \left(\frac{\sigma_{u,t}}{\sigma_f} \right)}{K_n \eta_0 + \beta} \tilde{y}_t \right),$$

and the scaling factor γ_{t+1} is given as

$$\gamma_{t+1} = \sqrt{\frac{P}{\left(1 + \frac{K_n S}{K_n \eta_0 + \beta}\right)^2 + \frac{K_n^2 S \sigma_{u,t}^2}{(K_n \eta_0 + \beta)^2} (1 + \beta)}}$$

for n = T - t - 1, and the parameters K_n , β , S and η s are defined in Lemma 2.

Proof. As shown in (3), sequential linear encoders are in the form of $x_{t+1} = \gamma_{t+1}u_{t+1}$ where

$$u_{t+1} = u_t + c_t \tilde{y}_t$$
 and $\gamma_{t+1} = \sqrt{\frac{P}{\sigma_{u,t+1}^2}}$

under a peak power constraint. Thus, sequential linear encoders are fully defined by the weight c_t for $t \in [0, T + 1]$. In the formulation of the MDP in Section IV, we defined the control action as $a_t = c_t$ and derived the optimal control action in (16) of Lemma 2. Substituting c_t as \tilde{a}_t , and using (11), we obtain the optimal sequential linear encoding scheme.

Therefore, the optimal encoding scheme to generate the transmitted symbol x_t given feedback \tilde{y}_t and the previous encoder state u_t is as given as (19).

B. Interpretation of optimal sequential linear codes

It is important to interpret and analyze the behavior of the optimal sequential linear scheme. To this end, we first compare the optimal sequential linear codes against the SK scheme (See Section II-A for a description). The following corollary shows that our scheme specializes to the SK scheme for channels with noiseless output feedback.

Corollary 1 (Noiseless feedback). For AWGN channels with noiseless output feedback, the optimal policy $\tilde{\phi}_t(u_t, \tilde{y}_t)$ derived in Theorem 3 coincides with the SK scheme under a peak power constraint *P*.

Proof. This corollary immediately follows from the optimal encoding scheme obtained in Theorem 3.

We considers a noiseless feedback channel with the forward noise variance $\sigma_f^2 = 1$. We have $\sigma_b^2 = 0$ i.e. $\beta := \sigma_b^2 / \sigma_f^2 = 0$.

Using (19) and substituting $\beta = 0$, we get

$$\tilde{\phi}_t \left(u_t, y_t \right) = \sqrt{\frac{P}{\sigma_{u,t+1}^2}} \left(u_t - \frac{\sqrt{P}\sigma_{u,t}}{1+P} y_t \right)$$
(20)

$$= \sqrt{\frac{P}{\sigma_{u,t+1}^2} \left(u_t - \mathbb{E}[U_t|y_t] \right)}, \qquad (21)$$

which coincides with the SK scheme as in [20].

The fact that the optimal sequential linear scheme coincides with the SK scheme is not surprising given that the SK scheme belongs to the family of sequential linear schemes. An interesting question is how/if the optimal sequential linear scheme is different from the SK scheme for channels with noisy output feedback. To answer this question, we begin by characterizing a natural extension of the SK scheme that follows the philosophy of the SK scheme.

The SK encoding scheme for noiseless feedback involves repeatedly sending the error in the receiver's estimate of the transmitted message at all subsequent steps. However, for noisy feedback channels, the encoder has access to only a *noisy* version of the estimate. An intuitive method would be to estimate the message at the encoder from the received noisy feedback, compute the error, and then send it as the new transmitted symbol, i.e.

$$\phi_t^{SK}\left(u_t, \tilde{y}_t\right) = \sqrt{\frac{P}{\sigma_{u,t+1}^2}} \left(u_t - \mathbb{E}[U_t | \tilde{y}_t]\right).$$

The remaining question is whether the optimal sequential linear scheme coincides with the extended SK scheme. In the following, we show that the two schemes are different; the optimal sequential linear

scheme strictly outperforms the extended SK scheme, bolstering the fact that our sequential linear scheme is non-trivial.

Corollary 2 (Noisy feedback). For AWGN channels with noisy output feedback, where forward and feedback channel variances are given as σ_f^2 and σ_b^2 respectively, the optimal policy $\tilde{\phi}_t(u_t, \tilde{y}_t)$ derived in Theorem 3 is not the same as the direct generalization of the SK scheme ϕ^{SK} , i.e. $\tilde{\phi}_t \neq \phi_t^{SK}$, under the peak power constraint *P*.

Proof. A strict generalization to the SK scheme implies that the update of the encoder state U_t depends on the estimate made by the encoder on the expected value for U_t based on the noisy received feedback \tilde{y}_t i.e.

$$u_{t+1} = u_t - \mathbb{E}\left[U_t | \tilde{y}_t\right] \tag{22}$$

$$= u_t - \frac{\sqrt{P(\sigma_{u,t})}}{P + \sigma_t^2 + \sigma_b^2} \tilde{y}_t.$$
(23)

Therefore, the direct generalization of the SK scheme is given as

$$\phi^{SK}\left(u_{t},\tilde{y}_{t}\right) = \sqrt{\frac{P}{\sigma_{u,t+1}^{2}}} \left(u_{t} - \frac{\sqrt{P}\sigma_{u,t}}{P + \sigma_{f}^{2} + \sigma_{b}^{2}}\tilde{y}_{t}\right).$$

To compare the extended SK scheme, denoted by ϕ^{SK} , and the optimal sequential linear encoding scheme $\tilde{\phi}$ derived in Theorem 3, in Figure 5, we plot the coefficient that is multiplied by feedback \tilde{y}_0 , from the first forward transmission, for the generalized scheme ϕ^{SK} and the proposed scheme $\tilde{\phi}_t$ as a function of the total number of transmissions T. The plot stays constant for scheme ϕ^{SK} for different values of T, unlike our derived scheme. This dependence on T proves that our proposed scheme is not the same as the generalized SK scheme. Also, the dependence shows that our scheme is non-trivial for noisy feedback and cannot be derived as a generalization from the SK scheme for which the derived coefficients are constant with T.



Fig. 5: The first coefficient that is multiplied to \tilde{y}_0 (i.e., c_0) over different values for the total number of transmissions T with $\sigma_f = 0dB$ and $\sigma_b = -20dB$. It can be deduced that unlike the generalized SK scheme ϕ^{SK} , the coefficients derived from the DP solution $\tilde{\phi}$ are a function of the total intended transmissions.

We now turn our attention to the achievable MSE. In the following, we derive the MSE achieved by the optimal sequential linear schemes by establishing the relationship between the value functions and the final MSE.

C. MMSE of the Optimal Sequential Linear schemes

We provide a theorem to compute the closed-form expression for the MMSE at the end of T + 1 transmissions with noisy feedback. We establish a relationship between the value functions computed in Lemma 2 using the DP algorithm in Section IV with the intended final MSE.

Theorem 4. Consider an AWGN channel with feedback with a peak power constraint of P, forward channel variance of σ_f^2 and feedback channel variance of σ_b^2 , then MSE after T + 1 transmissions in the estimate of the message is given by

$$MSE = \frac{\sigma_w^2}{\zeta_T},$$
(24)

where

$$\zeta_T = \left(1 + S + \frac{1}{K_T}\right) \left(\frac{\frac{\eta_2}{K_T} + \eta_4}{\frac{\eta_2}{K_T} + \eta_1}\right),\tag{25}$$

with parameters S, K_T and η 's being described in terms of the system parameters P, σ_f^2 , σ_b^2 and T as in (15)

Proof. A message with variance σ_w^2 can be transmitted with a peak power constraint of P over a forward and feedback channel with variances σ_f^2 and σ_b^2 , respectively, with the MSE after T + 1 transmissions given by

$$\mathsf{MSE} = V_0\left(\Sigma_0, \sigma_{u,0}^2\right).$$

This is straightforward from the assumption that all costs are 0 except the terminal cost, i.e, MSE, and from the fact that the value function captures the sum of costs till the terminal state.

Without loss of generality, we can assume the first transmission to be $x_0 = \sqrt{Pw}$. Therefore, the receiver error variance and the encoder variance at the conclusion of the raw transmission can be expressed as

$$\Sigma_0 = \begin{bmatrix} \frac{\sigma_w^2}{S+1} & \frac{\sigma_w^2}{S+1} \\ \frac{\sigma_w^2}{S+1} & \frac{\sigma_w^2}{S+1} \end{bmatrix} \text{ and } \sigma_{u,0}^2 = \sigma_w^2.$$

Using the expression in (15) and then substituting the value function in terms of the system parameters, we obtain

$$V_0\left(\Sigma_0, \sigma_{u,0}^2\right) \cdot = \frac{\sigma_w^2}{\zeta_T},$$

where

$$\zeta_T = \frac{(1 + K_T + SK_T) (K_T + \beta + \beta K_T + SK_T + 2S\beta K_T)}{(K_T) (K_T + \beta + \beta K_T + S\beta K_T)} = \left(1 + S + \frac{1}{K_T}\right) \left(\frac{\frac{\eta_2}{K_T} + \eta_4}{\frac{\eta_2}{K_T} + \eta_1}\right).$$

In Fig. 6, we plot the MSE (dB) in the estimation of the message w, for $\sigma_w = 1$, P = 10, $\sigma_f^2 = 1$ and varying values of σ_b^2 , against the number of transmissions T. We observe that the MSE drops exponentially for the noiseless feedback case ($\sigma_b = 0$). The exponential nature is evident as the graph is linear with respect to the T with the MSE in dB scale which we also show analytically (Corollary 3).



Fig. 6: Plots showing the MSE in the estimation of the original message w for $\sigma_w = 1$, P = 10, $\sigma_f^2 = 1$ and different values of feedback variance σ_b^2 against varying number of total transmissions T. The exponential nature of MSE with T is evident for the noiseless case. It will be shown later that the reduction of MSE in polynomial with T for noisy output feedback

The MMSE obtained using our proposed scheme with noisy feedback varies polynomial with T, but this dependence is not apparent. Therefore, we use an approximate expression to understand this dependence better in terms of T for both noiseless and noisy feedback cases. The obtained approximate asymptotic expression matches the bounds that were given in [2], [21].

Corollary 3 (MSE for noiseless feedback). Let P and σ_f^2 be the peak power constraint and forward channel variance respectively for AWGN channels with noiseless feedback. The MSE in the estimate of the message of variance σ_w^2 at the receiver decays exponentially in transmissions T for noiseless feedback, and the MSE is given as

$$MSE = \frac{\sigma_w^2}{(1+S)^T}, \text{ where } S = \frac{P}{\sigma_f^2}.$$
 (26)

Proof. See Appendix IX-D.

Corollary 4 (MSE for noisy feedback). Let P, σ_f^2 and σ_b^2 be the peak power constraint, forward and feedback channel variance respectively for AWGN channel with noisy feedback. The drop in the MSE of the optimal sequential linear schemes in estimating a message with variance σ_w^2 is approximately polynomial given as

$$MSE \approx \frac{\sigma_w^2}{\zeta^*(T)},\tag{27}$$

where $\zeta^{*}(T)$ is a polynomial function in T and is given as

$\zeta^{\star} = S\left(1 + \frac{1}{\beta}\right)T$

. with $S = \frac{P}{\sigma_f^2}$ and $\beta = \frac{\sigma_b^2}{\sigma_f^2}$.

Proof. See Appendix IX-E

Fig. 7 shows the comparison of the approximated expression for MSE in (27) with the actual closed form value of the MSE as a function of T obtained in (24). We empirically verify that the approximation is quite tight, and the tightness improves with the increasing value of the feedback variance σ_b^2 .

 \square



Fig. 7: The plot shows the actual MSE and the MSE approximation for $\sigma_w = 1$, P = 10, $\sigma_f^2 = 1$ and different values of feedback variance σ_b^2 against varying number of total transmissions T. It can be seen that the polynomial approximation of MSE in (27) is tight and is better with the higher values of T or with higher feedback noise variance

VI. TOTAL POWER CONSTRAINT

In the previous sections, we derived a closed-form solution for the optimal sequential linear scheme and the corresponding MSE under the *peak power* constraint. In this section, we derive the optimal sequential linear encoding scheme under the *total power* constraint. We begin with the MDP formulated for the peak power constraint but modify the action spaces; we introduce additional parameters to represent the power allocations for each of the transmissions. We then propose a dynamic program to optimize the parameters of the encoding function and the power allocation.

A. State and Action

In Section IV, we introduced an MDP to represent our communication system with feedback as tuple $\{s_t, a_t, \tau_t, r_t\}$ which was summarized in Table I. For the total power constraint, we introduce two new parameters in the state vector, P_t and Q_t , where the parameter P_t represents the instantaneous power used in the current transmission, while Q_t is the unallocated power budget for the remaining steps, including the current, i.e.

$$Q_t = \sum_{t'=t}^T P_{t'}.$$

We also introduce a parameter α_t into the action vector, defined as the fraction of available power Q_t that is allocated to the current transmission, i.e., $P_t = \alpha_t Q_t$.

As illustrated in Fig. 8, at any time t, given P_t and Q_t , the power budget for the remaining transmissions [t+1,T] is $Q_{t+1} = Q_t - P_t$. Now, the objective is to optimize the parameter α_{t+1} in $P_{t+1} = \alpha_{t+1}Q_{t+1}$ to determine the appropriate power constraint P_{t+1} for the $t + 1^{th}$ transmission. This is repeated at every time instants $t \in [0, T-1]$ except at time t = T where α_T is unity as all the remaining power are allocated to the final transmission. Thus, the extended state and the action vectors for the MDP are given as $s_t = \{\Sigma_t, \sigma_{u,t}^2, P_t, Q_t\}$ and $a_t = \{c_t, \alpha_{t+1}\}$.

Fig. 8: Power allocation in terms of the action parameter α_{t+1} . At time t + 1, the power constraint is determined as $P_{t+1} = \alpha_{t+1}Q_{t+1}$ where α_{t+1} is optimized MDP under the total power constraint through the DP algorithm.

Parameter	Symbol	Value
State	st	$\left\{\Sigma_t, \sigma_{u,t}^2, P_t, Q_t\right\}$
Action	a_t	c_t, α_{t+1}
Transition	$ au_t$	$g_{6}(\cdot), g_{7}(\cdot), g_{8}(\cdot)$

TABLE II: Table showing the elements of the proposed

Let us now provide a theorem to derive the transition functions that govern the altered definitions of the state and action vectors in line with the framework provided in Theorem 2.

 P_T

Theorem 5. Let s_t denote the pair of the error covariance matrix, the variance of the encoder's state and the power allocation parameters, $s_t := \{\Sigma_t, \sigma_{u,t}^2, P_t, Q_t\}$, and the control action a_t being the parameter c_t, α_{t+1} . Then, s_t satisfies the MDP as

$$s_{t+1} = \tau_t(s_t, a_t),$$

where the transition function $\tau_t(\cdot)$ is obtained from (28), (30) and (31).

Proof. We derive recursive equations for the Σ_t , $\sigma_{u,t}^2$ and the power parameters P_t and Q_t . These equations eventually constitute the function τ_t that govern the transition of s_t .

From (3), we have $u_{t+1} = u_t + c_t \tilde{y}_t$. Therefore, $\sigma_{u,t+1}^2$ can be expressed in terms of $\sigma_{u,t}^2$, P_t and the encoding function as

$$\sigma_{u,t+1}^2 = (1 + \gamma_t c_t)^2 \, \sigma_{u,t}^2 + c_t^2 \left(\sigma_f^2 + \sigma_b^2 \right), \tag{28}$$

where $\gamma_t = \sqrt{\frac{P_t}{\sigma_{u,t}^2}}$. From (28), we obtain a function $g_6(\cdot)$ such that $\sigma_{u,t+1}$ satisfies $\sigma_{u,t+1}^2 = g_4(\sigma_{u,t}^2, c_t)$. We note that γ_{t+1} is chosen to satisfy the power constraint P_{t+1} at time instant t+1 as

$$\gamma_{t+1} = \sqrt{\frac{P_{t+1}}{\sigma_{u,t+1}^2}} = \sqrt{\frac{\alpha_{t+1} (Q_t - P_t)}{\sigma_{u,t+1}^2}}.$$
(29)

We already have, $\Sigma_{t+1} = g_2(\Sigma_t, \gamma_t, \gamma_{t+1}, c_t)$ as shown in (5). From (28) and (29), Σ_t can be expressed recursively in terms of some deterministic function $g_7\left(\cdot
ight)$ as

$$\Sigma_{t+1} = g_2 \left(\Sigma_t, \gamma_t, \gamma_{t+1}, c_t \right) = g_2 \left(\Sigma_t, \sqrt{\frac{P_t}{\sigma_{u,t}^2}}, \sqrt{\frac{\alpha_{t+1(Q_t - P_t)}}{\sigma_{u,t+1}^2}}, c_t \right) = g_7 \left(\Sigma_t, \sigma_{u,t}^2, c_t, \alpha_{t+1} \right).$$
(30)

In addition, from our definitions of P_t , Q_t and α_{t+1} , we have $Q_{t+1} = Q_t - P_t$ and $P_{t+1} = \alpha_{t+1} (Q_t - P_t)$ which we can encapsulate in a function $g_8(\cdot)$ as

$$\{Q_{t+1}, P_{t+1}\} = g_8(Q_t, P_t, \alpha_{t+1})$$
(31)

Given that $s_t = \{\Sigma_t, \sigma_{u,t}^2, P_t, Q_t\}$, and ϕ_t includes $(\alpha_{t+1}, c_t) = a_t$, we conclude $s_{t+1} = \tau_t (s_t, a_t)$ from functions $g_6(\cdot), g_7(\cdot)$ and $g_8(\cdot)$.

The new definition of the MDP is summarized in Table II. The cost function is defined in the same way as for the constant peak power constraint case given in (13). In the later part of the section, we present this modified DP algorithm as a solution to the MDP to obtain the optimal sequential linear encoder with total power constraint and then analyze the solution in noiseless and noisy feedback cases.

B. Modified DP Algorithm

The DP algorithm was presented in Section IV-C to compute the optimal sequential linear encoder for the peak power constraint. The value function optimization was done over the action vector, which consisted of the coefficient c_t of the received feedback. In the modified algorithm for the total power constraint, we optimize over both the coefficients c_t and the fractional power allocation α_t . We note that the action is defined as $a_t = (c_t, \alpha_t)$, and we let $a_t(1)$ and $a_t(2)$ refer to c_t and α_t , respectively.

Algorithm 3: Proposed Dynamic Program
Output: Optimal action $\tilde{a}_{0:T-1}$, Value function $V_{0:T}$
Initialization : At $t = T$, $\forall s_T$, $V_T(s_T) = \epsilon_{w,T}^2$
for $t = T - 1, \cdots, 0$ do
$\tilde{a}_{t}(1) = \operatorname{argmin} V_{t+1}\left(\tau_{t}\left(s_{t}, a_{t}(1), a_{t}(2)\right)\right)$
$a_t(1)$
$\tilde{a}_t(2) = \operatorname{argmin} V_{t+1}\left(\tau_t\left(s_t, \tilde{a}_t(1), a_t(2)\right)\right)$
$\begin{bmatrix} a_t(2) \\ V(z) \\ V(z$
$V_t(s_t) = V_{t+1}(\tau_t(s_t, a_t))$
end

We solve the dynamic program for the total power constraint in Algorithm 3 and obtain expressions for the value functions V_t and the optimal actions \tilde{a}_t for every $t \in [0, T]$. The value function V_T is initialized at time instant t = T and the subsequent steps were followed similar to the steps in the DP algorithm for the peak power constraint (Algorithm 2). We derive the optimal coefficient ($\tilde{a}_t(1)$) and the power allocation ($\tilde{a}_t(2)$) through alternate optimization where they are determined at two separate steps. We use MATLAB and algebraic rearrangements to obtain solutions to the optimizations as expressions depicted in the Theorem described below.

Theorem 6. Let (T+1)P be the total power available for T+1 transmissions, and σ_f^2 and σ_b^2 be the forward and feedback channel variance, respectively, for AWGN channels with feedback. The value function is expressed as

$$V_{t}(s_{t}) = \frac{-\epsilon_{uw,t}^{4} + K_{T-t}\epsilon_{w,t}^{2}\sigma_{u,t}^{2} + \epsilon_{w,t}^{2}\epsilon_{u,t}^{2}}{K_{T-t}\sigma_{u,t}^{2} + \epsilon_{u,t}^{2}},$$

which is similar to the value function that was obtained for the peak power constraint.

The optimal coefficient c_t is given as

$$c_t = -\frac{K_n \sqrt{S_i} \left(\frac{\sigma_{u,t}}{\sigma_f}\right)}{K_n \eta_0 + \beta},$$

where n = T - t is the number of remaining transmissions and $K_n = f_{n-1}(f_{n-2}(\cdots f_1(K_1)))$ with $K_1 = \frac{1+\beta+S_T\beta}{S_T(1+\beta)(1+S_T)}$, $\eta_0 = (1+S_i)(1+\beta)$, $S_i = \frac{P_i}{\sigma_f^2}$ and f_i 's are separate functions determined from the computed power constraints P_i 's. (Note that structure of the optimal coefficient is similar as in the peak power constraint (Lemma 2)).

The optimal power allocation α_{t+1} is obtained in a closed form for the noiseless case as

$$\alpha_{t+1} = \frac{1}{T-t},\tag{32}$$

and for the noisy case, the solutions of α_{t+1} are not tractable for a general t. Therefore, we rely on numerical solutions for the computation of the optimal α_{t+1} .

Proof. The proof follows from Appendix IX-C. The solutions are obtained in symbolic MATLAB by solving the optimization equations individually for the coefficients and the power allocation. \Box

C. Interpretation

Let us now interpret the results we obtained above regarding the power allocation for the total power constraint. We begin with an interpretation of noiseless feedback scenario. In the following, we show that the power allocation for the noiseless case is uniform across all transmissions.

Corollary 5. For the AWGN channel with T noiseless feedback transmissions of a Gaussian message, with a total power constraint of (T + 1) P, the individual power constraint is given as P, i.e., (T + 1) P power uniformly spread across all T + 1 transmissions.

Proof. We determine the individual power allocations for the noiseless case using the optimal α from Theorem 6 as $\alpha_t = \frac{1}{T-t+1}$.

From (32), we obtain the fractional power allocation for the first raw transmission $\alpha_0 = \frac{1}{T+1}$ by substituting t = 0. The total budgeted power at the beginning of the transmissions is given as $Q_0 = (T+1)P$. Therefore the power allocation P_0 is given as

$$P_0 = \alpha_0 Q_0$$
$$\alpha_0 = \frac{1}{T+1}$$
$$P_0 = P.$$

From (31), the remaining power for the next transmission is given as

$$Q_1 = Q_0 - P_0 = TP.$$

$$\alpha_1 = \frac{1}{T}$$

$$P_1 = P.$$

These steps can be repeated to obtain Q_i and P_i , $i = 2 \dots T$ and thereby the power allocations for all transmissions. These power allocations are found to be uniform as P.

The Corollary 5 holds as we assumed that the message w is Gaussian. We note that the uniform power allocations for the noiseless case are not optimal when non-Gaussian messages like Pulse Amplitude Modulation (PAM) symbols are considered. The power allocation with M-PAM messages for any number of feedback transmissions, T is obtained in [20]. It provides a non-uniform power allocation between the first raw transmission of the PAM message and the rest of the transmissions.

For noisy feedback, we use numerical techniques to solve for the power allocation for each of the transmission. This is obtained by solving the optimization $\tilde{a}_t(2) = \underset{a_t(2)}{\operatorname{argmin}} V_{t+1}(\tau_t(s_t, \tilde{a}_t(1), a_t(2)))$ in

Algorithm 3 for the values of α_t which is found to be different across all transmissions. It was observed that the power allocations obtained were non-uniform across the transmissions.



Fig. 9: Comparison of the received SNR for channels with feedback between our DP scheme under peak/total power constraints and CL scheme under total power constraint [2]. A forward noise variance of $\sigma_f^2 = 1$ and a feedback noise variance of $\sigma_b^2 = 0.01$ were assumed. At low SNR, we outperform the CL scheme in both the peak power as well as the total power constraint. At high SNR, the peak power constraint results are slightly inferior but the total power constraint results match the CL scheme.

D. Comparison to the state-of-the-art linear schemes by Chance and Love

The authors in [2] have provided a linear scheme to transmit messages over an AWGN channel with noisy output feedback with a constraint on the total power which to our best of knowledge is the state of the art in the class of linear codes for such channels. The details of the scheme are discussed in Section II-B. In the following, we show that the analytical scheme with closed-form solutions that we derived in this paper outperforms or match the performance of the Chance and Love scheme.

In Fig. 9, we show the comparison of the received SNR between our DP scheme with both peak power constraint and the total power constraint, and with the implementation of Chance and Love scheme under the total power constraint [2]. We assume an AWGN channel with a forward noise variance of $\sigma_f^2 = 1$ and a feedback noise variance $\sigma_b^2 = .01$. The plots were generated by varying the power constraint P over T = 3 feedback transmissions such that the peak power constraint for each transmission was P while the total power constraint was (T + 1) P.

Under the total power constraint, our scheme (labeled as DP, Total Power) outperforms the CL scheme (also with the total power assumption) in the low SNR regime and has the same performance as the CL scheme at high SNR. The received SNR of our scheme under the peak power constraint (labeled as DP, Peak Power) is also shown for comparison. We can see that the performance of our scheme under the peak power constraint does not degrade much with respect to the one under the total power constraint.

In summary, we show that the scheme that we propose can be obtained in closed-form for the peak power constraint which closely matches (slightly inferior) to the CL scheme while outperforming it in the low SNR regime. Our performance for the total power constraint clearly outperforms the CL scheme in the low SNR and matches the performance at high SNR.

VII. PAM MESSAGE TRANSMISSION

In the previous sections, we considered Gaussian messages to transmit over the AWGN channel with feedback. The principal motivating factor was the simplicity in determining the *optimal* encoding scheme within the class of sequential linear schemes with the use of the Kalman filter at the receiver as the decoder.

In this section, we consider a more practical scenario, where the transmitter has message *bits* (or an M-PAM symbol) to communicate. We empirically demonstrate that the sequential linear coding scheme that we obtained via DP is not optimal within the class of sequential linear schemes for such scenarios. The analysis and derivation of the optimal scheme are not straightforward when we consider messages which are not Gaussian in nature. Therefore, we use a learning-based approach; we optimize the weights of the linear sequential scheme using the backpropagation instead of DP. We show that the sequential linear codes optimized for Gaussian messages, albeit by a small margin.

Concretely, we consider a setup where w is the intended m-PAM message to be transmitted across an AWGN channel with noisy output feedback. The linear sequential encoding scheme can be represented using (1) as

$$u_0 = w, \tag{33a}$$

$$u_{t+1} = u_t + c_t \tilde{y}_t, \tag{33b}$$

where c_t 's are the parameters that need to obtained. The final transmitted symbol, in line with our scheme in Section III, is a scaled version of u_t as

$$x_{t+1} = \gamma_{t+1} u_{t+1}, \tag{34a}$$

where the parameters γ_{t+1} are normalized such that the total power across (T+1) transmissions is (T+1) P. In a similar manner, we represent the operation at the decoder to obtain the decoded message as a linear combination of the received symbols as

$$\hat{w}_T = \sum_{t=0}^T e_t y_t,\tag{35}$$

where e_t 's are the unknown parameters to be obtained.

The collection of parameters from the encoder and the decoder are jointly optimized through a learningbased approach using backpropagation. A batch of m-PAM messages (w's) are generated and are passed through the system of equations in (33a)-(34a) and (35) to generate the corresponding decoded messages \hat{w} 's in terms of the parameters. A gradient descent algorithm is used to optimize these parameters such that the MSE E $\left[\left(W - \hat{W} \right)^2 \right]$ is minimized. The update of the weight parameters is repeated for multiple epochs till the MSE converges to the minimal value.

In Fig. 10, we compare the performance of the sequential linear encoding scheme obtained using the proposed DP algorithm and the learning-based approach introduced in this section. We consider a set of 2–PAM messages sent over the channel with T = 2 noisy feedback transmissions. In the learning-based approach, we trained a batch of 2000 randomly generated 2–PAM messages and optimized using an Adam optimizer with a learning rate of 1e - 3. The performance for the scheme obtained using the DP approach was evaluated by generating a set of random 2–PAM messages and then using the closed-form coefficients derived in Theorem 3 to obtain the final MSE. We compute the received SNR as the reciprocal of the final MSE at the end of three transmissions. The plot shows the received SNR at the decoder for different total power constraints with the forward and the backward variances as $\sigma_f^2 = 1$ and $\sigma_b^2 = 1$. The results show a performance improvement obtained in the received SNR at high SNR under a total power constraint.

VIII. CONCLUSION

We provided a novel approach using dynamic programming to design optimal sequential linear schemes for communicating a Gaussian message over AWGN channels with noisy (and noiseless) output feedback. The sequential linear scheme we introduced in this paper is a class of linear coding schemes for which the encoder maintains a state, which is updated based on the output feedback, and generates the transmission



Fig. 10: The plots compare the received SNR with 2–PAM symbols as messages against varying power constraint P in dB for an AWGN channel with feedback with forward and noise variance, $\sigma_f^2 = 1$ and $\sigma_b^2 = 1$ respectively. The improvement in the received SNR is evident when a learning based approach is used for PAM messages

symbol based on the state. We showed that existing linear schemes for AWGN channels with output feedback, namely, the SK scheme and the CL scheme, all belong to the class of sequential linear schemes.

We then derived a closed-form expression for the optimal sequential linear scheme by formulating a novel MDP and solving it using DP. We showed that our derived optimal sequential linear scheme outperforms the state-of-the-art CL scheme [2] under some channel conditions for channels with *noisy* output feedback while matching the SK scheme for channels with *noiseless* output feedback.

We provided several interpretation results for the optimal sequential linear scheme and its estimation error. We showed that the derived scheme is *not* a straightforward generalization of the SK scheme. We represented the estimation error as a function of a number of transmissions for channels with various levels of noisy feedback.

We also considered communicating message *bits* instead of a Gaussian message, for which we cannot analytically find the optimal sequential linear scheme. We presented a *learning-based* approach to optimize the coefficients of the sequential linear scheme directly for message *bits* and showed that by doing so, we can outperform the scheme with coefficients analytically optimized for a Gaussian message.

Finally, extending our approach to multi-user scenarios and using a learning-based method to directly optimize the bit error rate instead of the mean squared error would be interesting. We leave them as future work.

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IX. APPENDIX

A. Proof of Remark 2

Proof. We show that the SK scheme belongs to the family of sequential linear schemes.

The sequence of receiver's MMSE estimates $\{\hat{w}_0, \hat{w}_1, \hat{w}_2, \dots\}$ satisfies the following recursive equation

$$\hat{w}_t = \hat{w}_{t-1} + \mathsf{E}[U_t|y_t],$$

where $u_t = w - \hat{w}_{t-1}$ denotes the error in the most recent estimate of w and the estimation error u_t and y_t is the received signal observed as feedback at the encoder. This is derived from the fact that the error in estimate u_t is orthogonal to all the observations till time t, $y^{0:t-1}$. Therefore, the encoding process is sequential, $u_{t+1} = u_t - \mathsf{E}[U_t|y_t]$. The transmitted symbol x_{t+1} is a scaled version of u_t .

$$u_{t+1} = w - \hat{w}_t$$

= $(w - \hat{w}_{t-1}) - (\hat{w}_t - \hat{w}_{t-1})$
= $u_t - \mathsf{E} [U_t | y_t],$

The transmitter sends $x_t = \gamma_t u_t$, where γ_t denotes the power normalization constant. Under the average peak power assumption, $\gamma_t = \sqrt{P/E[U_t^2]}$. The estimation error can be also represented in a recursive equation; after t transmissions, the error in the estimate of w reduces is $E[(W - \hat{W})^2] = \sigma_w^2/(1 + P)^T$, which decays exponentially in P (See [20] for a detailed derivation). This scheme is analytically shown to be *optimal* for communication of Gaussian messages [19].

B. Proof of Remark 3

Proof. The authors in [2] propose a scheme for channels with noisy feedback as

$$x_{t+1} = \mathbf{F}_t \tilde{\mathbf{z}} + \mathbf{g} w, \tag{36}$$

where x_{t+1} is the transmitted symbol, $\tilde{\mathbf{z}}_t$ is the noise vector, $\mathbf{n}_t + \tilde{\mathbf{n}}_t$, and w is the intended message. The scheme is parameterized through the encoding matrix \mathbf{F} which is a lower triangular Toeplitz matrix given as

$$\begin{bmatrix} 0 & \cdots & 0 \\ -\frac{1-\beta_0^2}{(1-\sigma_b^2)\beta_0} & 0 & & \\ -\frac{1-\beta_0^2}{1+\sigma_b^2} & \ddots & \ddots & \\ \vdots & & \\ -\frac{1-\beta_0^2}{1+\sigma_b^2}\beta_0^{T-3} & \cdots & -\frac{1-\beta_0^2}{1+\sigma_b^2} & -\frac{1-\beta_0^2}{(1-\sigma_b^2)\beta_0} & 0 \end{bmatrix}$$

while g is given as

$$= \sqrt{\frac{1 - \beta_0^2}{1 - \beta_0^{2(T+1)}} \begin{bmatrix} 1 & \beta_0 & \beta_0^2 & \cdots & \beta_0^T \end{bmatrix}^T}$$

The value of β_0 is obtained as a solution to (31) of the paper. Here, we show that the scheme is sequential and linear in terms of the feedback and the past symbols.

From (36), we can represent the transmitted symbol at any instant t as

$$\begin{split} x_{t+1} &= \sqrt{\frac{1-\beta_0^2}{1-\beta_0^{2(T+1)}}} \beta_0^{t+1} w + \left[-\frac{1-\beta_0^2}{1+\sigma_b^2} \beta_0^t - \frac{1-\beta_0^2}{1+\sigma_b^2} \beta_0^{t-1} \cdots - \frac{1-\beta_0^2}{(1-\sigma_b^2)\beta_0} \right] \tilde{\mathbf{z}}^t \\ &= \sqrt{\frac{1-\beta_0^2}{1-\beta_0^{2(T+1)}}} \beta_0^{t+1} w + \left[-\frac{1-\beta_0^2}{1+\sigma_b^2} \beta_0^t - \frac{1-\beta_0^2}{1+\sigma_b^2} \beta_0^{t-1} \cdots - \frac{1-\beta_0^2}{(1-\sigma_b^2)} \right] \tilde{\mathbf{z}}^{t-1} + -\frac{1-\beta_0^2}{(1-\sigma_b^2)\beta_0} \tilde{z}_t \\ &= \beta_0 \left(\sqrt{\frac{1-\beta_0^2}{1-\beta_0^{2(T+1)}}} \beta_0^t w + \left[-\frac{1-\beta_0^2}{1+\sigma_b^2} \beta_0^{t-1} - \frac{1-\beta_0^2}{1+\sigma_b^2} \beta_0^{t-2} \cdots - \frac{1-\beta_0^2}{(1-\sigma_b^2)} \right] \tilde{\mathbf{z}}^{t-1} \right) + -\frac{1-\beta_0^2}{(1-\sigma_b^2)\beta_0} \tilde{z}_t \\ &= \beta_0 x_t + -\frac{1-\beta_0^2}{(1-\sigma_b^2)\beta_0} (\tilde{y}_t - x_t) \\ &= \left(\beta_0 + -\frac{1-\beta_0^2}{(1-\sigma_b^2)\beta_0} \right) x_t + -\frac{1-\beta_0^2}{(1-\sigma_b^2)\beta_0} \tilde{y}_t \end{split}$$

We represent the current transmitted symbol x_{t+1} in terms of the past symbol x_t and the received feedback \tilde{y}_t . The analysis establishes the sequential nature of the scheme.

C. Proof of Lemma 2

Proof. At time t = T - 1, from the optimization step in Algorithm 2, $\tilde{a}_t = \underset{a_t}{\operatorname{argmin}} V_{t+1}(\tau_t(s_t, a_t))$, we find \tilde{a}_{T-1} through $\frac{\partial V_T}{\partial a_{T-1}} = 0$ which leads to

$$\tilde{a}_{T-1} = -\frac{\sqrt{S}\left(\frac{\sigma_{u,T-1}}{\sigma_f}\right)}{\eta_0},$$

where all the symbols have been defined in Lemma 2. Therefore, the optimal transmission scheme is given as,

$$\tilde{\phi}_{T-1}\left(u_{T-1}, \tilde{y}_{T-1}\right) = \sqrt{\frac{P}{\sigma_{u,T}^2}} \left(u_{T-1} - \frac{\sqrt{S}\left(\frac{\sigma_{u,T-1}}{\sigma_f}\right)}{\eta_0} \tilde{y}_{T-1}\right).$$

The value function that captures the MSE if we use the optimal policy from t = T - 1 till T which is given as

$$V_{T-1}(s_{T-1}) = \frac{-(\epsilon_{uw,T-1})^4 + K_1(\epsilon_{w,T-1}\sigma_{u,T-1})^2 + (\epsilon_{w,T-1}\epsilon_{u,T-1})^2}{K_1\sigma_{u,T-1}^2 + (\epsilon_{u,T-1})^2}$$

where

$$K_{1} = \frac{1 + \beta + S\beta}{(1 + \beta) S (1 + S)} = \frac{\eta_{1}}{\eta_{3}}$$

We repeat the same procedure for t = T - 2, t = T - 3 and for any general t and obtain corresponding $\tilde{\phi}$ and V as

$$\begin{split} \tilde{\phi}_{T-2} &= \frac{\sqrt{P}}{\sigma_{u,T-1}} \left(u_{T-2} - \frac{K_2 \sqrt{S} \left(\frac{\sigma_{u,T-2}}{\sigma_f}\right)}{K_2 \eta_0 + \beta} \tilde{y}_{T-2} \right) \\ V_{T-2} &= \frac{-\left(\epsilon_{uw,T-2}\right)^4 + K_2 \left(\epsilon_{w,T-2} \sigma_{u,T-2}\right)^2 + \left(\epsilon_{w,T-2} \epsilon_{u,T-2}\right)^2}{K_2 \sigma_{u,T-2}^2 + \left(\epsilon_{u,T-2}\right)^2}, \\ \tilde{\phi}_{T-3} &= \frac{\sqrt{P}}{\sigma_{u,T-2}} \left(u_{T-3} - \frac{K_3 \sqrt{S} \left(\frac{\sigma_{u,T-3}}{\sigma_f}\right)}{K_3 \eta_0 + \beta} \tilde{y}_{T-3} \right) \\ V_{T-3} &= \frac{-\left(\epsilon_{uw,T-3}\right)^4 + K_3 \left(\epsilon_{w,T-3} \sigma_{u,T-3}\right)^2 + \left(\epsilon_{w,T-3} \epsilon_{u,T-3}\right)^2}{K_3 \sigma_{u,T-3}^2 + \left(\epsilon_{u,T-3}\right)^2}, \\ \tilde{\phi}_t &= \frac{\sqrt{P}}{\sigma_{u,t+1}} \left(u_t - \frac{K_{T-t} \sqrt{S} \left(\frac{\sigma_{u,t}}{\sigma_f}\right)}{K_{T-t} \eta_0 + \beta} \tilde{y}_t \right) \\ V_t &= \frac{-\left(\epsilon_{uw,t}\right)^4 + K_{T-t} \left(\epsilon_{w,t} \sigma_{u,t}\right)^2 + \left(\epsilon_{w,t} \epsilon_{u,t}\right)^2}{K_{T-t} \sigma_{u,t}^2 + \left(\epsilon_{u,t}\right)^2}, \end{split}$$

where

$$\begin{split} K_2 &= \frac{\eta_1 K_1^2 + \eta_2 K_1}{\eta_3 K_1^2 + \eta_4 K_1 + \eta_2} = f\left(K_1\right).\\ K_3 &= \frac{\eta_1 K_2^2 + \eta_2 K_2}{\eta_3 K_2^2 + \eta_4 K_2 + \eta_2} = f\left(K_2\right).\\ K_t &= \frac{\eta_1 K_{t-1}^2 + \eta_2 K_{t-1}}{\eta_3 K_{t-1}^2 + \eta_4 K_{t-1} + \eta_2} = f\left(K_{t-1}\right) = f^{t-1}\left(K_1\right). \end{split}$$

Significance of K_n: We observe that the structure of the value expression V_t with respect to the state s_t remain similar barring the value of K which is a function of system parameters P, σ_f^2 and σ_b^2 and T. Therefore, the value function V_t and the optimal policy at any instant t can be obtained by computing K_1 and then applying the function $f(\cdot)$, T - t - 1 times to obtain K_{T-t} . The expressions of K_1 , K_n , and $f(\cdot)$ can be obtained from Lemma 2. This formulation helps us determine the value function (=MSE) and the optimal policy for any number of iterations and any value of feedback variance without worrying about the increase in complexity.

D. Proof to Corollary 3

Proof. We provided a solution to the dynamic program in Section IV-C and obtained the value function expression in Lemma 2. In Theorem 4, we obtained the exact closed-form expressions for the MSE. The parameter K_n in the expression is defined through the system parameters, the power constraint P, the forward noise variance σ_f^2 , and the feedback noise variance σ_b^2 . In order to study the variation of MSE with respect to T, we study the progression of K_1 through K_T .

We begin by studying the series K_n , $n = 1 \dots t$ generated from the recursion function f. We establish that K_n is a geometric series for the noiseless case where $\sigma_b = 0$ and for large n, where

$$K_n = \frac{\eta_1 K_{n-1}^2 + \eta_2 K_{n-1}}{\eta_3 K_{n-1}^2 + \eta_4 K_{n-1} + \eta_2}.$$
(37)

as defined in (17). We get

$$K_n = \frac{1}{1+S} K_{n-1}$$

or $\frac{1}{K_T} = (1+S)^T \frac{1}{K_1}$
 $= (1+S)^T \frac{\eta_3}{\eta_4}.$

By substituting in (24), we get,

$$MSE = \frac{\sigma_w^2}{(1+S)^T}.$$

E. Proof to Corollary 4

Proof. We extend the results of the noiseless case to study the variation of MSE with transmissions T with noisy feedback case. We establish that K_n is a harmonic series for the noisy feedback case.

$$K_n = \frac{\eta_1 K_{n-1}^2 + \eta_2 K_{n-1}}{\eta_3 K_{n-1}^2 + \eta_4 K_{n-1} + \eta_2}.$$
(38)

as defined in (18). Now, we show that the common difference between the reciprocal terms of K_n is a constant. From (38), we have,

$$\frac{1}{K_n} - \frac{1}{K_{n-1}} = \frac{\eta_3 K_{n-1} + \eta_4 - \eta_1}{\eta_1 K_{n-1} + \eta_2}.$$

We observe that K_n is a monotonically decreasing function. Assuming that K_n becomes much smaller with increasing n we get,

$$\frac{1}{K_n} - \frac{1}{K_{n-1}} = \frac{\eta_4 - \eta_1}{\eta_2}$$

i.e. $\frac{1}{K_T} = \frac{1}{K_1} + \frac{\eta_4 - \eta_1}{\eta_2}T.$ (39)

This approximation holds for any value of T. In fact, with a high value of T, the difference between the exact and the approximate expression goes down to zero.

The exact MSE expression is given in (24) which can be approximated by using the harmonic progression series for K_n from (39). Thus we can obtain a very close approximation for the error variance after T transmissions without the need to solve the DP or use the function f recursively, given by

$$\mathsf{MSE} = \frac{\sigma_w^2}{\zeta_T},$$

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where

$$\zeta_{T} = \frac{\left(\left(\eta_{1}\eta_{4} - \eta_{1}^{2}\right)T + \eta_{2}\eta_{3} + \eta_{1}\eta_{4}\right)\left(\left(\eta_{1}\eta_{4} - \eta_{1}^{2}\right)T + \eta_{2}\eta_{3} + \eta_{1}\eta_{2}\left(S + 1\right)\right)}{\eta_{1}\eta_{2}\left(\left(\eta_{1}\eta_{4} - \eta_{1}^{2}\right)T + \eta_{2}\eta_{3} + \eta_{1}^{2}\right)}\\ = \frac{\left(\frac{\eta_{4} - \eta_{1}}{\eta_{2}}T + \frac{\eta_{2}\eta_{3} + \eta_{1}\eta_{4}}{\eta_{1}\eta_{2}}\right)\left(\frac{\eta_{4} - \eta_{1}}{\eta_{2}}T + \frac{\eta_{2}\eta_{3}}{\eta_{1}\eta_{2}} + S + 1\right)}{\frac{\eta_{4} - \eta_{1}}{\eta_{2}}T + \frac{\eta_{2}\eta_{3} + \eta_{1}^{2}}{\eta_{1}\eta_{2}}}.$$

Upon further simplifications, the approximate value for MMSE for high value of T can be expressed as

$$\text{MSE} \approx \frac{\sigma_w^2}{\zeta^{\star}}, \text{ where }$$

$$\zeta^{\star} = \left(\frac{\eta_4 - \eta_1}{\eta_2}\right) T$$
$$= \left(\frac{S + S\beta}{\beta}\right) T = S\left(1 + \frac{1}{\beta}\right) T.$$

The bounds obtained match the results in [21] and in [2].

It is worth noting that the resulting MMSE expression is described only through the system parameters. The definition for the η parameters is provided in Theorem 4. This expression helps us (a) establish the bounds for our expression as was provided by other authors in their work like Weissman et al. in [21] and Chance and Love in [2], (b) visualize the progression of the MMSE with the increase in the parameter T which was not apparent from the exact expression in (24).