# Linear Programming Bounds for Almost-Balanced Binary Codes 

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#### Abstract

We revisit the linear programming bounds for the size vs. distance trade-off for binary codes, focusing on the bounds for the almost-balanced case, when all pairwise distances are between $d$ and $n-d$, where $d$ is the code distance and $n$ is the block length. We give an optimal solution to Delsarte's LP for the almost-balanced case with large distance $d \geq(n-\sqrt{n}) / 2+1$, which shows that the optimal value of the LP coincides with the Grey-Rankin bound for self-complementary codes.

We also show that a limitation of the asymptotic LP bound shown by Samorodnitsky, namely that it is at least the average of the first MRRW upper bound and Gilbert-Varshamov bound, continues to hold for the almost-balanced case.


## I. Introduction

This paper concerns the standard size-versus-distance tradeoff for binary error-correcting codes. A binary code $C$ with block length $n$ and distance $d$ is a subset of the $n$-dimensional Hamming cube $\{0,1\}^{n}$, such that the Hamming distance between any two elements from $C$ is at least $d$. In this paper such codes will be referred to as min-distance codes, and we denote $A(n, d)$ to be the maximal size of such a code. Understanding the behavior of $A(n, d)$ is one of the most fundamental challenges in combinatorial coding theory. For the case when $d$ is linear in $n$, one is interested in asymptotic rates of the code rather than their sizes:

$$
d
$$

where $\delta \approx d / n \leq 1 / 2$ is the relative distance of the code.
The best known asymptotic bounds as well as finitelength bounds on $A(n, d)$ for many parameters are given by Delsarte's linear programming approach [1]. Specifically, McEliece, Rodemich, Rumsey, and Welch [2] obtain the best known upper bounds on $R(\delta)$ for the entire region $\delta \in(0,1 / 2)$ by finding a good feasible solution to Delsarte's LP. There is, however, a sizeable gap between the MRRW upper bound on $R(\delta)$ and the Gilbert-Varshamov lower bound $R(\delta) \geq 1-h(\delta)$ where $h(\cdot)$ is the binary entropy function, and shrinking this gap has remained a major challenge for almost 45 years.

In this paper, we are interested in the performance of the linear programming approach for bounding the size of almost-balanced codes, which obey the stronger condition that the Hamming distance between any two distinct codewords lies between $d$ and $(n-d)$. Almost-balanced codes when $d=(1-\epsilon) n / 2$ are closely related to $\epsilon$-biased spaces that are
of fundamental interest in pseudorandomness and derandomization, starting with the seminal work of Naor and Naor [3]. The recent breakthrough explicit construction of high distance codes approaching the Gilbert-Varshamov bound by Ta-Shma crucially proceeds by constructing almost-balanced codes [4].

Almost-balanced codes are closely related to selfcomplementary codes. The code $C^{\prime}$ is self-complementary if $u \in C^{\prime}$ implies that the complementary vector $\bar{u}$ is also in $C^{\prime}$. The connection between almost-balanced and selfcomplementary codes is two-way:

- any almost-balanced code $C$ with distance $d$ (i.e. all pairwise distances lie in the interval $[d, n-d]$ ) corresponds to a self-complementary code of size $2|C|$. Indeed, the code $C^{\prime}=\{u \mid u$ or $\bar{u}$ is in $C\}$ is self-complementary and has size $2|C|$.
- any self-complementary code $C^{\prime}$ with distance $d$ corresponds to a family of almost-balanced codes of size $\left|C^{\prime}\right| / 2$ and distance $d$. Such codes can be constructed by taking one codeword from every complementary pair $u, \bar{u} \in C^{\prime}$.
Thus to translate the bounds for almost-balanced codes to self-complementary codes, one just needs to multiply the size-versus-distance trade-off (for the same distance $d$ ) by 2 .

Let us denote by $B(n, d)$ the maximal size of an almostbalanced code with block length $n$ and distance $d$. Once again, when $d$ is linear in $n$, we can study the asymptotic rate of the codes rather than their sizes:

$$
R^{\mathrm{bal}}(\delta)=\limsup _{n \rightarrow \infty} \frac{\log B(n,\lfloor\delta n\rfloor)}{n}
$$

The driving force behind this work is to study upper bounds obtained for almost-balanced codes via the linear programming method, and compare them to the ones for the min-distance codes. Our results consists of two parts:

1) For almost-balanced codes with large distances $d \geq \frac{n-\sqrt{n}}{2}+1$ we find an optimal solution to Delsarte's linear program. This solution gives an upper bound on $B(n, d)$ equivalent to the Gray-Rankin bound for self-complementary codes [5]-[8]. Since the solution we obtain is optimal, this proves a matching lower bound on Delsarte's LP problem, showing that LP approach cannot prove better bounds for the almost-balanced case. While this was already implied for a very limited range of
pairs $(n, d)$ for which Grey-Rankin bound is achievable ([9]), our bound works for arbitrary even $n, d$ satisfying $d \geq \frac{n-\sqrt{n}}{2}+1$.
2) We show (Theorem 2) that the asymptotic LP bound for almost-balanced codes is at least the average of the first MRRW bound and Gilbert-Varshamov lower bound. This is the analog of Samorodnitsky's result [10] for min-distance codes, and indicates that the (direct) linear programming approach cannot attain the best known lower bound on $R^{\mathrm{bal}}(\delta)$. The proof is a simple adaptation of Samorodnitsky's argument.

## II. Krawtchouk polynomials and Delsarte's LP

For fixed $n$, the Krawtchouk polynomials are defined as

$$
K_{s}(x)=\sum_{j=0}^{s}(-1)^{j}\binom{x}{j}\binom{n-x}{s-j}, \quad s=0,1, \ldots, n .
$$

$K_{s}(x)$ is a degree-s polynomial, and $K_{0}, K_{1}, \ldots, K_{n}$ form an orthogonal family with respect to a measure $\mu_{i}=\binom{n}{i} / 2^{n}$ :

$$
\sum_{j=0}^{n} \frac{\binom{n}{j}}{2^{n}} K_{s}(j) K_{t}(j)=\delta_{s t}\binom{n}{s}
$$

Notice that $K_{0}(x) \equiv 1$ and $K_{i}(0)=\binom{n}{i}$. Below are some of the properties of Krawtchouk polynomials we will need.

Fact 1: $K_{s}(n / 2-x)$ is an even function for even $s$, and is an odd function for odd $s$.

Fact 2: $K_{n-s}(i)=K_{s}(i)$ for even $i, \quad$ and $K_{n-s}(i)=-K_{s}(i)$ for odd $i$.

Fact 3: Reciprocity property: $\binom{n}{s} K_{i}(s)=\binom{n}{i} K_{s}(i)$.
The Delsarte's upper bound on $A(n, d)$ can be formulated as the following Linear Programming optimization problem:

$$
\begin{align*}
& \max _{a_{i} \geq 0} \sum_{k=0}^{n} a_{k} \\
& \text { s.t. } \sum_{k=0}^{n} a_{k} K_{s}(k) \geq 0, \quad s=0,1, \ldots, n,  \tag{P}\\
& a_{0}=1, \\
& a_{k}=0 \quad 1 \leq k \leq(d-1)
\end{align*}
$$

Denote $\boldsymbol{\beta}(x)=\sum_{k=0}^{n} \beta_{k} K_{k}(x)$. Then the the dual of the above LP can be written as

$$
\begin{array}{cr}
\min _{\beta_{i} \geq 0} & \boldsymbol{\beta}(0) \\
\text { s.t. } & \boldsymbol{\beta}(u) \leq 0, \quad u=d, d+1, \ldots, n  \tag{D}\\
& \beta_{0}=1
\end{array}
$$

Denoting by $A_{L P}(n, d)$ the optimum values of the above two problems, the LP bound claims that $A(n, d) \leq A_{L P}(n, d)$.

The optimal values of the LPs and asymptotic upper bound $R_{L P}(\delta)=\limsup _{n \rightarrow \infty} \frac{1}{n} \log A_{L P}(n,\lfloor\delta n\rfloor)$ are not known yet, however upper bounds on $A_{L P}(n, d)$ (and $R_{L P}(\delta)$ ) can be obtained by finding feasible solutions to the dual LP (D). In particular, the (first) MRRW upper bound $R_{M R R W}(\delta)=$
$H\left(\frac{1}{2}-\sqrt{\delta(1-\delta)}\right)$ is obtained in such a way in [2] (in this paper we do not address the second MRRW bound).

To formulate the Delsarte's LP for the almost-balanced case, we point out that the variables $a_{i}, i=0,1, \ldots, n$ in the primal form of LP $(\mathcal{P})$ correspond to the distance distribution of the code $C$, i.e. $a_{i}=\frac{1}{|C|}|\{(x, y) \in C \mid \Delta(x, y)=i\}|$. The first set of constraints in the LP $(\mathcal{P})$ then correspond to DelsarteMacWilliam's inequalities for the transformed (dual) distance distribution, while the constraints $a_{k}=0$ for $1 \leq k<d$ denote that there are no codewords at distance below $d$.

For an almost-balanced code $C$ there is no two codewords in $C$ at distance below $d$ or above $(n-d)$, i.e. $a_{k}=0$ for $k<d$ or $k>(n-d)$. Therefore, the Delsarte's linear program for almost-balanced binary codes with distance $d$ is

$$
\begin{align*}
& \max _{a_{i} \geq 0} \sum_{k=0}^{n} a_{k} \\
& \text { s.t. } \sum_{k=0}^{n} a_{k} K_{s}(k) \geq 0, \quad s=0,1, \ldots, n,  \tag{BP}\\
& a_{0}=1, \\
& a_{k}=0 \quad 1 \leq k \leq d-1 \text { and } k>(n-d)
\end{align*}
$$

The dual to the above is the following linear program:

$$
\begin{array}{cr}
\min _{i \geq 0} & \boldsymbol{\beta}(0) \\
\text { s.t. } & \boldsymbol{\beta}(u) \leq 0, \quad u=d, d+1, \ldots, n-d  \tag{BD}\\
& \beta_{0}=1
\end{array}
$$

(the difference from the LP $(\mathcal{D})$ is that $\boldsymbol{\beta}(u)$ for $u>(n-d)$ are no longer required to be non-positive.)

We denote by $B_{L P}(n, d)$ the optimal value of the above pair of LPs, and thus $B(n, d) \leq B_{L P}(n, d)$. In analogy with $R_{L P}$, define by $R_{L P}^{\mathrm{bal}}(\delta)=\limsup _{n \rightarrow \infty} \frac{1}{n} \log B_{L P}(n,\lfloor\delta n\rfloor)$ the LP upper bound on the maximal asymptotic rate for almostbalanced codes of relative distance $\delta: R^{\text {bal }}(\delta) \leq R_{L P}^{\text {bal }}(\delta)$.

## III. LP BOUND FOR ALMOST-BALANCED CODES WITH LARGE DISTANCES

In this section we find the exact optimal solutions to Delsarte's linear programs $(\sqrt{\mathcal{B} \mathcal{D}}$ ) and $(\sqrt{\mathcal{P}})$ for almost-balanced codes with large distance. Specifically, we prove

Theorem 1: Let $n, d$ be even such that $d \geq \frac{n-\sqrt{n}}{2}+1$. Then the optimal value of the linear programs $\widehat{\mathcal{B D}}$ and $\widehat{\mathcal{B P}}$ is $B_{L P}(n, d)=\frac{4 d(n-d)}{n-(n-2 d)^{2}}$.

The upper bound on the size of almost-balanced linear codes $B(n, d) \leq B_{L P}(n, d)$ for this regime is equivalent to the Grey-Rankin bound [5]-[7] on the size of self-complementary codes. This bound was also obtained by Delsarte by providing a feasible solution to $\mathcal{B D}$, see, for instance, Problem (18) in Chapter 17 (p. 544) in [8], or [11] for a more general case. What we show below is that this solution is in fact optimal for $\overline{\mathcal{D} \mathcal{D}}$ and therefore for $\overline{\mathcal{B} \mathcal{D}}$ via duality.

This implies that the bound in Theorem 1 is the best upper bound on $B(n, d)$ one can obtain using (a straightforward
application of) Delsarte's LP, so the Grey-Rankin bound cannot be improved using this approach. It was already shown in the literature that the Grey-Rankin bound can sometimes be achieved [9], however the range of pairs $(n, d)$ for which this happens is limited, and is related to existence of designs with certain parameters. Theorem 11 on the other hand, shows that the LP approach cannot improve the Grey-Rankin bound for any even $n, d$. Besides that, we think that finding optimal (instead of feasible) solutions to Delsarte's LP is of independent interest, even for this narrow regime and the special case of almost-balanced codes.

To prove Theorem 1 we start with the following claim
Lemma 1: Let $\theta=\left(\theta_{0}, \theta_{1}, \ldots, \theta_{\underline{n}}\right) \in \mathbb{R}^{n+1}$ be a feasible solution to the LP $(\overline{B D})$. Define $\bar{\theta} \in \mathbb{R}^{n+1}$ which coincides with $\theta$ on all even coordinates, and has 0 elsewhere: $\bar{\theta}=\left(\theta_{0}, 0, \theta_{2}, 0, \theta_{4}, \ldots\right)$. Then $\bar{\theta}$ is a feasible solution to the LP ( $\overline{\mathcal{B D}}$ ) for almost-balanced codes.

Proof: Denote $\boldsymbol{\theta}(x)=\sum_{k=0}^{n} \theta_{k} K_{k}(x)$, then $\boldsymbol{\theta}(u) \leq 0$ for any integer $u$ s.t. $d \leq u \leq(n-d)$. Consider then

$$
\overline{\boldsymbol{\theta}}(x)=\frac{\boldsymbol{\theta}(x)+\boldsymbol{\theta}(n-x)}{2}=\sum_{k=0}^{n} \theta_{k} \frac{K_{k}(x)+K_{k}(n-x)}{2}
$$

By Fact 1 K $K_{k}(n / 2-y)$ is an even polynomial for even $k$, and so $\frac{K_{k}(x)+K_{k}(n-x)}{2}=K_{k}(x)$. On the other hand, $\frac{K_{k}(x)+K_{k}(n-x)}{2}=0$ for odd $k$, and therefore $\overline{\boldsymbol{\theta}}(x)=\sum_{k=0}^{n} \bar{\theta}_{k} K_{k}(x)$ by our definition of $\bar{\theta}$.

For any integer $d \leq u \leq n-d$, we also have $d \leq(n-u) \leq$ $(n-d)$. It means that $\boldsymbol{\theta}(u) \leq 0$ and $\boldsymbol{\theta}(n-u) \leq 0$, so $\overline{\boldsymbol{\theta}}(u) \leq 0$ for any such $u$. Therefore $\overline{\bar{\theta}}$ if feasible for $(\overline{\mathcal{B D}})$.

Corollary 1: Optimal solution to $\overline{\mathcal{B D}}$ has $\beta_{i}=0$ for odd $i$.
Proof: $\theta_{i} \geq 0$ and $K_{k}(0)=\binom{n}{k} \geq 0$ in Lemma 11 thus $\overline{\boldsymbol{\theta}}(0)=\sum_{k=0}^{n} \bar{\theta}_{k} K_{k}(0) \leq \sum_{k=0}^{n} \theta_{k} K_{k}(0)=\boldsymbol{\theta}(0)$. So nullifying all the odd-indexed coordinates doesn't increase the objective value in $\overline{\mathcal{B} \mathcal{D}}$

Now, for every even distance $d$ such that $\frac{n-\sqrt{n}}{2}+1 \leq$ $d \leq n / 2$ we find an optimal solution to $\overline{\mathcal{B D}}$ and its dual $(\mathcal{B P})$. The condition on $d$ comes from a restriction $\left|K_{3}(d)\right| \leq\left|K_{2}(d)\right|(n-2 d) / 3$, which we use in the proof and prove in Lemma 4

Observe that $K_{2}(x)=2 x^{2}-2 n x+\binom{n}{2}$ has roots $\frac{n \pm \sqrt{n}}{2}$, and so $K_{2}(d)<0$ for distances $d$ of our interest. We obtain the optimal solution to $(\mathcal{B D}$ with a degree-2 function $\boldsymbol{\beta}(x)$ (i.e. $\beta_{i}=0$ for $i>2$ ) such that $\beta_{1}=0$ (due to Corollary (1). Clearly then, the problem reduces to minimizing $\beta_{2}$ with the condition that $\boldsymbol{\beta}(u) \leq 0$ for $u$ between $d$ and $(n-d)$. Since the function $\boldsymbol{\beta}(x)=1+\beta_{2} K_{2}(x)$ is quadratic symmetric around $n / 2$, it is clear that this is equivalent to the condition $\boldsymbol{\beta}(d)=1+\beta_{2} K_{2}(d) \leq 0$. Thus the best degree- 2 solution to $\sqrt{\mathcal{B D}}$ is exactly $\beta_{0}=1, \beta_{2}=-\frac{1}{K_{2}(d)}$, and $\beta_{i}=0$ for all other $i$. We now prove that this is actually the overall optimal solution to $(\overline{\mathcal{B D}}$ (for even $d$ and $n$ ).

Lemma 2: $\beta_{0}=1, \beta_{2}=-\frac{1}{K_{2}(d)}$, and $\beta_{i}=0$ for all other $i$ is an optimal solution for $(\mathcal{B D})$ when $\frac{n-\sqrt{n}}{2}+1 \leq d \leq n / 2$ and $n, d$ are even.

Proof: We use duality of linear programming to prove optimally. Namely, we use the fact that if $\beta$ is feasible for the LP $(\sqrt{\mathcal{D} \mathcal{D}})$, some $\alpha$ is feasible for its dual LP $(\mathcal{B P})$, and complementary slackness conditions are satisfied, then $\beta$ and $\alpha$ are optimal for the LP and its dual.

Complementary slackness conditions for our case are:

$$
\begin{array}{rlr}
\boldsymbol{\beta}(u) \cdot \alpha_{u} & =0 & u=d, d+1, \ldots,(n-d) \\
\beta_{s} \cdot\left(\sum_{k=0}^{n} \alpha_{k} K_{s}(k)\right) & =0 & s=1,2, \ldots, n
\end{array}
$$

Since $\boldsymbol{\beta}(x)=1-K_{2}(x) / K_{2}(d)$ only has roots at $d$ and $(n-d)$, we immediately see that $\alpha_{u}=0$ for all $u$ other than $d$ and $(n-d)$. Further, since $\beta_{2} \neq 0$, we have that $\binom{n}{2}+\alpha_{d} \cdot K_{2}(d)+$ $\alpha_{n-d} \cdot K_{2}(n-d)=0$. We claim that taking

$$
\alpha_{d}=\alpha_{n-d}=-\frac{\binom{n}{2}}{2 K_{2}(d)}
$$

and $\alpha_{i}=0$ for every other coordinate gives a feasible solution $\alpha$ to the LP $(\widehat{\mathcal{B} \mathcal{P}})$. Notice that all the complementary slackness conditions are satisfied for such $\beta$ and $\alpha$.

We now prove that Delsarte-MacWilliam's inequalities in $(\overline{\mathcal{B P}})$ are satisfied. Specifically, we need to show

$$
\binom{n}{s}+\alpha_{d} K_{s}(d)+\alpha_{n-d} K_{s}(n-d) \geq 0, \quad s=0,1, \ldots, n
$$

The case $s=0$ is straightforward as $\alpha \geq 0$, and for $s=2$ the equality holds by the choice of $\alpha$. Next, for odd $s, K_{s}(d)=$ $-K_{s}(n-d)$ by Fact 1 and therefore $\binom{n}{s}+\alpha_{d} \cdot K_{s}(d)+$ $\alpha_{n-d} \cdot K_{s}(n-d)=\binom{n}{s} \geq 0$. Since $n$ is even, this applies to $s=(n-1)$. Moreover, for $s=n, K_{n}(d)=K_{0}(d)$ and $K_{n}(n-d)=K_{0}(n-d)$ (using Fact 2 and since $n, d$ are even), so the inequality also holds.

For every even $s$ we have $K_{s}(d)=K_{s}(n-d)$. So it remains to prove for every even $2<s<n-1$ that

$$
\begin{equation*}
\binom{n}{s} \geq-2 \cdot \alpha_{d} \cdot K_{s}(d)=\frac{\binom{n}{2}}{K_{2}(d)} \cdot K_{s}(d) \tag{1}
\end{equation*}
$$

Denote for convenience $C=\left|K_{2}(d) /\binom{n}{2}\right|$. We prove the following statement, from which (1) clearly follows

$$
\begin{equation*}
\left|K_{s}(d)\right| \leq C \cdot\binom{n}{s}, \quad s=2,3, \ldots,(n-2) \tag{2}
\end{equation*}
$$

Notice that it is sufficient to show (2) only for $2 \leq s \leq n / 2$, as the inequality for all other values of $s$ will follow from Fact 1

We are going to need the following
Lemma 3: Let $\left|K_{q-1}(d)\right| \leq \delta \cdot\binom{n}{q-1}$ and $\left|K_{q}(d)\right| \leq \delta \cdot\binom{n}{q}$ for some $\delta>0, d<n / 2$, and positive integer $q<n$. Then

$$
\left|K_{q+1}(d)\right| \leq \delta \cdot\binom{n}{q+1} \cdot \frac{n-2 d+q}{n-q}
$$

For clarity of exposition, we defer its proof until the end of this section.

We show (2) in two steps. First, we prove the following:
Hypothesis: For every $1 \leq t \leq d / 2$,

$$
\begin{gather*}
\left|K_{2 t}(d)\right| \leq C \cdot\binom{n}{2 t} \cdot \prod_{i=2}^{t} \frac{n-2 d+2 i}{n-2 i} \\
\left|K_{2 t+1}(d)\right| \leq C \cdot\binom{n}{2 t+1} \cdot \prod_{i=2}^{t} \frac{n-2 d+2 i}{n-2 i} \tag{3}
\end{gather*}
$$

where empty products are treated as 1 .
Base: For $t=1$, we already know $\left|K_{2}(d)\right|=C \cdot\binom{n}{2}$. The proof of above inequality for $\left|K_{3}(d)\right|$ is defered to Lemma 4 Step: Denote $\eta_{t}=\prod_{i=2}^{t} \frac{n-2 d+2 i}{n-2 i}$ for brevity. So, suppose (3) holds for some $(t-1)$, where $1 \leq t \leq d / 2$. First, we derive from Lemma 3 for $q=2 t-1$ :
$\left|K_{2 t}(d)\right| \leq C \cdot \eta_{t-1} \cdot\binom{n}{2 t} \cdot \frac{n-2 d+(2 t-1)}{n-(2 t-1)} \leq C \cdot \eta_{t} \cdot\binom{n}{2 t}$, where we use $\frac{n-2 d+(2 t-1)}{n-(2 t-1)} \leq \frac{n-2 d+2 t}{n-2 t}$. Notice also that $\frac{n-2 d+2 t}{n-2 t} \leq 1$ for $t \leq d / 2$, and so $\eta_{t} \leq \eta_{t-1}$. Then apply Lemma 3 for $q=2 t$ and $\delta=C \cdot \eta_{t-1}$ again:
$\left|K_{2 t+1}(d)\right| \leq C \eta_{t-1} \cdot\binom{n}{2 t+1} \frac{n-2 d+2 t}{n-2 t}=C \eta_{t}\binom{n}{2 t+1}$.
Therefore, (3) holds for any $1 \leq t \leq d / 2$. Denote $\Phi=$ $\eta_{d / 2}=\prod_{i=2}^{d / 2} \frac{n-2 d+2 i}{n-2 i}$. We now prove the following
Hypothesis: For every $s$ such that $d \leq s \leq n / 2$,

$$
\begin{equation*}
\left|K_{s}(d)\right| \leq C \cdot \Phi \cdot\binom{n}{s} \cdot \prod_{k=d+1}^{s-1} \frac{n-2 d+k}{n-k} \tag{4}
\end{equation*}
$$

Base: The cases $s=d,(d+1)$ follow from (3) for $t=d / 2$. Step: Denote $\mu_{s}=\prod_{k=d+1}^{s-1} \frac{n-2 d+k}{n-k}$, and notice that $\mu_{s-1} \leq \mu_{s}$ for any $s$ within the range of interest. Suppose (4) holds for $(s-2)$ and $(s-1)$, and $(d+2) \leq s \leq n / 2$. Then apply Lemma 3 for $q=(s-1)$ and $\delta=C \cdot \Phi \cdot \mu_{s-1}$ :

$$
\left|K_{s}(d)\right| \leq C \Phi \cdot \mu_{s-1} \cdot\binom{n}{s} \frac{n-2 d+(s-1)}{n-(s-1)}=C \cdot \Phi \cdot \mu_{s} \cdot\binom{n}{s}
$$

We are finally ready to prove (2) for every $s$ between 2 and $n / 2$. For $s$ such that $2 \leq s \leq(d+1)$, (3) implies $\left|K_{s}(d)\right| \leq$ $C \cdot \eta_{\lfloor s / 2\rfloor} \cdot\binom{n}{s}$. Clearly $\eta_{\lfloor s / 2\rfloor} \leq 1$ as every term in the product is at most 1 , so (2) holds for such $s$.

Next, we have from (4) that $\left|K_{s}(d)\right| \leq C \cdot \Phi \cdot \mu_{s} \cdot\binom{n}{s} \leq$ $C \cdot \Phi \cdot \mu_{n / 2} \cdot\binom{n}{s}$ for every s between $(d+2)$ and $n / 2$, as $\mu_{s}$ is an increasing sequence. So it is sufficient to show that $\Phi \cdot \mu_{n / 2} \leq 1$. Denote $w=n / 2-d-1$, so $w<\sqrt{n} / 2$, since $d>(n-\sqrt{n}) / 2$. Recall that $\Phi=\eta_{d / 2} \leq \eta_{w+1}$, and so

$$
\begin{aligned}
\Phi \cdot \mu_{n / 2} & \leq \prod_{i=2}^{w+1} \frac{n-2 d+2 i}{n-2 i} \cdot \prod_{v=1}^{w} \frac{n-d+v}{n-d-v} \\
& \leq\left(\frac{n-2 d+2 w+2}{n-2 w-2}\right)^{w}\left(\frac{n-d+w}{n-d-w}\right)^{w}
\end{aligned}
$$

$$
=\left(\frac{4 w+4}{n-2 w-2} \cdot \frac{n / 2+2 w+1}{n / 2+1}\right)^{w} \leq 1
$$

The final inequality clearly holds because $w<\sqrt{n} / 2$.
This completes the proof of (2) for the whole range of $2 \leq$ $s \leq(n-2)$, and together with our arguments that (1) holds for $s \in\{0,1,(n-1), n\}$, this means that all Delsarte-McWilliam's inequalities from $(\widehat{\mathcal{B P}})$ are satisfied. Therefore, we conclude that $\alpha$ is optimal for $(\overline{\mathcal{P}})$ and $\beta$ is optimal for $(\overline{\mathcal{D} D}$.

Proof of Lemma 3. We use the following recurrence for Krawtchouk polynomials:

$$
\begin{equation*}
(q+1) K_{q+1}(x)=(n-2 x) K_{q}(x)-(n-q+1) K_{q-1}(x) \tag{5}
\end{equation*}
$$

for any positive integer $q<n$, where $K_{0}(x)=1$, and $K_{1}(x)=n-2 x$. The proof now follows from a simple inductive calculation.

Lemma 4: Let $\frac{n-\sqrt{n}}{2}+1 \leq d \leq n / 2$. Then

$$
\left|K_{3}(d)\right| \leq \frac{\left|K_{2}(d)\right|}{\binom{n}{2}}\binom{n}{3}=\frac{\left|K_{2}(d)\right| \cdot(n-2)}{3}
$$

Proof: We know $K_{1}(d) \geq 0$ and $K_{2}(d)<0$ for such $d$. Using recurrence (5) and this sign information, obtain

$$
\left|K_{3}(d)\right|=\frac{-(n-2 d) K_{2}(d)+(n-1)(n-2 d)}{3}
$$

We need to show that the above is at most $-\frac{K_{2}(d) \cdot(n-2)}{3}$. Equivalently, we need to prove

$$
\begin{equation*}
-2(d-1) K_{2}(d) \stackrel{?}{\geq}(n-2 d)(n-1) \tag{6}
\end{equation*}
$$

Decompose $K_{2}(d)=2\left(d-\frac{n-\sqrt{n}}{2}\right)\left(d-\frac{n+\sqrt{n}}{2}\right)$, and using the conditions on $d$, we have $-K_{2}(d) \geq \sqrt{n}$.

Further, using $\sqrt{n}-2 \geq(n-2 d)$ for (6) we finally get

$$
\begin{aligned}
-2(d-1) K_{2}(d) \geq(n-\sqrt{n}) \sqrt{n} & >(\sqrt{n}-2) n \\
& \geq(n-2 d)(n-1)
\end{aligned}
$$

This finally brings us the the proof of our main result.
Proof of Theorem [7: Observe that $K_{2}(d)=2 d^{2}-2 n d-$ $\binom{n}{2}=-\frac{1}{2}\left(n-(n-2 d)^{2}\right)$. Applying Lemma 2 obtain

$$
\begin{aligned}
B_{L P}(n, d)=1-\frac{\binom{n}{2}}{K_{2}(d)} & =1+\frac{n(n-1)}{n-(n-2 d)^{2}} \\
& =\frac{4 d(n-d)}{n-(n-2 d)^{2}}
\end{aligned}
$$

## IV. LOWER BOUND ON LP BOUND FOR ALMOST-BALANCED CODES

Consider asymptotic lower and upper bounds on $R(\delta)$

$$
R_{G V}(\delta) \leq R(\delta) \leq R_{M R R W}(\delta)
$$

where $R_{G V}(\delta)=1-H(\delta)$ is a Gilbert-Varshamov bound obtained using a standard packing argument, which is currently the best known lower bound on $R(\delta)$. The bound $R_{M R R W}(\delta)=H\left(\frac{1}{2}-\sqrt{\delta(1-\delta)}\right)$ is the first MRRW [2]
bound, which is the best known upper bound for $\delta>0.273$ (the second MRRW bound is the best known for the remaining range of $\delta$ ).

In [10] Samorodnitsky proved an integrality gap of at most 2 for the MRRW bound with respect to the true LP bound:

$$
\frac{R_{G V}(\delta)+R_{M R R W}(\delta)}{2} \leq R_{L P}(\delta) \leq R_{M R R W}(\delta)
$$

Combined with the fact that $R_{M R R W}(\delta)>R_{G V}(\delta)$ for any $\delta \in\left(0, \frac{1}{2}\right)$, the above proved that Delsarte's linear programming bound cannot attain the currently best known lower bound on $R(\delta)$.

In this section we prove an analogous result for the linear programming bound for the almost-balanced codes. Our proof is a slight modification of a proof from [10].

Theorem 2: $R_{L P}^{\mathrm{bal}}(\delta) \geq \frac{R_{G V}(\delta)+R_{M R R W}(\delta)}{2}$ for any $\delta \in\left(0, \frac{1}{2}\right)$.

To obtain a lower bound, we derive a feasible solution to $(\overline{\mathcal{B P}}$, closely following [10].

Lemma 5: Let $\varepsilon=\frac{1}{4 n} \sqrt{\frac{\binom{n}{x_{d}}}{2^{n} \cdot\binom{n}{d}}}$, where $x_{d}$ is the first (smallest) root of the polynomial $K_{d}(x)$, and let

- $a_{0}=1$
- $a_{k}=0$ for $0 \leq k \leq d-1$ and $(n-d+1) \leq k \leq n$
- $a_{d}=a_{n-d}=\varepsilon \cdot(d+1) \cdot\binom{n}{d}$
- $a_{k}=\varepsilon \cdot\binom{n}{k}$ for $d<k<(n-d)$.

Then $a_{0}, \ldots, a_{n}$ is a feasible solution to the LP $(\overline{\mathcal{P} P}$.
Proof: Clearly, we only need to verify the first set of constraints in $\overline{\mathcal{B} \mathcal{P}}$ (Delsarte-MacWilliam's inequalities). The case $s=0$ is immediate, so assume $s \geq 1$. We have

$$
\begin{align*}
& \sum_{k=0}^{n} a_{k} K_{s}(k)=K_{s}(0)+\varepsilon \cdot \sum_{k=d}^{n-d}\binom{n}{k} K_{s}(k)  \tag{7}\\
& +\varepsilon \cdot d\binom{n}{d}\left(K_{s}(d)+K_{s}(n-d)\right)  \tag{8}\\
& =\binom{n}{s}+\varepsilon \sum_{k=d}^{n-d}\binom{n}{s} K_{k}(s)+\varepsilon d\binom{n}{s}\left(K_{d}(s)+K_{n-d}(s)\right) \tag{9}
\end{align*}
$$

where we used reciprocity property from Fact 3
Next we use Facts [1]2 and consider two cases. When $s$ is odd, everything except the first summand in the RHS of (9) cancels out. Indeed, $K_{d}(s)=-K_{n-d}(s)$, and the summation in the middle can be written as

$$
\begin{aligned}
\sum_{k=d}^{n-d}\binom{n}{s} K_{k}(s)= & \binom{n}{s} \sum_{k=d}^{\frac{n}{2}-1}\left(K_{k}(s)+K_{n-k}(s)\right) \\
& +\binom{n}{s} \cdot K_{n / 2}(s) \cdot[(n+1) \bmod 2]
\end{aligned}
$$

Observe that $K_{k}(s)+K_{n-k}(s)=0$ for any $k$ within the summation range. Finally, for even $n$ we have $\binom{n}{s} K_{n / 2}(s)=$ $\binom{n}{n / 2} K_{s}(n / 2)=0$, as $K_{s}(n / 2-x)$ is an odd function. Therefore, $\sum_{k=0}^{n} a_{k} K_{s}(k)=K_{s}(0)=\binom{n}{s} \geq 0$ for odd $s$.

Now consider the case of even $s$. Using the fact that Krawtchouk polynomials are orthogonal with respect to the binomial measure $\mu(k)=\binom{n}{k} / 2^{n}$ and that $K_{0}(k) \equiv 1$, obtain $2^{n} \cdot \sum_{k=0}^{n} \mu(k) K_{s}(k) \cdot K_{0}(k)=\sum_{k=0}^{n}\binom{n}{k} K_{s}(k)=0$. Then in (7)-(8) for the summation in the RHS we can write

$$
\begin{aligned}
\sum_{k=d}^{n-d} & \binom{n}{k} K_{s}(k)=-\sum_{k=0}^{d-1}\binom{n}{k} K_{s}(k)-\sum_{k=n-d+1}^{n}\binom{n}{k} K_{s}(k) \\
& =-\sum_{k=0}^{d-1}\binom{n}{s} K_{k}(s)-\sum_{k=n-d+1}^{n}\binom{n}{s} K_{k}(s) \\
& =-\binom{n}{s} \sum_{k=0}^{d-1}\left(K_{k}(s)+K_{n-k}(s)\right)=-2\binom{n}{s} \sum_{k=0}^{d-1} K_{k}(s)
\end{aligned}
$$

where we used reciprocity and Fact 2 for even $s$. Using these properties again for last part of RHS in (7)-(8), we obtain

$$
\sum_{k=0}^{n} a_{k} K_{s}(k)=\binom{n}{s}\left(1-2 \varepsilon \sum_{k=0}^{d-1} K_{k}(s)+2 \varepsilon \cdot d K_{d}(s)\right)
$$

Finally, we notice that the RHS of the above equation is exactly the expression derived by Samorodnitsky in [10, eq. (22)] (we took $\varepsilon$ exactly two times smaller than in [10] for these expressions to coincide), where it was proven to be non-negative. Therefore, $a_{0}, \ldots, a_{n}$ is feasible for $(\mathcal{B P})$.

Proof of Theorem 2. Taking the feasible solution $a_{0}, \ldots, a_{n}$ for $(\overline{\mathcal{B P}})$ from Lemma 5 we obtain

$$
B_{L P}(n, d) \geq \sum_{k=0}^{n} a_{k} \geq \varepsilon \sum_{k=d}^{n-d}\binom{n}{k}
$$

Consider some fixed $\delta \in(0,1 / 2)$ and $d=\lfloor\delta n\rfloor$ as $n$ increases. Standard concentration properties of Binomial distribution (e.g. Chernoff bound) then imply that for large enough $n$, most of the weight will lie between $\binom{n}{d} / 2^{n}$ and $\binom{n}{n-d} / 2^{n}$. Then for such large $n$ we write

$$
B_{L P}(n, d) \geq \varepsilon \cdot 2^{n-1} \geq \frac{1}{8 n} \sqrt{\frac{\binom{n}{\left\lfloor x_{d}\right\rfloor} \cdot 2^{n}}{\binom{n}{d}}}
$$

Finally, we use an asymptotic for the first root of Krawtchouk polynomial $K_{d}$ as $n$ goes to infinity: $x_{d}=n\left(\frac{1}{2}-\sqrt{\frac{d}{n}\left(1-\frac{d}{n}\right)}\right)+o(n)$. Together with an asymptotic $\lim _{n \rightarrow \infty} \frac{1}{n} \log _{2}\binom{n}{\gamma n}=H(\gamma)$, we derive

$$
\begin{aligned}
& R_{L P}^{\mathrm{bal}}(\delta) \geq \lim _{n \rightarrow \infty} \frac{1}{2 n}\left[\log _{2}\binom{n}{\left\lfloor x_{d}\right\rfloor}+n-\log _{2}\binom{n}{\delta n}\right] \\
& =\frac{1-H(\delta)+H(1 / 2-\sqrt{\delta(1-\delta)})}{2}=\frac{R_{G V}(\delta)+R_{M R R W}(\delta)}{2} .
\end{aligned}
$$

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