Secure Modulo Sum via Multiple Access Channel

Masahito Hayashi, Fellow, IEEE

Abstract

We discuss secure computation of modular sum when multiple access channel from distinct players A_1, \ldots, A_c to a third party (Receiver) is given. Then, we define the secure modulo sum capacity as the supremum of the transmission rate of modulo sum without information leakage of other information. We derive its useful lower bound, which is numerically calculated under a realistic model that can be realizable as a Gaussian multiple access channel (MAC).

Index Terms

modulo sum, secure communication, multiple access channel, secrecy analysis

I. INTRODUCTION

Recently, secure multiparty computation has been actively studied even from the information theoretical viewpoint. One of its simple examples is secure computation of modular sum [1], [2]. This problem has been often discussed from the Shannon-theoretic viewpoint [3], [4]. If we assume secure communication channels between a part of players, secure computation of modular sum is possible. Secure communication channels consumes cryptographic resources. In this paper, instead of such resources, we focus on multiple access channel (MAC), which has been actively studied for a long time [5], [6], [7]. That is, we study how to realize secure computation via MAC instead of conventional secure communication channels. As a typical example, we investigate secure computation of modular sum when a MAC from distinct players A_1, \ldots, A_c to a third party (Receiver) is given. That is, each player A_i sends a sequence of their secure random number $M_i = (M_{i,1}, \ldots, M_{i,k}) \in \mathbb{F}_q^k$ that is subject to the uniform distribution, and Receiver recovers only the modulo sums $M_{[c]} := (\bigoplus_{i=1}^c M_{i,1}, \ldots, \bigoplus_{i=1}^c M_{i,k})$ as Fig. 1, where $\bigoplus_{i=1}^c M_{i,1} := M_{1,j} \oplus \cdots \oplus M_{c,j}$ and $[c] := \{1, \ldots, c\}$. Here, to keep the secrecy, it is required that Receiver cannot obtain any information except for the modulo sums. In this way, we can compute the modulo sums. As a simple case [8], [9], we may consider the channel where the output signal Y of the channel is given as

$$Y = X_1 \oplus \cdots \oplus X_c \oplus N \tag{1}$$

with the *i*-th input variables X_i and the noise variable N, where \oplus is the modulo sum.

When all senders encode their message by using the same linear code, the receiver cannot obtain any information with respect to each sender's message because each message is subject to the uniform distribution. Hence, the code is decodable under the noise N, the receiver can recover the modulo sums $(\bigoplus_{i=1}^c M_{i,1}, \ldots, \bigoplus_{i=1}^c M_{i,k})$. However, the real channel does not has such a form. A realistic example is a Gaussian MAC, whose typical case is given as follows. The receiving signal Y takes values in the set of real numbers \mathbb{R} , and the i-th sender's signal X_i takes values in discrete values $\{0,1,\ldots,p-1\}$. Then, a typical channel is given as

$$Y = E(X_1 + \dots + X_c) + N \in \mathbb{R},\tag{2}$$

where E is a constant and N is the Gaussian variable with average 0 and variance v. In this case, a part of information for the message of each sender might be leaked to the receiver. To cover Gaussian MAC, this paper addresses a general MAC.

The work reported here was supported in part by Guangdong Provincial Key Laboratory (Grant No. 2019B121203002), the JSPS Grant-in-Aid for Scientific Research (A) No.17H01280, (B) No. 16KT0017, and Kayamori Foundation of Informational Science Advancement.

Masahito Hayashi is with Shenzhen Institute for Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen, 518055, China, Guangdong Provincial Key Laboratory of Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen 518055, China, Shenzhen Key Laboratory of Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen 518055, China, and the Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan (e-mail:hayashi@sustech.edu.cn).

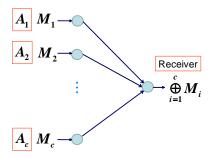


Fig. 1. Secure computation of modulo sum via multiple access channel.

If the secrecy condition is not imposed and the channel is a Gaussian MAC, this problem is a simple example of computation-and-forward [10], [11], [12], [13], [14], [15], [16], [17]. Due to the secrecy condition, we cannot realize this task by their simple application. In the case of c=2, since the modulo sums give the information of the other player, the protocol enables the two players to exchange their messages without information leakage to Receiver [18], [19], [20], [21], [22], [23] when Receiver broadcasts the modulo sum to both players. Application of such a protocol is discussed in several network models [26]. Our protocol can be regarded as an multi-player extension with a generic setting of the protocol given in [18], [19], [20], [21], [23].

Further, when we can realize secure transmission of modulo sum with k transmitter, multiplying -1 to the receiver's variable, we can realize secure modulo zero-sum randomness among k+1 players, that is the k+1 random variables with zero-sum condition and certain security conditions [24, Section 2][25, Section II]. As is explained in [24], [25], secure modulo zero-sum randomness can be regarded as a cryptographic resource for several cryptographic tasks including secure multi-party computation of homomorphic functions, secret sharing without secure communication channel, and a cryptographic task, multi-party anonymous authentication. In this sense, secure transmission of modulo sum is needed in the viewpoint of cryptography.

In this paper, we formulate the secure modulo sum capacity as the maximum of the secure transmission of modulo sum via multiple access channel. When the multiple access channel satisfies a symmetric condition, the secure modulo sum capacity equals the capacity of a certain single access channel. In a general setting, we derive the lower bound of the secure modulo sum capacity. Further, we give several realistic examples by using Gaussian MAC, which can be realized in wireless communication.

The remaining part of this paper is organized as follows. Section II prepares notations used in this paper. Section III defines the secure modulo sum capacity, and gives two theorems, which discusses the symmetric case and the general case. Section IV gives the proof for the semi-symmetric case. Section V discusses the Gaussian MAC.

II. NOTATIONS

In this section, we prepare notations and information quantities used in this paper. Given a joint distribution channel P_{Y,Z_1,Z_2} over the product system of a finite discrete set $\mathcal{Z}_1 \times \mathcal{Z}_2$ and a continuous set \mathcal{Y} , we denote the conditional probability density function of $P_{Y|Z_1,Z_2}$ by $p_{Y|Z_1,Z_2}(y|z_1,z_2)$. Then, we define the conditional distribution $P_{Y|Z_2}$ over a continuous set \mathcal{Y} conditioned in the discrete set \mathcal{Z}_2 by the conditional probability density function $p_{Y|Z_2}(y|z_2) := \sum_{z_2 \in \mathcal{Z}_2} P_{Z_2}(z_2) p_{Y|Z_1,Z_2}(y|z_1,z_2)$. Then, we define two types of the Rényi conditional mutual information $I_{1+s}(Y;Z_1|Z_2)$

$$sI_{1+s}(Y; Z_1|Z_2)$$

$$:= \log \sum_{z_1, z_2} P_{Z_1, Z_2}(z_1, z_2) \int_{\mathcal{Y}} \frac{p_{Y|Z_1, Z_2}(y|z_1, z_2)^{1+s}}{p_{Y|Z_2}(y|z_2)^s} dy$$

$$\frac{s}{1+s} I_{1+s}^{\downarrow}(Y; Z_1|Z_2)$$

$$:= \log \sum_{z_1, z_2} P_{Z_2}(z_2) \int_{\mathcal{Y}} \left(\sum_{z_1, z_2} P_{Z_1|Z_2}(z_1|z_2) p_{Y|Z_1, Z_2}(y|z_1, z_2)^{1+s} \right)^{\frac{1}{1+s}} dy$$

$$(4)$$

for s>0. Since $\lim_{s\to 0} sI_{1+s}(Y;Z_1|Z_2)=\lim_{s\to 0} sI_{1+s}^{\downarrow}(Y;Z_1|Z_2)=0$, taking the limit $s\to 0$, we have

$$\lim_{s \to 0} \frac{sI_{1+s}(Y; Z_1|Z_2)}{s} = \lim_{s \to 0} \frac{sI_{1+s}^{\downarrow}(Y; Z_1|Z_2)}{s} = I(Y; Z_1|Z_2), \tag{5}$$

where $I(Y; Z_1|Z_2)$ expresses the conditional mutual information. Also, we have

$$I_{1+s}^{\downarrow}(Y; Z_1|Z_2) \le I_{1+s}(Y; Z_1|Z_2) \le I_{\frac{1}{1-s}}^{\downarrow}(Y; Z_1|Z_2).$$
 (6)

The concavity of the function $x \mapsto x^{\frac{1}{1+s}}$ yields

$$e^{\frac{s}{1+s}I_{1+s}^{\downarrow}(Y;Z_1|Z_2,Z_3)} \le e^{\frac{s}{1+s}I_{1+s}^{\downarrow}(Y;Z_1,Z_2|Z_3)}.$$
(7)

Given a channel $P_{Y|Z_1,Z_2,Z_3}$ from the finite discrete set $\mathcal{Z}_1 \times \mathcal{Z}_2 \times \mathcal{Z}_3$ to a continuous set \mathcal{Y} , when the random variables Z_1,Z_2,Z_3 are generated subject to the uniform distributions, we have a joint distribution among Y,Z_1,Z_2 . In this case, we denote the Renyi conditional mutual information and the conditional mutual information by $I_{1+s}(Y;Z_1|Z_2)[P_{Y|Z_1,Z_2,Z_3}]$ and $I(Y;Z_1|Z_2)[P_{Y|Z_1,Z_2,Z_3}]$, respectively.

Further, we denote the vector $(x_i)_{i\in\mathcal{I}}$ by $\vec{x}_{\mathcal{I}}$. When each element x_i belongs to the same vector space, we use the notation

$$x_{\mathcal{I}} := \bigoplus_{i \in \mathcal{I}} x_i. \tag{8}$$

III. SECURE MODULO SUM CAPACITY

Consider c players and Receiver. Player A_i has a random variable $M_i \in \mathbb{F}_q^k$ and sends it to Receiver, where M_i is independently subject to the uniform distribution. The task of Receiver is computing the modulo sum $M_{[c]} \in \mathbb{F}_q^k$. Also, it is required that Receiver obtains no other partial information of $\{M_i\}_{i=1}^c$. That is, when Receiver's information is denoted by $\mathbf{Y} \in \mathcal{Y}^n$, we impose the following condition

$$I(\mathbf{Y}; \vec{M}_{\mathcal{J}}) \cong 0 \tag{9}$$

for any non-empty set $\mathcal{J} \subsetneq [c]$, where $\vec{M}_{\mathcal{J}} := (M_i)_{i \in \mathcal{J}}$. We call this task secure transmission of common reference string.

To realize this task, we employ a multiple access channel (MAC) W with c input alphabets $\mathcal{X}_1,\ldots,\mathcal{X}_c$ and an output alphabet \mathcal{Y} , which might be a continuous set. Here, the i-th input alphabet \mathcal{X}_i is under the control of Player A_i . That is, given an element $(x_1,\ldots,x_c)\in\mathcal{X}_1\times\ldots\times\mathcal{X}_c$, we have the output distribution $W_{(x_1,\ldots,x_c)}$ on \mathcal{Y} . When Receiver is required to compute the modulo sum of elements of $\mathbb{F}_q^{k_n}$ with the above security condition, we employ n uses of the multiple access channel W. Here, we use the bold font like $\mathbf{Y},\mathbf{x}_j,\mathbf{X}_j,\mathbf{Y}_{\mathcal{J}}$ to express n symbols while the italic font like $Y,x_j,X_j,\vec{Y}_{\mathcal{J}}$ expresses single symbol. Receiver's variable $\mathbf{Y}\in\mathcal{Y}^n$ obeys the output distribution of the channel $W^{(n)}$, which is defined as the distributions $W_{(\mathbf{x}_1,\ldots,\mathbf{x}_c)}^{(n)}:=W_{(x_{1,1},\ldots,x_{c,1})}\times\cdots\times W_{(x_{1,n},\ldots,x_{c,n})}$ on \mathcal{Y}^n for $(\mathbf{x}_1,\ldots,\mathbf{x}_c)\in\mathcal{X}_1^n\times\ldots\times\mathcal{X}_c^n$.

The encoder is given as a set of stochastic maps $\Phi_{i,e}^{(n)}$ from $\mathbb{F}_q^{k_n}$ to \mathcal{X}_i^n with $i=1,\ldots,c$. The decoder is given as a map $\Phi_d^{(n)}$ from \mathcal{Y}^n to $\mathbb{F}_q^{k_n}$. The tuple $((\Phi_{i,e}^{(n)})_{i=1}^c,\Phi_d^{(n)})$ is simplified to $\Phi^{(n)}$ and is called a code. The performance of code $\Phi^{(n)}$ is given by the following two values. One is the decoding error probability

$$\epsilon(\Phi^{(n)}) := \frac{1}{q^{ck_n}} \sum_{\substack{m_1, \dots, m_c, m' \\ m' \neq \bigoplus_{j=1}^c m_j}} \mathbb{E}_{(\Phi_{i,e}^{(n)})_{i=1, \dots, c}} \left[W_{(\Phi_{i,e}^{(n)}(m_i))_{i=1, \dots, c}}^{(n)} ((\Phi_d^{(n)})^{-1}(m')) \right], \tag{10}$$

where the sum is taken with respect to $m_1, \ldots, m_c, m' \in \mathbb{F}_q^{k_n}$ with the condition $m' \neq \bigoplus_{j=1}^c m_j$, and $\mathbb{E}_{(\Phi_{i,e}^{(n)})_{i=1,\ldots,c}}$ expresses the expectation with respect to the random variables $(\Phi_{i,e}^{(n)})_{i=1,\ldots,c}$. The other is the leaked information

$$I_{\mathcal{J}}(\Phi^{(n)}) := I(\mathbf{Y}; \vec{M}_{\mathcal{J}}) \tag{11}$$

for a non-empty set $\mathcal{J} \subseteq [c]$. The transmission rate is given as $\frac{k_n \log q}{n}$.

When a sequence of codes $\{\Phi^{(n)}\}$ satisfies the conditions

$$\epsilon(\Phi^{(n)}) \to 0 \tag{12}$$

and

$$I_{\mathcal{J}}(\Phi^{(n)}) \to 0 \tag{13}$$

for a non-empty set $\mathcal{J} \subsetneq [c]$, the limit $\limsup_{n \to \infty} \frac{k_n \log q}{n}$ is called an achievable rate. The *secure modulo sum capacity* $C_{\text{SMS}}(W)$ is defined as the supremum of achievable rates with respect to the choice of the sequence codes as well as the prime power q.

Now, we assume that \mathcal{X}_i is an l-dimensional vector space over a finite field \mathbb{F}_q . Also, \mathcal{Y} is a symmetric space for \mathbb{F}_q , i.e., for an element $x \in \mathbb{F}_q^l$, there is a function f_x on \mathcal{Y} such that $f_{x_1} \circ f_{x_2} = f_{x_1 \oplus x_2}$. Here, the collection of functions $f = (f_x)$ is called an action of \mathbb{F}_q^l on \mathcal{Y} . A multiple access channel W is called *symmetric* when the relation

$$W_{x_1,...,x_c}(B_{f,x}) = W_{x_1 \oplus x,...,x_c}(B) = \dots = W_{x_1,...,x_c \oplus x}(B)$$
 (14)

holds for a measurable subset $B \subset \mathcal{Y}$, where $B_{f,x} := \{f_x(y)\}_{y \in B}$.

For a symmetric multiple access channel W, we have $W_{x_1,\dots,x_c}=W_{x_1+\dots+x_c,0,\dots,0}$, and define the symmetric single access channel W_S as

$$W_{S,x} := W_{x,0,\dots,0},\tag{15}$$

which includes the channel (1).

Theorem 1: For a symmetric multiple access channel W, we have

$$C_{\text{SMS}}(W) = C(W_S),\tag{16}$$

where $C(W_S)$ is the channel capacity of the channel W_S .

Its proof is very simple. In this case, the transmission of modulo sum is the same as the information transmission under the symmetric single access channel W_S when the senders employ a common algebraic encoder. The symmetric condition guarantees the secrecy condition when the senders employ a common algebraic encoder because the two distributions W_{x_1,\dots,x_c} and $W_{x_1',\dots,x_c'}$ cannot be distinguished when $x_1\oplus\dots\oplus x_c=x_1'\oplus\dots\oplus x_c'$. Hence, the rate $C(W_S)$ is achieved. The converse part can be shown as follows. Consider the case when the messages by Senders A_2,\dots,A_c are fixed. In this case, the problem is reduced to the message transmission from Sender A_1 to Receiver. Due to the above symmetric condition, the channel with this case from Sender A_1 to Receiver is also W_S . Hence, it is impossible to exceed the rate $C(W_S)$. Hence, the converse part follows.

However, it is not so easy to realize a symmetric multiple access channel W. Here, we remark the relation with the existing papers [8], [9], which address a similar channel. However, they consider the information leakage to the third party with respect to the modulo sum. This paper considers the information leakage to the receiver with respect to the message of each player etc. To address this problem, we denote $I(Y; X_{[c]})[W]$ and $I(Y; X_i | \vec{X}_{\mathcal{J}_i})[W]$ by the mutual information $I(Y; X_{[c]})$ and $I(Y; X_i | \vec{X}_{\mathcal{J}_i})$ when X_1, \ldots, X_c is subject to the uniform distribution and Y is their output of the multiple access channel W. We define $I(Y; (X_j \ominus X_{j'})_{j,j' \in \mathcal{J}^c} | \vec{X}_{\mathcal{J}_i})[W]$ and $I(Y; (X_j \ominus X_{j'})_{j,j' \in \mathcal{J}^c} | \vec{X}_{\mathcal{J}_i})[W]$ in the same way, where Θ expresses the minus in the sense of the finite field \mathbb{F}_q . We prepare notations $I(W) := I(Y; X_{[c]})[W]$, $\mathcal{J}_j := \{i \in \mathcal{J} | i \neq j\}$, and $\vec{X}_{\mathcal{J}} := (X_i)_{i \in \mathcal{J}}$.

For the general case, instead of Theorem 1, we have the following theorem.

Theorem 2: Assume that \mathcal{X}_i is an l-dimensional vector space over \mathbb{F}_q as \mathcal{X} . When a multiple access channel W satisfies

$$I(W) > I(Y; X_i | \vec{X}_{\mathcal{J}_i})[W] \tag{17}$$

$$I(W)(1+|\mathcal{J}^c|) > I(Y; X_i \vec{X}_{\mathcal{J}^c} | \vec{X}_{\mathcal{J}_i})[W]$$

$$\tag{18}$$

$$I(W)|\mathcal{J}^c| > I(Y; (X_j \ominus X_{j'})_{j,j' \in \mathcal{J}^c} | \vec{X}_{\mathcal{J}_i})[W]$$
(19)

for $\mathcal{J} \subsetneq [c]$ and $i \in \mathcal{J}$, we have

$$C_{\text{SMS}}(W) \ge \min_{i \in \mathcal{J} \subsetneq [c]} \min \left(I(W) - I(Y; X_i | \vec{X}_{\mathcal{J}_i})[W], \right.$$

$$(|\mathcal{J}^c| + 1)I(W) - I(Y; X_i, \vec{X}_{\mathcal{J}^c} | \vec{X}_{\mathcal{J}_i})[W],$$

$$|\mathcal{J}^c| I(W) - I(Y; X_i, (X_j \ominus X_{j'})_{j,j' \in \mathcal{J}^c} | \vec{X}_{\mathcal{J}_i})[W] \right). \tag{20}$$

Under the notation $I(Y; X_i | \vec{X}_{\mathcal{J}_i})[W]$, the symbols $W, Y, X_i, \vec{X}_{\mathcal{J}_i}$, and $\vec{X}_{[c] \setminus \mathcal{J}}$ correspond to $P_{Y|Z_1,Z_2,Z_3}, Y, Z_1, Z_2$, and Z_3 in the definition of $I(Y; Z_1 | Z_2)[P_{Y|Z_1,Z_2,Z_3}]$, respectively.

Due to Lemma 2 of Appendix, the maximum $\max_{j \in \mathcal{J}_{\downarrow}^{\subseteq}[c]} I(Y; X_j | \vec{X}_{\mathcal{J}_j})[W]$ is realized when $|[c] \setminus \mathcal{J}| = 1$. The mutual information I(W) is a generalization of the achievable rate with \mathbb{F}_2 given in [28].

To hide message M_i , other messages work as scramble variables for M_i . If the variables of other senders are subject to the uniform distribution on \mathcal{X}^n , Condition (17) is sufficient for the secrecy. However, the variables of other senders are not subject to the above uniform distribution on \mathcal{X}^n in our code. Moreover, when we fix the message M_i and we take the average for other messages, the output cannot be regarded as the output under the above uniform distribution in general because the message size is not sufficiently large. Hence, we need to care the deviation of the inputs of other senders. Conditions (18) and (19) are the additional conditions to cover the secrecy even with such deviation.

IV. PROOF OF THEOREM 2

Step (1): We construct our code randomly as follows. As the first step, we choose a sequence of integers k_n such that the minimum among the differences between LHS and RHS in inequalities (17) – (19) equals $\epsilon + \lim_{n \to \infty} \frac{k_n}{n} \log q$, where $\epsilon > 0$ is sufficiently small real number. Then, we choose a sequence of integers k'_n such that $I(W) - \frac{\epsilon}{2} = \lim_{n \to \infty} \frac{k_n + k'_n}{n} \log q$. Then, we have the following inequalities;

$$\lim_{n \to \infty} \frac{k_n + k_n'}{n} \log q < I(W) \tag{21}$$

$$\lim_{n \to \infty} \frac{k'_n + (k_n + k'_n)|\mathcal{J}^c|}{n} \log q > I(Y; X_i, \vec{X}_{\mathcal{J}^c}|\vec{X}_{\mathcal{J}_i})[W]$$
(22)

$$\lim_{n \to \infty} \frac{k'_n}{n} \log q > I(Y; X_i | \vec{X}_{\mathcal{J}_i})[W], \tag{23}$$

and

$$\lim_{n \to \infty} \frac{k'_n + (k_n + k'_n)(|\mathcal{J}^c| - 1)}{n} \log q > I(Y; X_i, (X_j \ominus X_{j'})_{j,j' \in \mathcal{J}^c} | \vec{X}_{\mathcal{J}_i})[W]$$
(24)

for $\mathcal{J} \subsetneq [c]$ and $i \in \mathcal{J}$. Let L_i be a scramble random variable with the dimension k'_n for $i = 1, \ldots, c$, which is assumed to be subject to the uniform distribution.

Then, we randomly choose a pair of an invertible linear map G_1 from $\mathbb{F}_q^{k_n+k'_n}\times\mathbb{F}_q^{nl-(k_n+k'_n)}$ to \mathcal{X}^n and a linear map G_2 from \mathcal{X}^n to $\mathbb{F}_q^{nl-(k_n+k'_n)}$ such that $G_2\circ G_1(F,E)=E$ and the universal condition

$$\Pr\{G_2(x) = G_2(x')\} \le \frac{1}{q^{nl - (k_n + k'_n)}}$$
(25)

holds for $x \neq x' \in \mathcal{X}^n$. Also, we randomly choose a pair of an invertible linear map G_3 from $\mathbb{F}_q^{k_n} \times \mathbb{F}_q^{k'_n}$ to $\mathbb{F}_q^{k_n+k'_n}$ and a linear map G_4 from $\mathbb{F}_q^{k_n+k'_n}$ to $\mathbb{F}_q^{k_n}$ such that $G_4 \circ G_3(M,L) = M$ and the universal condition

$$\Pr\{G_4(f) = G_4(f')\} \le \frac{1}{q^{k'_n}} \tag{26}$$

holds for $f \neq f' \in \mathbb{F}_q^{k_n + k'_n}$. Hence, G_1 and G_3 uniquely determine G_2 and G_4 , respectively.

We also randomly and independently choose the random variable E_i subject to the uniform distribution on $\mathbb{F}_q^{nl-(k_n+k'_n)}$ for $i=1,\ldots,c$. Hence, G_1,G_3 , and E_1,\ldots,E_c are priorly shared among players and Receiver. Then, we consider the following protocol. That is, using the random variable G_1,G_3 , and E_i , we define the encoder $\Phi_{i,e}^{(n)}$ as follows. For a given M_i , Player A_i randomly chooses the scramble random variable L_i . That is, each player

 A_i transmits $G_1(G_3(L_i, M_i), E_i)$. In this case, since G_1 , G_3 , and E_i are shared among Senders and Receiver, the encoder of Sender A_i is given as an affine map $(L_i, M_i) \mapsto G_1(G_3(L_i, M_i), E_i)$. The detail analysis for affine maps are explained in Appendix A of [32]. Here, the variables $G_1, G_3, \vec{M}, \vec{L}, \vec{E}$ are assumed to be independent of each other. Then, we have the relation

$$F_i = G_3(L_i, M_i), \quad \mathbf{X}_i = G_1(F_i, E_i).$$
 (27)

Since \vec{E} is subject to the uniform distribution on $\mathbb{F}_q^{c(nl-(k_n+k'_n))}$, even when the maps G_1 and G_3 are fixed, \vec{X} is subject to the uniform distribution on \mathcal{X}^{cn} . Receiver receives the random variable $\mathbf{Y} \in \mathcal{Y}^n$ that depends only on \vec{X} . That is, we have the Markov chain $(G_1,G_3,\vec{M},\vec{L},\vec{E})-\vec{X}-\mathbf{Y}$. Due to (21), Receiver can decode $\bigoplus_{j=1}^c F_j = G_3(L_{[c]},M_{[c]})$, i.e., $M_{[c]}$ and $L_{[c]}$ from \mathbf{Y} by using the knowledge $E_{[c]} = \bigoplus_{i=1}^c E_i$ for the coset. The proof is similar to the same way as Appendix B of [32] and is different from [33]. The detail is explained in Appendix C.

Step (2): We divide the leaked information to several parts that can be bounded. Given $\mathcal{J} \subset [c]$, we discuss the leaked information for $M_{\mathcal{J}} := \sum_{i \in \mathcal{J}} M_i$. For $\vec{M}_{\mathcal{J}} := (M_i)_{i \in \mathcal{J}}$, using $\tilde{\mathcal{J}}_i := \{j \in \mathcal{J} | j < i\}$, we have

$$I(\mathbf{Y}; \vec{M}_{\mathcal{J}}|G_{1}, G_{3}, \vec{E})$$

$$= \sum_{i \in \mathcal{J}} I(\mathbf{Y}; M_{i}|\vec{M}_{\tilde{\mathcal{J}}_{i}}, G_{1}, G_{3}, \vec{E})$$

$$\stackrel{(a)}{\leq} \sum_{i \in \mathcal{J}} I(\mathbf{Y}; M_{i}|\vec{M}_{\mathcal{J}_{i}}, \vec{L}_{\mathcal{J}_{i}}, G_{1}, G_{3}, \vec{E})$$

$$= \sum_{i \in \mathcal{J}} I(\mathbf{Y}; M_{i}|\vec{M}_{\mathcal{J}_{i}}, \vec{L}_{\mathcal{J}_{i}}, G_{1}, G_{3}, \vec{E}_{\mathcal{J}_{i}^{c}})$$

$$= \sum_{i \in \mathcal{J}} I(\mathbf{Y}; M_{i}|\vec{\mathbf{X}}_{\mathcal{J}_{i}}, G_{1}, G_{3}, \vec{E}_{\mathcal{J}_{i}^{c}})$$

$$= \sum_{i \in \mathcal{J}} I(\mathbf{Y}; M_{i}|\vec{E}_{\mathcal{J}^{c}}, \vec{\mathbf{X}}_{\mathcal{J}_{i}}, G_{1}, G_{3}, E_{i}), \tag{28}$$

where (a) follows from Eq. (35) in Appendix.

Step (3): We focus on the randomness of the choice of G_3 . Then, for $s \in [0, \frac{1}{2}]$, we have

$$I(\mathbf{Y}; M_{i}|G_{3}, G_{1}, E_{i}, \vec{E}_{\mathcal{J}^{c}}, \vec{\mathbf{X}}_{\mathcal{J}_{i}})$$

$$\stackrel{(a)}{\leq} q^{-sk'_{n}} e^{sI_{1+s}(\mathbf{Y}; F_{i}|G_{1}, E_{i}, \vec{E}_{\mathcal{J}^{c}}, \vec{\mathbf{X}}_{\mathcal{J}_{i}})}$$

$$\stackrel{(b)}{\leq} q^{-sk'_{n}} e^{sI_{\frac{1}{1-s}}} (\mathbf{Y}; F_{i}|G_{1}, E_{i}, \vec{E}_{\mathcal{J}^{c}}, \vec{\mathbf{X}}_{\mathcal{J}_{i}})$$

$$\stackrel{(c)}{\leq} q^{-sk'_{n}} e^{sI_{\frac{1}{1-s}}} (\mathbf{Y}; F_{i}, E_{i}|G_{1}, \vec{E}_{\mathcal{J}^{c}}, \vec{\mathbf{X}}_{\mathcal{J}_{i}})$$

$$\stackrel{(d)}{\leq} q^{-sk'_{n}} e^{sI_{\frac{1}{1-s}}} (\mathbf{Y}; \mathbf{X}_{i}|G_{1}, \vec{E}_{\mathcal{J}^{c}}, \vec{\mathbf{X}}_{\mathcal{J}_{i}})$$

$$\stackrel{(e)}{\leq} q^{-sk'_{n}} e^{sI_{\frac{1}{1-s}}} (\mathbf{Y}; \mathbf{X}_{i}|G_{1}, \vec{E}_{\mathcal{J}^{c}}, \vec{\mathbf{X}}_{\mathcal{J}^{c}})$$

$$+ q^{-sk'_{n}} e^{sI_{\frac{1}{1-s}}} (\mathbf{Y}; \mathbf{X}_{i}|\vec{\mathbf{X}}_{\mathcal{J}_{i}})$$

where (a) follows from Theorem 4 of [30] and (26); (b) follows from the second inequality in (6); (c) follows from (7); (d) follows from the fact that the pair (E_i, F_i) and \mathbf{X}_i uniquely determine each other; and (e) follows from Appendix B. Due to Conditions (22) – (24), all the terms in (29) go to zero exponentially. Hence, we obtain Theorem 2.

Remark 1: Theorem 2 cannot be shown by simple application of the result of wire-tap channel [29] as follows. Consider the secrecy of the message M_1 of player A_1 . In this case, if other players transmit elements of \mathcal{X}^n with equal probability, the channel from player A_1 to Receiver is given as n-fold extension of the channel $x \mapsto \frac{1}{q} \sum_{x_2,...,x_c} W_{x,x_2,...,x_c}$, which enables us to directly apply the result of wire-tap channel. However, other players



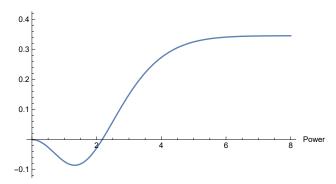


Fig. 2. Our lower bound of secure modulo sum capacity with p=2 when three senders send information (c=3) and the variance v is 1. The base of logarithm is chosen to be e. The horizontal axis expresses the power E. The vertical axis expresses our lower bond $h(\frac{P_0+2P_E+2P_2E+P_3E}{6})+h(\frac{P_0+2P_E}{2})-h(\frac{P_0+2P_2E+P_2E}{3})-h(\frac{P_0+2P_E+P_2E}{4})$ This lower bound approaches $\frac{1}{2}\log 2$ and is positive with $E\geq 2.1715$.

transmit elements of the image of G, which is a subset of \mathcal{X}^n , with equal probability. Hence, the channel from player A_1 to Receiver does not have the above simple form. Therefore, we need more careful discussion.

Remark 2: In this proof, Receiver does not use the knowledge E_1, \ldots, E_c except for $E_{[c]} := \bigoplus_{i=1}^c E_i$. Hence, there is a possibility that the transmission rate can be improved by using this knowledge.

V. Gaussian MAC

A. Real case

First, we consider the Gaussian MAC (2), in which \mathcal{Y} is \mathbb{R} and \mathcal{X}_i is \mathbb{F}_p . Using the Gaussian distribution P_a with the average a and variance v on \mathbb{R} , we have the channel as $W_{x_1,\dots,x_c} := P_{E(x_1+\dots+x_c)}$. In this case, we have

$$I(W) = h(\frac{\sum_{x_1, \dots, x_c} P_{(x_1 + \dots + x_c)E}}{p^c})$$

$$- \sum_{j=0}^{p} \frac{1}{p} h(\sum_{i=1}^{c} \frac{\alpha(i, j)}{p^{c-1}} P_{(j+ip)E})$$
(30)

$$\max_{j \in \mathcal{J}_{\subseteq}^{\subseteq}[c]} I(Y; X_j | \vec{X}_{\mathcal{J}_j})[W] \stackrel{(a)}{=} I(Y; X_1 | \vec{X}_{\{2, \dots, c-1\}})[W]$$
(31)

$$=h(\frac{\sum_{j,j'=0}^{p-1} P_{jj'E}}{p^2}) - h(\frac{\sum_{j=0}^{p-1} P_{jE}}{p}), \tag{32}$$

where h is the differential entropy and $\alpha(i,j)$ is the number of elements $(x_1,\ldots,x_c)\in\{0,\ldots,p-1\}^c$ satisfying

where h is the differential entropy and $\alpha(i,j)$ is the number of elements $(x_1,\ldots,x_c)\in\{0,\ldots,p-1\}$ satisfying $\sum_{i'=1}^c x_{i'}=j+ip$. Here, (a) follows the discussion after Theorem 2. When p=2 and c=3, our lower bound is $h(\frac{P_0+2P_E+2P_2+P_{3E}}{6})-h(\frac{P_0+2P_2+P_{2E}}{3})-h(\frac{P_0+2P_2+P_{2E}}{4})+h(\frac{P_0+P_E}{2})$, which is numerically calculated as Fig. 2. When E goes to infinity, the distributions $\{P_{iE}\}_{i=0}^{c(p-1)}$ can be distinguished so that I(W) goes to $\log p$ and $h(\frac{\sum_{j=0}^{p-1}P_{jj'E}}{p^2})-H(\frac{\sum_{j=0}^{p-1}P_{jE}}{p})$ goes to $H(Q_p)-\log p=\sum_{j=-p+1}^{p-1}\frac{|j-p|}{p^2}\log\frac{p}{|j-p|}$, where the distribution Q_p on $\{-p+1,2,\ldots,p-1\}$ is defined as $Q_p(j):=\frac{|j-p|}{p^2}$. Hence, our lower bound of the secure modulo sum capacity goes to $\sum_{j=-p+1}^{p-1}\frac{|j-p|}{p^2}\log|j-p|$. For example, when p=2, it is $\frac{1}{2}\log 2$.

B. Complex case

Next, we consider the case when \mathcal{Y} is \mathbb{C} , and discuss the case with $\mathcal{X}_i = \mathbb{F}_7$ as a typical case [15], [17]. Let P_a be the Gaussian distribution with the average a and variance v on \mathbb{C} . We define the map u from \mathbb{F}_7 to \mathbb{C} as follows.

$$u(0) = 0, \ u(6) = E, \ u(7) = Ee^{\frac{2\pi}{3}i}, \ u(8) = Ee^{\frac{\pi}{3}i},$$

$$u(1) = Ee^{\frac{4\pi}{3}i}, \ u(3) = Ee^{\frac{5\pi}{3}i}, \ u(2) = -E.$$
(33)

Then, we define $W_{x_1,...,x_c} := P_{(u(x_1)+...+u(x_c))E}$. Hence, we have (31). Due to the same reason as the above case, when E goes to infinity, I(W) goes to $\log 7$ and $I(Y; X_1 | \vec{X}_{\{2,\dots,c-1\}})[W]$ goes to $\sum_{j=-6}^6 \frac{|j-7|}{7^2} \log \frac{7}{|j-7|}$. Hence, our lower bound of the secure modulo sum capacity goes to $\sum_{j=-6}^6 \frac{|j-7|}{7^2} \log |j-7|$.

VI. CONCLUSION

To discuss the generation of common reference string via a multiple access channel, we have introduced the secure modulo sum capacity for a multiple access channel. We have shown that the secure modulo sum capacity equals the channel capacity when the multiple access channel satisfies the symmetric condition. Since the symmetric condition does not hold in a natural setting, we have derived a lower bound for the secure modulo sum capacity in a general setting. We have examined this lower bound under the real and complex Gaussian MAC with numerical analysis.

ACKNOWLEDGMENTS

The author is very grateful to Professor Ángeles Vázquez-Castro and Professor Takeshi Koshiba for their helpful discussions.

APPENDIX A CONDITIONAL MUTUAL INFORMATION

We prepare a lemma for conditional mutual information.

Lemma 1:

$$I(A; B|CD) = -I(B; C|D) + I(A; B|D) + I(B; C|AD).$$
(34)

In particular, when I(B; C|D) = 0, we have

$$I(A;B|D) \le I(A;B|CD). \tag{35}$$

This equation can be shown as follows. Proof.

$$I(A; B|CD) + I(B; C|D) = I(AC; B|D)$$

= $I(A; B|D) + I(B; C|AD)$. (36)

APPENDIX B PROOF OF STEP (e) IN (29)

Given $\vec{\mathbf{x}}_{\mathcal{J}^c} \in \mathcal{X}^{|\mathcal{J}^c|}$ and G_1 , we introduce the following conditions for $\vec{\mathbf{x}}'_{\mathcal{I}^c} \in \mathcal{X}^{|\mathcal{J}^c|}$.

- (C1) $\vec{\mathbf{x}}'_{\mathcal{J}^c} \neq \vec{\mathbf{x}}_{\mathcal{J}^c}$.

- (C2) $(G_2(\mathbf{x}'_j))_{j\in\mathcal{J}^c} = (G_2(\mathbf{x}_j))_{j\in\mathcal{J}^c}.$ (C3) $(\mathbf{x}'_j \ominus \mathbf{x}'_{j'})_{j,j'\in\mathcal{J}^c} \neq (\mathbf{x}_j \ominus \mathbf{x}_{j'})_{j,j'\in\mathcal{J}^c}.$ (C4) $(\mathbf{x}'_j \ominus \mathbf{x}'_{j'})_{j,j'\in\mathcal{J}^c} = (\mathbf{x}_j \ominus \mathbf{x}_{j'})_{j,j'\in\mathcal{J}^c}.$

In the following, we denote the sum with respect to $\vec{\mathbf{x}}'_{\mathcal{J}^c} \in \mathcal{X}^{|\mathcal{J}^c|}$ under the conditions (C1) and (C2) by $\sum_{\vec{\mathbf{x}}'_{\mathcal{J}^c}:(\mathrm{C1})(\mathrm{C2})}$, etc. Using the condition (25), we have

$$\mathbb{E}_{G_{1}} \sum_{\vec{\mathbf{x}}'_{\mathcal{J}^{c}}:(C1)(C2)(C3)} P_{Y|\mathbf{X}_{i}=\mathbf{x}_{i},\vec{\mathbf{X}}_{\mathcal{J}^{c}}=\vec{\mathbf{x}}'_{\mathcal{J}^{c}},\vec{\mathbf{X}}_{\mathcal{J}_{i}}}(y) \leq \sum_{\vec{\mathbf{x}}'_{\mathcal{J}^{c}}:(C1)(C3)} q^{(-nl+(k_{n}+k'_{n}))|\mathcal{J}^{c}|} P_{Y|\mathbf{X}_{i}=\mathbf{x}_{i},\vec{\mathbf{X}}_{\mathcal{J}^{c}}=\vec{\mathbf{x}}'_{\mathcal{J}^{c}},\vec{\mathbf{X}}_{\mathcal{J}_{i}}}(y)$$

$$\leq \sum_{\vec{\mathbf{x}}'_{\mathcal{J}^{c}}:(C1)(C3)} q^{(-nl+(k_{n}+k'_{n}))|\mathcal{J}^{c}|} P_{Y|\mathbf{X}_{i}=\mathbf{x}_{i},\vec{\mathbf{X}}_{\mathcal{J}^{c}}=\vec{\mathbf{x}}'_{\mathcal{J}^{c}},\vec{\mathbf{X}}_{\mathcal{J}_{i}}}(y)$$

$$= q^{(k_{n}+k'_{n})|\mathcal{J}^{c}|} P_{Y|\mathbf{X}_{i}=\mathbf{x}_{i},\vec{\mathbf{X}}_{\mathcal{J}^{c}}=\vec{\mathbf{x}}'_{\mathcal{J}^{c}},\vec{\mathbf{X}}_{\mathcal{J}_{i}}}(y)$$

$$\leq \sum_{\vec{\mathbf{x}}'_{\mathcal{J}^{c}}:(C1)(C4)} q^{-nl+(k_{n}+k'_{n})} P_{Y|\mathbf{X}_{i}=\mathbf{x}_{i},\vec{\mathbf{X}}_{\mathcal{J}^{c}}=\vec{\mathbf{x}}'_{\mathcal{J}^{c}},\vec{\mathbf{X}}_{\mathcal{J}_{i}}}(y)$$

$$\leq \sum_{\vec{\mathbf{x}}'_{\mathcal{J}^{c}}:(C4)} q^{-nl+(k_{n}+k'_{n})} P_{Y|\mathbf{X}_{i}=\mathbf{x}_{i},\vec{\mathbf{X}}_{\mathcal{J}^{c}}=\vec{\mathbf{x}}'_{\mathcal{J}^{c}},\vec{\mathbf{X}}_{\mathcal{J}_{i}}}(y)$$

$$\leq \sum_{\vec{\mathbf{x}}'_{\mathcal{J}^{c}}:(C4)} q^{-nl+(k_{n}+k'_{n})} P_{Y|\mathbf{X}_{i}=\mathbf{x}_{i},\vec{\mathbf{X}}_{\mathcal{J}^{c}}=\vec{\mathbf{x}}'_{\mathcal{J}^{c}},\vec{\mathbf{X}}_{\mathcal{J}_{i}}}(y)$$

$$= q^{(k_{n}+k'_{n})} P_{Y|\mathbf{X}_{i}=\mathbf{x}_{i},(\mathbf{X}_{j}-\mathbf{X}_{j'}=\mathbf{x}_{j}-\mathbf{x}_{j'})_{j,j'\in\mathcal{J}^{c}},\vec{\mathbf{X}}_{\mathcal{J}_{i}}}(y). \tag{38}$$

To show the step (e) in (29), we define the set $\mathcal{S}(G_2, \vec{E}_{\mathcal{J}^c}) := \{\vec{\mathbf{x}}_{\mathcal{J}^c} \in \mathcal{X}^{|\mathcal{J}^c|} | G_2(\mathbf{x}_j) = E_j \text{ for } j \in \mathcal{J}^c\}$. Using

(37) and (38), for $s \in [0, \frac{1}{2}]$, we have the following relations, where the explanations for steps is explained later.

$$\begin{split} &e^{sI_{-L}^{l}}(\mathbf{Y}|\mathbf{X}_{i}|G_{l},\tilde{E}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{l}})} \\ &= \mathbb{E}_{\tilde{\mathbf{X}}_{\mathcal{J}_{i}},\tilde{E}_{\mathcal{J}^{c}},G_{l}}(\int_{\mathcal{Y}}\sum_{\mathbf{X}_{i}}P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{E}_{\mathcal{J}^{c}},G_{l},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y)^{\frac{1}{1-c}}dy)^{1-s} \\ &= \mathbb{E}_{\tilde{\mathbf{X}}_{\mathcal{J}_{i}},\tilde{E}_{\mathcal{J}^{c}},G_{l}}(\int_{\mathcal{Y}}\sum_{\mathbf{X}_{i}}(q^{-(k_{n}+k_{n}^{l})|\mathcal{J}^{c}|}\sum_{\tilde{\mathbf{X}}_{\mathcal{J}^{c}}\in\mathcal{S}(G_{2},\tilde{E}_{\mathcal{J}^{c}})}P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y))^{\frac{1}{1-c}}dy)^{1-s} \\ &\leq \mathbb{E}_{\tilde{\mathbf{X}}_{\mathcal{J}_{i}}}(\int_{\mathcal{Y}}\sum_{\mathbf{X}_{i}}\mathbb{E}_{\tilde{E}_{\mathcal{J}^{c}},G_{l}}(q^{-(k_{n}+k_{n}^{l})|\mathcal{J}^{c}|}\sum_{\tilde{\mathbf{X}}_{\mathcal{J}^{c}}\in\mathcal{S}(G_{2},\tilde{E}_{\mathcal{J}^{c}})}P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y))^{\frac{1}{1-c}}dy)^{1-s} \\ &= \mathbb{E}_{\tilde{\mathbf{X}}_{\mathcal{J}_{i}}}(\int_{\mathcal{Y}}\sum_{\mathbf{X}_{i}}q^{-\frac{(k_{n}+k_{n}^{l})}{1-s}}|\mathcal{J}^{c}|}\mathbb{E}_{\tilde{E}_{\mathcal{J}^{c}},G_{l}}(\sum_{\tilde{\mathbf{X}}_{\mathcal{J}^{c}}\in\mathcal{S}(G_{2},\tilde{E}_{\mathcal{J}^{c}})}P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y)\\ &\cdot \left(P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y)+\sum_{\tilde{\mathbf{X}}_{\mathcal{J}^{c}}\in\mathcal{S}(G_{2},\tilde{E}_{\mathcal{J}^{c}})}P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y)\\ &\cdot \left(P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y)+\sum_{\tilde{\mathbf{X}}_{\mathcal{J}^{c}}\in\mathcal{S}(G_{2},\tilde{E}_{\mathcal{J}^{c}})}P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y)\\ &\cdot \left(P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y)+\sum_{\tilde{\mathbf{X}}_{\mathcal{J}^{c}}\in\mathcal{S}(G_{2},\tilde{E}_{\mathcal{J}^{c}})}P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y)\\ &\cdot \left(P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y)+\sum_{\tilde{\mathbf{X}}_{\mathcal{J}^{c}}\in\mathcal{S}(G_{2},\tilde{E}_{\mathcal{J}^{c}})}P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y)\\ &\cdot \left(P_{Y}|\mathbf{X}_{i}=\mathbf{x}_{i},\tilde{\mathbf{X}}_{\mathcal{J}^{c}}=\tilde{\mathbf{X}}_{\mathcal{J}^{c}},\tilde{\mathbf{X}}_{\mathcal{J}_{i}}(y)}\right)^{1-s}\\ &=\mathbb{E}_{\tilde{\mathbf{X}}_{\mathcal{J}_{i}$$

$$=\mathbb{E}_{\vec{\mathbf{X}}_{\mathcal{I}_{i}}}\left(\int_{\mathcal{Y}}\sum_{\mathbf{x}_{i}}q^{(-nl-\frac{s}{1-s}(k_{n}+k'_{n}))|\mathcal{I}^{c}|}\sum_{\vec{\mathbf{x}}_{\mathcal{I}^{c}}}P_{Y|\mathbf{X}_{i}=\mathbf{x}_{i},\vec{\mathbf{X}}_{\mathcal{I}^{c}}=\vec{\mathbf{x}}_{\mathcal{I}^{c}},\vec{\mathbf{X}}_{\mathcal{I}_{i}}}(y)^{\frac{1}{1-s}}\right.$$

$$+P_{Y|\mathbf{X}_{i}=\mathbf{x}_{i},\vec{\mathbf{X}}_{\mathcal{I}_{i}}}(y)^{\frac{1}{1-s}}$$

$$+q^{(-\frac{s}{1-s}(k_{n}+k'_{n})-nl)(|\mathcal{I}^{c}|-1)}\sum_{\vec{\mathbf{x}}_{\mathcal{I}^{c}}:(2)}P_{Y|\mathbf{X}_{i}=\mathbf{x}_{i},(\mathbf{X}_{j}-\mathbf{X}_{j'}=\mathbf{x}_{j}-\mathbf{x}_{j'})_{j,j'\in\mathcal{I}^{c}},\vec{\mathbf{X}}_{\mathcal{I}_{i}}}(y)^{\frac{s}{1-s}}dy\right)^{1-s}$$

$$=q^{-s(k_{n}+k'_{n})|\mathcal{I}^{c}|}e^{sI^{\downarrow}_{\frac{1}{1-s}}(\mathbf{Y};\mathbf{X}_{i},\vec{\mathbf{X}}_{\mathcal{I}^{c}}|\vec{\mathbf{X}}_{\mathcal{I}_{i}})}+e^{sI^{\downarrow}_{\frac{1}{1-s}}(\mathbf{Y};\mathbf{X}_{i}|\vec{\mathbf{X}}_{\mathcal{I}_{i}})}+q^{-s(k_{n}+k'_{n})(|\mathcal{I}^{c}|-1)}e^{sI^{\downarrow}_{\frac{1}{1-s}}(\mathbf{Y};\mathbf{X}_{i},(\mathbf{X}_{j}-\mathbf{X}_{j'})_{j,j'\in\mathcal{I}^{c}}|\vec{\mathbf{X}}_{\mathcal{I}_{i}})},$$
(39)

where each step can be shown as follows. (a) follows from the concavity of $x \mapsto x^{1-s}$ with $x \ge 0$. (b) follows from the concavity of $x \mapsto x^{\frac{s}{1-s}}$ with $x \ge 0$. (c) follows from the inequalities (37) and (38). (d) follows from the inequality $(x+y+z)^{\frac{s}{1-s}} \le x^{\frac{s}{1-s}} + y^{\frac{s}{1-s}} + z^{\frac{s}{1-s}}$ with $x,y,z \ge 0$.

APPENDIX C

ACHIEVABILITY OF MUTUAL INFORMATION RATE

In this appendix, generalizing the method of [32] with c=2 to the case with a general positive integer c, we show that the mutual information rate $I(W)=I(Y;X_{[c]})[W]$ can be achieved in the sense of computation-and-forward under the code construction presented in **Step (1)** in Section IV. That is, we show that Receiver can decode $F_{[c]}:=\bigoplus_{j=1}^c F_j=G_3(L_{[c]},M_{[c]})$ with an error probability approaching to zero when the rate satisfies the condition (21). In this derivation, we employ basic knowledge for affine maps that is presented in Appendix A of [32].

For this aim, we define the degraded channel W_D from \mathcal{X} to \mathcal{Y} as

$$W_{D,x} := \sum_{(x_1,\dots,x_{c-1})\in\mathbb{F}_q^{l(c-1)}} q^{-l(c-1)} W_{(x_1,x_2,\dots,x_{c-1},x\ominus(\bigoplus_{i=1}^{c-1} x_i))}.$$

$$(40)$$

Then, we find that I(W) is the mutual information between the input X and the output of the degraded channel W_D when X is subject to the uniform distribution on \mathbb{F}_q^l .

In the presented encoder in **Step** (1) in Section IV, the sender A_i encodes the message F_i to $G_1(F_i, E_i)$, and $\vec{E} = (E_1, \dots, E_c)$ is subject to the uniform distribution on $\mathbb{F}_q^{c(nl-(k_n+k'_n))}$. In the following, we assume that Receiver makes the decoder dependently only of G_1 and $E_{[c]} := \bigoplus_{i=1}^c E_i$. Hence, G_1 and $E_{[c]}$ are fixed to g_1 and $e_{[c]}$. When we denote $F_{[c]} \in \mathbb{F}_q^{k_n+k'_n}$ by m, Receiver's receiving signal is subject to the following distribution;

$$W_{(g_{1}(F_{1},E_{1}),...,g_{1}(F_{c-1},E_{c-1}),g_{1}(F_{c},E_{c}))}^{(n)} = W_{(g_{1}(F_{1},E_{1}),...,g_{1}(F_{c-1},E_{c-1}),g_{1}(m\ominus(\bigoplus_{i=1}^{c-1}F_{i}),e_{[c]}\ominus(\bigoplus_{i=1}^{c-1}E_{i})))}$$

$$= W_{(g_{1}(F_{1},E_{1}),...,g_{1}(F_{c-1},E_{c-1}),g_{1}(m,e_{[c]})\ominus(\bigoplus_{i=1}^{c-1}g_{1}(F_{i},E_{i}))}^{c-1}$$

$$(41)$$

Here, $(F_1, E_1), \ldots, (F_{c-1}, E_{c-1})$ are independently subject to the uniform distribution on \mathbb{F}_q^{nl} , and the decoder to recover $F_{[c]}$ does not depend on them. Taking the expectation with respect to these variables, we have

$$\mathbb{E}_{(F_1,E_1),\dots,(F_{c-1},E_{c-1})}W_{(g_1(F_1,E_1),\dots,g_1(F_{c-1},E_{c-1}),g_1(m,e_{[c]})\ominus(\bigoplus_{i=1}^{c-1}g_1(F_i,E_i))}^{(n)} = W_{D,g_1(m,e_{[c]})}^{(n)}.$$

$$(42)$$

When Receiver applies the maximum likelihood decoder to the degraded channel $W^{(n)}$, the randomized choice of the affine code $G_1(m)+E_{[c]}$ satisfies the condition of Proposition 2 of [32] because the condition (25) and $G_2\circ G_1(F,E)=E$ imply the condition (38) in [32, Appendix A]. In the application of Proposition 2 of [32], G_1 of this paper corresponds to F in [32, Appendix A]. Hence, the average decoding error probability for $F_{[c]}$ goes to zero under a rate smaller than $I(W)=I(Y;X_{[c]})[W]$, i.e., under the condition (21).

REFERENCES

- [1] B. Chor and E. Kushilevitz, "A communication-privacy tradeoff for modular addition," *Information Processing Letters*, vol. 45, no. 4, 205 210, (1993).
- [2] B. Chor and N. Shani, "The privacy of dense symmetric functions," Computational Complexity, vol 5, no. 1, 43 59, (1995.)
- [3] E. J. Lee and E. Abbe, "A Shannon Approach to Secure Multi-party Computations," arXiv:1401.7360v3
- [4] D. Data, B. Kumar Dey, M. Mishra, V. M. Prabhakaran, "How to Securely Compute the Modulo-Two Sum of Binary Sources," Proceeding of IEEE Information Theory Workshop (ITW) 2014, Hobart, TAS, Australia, 2-5 Nov. 2014.
- [5] C. E. Shannon, "Two-way communication channels", Proc. 4th Berkeley Symp. Math. Statist. Probability, Berkeley, CA, vol. 1, 611–644 (1961).
- [6] R. Ahlswede, "Multiway communication channels," Proceedings of 2nd International Symposium on Information Theory, Thakadsor, Armenian SSR, Akademiai Kiado, Budapest, 2352 (1971).
- [7] H. H.-J. Liao, Multiple access channels, Ph.D. dissertation, University of Hawaii, Honolulu, HI, (1972).
- [8] M. Goldenbaum, H. Boche, and H. V. Poor, "On Secure Computation Over the Binary Modulo-2 Adder Multiple-Access Wiretap Channel," *Proc. IEEE Information Theory Workshop (ITW' 16)*, Cambridge, UK, September 2016, pp. 21-25.
- [9] M. Goldenbaum, H. Boche, and H. V. Poor, "Secure Computation of Linear Functions Over Linear Discrete Multiple-Access Wiretap Channels," Proc. 50th Asilomar Conference on Signals, Systems and Computers (ACSSC'16), Pacific Grove, CA, USA, November 2016, pp. 1670-1674.
- [10] B. Nazer, M. Gastpar, "Compute-and-forward: A novel strategy for cooperative networks," Asilomar Conf. on Signals, Syst. and Computers 2008, 69–73 (2008).
- [11] B. Nazer and M. Gastpar, "Compute-and-forward: harnessing interference through structured codes," *IEEE Trans. Inform. Theory*, vol. 57 no.10, 6463–6486 (2011).
- [12] B. Nazer, V.R. Cadambe, V. Ntranos and G. Caire, "Expanding the compute-and-forward framework: Unequal powers, signal levels, and multiple linear combinations," *IEEE Trans. Inform. Theory*, vol. 62, no. 9, 4879–4909 (2016).
- [13] U. Niesen and P. Whiting, "The degrees of freedom of compute-and-forward," IEEE Trans. Inform. Theory, vol. 58, 5214–5232 (2012).
- [14] C. Feng, D. Silva, and F. R. Kschischang, "An algebraic approach to physical-layer network coding," *IEEE Trans. Inform. Theory*, vol. 59, 7576–7596 (2013).
- [15] N. E. Tunali, Y.-C. Huang, J. Boutros, and K. Narayanan, "Lattices over Eisenstein integers for compute-and-forward," Conf. Allerton Comm., Control, and Computing 2012, 33–40 (2012).
- [16] Y.-C. Huang, K. Narayanan and P.-C. Wang, "Adaptive compute-and-forward with lattice codes over algebraic integers," Proc. of IEEE Int. Symp. on Info. Theory (ISIT 2015), 14-19 June 2015, Hong Kong, China, 566–570.
- [17] A. Vazquez-Castro, "Arithmetic geometry of compute and forward," IEEE Inf. Theory Workshop 2014, 2-5 Nov. 2014, Hobart, TAS, Australia, 122–126.
- [18] Z. Ren, J. Goseling, J. H. Weber, and M. Gastpar, "Secure Transmission Using an Untrusted Relay with Scaled Compute-and-Forward," Proc. of IEEE Inf. Theory Workshop 2015, 26 April – 1 May 2015, Jerusalem, Israel.
- [19] X. He and A. Yener, "End-to-end secure multi-hop communication with untrusted relays," *IEEE. Trans. on Wireless Comm.*, vol. 12, 1–11 (2013).
- [20] X. He and A. Yener, "Strong secrecy and reliable Byzantine detection in the presence of an untrusted relay," *IEEE Trans. Inform. Theory*, vol. 59, no. 1, 177–192 (2013).
- [21] S. Vatedka, N. Kashyap, and A. Thangaraj, "Secure Compute-and-Forward in a Bidirectional Relay," *IEEE Trans. Inform. Theory*, vol. 61, 2531 2556 (2015).
- [22] A. A. Zewail and A. Yener, "The Two-Hop Interference Untrusted-Relay Channel with Confidential Messages," Proc. of IEEE Inf. Theory Workshop 2015, 11-15 Oct. 2015, Jeju, South Korea, 322–326
- [23] M. Hayashi, T. Wadayama, and A. Vázquez-Castro, "Secure Computation-and-Forward Communication with Linear Codes," *IEEE Journal on Selected Areas in Information Theory*, Volume: 2, Issue: 1, 139 148 (2021).
- [24] M. Hayashi and T. Koshiba, "Secure Modulo Zero-Sum Randomness as Cryptographic Resource," https://eprint.iacr.org/2018/802
- [25] M. Hayashi and T. Koshiba, "Verifiable Quantum Secure Modulo Summation," arXiv:1910.05976 (2019).
- [26] M. Hayashi, "Secure physical layer network coding versus secure network coding," *Proc. IEEE Inf. Theory Workshop 2018 (ITW 2018)*, Guangzhou, China, November 25-29, 2018, pp. 430 434.
- [27] M. Hayashi and R. Matsumoto, "Secure Multiplex Coding with Dependent and Non-Uniform Multiple Messages," *IEEE Trans. Inform. Theory*, vol. 62, no. 5, 2355 2409 (2016).
- [28] S. S. Ullah, G. Liva, and S. C. Liew, "Physical-layer Network Coding: A Random Coding Error Exponent Perspective," *IEEE Inf. Theory Workshop 2017*, 5-10 Nov. 2017, Kaohsiung, Taiwan.
- [29] A. D. Wyner, "The wiretap channel," Bell. Sys. Tech. Jour., vol. 54, 1355-1387, 1975.
- [30] M. Hayashi, "Exponential decreasing rate of leaked information universal random privacy amplification," *IEEE Trans. Inf. Theory*, vol. 57, no. 6, pp. 3989 4001, 2011.
- [31] M. N. Wegman and J. Carter, "New hash functions and their use in authentication and set equality," *Journal of Computer Sciences and Systems*, vol. 22, pp. 265 279, 1981.
- [32] M. Hayashi and Á. Vázquez-Castro, "Physical Layer Computation as NOMA for Integrated Wireless Systems," *IEEE Transactions on Communications*; DOI: 10.1109/TCOMM.2021.3067907.
- [33] S.H. Lim, C. Feng, A. Pastore, B. Nazer, and M. Gastpar, "Compute-Forward for DMCs: Simultaneous Decoding of Multiple Combinations," *IEEE Trans. Inf. Theory*, vol. 66, no. 10, pp. 6242 6255, 2020