Asynchronous Guessing Subject to Distortion

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Abstract—The problem of guessing subject to distortion is considered, and the performance of randomized guessing strategies is investigated. A one-shot achievability bound on the guessing moment (i.e., moment of the number of required queries) is given. Applying this result to i.i.d. sources, it is shown that randomized strategies can asymptotically attain the optimal guessing moment. Further, a randomized guessing scheme which is feasible even when the block size is extremely large is proposed, and a singleletter characterization of the guessing moment achievable by the proposed scheme is obtained.

I. INTRODUCTION

Consider the problem of guessing the realized value x of a random variable X using a sequence of queries of the form "Is X = x?". We are interested in how many queries are required until an affirmative answer is obtained; the number is called the *guesswork*. This guessing problem was introduced by Massey [1], where the expectation of guesswork was investigated. Subsequently Arikan [2] investigated the ρ -th moment G_{ρ} of guesswork, which is called the *guessing moment*. Further, Arikan and Merhav [3] extended Arikan's result [2] to the lossy case, where the value x of X is not necessarily identified but it is required to find a value \hat{x} satisfying $d(x, \hat{x}) \leq \Delta$ for given distortion measure d and distortion level Δ .

Recently asynchronous guessing problem was introduced by Salamatian *et al.* [4] as an information-theoretic model for brute-force botnet attacks. In the asynchronous setting, the guesser is restricted so that it does not know which queries were already asked. In [4], a modified variation V_{ρ} of the guessing moment G_{ρ} attainable by asynchronous guessing strategies is investigated. Their result implies that the optimal asynchronous guessing is given by *randomized guessing*, where the guesser choses a query according to a certain probability distribution, and that the penalty of lack of synchronization is asymptotically negligible.

The primary motivation of this work is to extend the study of [4] to the lossy case. Specifically, randomized guessing subject to distortion is studied. Our model provides a simplified mathematical model for brute-force attacks against bio-metrics authentication, where attacker's task is to find a query which is sufficiently similar to the bio-metric data stored in the system.

A. Contributions

Our first contribution is to give an achievability bound on V_{ρ} in terms of a variation of the Rényi entropy (Theorem 1). In particular, when the order ρ of the moment is an integer, we directly evaluate the guessing moment G_{ρ} and give an achievability bound (Theorem 2 and its corollaries). Our achievability result reveals that there exists a deterministic

quantizer π which does not depend on the parameter ρ and the optimal guessing strategy is given by the tilted distribution of the quantized $\hat{X} = \pi(X)$.

Next we apply our achievability bound to independent and identically distributed (i.i.d.) sources, and then, show that synchronization is not necessary to achieve the asymptotically optimal guessing moment. Furthermore, for asymptotic case, we propose i.i.d. asynchronous guessing strategies, which are simple and feasible even when the block size n is extremely large. We investigate the asymptotic performance of i.i.d. asynchronous guessing strategies and give a single-letter characterization of the optimal guessing moment achievable by i.i.d. strategies (Theorem 3 and its corollary).

B. Related Work

The study of guessing was pioneered by Massey [1]. Arikan [2] demonstrated that the Rényi entropy [5] characterizes the guessing moment (up to some factor). Recently, tighter bounds on the guessing moment were given by Sason and Verdú [6].

The guessing problem has been studied in various contexts such as guessing allowing errors [7], guessing subject to distortion [3], [8], investigation of large deviation perspective of guessing [9], [10], guesswork in multi-user systems [11], and guesswork with distributed encoders [12] and so on.

Applications of guessing are around the information security, e.g., cracking passwords; See the introduction of [4] and Section II of [13] for review on guessing and security. To understand the impact of synchronization in botnet attacks, Salamatian *et al.* [4] proposed a simplified model for distributed brute-force attacks and introduced randomized guessing. Merhav and Cohen [13] studied randomized guessing under source uncertainty and proposed the universal randomized guessing strategy based on the LZ78 data compression algorithm [14]. The problem of randomized guessing under the individualsequence approach was also investigated by Merhav [15].

C. Organization

The rest of this paper is organized as follows. In Section II, we introduce a variation of the Rényi entropy and its property. Section III describes our main results; one-shot results are given in Section III-A and asymptotic results for i.i.d. sources are given in Section III-B. All theorems are proved in Section IV. Section V concludes the paper.

II. PRELIMINARY

Let \mathcal{X} and $\hat{\mathcal{X}}$ be finite alphabets. Let $d: \mathcal{X} \times \hat{\mathcal{X}} \to [0, \infty)$ be a distortion measure and fix the distortion level $\Delta \geq 0$. For each $x \in \mathcal{X}$, let $\mathcal{A}_{\Delta}(x) \triangleq \{\hat{x} \in \hat{\mathcal{X}} : d(x, \hat{x}) \leq \Delta\}$. We assume that $\mathcal{A}_{\Delta}(x) \neq \emptyset$ for any $x \in \mathcal{X}$.

In the discussion of one-shot guessing, we will use the quantity $H^{\Delta}_{\alpha}(X)$, which was introduced in [8]. Let \mathcal{W}_{Δ} be the set of conditional distributions $P_{\hat{X}|X}$ such that $\Pr\{d(X, \hat{X}) \leq \Delta\} = 1$ (or equivalently $P_{\hat{X}|X}(\hat{x}|x) = 0$ if $d(x, \hat{x}) > \Delta$). Then $H^{\Delta}_{\alpha}(X)$ is defined as follows.

Definition 1: For $\alpha \in (0, 1) \cup (1, \infty)$,

$$H^{\Delta}_{\alpha}(X) \triangleq \inf_{P_{\hat{X}|X} \in \mathcal{W}_{\Delta}} H_{\alpha}(\hat{X})$$
(1)

where $H_{\alpha}(\hat{X})$ is the Rényi entropy of $\hat{X} \sim P_{\hat{X}}$:¹

$$H_{\alpha}(\hat{X}) \triangleq \frac{1}{1-\alpha} \log \sum_{\hat{x} \in \hat{\mathcal{X}}} \left[P_{\hat{X}}(\hat{x}) \right]^{\alpha}.$$

As shown in Appendix A, the infimum in (1) can be achieved by a deterministic quantizer.

Proposition 1: There exists a deterministic function $\pi: \mathcal{X} \to \hat{\mathcal{X}}$ such that $\pi(x) \in \mathcal{A}_{\Delta}(x)$ for all $x \in \mathcal{X}$ and that $\hat{X} = \pi(X)$ satisfies, for all $\alpha \in (0, 1)$,

$$H_{\alpha}(\hat{X}) = H_{\alpha}^{\Delta}(X).$$
(2)

III. MAIN RESULTS

A. One-Shot Bounds

Let us consider a random variable $X \sim P_X$ on \mathcal{X} . We investigate the problem of guessing the realization value x of X subject to the distortion measure d.

An asynchronous guessing strategy is determined by a distribution $P_{\hat{X}}$ on \hat{X} , which is independent of the realization x of X but may depend on P_X . The guesser continues to emit i.i.d. sequence of random variables $\hat{X}_1, \hat{X}_2, \ldots$ according to $P_{\hat{X}}$ as long as $d(x, \hat{X}_i) > D$. The number of guesses $G(x|P_{\hat{X}})$ is given by the first index k such that $d(x, \hat{X}_k) \leq \Delta$ or equivalently $\hat{X}_k \in \mathcal{A}_{\Delta}(x)$. It should be emphasized that, even when X = x is fixed, the number $G(x|P_{\hat{X}})$ of guesses is a random variable. It is easily seen that the distribution of $G(x|P_{\hat{X}})$ is the geometric distribution with the parameter $P_{\hat{X}}(\mathcal{A}_{\Delta}(x))$, i.e.,

$$\Pr\{G(x|P_{\hat{X}}) = k\} = \left[1 - P_{\hat{X}}(\mathcal{A}_{\Delta}(x))\right]^{k-1} P_{\hat{X}}(\mathcal{A}_{\Delta}(x)).$$

Thus, for a given parameter $\rho > 0$, the ρ -th moment of the number of guesses can be written as

$$\mathbb{E}\left[G_{\rho}(X|P_{\hat{X}})\right] = \sum_{x \in \mathcal{X}} P_X(x)G_{\rho}(x|P_{\hat{X}})$$

where²

$$G_{\rho}(x|P_{\hat{X}}) \triangleq \sum_{k=1}^{\infty} k^{\rho} \left[1 - P_{\hat{X}}(\mathcal{A}_{\Delta}(x)) \right]^{k-1} P_{\hat{X}}(\mathcal{A}_{\Delta}(x)).$$

¹Throughout the paper, log denotes the natural logarithm.

²Note that $G_{\rho}(x|P_{\hat{X}})$ is not a random variable although $G(x|P_{\hat{X}})$ is.

While our main interest is $\mathbb{E}\left[G_{\rho}(X|P_{\hat{X}})\right]$, we first investigate the quantity³

$$V_{\rho}(x|P_{\hat{X}}) \triangleq \mathbb{E}\left[\begin{pmatrix} G(x|P_{\hat{X}}) + \rho - 1 \\ \rho \end{pmatrix} \right].$$

where the expectation \mathbb{E} is taken with respect to the random variable $G(x|P_{\hat{X}})$ and $\binom{a}{b}$ is the generalized binomial coefficient defined in terms of the gamma function Γ , i.e.,

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}.$$

The virtue of V_{ρ} is that its value can be explicitly given as follows.

Proposition 2: For any guessing strategy $P_{\hat{\chi}}$ and $\rho > 0$,

$$V_{\rho}(x|P_{\hat{X}}) = \left(\frac{1}{P_{\hat{X}}(\mathcal{A}_{\Delta}(x))}\right)^{\rho}, \quad \forall x \in \mathcal{X}.$$

Corollary 1: For any $P_{\hat{X}}$ and $\rho > 0$,

$$\mathbb{E}[V_{\rho}(X|P_{\hat{X}})] = \sum_{x \in \mathcal{X}} P_X(x) \left(\frac{1}{P_{\hat{X}}(\mathcal{A}_{\Delta}(x))}\right)^{\rho}.$$

The proposition is proved in Appendix B.

Our one-shot achievability bound on $\mathbb{E}[V_{\rho}(X|P_{\hat{X}})]$ is stated as follows.

Theorem 1: There exists a deterministic function $\pi: \mathcal{X} \to \hat{\mathcal{X}}$ such that, for all $\rho > 0$, the tilted distribution

$$P_{\hat{X}_{\rho}^{*}}(\hat{x}) \triangleq \frac{P_{\hat{X}}(\hat{x})^{\frac{1}{1+\rho}}}{\sum_{\hat{x}'} P_{\hat{X}}(\hat{x}')^{\frac{1}{1+\rho}}}$$
(3)

of the distribution $P_{\hat{X}}$ of $\hat{X} = \pi(X)$ satisfies

$$\log \mathbb{E}[V_{\rho}(X|P_{\hat{X}_{\rho}^*})] \le \rho H^{\Delta}_{\frac{1}{1+\rho}}(X).$$

The theorem is proved in Section IV-A.

Now we investigate our main interest, i.e., the ρ -th moment $\mathbb{E}[G_{\rho}(X|P_{\hat{X}})]$ of the number of guesses. In particular, we consider the case where $\rho = 1, 2, ...$ is a positive integer.⁴ In this case, we can directly evaluate $G_{\rho}(x|P_{\hat{X}})$ by using the moment generating function $M(t) = p_x e^t / (1 - (1 - p_x)e^t)$ of the geometric distribution with the parameter $p_x \triangleq P_{\hat{X}}(\mathcal{A}_{\Delta}(x))$; e.g., the first four moments are

$$G_1(x|P_{\hat{X}}) = 1/p_x,$$

$$G_2(x|P_{\hat{X}}) = (2 - p_x)/p_x^2,$$

$$G_3(x|P_{\hat{X}}) = (p_x^2 - 6p_x + 6)/p_x^3,$$

$$G_4(x|P_{\hat{X}}) = (-p_x^3 + 14p_x^2 - 36p_x + 24)/p_x^4.$$

Further, we have upper and lower bounds on $G_{\rho}(x|P_{\hat{X}})$ as follows.

³The idea of considering the modification V_{ρ} of the moment G_{ρ} was introduced by Salamatian *et al.* [4].

⁴For non-integer $\rho > 0$, we may numerically evaluate the ρ -th moment by using the technique recently developed in [16].

Theorem 2: For any guessing strategy $P_{\hat{X}}$, any $x \in \mathcal{X}$, and any positive integer ρ ,

$$V_{\rho}(x|P_{\hat{X}}) \le G_{\rho}(x|P_{\hat{X}}) \le (\rho!)V_{\rho}(x|P_{\hat{X}}),$$

or equivalently

$$\left(\frac{1}{P_{\hat{X}}(\mathcal{A}_{\Delta}(x))}\right)^{\rho} \le G_{\rho}(x|P_{\hat{X}}) \le (\rho!) \left(\frac{1}{P_{\hat{X}}(\mathcal{A}_{\Delta}(x))}\right)^{\rho}.$$

Corollary 2: For any $P_{\hat{X}}$ and positive integer ρ ,

$$\mathbb{E}[V_{\rho}(X|P_{\hat{X}})] \le \mathbb{E}[G_{\rho}(X|P_{\hat{X}})] \le (\rho!)\mathbb{E}[V_{\rho}(X|P_{\hat{X}})]$$

and

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$$\sum_{x \in \mathcal{X}} P_X(x) \left(\frac{1}{P_{\hat{X}}(\mathcal{A}_{\Delta}(x))} \right)^{\rho}$$

$$\leq \mathbb{E}[G_{\rho}(X|P_{\hat{X}})]$$

$$\leq (\rho!) \sum_{x \in \mathcal{X}} P_X(x) \left(\frac{1}{P_{\hat{X}}(\mathcal{A}_{\Delta}(x))} \right)^{\rho}.$$

The theorem is proved in Section IV-B.

From Theorems 1 and 2, we can obtain a one-shot achievability result in terms of $\mathbb{E}[G(X|P_{\hat{X}})^{\rho}]$ as follows.

Corollary 3: There exists a deterministic function $\pi: \mathcal{X} \to \hat{\mathcal{X}}$ such that, for any positive integer ρ , the tilted distribution $P_{\hat{X}_{*}}$ defined as (3) satisfies

$$\log \mathbb{E}[G_{\rho}(X|P_{\hat{X}_{\rho}^*})] \le \rho H^{\Delta}_{\frac{1}{1+\rho}}(X) + \log(\rho!).$$

Let us compare our result with that of synchronous case. A synchronous guessing strategy is determined by a bijection $\mathcal{G}: \hat{\mathcal{X}} \to \{1, 2, \dots, |\hat{\mathcal{X}}|\}$, and the number of guesses when X = x is given by

$$G^{\mathrm{sync}}(x|\mathcal{G}) \triangleq \min_{\hat{x} \in \mathcal{A}_{\Delta}(x)} \mathcal{G}(\hat{x}).$$

According to [8], the optimal ρ -th moment achievable by synchronous strategies satisfies

$$\rho H^{\Delta}_{\frac{1}{1+\rho}}(X) - \rho \log \log(1 + \min\{|\mathcal{X}|, |\hat{\mathcal{X}}|\}) \\
\leq \log \min_{\mathcal{G}} \mathbb{E}[G^{\text{sync}}(X|\mathcal{G})^{\rho}] \\
\leq \rho H^{\Delta}_{\frac{1}{1+\rho}}(X).$$
(4)

Comparing (4) with Corollary 3, we can see that the penalty of lack of synchronization is upper bounded by

$$\log(\rho!) + \rho \log \log(1 + \min\{|\mathcal{X}|, |\hat{\mathcal{X}}|\}).$$

B. Asymptotics for Stationary Memoryless Sources

In this subsection, we apply our one-shot results to i.i.d. sources and investigate the asymptotic behavior of the ρ -th moment of the number of guesses.⁵

Let \mathcal{X}^n (resp. $\hat{\mathcal{X}}^n$) is the *n*-fold Cartesian product of \mathcal{X} (resp. $\hat{\mathcal{X}}$). The distortion between $x \in \mathcal{X}^n$ and $\hat{x} \in \hat{\mathcal{X}}^n$ per

symbol is defined by $d_n(\boldsymbol{x}, \hat{\boldsymbol{x}}) = (1/n) \sum_{i=1}^n d(x_i, \hat{x}_i)$. We investigate the problem of guessing the realization value \boldsymbol{x} of $X^n = (X_1, X_2, \dots, X_n)$ subject to the distortion measure d_n , where X_1, \dots, X_n are independently generated according to an identical distribution P_X on \mathcal{X} .

As in the one-shot case, an asynchronous guessing strategy is determined by a distribution $P_{\hat{X}^n}$ on $\hat{\mathcal{X}}^n$. As a direct consequence of Corollary 3, there exists a deterministic function $\pi_n \colon \mathcal{X}^n \to \hat{\mathcal{X}}^n$ such that the strategy $P_{\hat{X}^{*n}_{\rho}}$ induced by π_n satisfies

$$\frac{1}{n}\log \mathbb{E}[G_{\rho}(X^n|P_{\hat{X}_{\rho}^{*n}})] \leq \frac{1}{n}\rho H_{\frac{1}{1+\rho}}^{\Delta}(X^n) + \zeta_n$$

where $\zeta_n \triangleq (\rho!)/n \to 0$ as $n \to \infty$. This fact indicates that synchronization is not necessary to achieve the asymptotically optimal guessing moment.

However, the strategy $P_{\hat{X}_{\rho}^{*n}}$ may be not feasible when n is large. In particular, it may not be easy to find and implement the function π_n . Hence, we restrict the class of guessing strategies.

Definition 2: An asynchronous guessing strategy $P_{\hat{X}^n}$ is said to be an *i.i.d. asynchronous guessing strategy* if there exists a distribution $Q_{\hat{X}}$ on $\hat{\mathcal{X}}$ satisfying

$$P_{\hat{X}^n}(\hat{\boldsymbol{x}}) = Q_{\hat{X}}^n(\boldsymbol{x}) \triangleq \prod_{i=1}^n Q_{\hat{X}}(\hat{x}_i), \quad \forall \hat{\boldsymbol{x}} \in \hat{\mathcal{X}}^n.$$

In the following, we investigate the optimal guessing moment asymptotically achievable by i.i.d. asynchronous guessing strategies. To state our result, we introduce some notation. We use the following standard information-theoretic quantities [17]. For a distribution P and a conditional distribution V, let H(P) be the entropy of P, $H(V|P) = \sum_{x} P(x)H(V(\cdot|x))$ be the conditional entropy, and I(P, V) = H(PV) - H(V|P)be the mutual information, where PV is the distribution such that $PV(\hat{x}) = \sum_{x} P(x)V(\hat{x}|x)$. For two distributions P and Q, let D(P||Q) be the divergence between P and Q. Let $\overline{W}_{\Delta}(Q_X)$ be the set of conditional distributions satisfying $\sum_{x,\hat{x}} Q_X(x)V(\hat{x}|x)d(x,\hat{x}) \leq \Delta$.

Definition 3: For distributions Q_X on \mathcal{X} and $Q_{\hat{X}}$ on $\hat{\mathcal{X}}$,

$$R(Q_X, Q_{\hat{X}} | \Delta) \triangleq \min_{V \in \overline{\mathcal{W}}_{\Delta}(Q_X)} [I(Q_X, V) + D(Q_X V || Q_{\hat{X}})]$$

where $Q_X V$ is the distribution such that $Q_X V(\hat{x}) = \sum_x Q_X(x) V(\hat{x}|x)$.

The next theorem, which is proved in Section IV-C, is our main result of this subsection.

Theorem 3: For any i.i.d. asynchronous guessing strategy $Q_{\hat{X}}$ and positive integer ρ ,

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}[G_{\rho}(X^n | Q_{\hat{X}}^n)] \\= \max_{Q_X} \left[\rho R(Q_X, Q_{\hat{X}} | \Delta) - D(Q_X \| P_X) \right]$$

where the maximum is taken over all distributions Q_X on \mathcal{X} .

⁵To simplify the argument, we assume that ρ is a positive integer. However, it is not hard to show that our argument is valid for non-integer $\rho > 0$; See Appendix C.

As a consequence of the theorem, we obtain the following result on the optimal ρ -th moment achievable by i.i.d. asynchronous guessing strategies.

Corollary 4: The exponent of the optimal ρ -th moment achievable by i.i.d. asynchronous guessing strategies is

$$E_{\rho}^{\text{i.i.d}}(P_X|\Delta) \triangleq \min_{Q_{\hat{X}}} \max_{Q_X} \left[\rho R(Q_X, Q_{\hat{X}}) - D(Q_X \| P_X) \right]$$

where min (resp. max) is taken over all distributions $Q_{\hat{X}}$ on $\hat{\mathcal{X}}$ (resp. Q_X on \mathcal{X}).

Remark 1: The corollary guarantees that we can find the optimal i.i.d. strategy by solving the minimization in the definition of $E_{\rho}^{\text{i.i.d}}(P_X|\Delta)$, which does not depend on n. So, our strategy is feasible even when n is extremely large.

Remark 2: It should be emphasized that Theorem 3 holds for any strategy $Q_{\hat{X}}$. In other words, $Q_{\hat{X}}$ is not necessarily depend on P_X . Hence, it can be easily applied to guessing under source uncertainty. Assume that the guesser does not know the source distribution P_X but it knows the fact that $P_X \in \mathcal{P}$ for a subset \mathcal{P} of distributions. Theorem 3 shows that, under this setting, the exponent of the optimal ρ -th guessing moment asymptotically achievable by i.i.d. strategies is

$$\min_{Q_{\hat{X}}} \max_{P_X \in \mathcal{P}} \max_{Q_X} \left[\rho R(Q_X, Q_{\hat{X}}) - D(Q_X \| P_X) \right].$$

Lastly, we investigate the penalty of restricting strategies to be i.i.d.

Let us define

$$E_{\rho}(P_X|\Delta) \triangleq \max_{Q_X} \left\{ \rho R(Q_X|\Delta) - D(Q_X||P_X) \right\}$$

where the maximum is taken over all distributions Q_X on \mathcal{X} and $R(Q_X|\Delta)$ is the *rate-distortion function*; i.e.,

$$R(Q_X|\Delta) \triangleq \min_{V \in \overline{\mathcal{W}}_\Delta(Q_X)} I(Q_X, V).$$

It is known that $E_{\rho}(P_X|\Delta)$ is the exponent of the optimal ρ th guessing moment asymptotically achievable by synchronous strategies [3]; i.e.,

$$E_{\rho}(P_X|\Delta) = \lim_{n \to \infty} \frac{1}{n} \log \min_{\mathcal{G}_n \text{ on } \hat{\mathcal{X}}^n} \mathbb{E}[G^{\mathsf{sync}}(X^n | \mathcal{G}_n)^{\rho}].$$

Further, results of [8] implies that

$$E_{\rho}(P_X|\Delta) = \lim_{n \to \infty} \frac{\rho}{n} H^{\Delta}_{\frac{1}{1+\rho}}(X^n).$$

On the other hand, since

$$\min_{Q_{\hat{X}}} R(Q_X, Q_{\hat{X}} | \Delta) = R(Q_X | \Delta),$$

we have

$$E_{\rho}^{\text{i.i.d}}(P_X|\Delta) = \min_{Q_{\hat{X}}} \max_{Q_X} \left[\rho R(Q_X, Q_{\hat{X}}|\Delta) - D(Q_X || P_X) \right]$$

$$\geq \max_{Q_X} \min_{Q_{\hat{X}}} \left[\rho R(Q_X, Q_{\hat{X}}|\Delta) - D(Q_X || P_X) \right]$$

$$= E_{\rho}(P_X |\Delta).$$

From this, we can see the suboptimality of i.i.d. strategies and evaluate the penalty as

$$\begin{split} \min_{Q_{\hat{X}}} \max_{Q_X} \left[\rho R(Q_X, Q_{\hat{X}} | \Delta) - D(Q_X \| P_X) \right] \\ - \max_{Q_X} \min_{Q_{\hat{X}}} \left[\rho R(Q_X, Q_{\hat{X}} | \Delta) - D(Q_X \| P_X) \right] \\ \text{IV. Proofs} \end{split}$$

A. Proof of Theorem 1

Let π be the function given in Proposition 1 and let $\hat{X} = \pi(X)$. Since $\pi(x) \in \mathcal{A}_{\Delta}(x)$, we have $P_{\hat{X}_{\rho}^{*}}(\mathcal{A}_{\Delta}(x)) \geq P_{\hat{X}_{\rho}^{*}}(\pi(x))$. Hence, letting $\pi^{-1}(\hat{x}) \triangleq \{x \in \mathcal{X} : \pi(x) = \hat{x}\}$, we have

$$\mathbb{E}[V_{\rho}(X|P_{\hat{X}_{\rho}^{*}})] = \sum_{x \in \mathcal{X}} P_{X}(x) \left(\frac{1}{P_{\hat{X}_{\rho}^{*}}(\mathcal{A}_{\Delta}(x))}\right)^{\rho}$$

$$\leq \sum_{x \in \mathcal{X}} P_{X}(x) \left(\frac{1}{P_{\hat{X}_{\rho}^{*}}(\pi(x))}\right)^{\rho}$$

$$= \sum_{\hat{x} \in \hat{\mathcal{X}}} \sum_{x \in \pi^{-1}(\hat{x})} P_{X}(x) \left(\frac{1}{P_{\hat{X}_{\rho}^{*}}(\hat{x})}\right)^{\rho}$$

$$= \sum_{\hat{x} \in \hat{\mathcal{X}}} P_{\hat{X}}(\hat{x}) \left(\frac{1}{P_{\hat{X}_{\rho}^{*}}(\hat{x})}\right)^{\rho}$$

$$= \exp\left[\rho H_{\frac{1}{1+\rho}}(\hat{X})\right]$$

$$= \exp\left[\rho H_{\frac{1}{1+\rho}}(X)\right]$$

where the last equality follows from (2).

B. Proof of Theorem 2

Since $\rho > 1$, using Jensen's inequality, we have

$$G_{\rho}(x|P_{\hat{X}}) = \sum_{k=1}^{\infty} \Pr\{G(x|P_{\hat{X}}) = k\}k^{\rho}$$
$$\geq \left\{\sum_{k=1}^{\infty} \Pr\{G(x|P_{\hat{X}}) = k\}k\right\}^{\rho}$$
$$= \{1/P_{\hat{X}}(\mathcal{A}_{\Delta}(x))\}^{\rho}$$
(5)

where the last equality follows from the fact that the distribution of $G(x|P_{\hat{X}})$ is the geometric distribution with the parameter $P_{\hat{X}}(\mathcal{A}_{\Delta}(x))$.

On the other hand, since $G(x|P_{\hat{X}})$ is an integer-valued random variable and ρ is an integer,

$$\begin{aligned} \frac{G(x|P_{\hat{X}})^{\rho}}{\rho!} &\leq \frac{1}{\rho!} G(x|P_{\hat{X}}) \times [G(x|P_{\hat{X}}) + 1] \\ &\times [G(x|P_{\hat{X}}) + 2] \times \dots \times [G(x|P_{\hat{X}}) + \rho - 1] \\ &= \binom{G(x|P_{\hat{X}}) + \rho - 1}{\rho}. \end{aligned}$$

Taking the expectation with respect to $G(x|P_{\hat{X}})$ and multiply both sides by ρ !, we have

$$G_{\rho}(x|P_{\hat{X}}) \le (\rho!)V_{\rho}(x|P_{\hat{X}}).$$
 (6)

Combining (5) and (6) with Proposition 2, we have the theorem. $\hfill \Box$

C. Proof of Theorem 3

Before giving the proof, we introduce some notation.

For two positive sequences a_n and b_n , we will write $a_n \doteq b_n$ if $\lim_{n\to\infty} (1/n) \log(a_n/b_n) = 0$. Similarly, when a_n and b_n depend on a sequence x, the notation $a_n(x) \doteq b_n(x)$ means that

$$\lim_{n \to \infty} \max_{\boldsymbol{x} \in \mathcal{X}^n} \left| \frac{1}{n} \log \frac{a_n(\boldsymbol{x})}{b_n(\boldsymbol{x})} \right| = 0.$$

In our proof, we use the *method of types* [17]. For $\boldsymbol{x} \in \mathcal{X}^n$, the *type* $P_{\boldsymbol{x}}$ is the empirical distribution of $\boldsymbol{x} = (x_1, \ldots, x_n)$; i.e., $P_{\boldsymbol{x}}(a) = (1/n)|\{1 \le i \le n : x_i = a\}|$ for all $a \in \mathcal{X}$. Let \mathcal{P}_n be the possible types of length n sequences. For $Q \in \mathcal{P}_n$, \mathcal{T}_Q is the set of sequences \boldsymbol{x} such that $P_{\boldsymbol{x}} = Q$. For a conditional distribution $V: \mathcal{X} \to \hat{\mathcal{X}}$ and $\boldsymbol{x} \in \mathcal{X}^n$, $\mathcal{T}_V(\boldsymbol{x})$ denotes the set of sequences $\hat{\boldsymbol{x}} = (\hat{x}_1, \ldots, \hat{x}_n) \in \hat{\mathcal{X}}^n$ satisfying $V(b|a) = |\{j: (x_j, \hat{x}_j) = (a, b)\}|/|\{i: x_i = a\}|$ for all $(a, b) \in \mathcal{X} \times \hat{\mathcal{X}}$.

Proof of Theorem 3: From Corollary 2, we have

$$\mathbb{E}[G_{\rho}(X^{n}|Q_{\hat{X}}^{n})] \doteq \sum_{\boldsymbol{x}\in\mathcal{X}^{n}} P_{X^{n}}(\boldsymbol{x})[Q_{\hat{X}}^{n}(\mathcal{A}_{\Delta}(\boldsymbol{x}))]^{-\rho} \quad (7)$$

where $\mathcal{A}_{\Delta}(\boldsymbol{x}) \triangleq \{ \hat{\boldsymbol{x}} \in \hat{\mathcal{X}}^n : d_n(\boldsymbol{x}, \hat{\boldsymbol{x}}) \leq \Delta \}$. So, we evaluate the exponent of the right-hand side of (7).

For any \hat{x} and V, we have

$$\begin{aligned} Q_{\hat{X}}^{n}(\hat{\boldsymbol{x}}) &= \exp\left\{-n[H(P_{\hat{\boldsymbol{x}}}) + D(P_{\hat{\boldsymbol{x}}} \| Q_{\hat{X}})]\right\},\\ |\mathcal{T}_{V}(\boldsymbol{x})| &\doteq \exp\{nH(V|P_{\boldsymbol{x}})\} \end{aligned}$$

and thus,

$$Q_{\hat{X}}^{n}(\mathcal{T}_{V}(\boldsymbol{x})) \doteq \exp\left\{-n[I(P_{\boldsymbol{x}}, V) + D(P_{\boldsymbol{x}}V \| Q_{\hat{X}})]\right\}$$

where $P_{\boldsymbol{x}}V$ is the distribution on $\hat{\mathcal{X}}$ such that $P_{\boldsymbol{x}}V(\hat{x}) = \sum_{x} P_{\boldsymbol{x}}(x)V(\hat{x}|x).$

Further, $\mathcal{A}_{\Delta}(\boldsymbol{x})$ can be written as

$$\mathcal{A}_{\Delta}(oldsymbol{x}) = igcup_{V:X_{x,\hat{x}}} P_{oldsymbol{x}}(x) V(\hat{x}|x) d(x,\hat{x}) {\leq} \Delta \mathcal{T}_V(oldsymbol{x})$$

Hence, we have

$$Q_{\hat{X}}^{n}(\mathcal{A}_{\Delta}(\boldsymbol{x})) \doteq \exp\left\{-n\min_{V}[I(P_{\boldsymbol{x}},V) + D(P_{\boldsymbol{x}}V \| Q_{\hat{X}})]\right\}$$
$$= \exp\left\{-nR(P_{\boldsymbol{x}},Q_{\hat{X}}|\Delta)\right\}.$$
(8)

On the other hand, we have

$$P_X(\mathcal{T}_{Q_X}) \doteq \exp\{-nD(Q_X \| P_X)\}$$

for all $Q_X \in \mathcal{P}_n$. Combining this with (8), we have

$$\sum_{\boldsymbol{x}\in\mathcal{X}^{n}} P_{X^{n}}(\boldsymbol{x})[Q_{\hat{X}}^{n}(\mathcal{A}_{\Delta}(\boldsymbol{x}))]^{-\rho}$$

$$= \sum_{Q_{X}\in\mathcal{P}_{n}} \sum_{\boldsymbol{x}\in\mathcal{T}_{Q_{X}}} P_{X^{n}}(\boldsymbol{x})[Q_{\hat{X}}^{n}(\mathcal{A}_{\Delta}(\boldsymbol{x}))]^{-\rho}$$

$$\doteq \sum_{Q_{X}\in\mathcal{P}_{n}} \exp\left\{n\left[\rho R(Q_{X},Q_{\hat{X}}|\Delta) - D(Q_{X}||P_{X})\right]\right\}$$

$$\doteq \exp\left\{n\max_{Q_{X}}\left[\rho R(Q_{X},Q_{\hat{X}}|\Delta) - D(Q_{X}||P_{X})\right]\right\}.$$

V. CONCLUDING REMARKS

In this paper, randomized strategies for guessing subject to distortion was studied. A one-shot achievability bound on the guessing moment was given. Further, feasible i.i.d. asynchronous guessing scheme was proposed, and its asymptotic performance was investigated.

Lastly, we give some comments regarding generalizations of our results.

- It is not hard to extend the result to the case where sideinformation is available at the guesser.
- Our result shows that the behavior of

$$-\frac{1}{n}\log Q_{\hat{X}}^{n}(\mathcal{A}_{\Delta}(\boldsymbol{x})) \simeq R(P_{\boldsymbol{x}}, Q_{\hat{X}}|\Delta)$$

determines the guessing moment (See (8) and (7) in the proof of Theorem 3). Since the behavior of $(-1/n) \log Q_{\hat{X}}^n(\mathcal{A}_{\Delta}(\boldsymbol{x}))$ for sources with memory is well studied in the context of the rate-distortion theory (see [18] and references there in), we can apply those results. For example, our argument can also be applied to stationary ergodic sources by using Theorem 3 of [19].

Appendix A

PROOF OF PROPOSITION 1

First we introduce the concept of majorization and Schur concavity, which play important role in the proof.

Let \mathbb{R}^m_+ be the set of vectors with m nonnegative components. Given $\mathbf{p} = (p_1, p_2, \dots, p_m) \in \mathbb{R}^m_+$, denote by $p_{[1]} \ge p_{[2]} \ge \dots \ge p_{[m]}$ the permutation of the components of \mathbf{p} in the nonincreasing order.

Definition 4: We say that $\mathbf{q} \in \mathbb{R}^m_+$ majorizes $\mathbf{p} \in \mathbb{R}^m_+$ (and write $\mathbf{p} \prec \mathbf{q}$) if

$$\sum_{i=1}^{j} p_{[i]} \le \sum_{i=1}^{j} q_{[i]}, \quad \forall j = 1, 2, \dots, m-1$$

and

$$\sum_{i=1}^{m} p_{[i]} \le \sum_{i=1}^{m} q_{[i]}.$$
(9)

Definition 5: A real valued function h on \mathbb{R}^m_+ is said to be Schur concave if $h(\mathbf{p}) \ge h(\mathbf{q})$ for any $\mathbf{p}, \mathbf{q} \in \mathbb{R}^m_+$ satisfying $\mathbf{p} \prec \mathbf{q}$.

It is well known that $\phi(\mathbf{p}) = \sum_{i=1}^{m} (h_i)^{\alpha}$ for $\alpha \in (0, 1)$ is Schur concave; See, e.g. [20]. Thus, we can easily see that the Rényi entropy of order $\alpha \in (0, 1)$ is also Schur concave. Hence, to prove Proposition 1, it is sufficient to prove the following lemma. Although the same argument is given in the last page of [21], we give a proof for the completeness.

Lemma 1: There exists a deterministic function $\pi: \mathcal{X} \to \mathcal{X}$ such that (i) $d(x, \pi(x)) \leq \Delta(x)$ for all $x \in \mathcal{X}$ and (ii) the distribution $P_{\hat{X}}$ of $\hat{X} = \pi(X)$ majorizes any $P_{\tilde{X}}$ induced by $P_{\tilde{X}|X} \in \mathcal{W}_{\Delta}$.

Proof: Let $m \triangleq |\mathcal{X}|$ in this proof. For each $\hat{x} \in \mathcal{X}$, let

$$\mathcal{B}_{\Delta}(\hat{x}) \triangleq \{ x \in \mathcal{X} : d(x, \hat{x}) \le \Delta \}.$$

We define the order $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_m$ of symbols in $\hat{\mathcal{X}}$ as follows. Let \hat{x}_1 be a symbol satisfying

$$\Pr\{X \in \mathcal{B}_{\Delta}(\hat{x}_1)\} = \max_{\hat{x} \in \hat{\mathcal{X}}} \Pr\{X \in \mathcal{B}_{\Delta}(\hat{x})\}.$$

Then, for i = 2, 3, ..., m, let $\hat{x}_i \in \hat{\mathcal{X}} \setminus {\{\hat{x}_1, \ldots, \hat{x}_{i-1}\}}$ be a symbol such that

$$\Pr\left\{X \in \mathcal{B}_{\Delta}(\hat{x}_{i}) \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_{\Delta}(\hat{x}_{j})\right\}$$
$$= \max_{\hat{x} \in \hat{\mathcal{X}}} \Pr\left\{X \in \mathcal{B}_{\Delta}(\hat{x}) \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_{\Delta}(\hat{x}_{j})\right\}.$$

Let $\mathcal{X}_i \triangleq \mathcal{B}_{\Delta}(\hat{x}_i) \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_{\Delta}(\hat{x}_j)$ $(i = 1, \ldots, m)$. Then $\mathcal{X}_1, \ldots, \mathcal{X}_m$ give a partition of \mathcal{X} , and thus, we can define $\pi \colon \mathcal{X} \to \hat{\mathcal{X}}$ so that $\pi(x) = \hat{x}_i$ if $x \in \mathcal{X}_i$. It is apparent that $d(x, \pi(x)) \leq \Delta$ for all $x \in \mathcal{X}$. Further, from the construction, the distribution $P_{\hat{X}}$ of $\hat{X} = \pi(X)$ satisfies

$$P_{\hat{X}}(\hat{x}_1) \ge P_{\hat{X}}(\hat{x}_2) \ge \dots \ge P_{\hat{X}}(\hat{x}_m).$$

We will prove that π satisfies (ii) by contradiction. Assume that there exists $P_{\tilde{X}|X} \in W_{\Delta}$ such that $P_{\tilde{X}}$ induced by $P_{\tilde{X}|X}$ is *not* majorized by $P_{\hat{X}}$. Let us define another order $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m$ in $\hat{\mathcal{X}}$ so that

$$P_{\tilde{X}}(\tilde{x}_1) \ge P_{\tilde{X}}(\tilde{x}_2) \ge \dots \ge P_{\tilde{X}}(\tilde{x}_m).$$

Since $P_{\tilde{X}} \not\prec P_{\hat{X}}$, there exists k such that

$$\sum_{i=1}^{k} P_{\tilde{X}}(\tilde{x}_i) > \sum_{i=1}^{k} P_{\hat{X}}(\hat{x}_i).$$

Hence, we have

$$\begin{aligned} &\Pr\{d(X, \hat{X}) > \Delta\} \\ &\geq \sum_{x \in \mathcal{X}} \sum_{i=1}^{k} P_X(x) P_{\tilde{X}|X}(\tilde{x}_i|x) \mathbf{1}[d(x, \tilde{x}_i) > \Delta] \\ &= \sum_{i=1}^{k} P_{\tilde{X}}(\tilde{x}_i) \\ &- \sum_{x \in \mathcal{X}} P_X(x) \sum_{i=1}^{k} P_{\tilde{X}|X}(\tilde{x}_i|x) \mathbf{1}[x \in \mathcal{B}_{\Delta}(\tilde{x}_i)] \\ &\stackrel{\text{(a)}}{\geq} \sum_{i=1}^{k} P_{\tilde{X}}(\tilde{x}_i) - \Pr\left\{X \in \bigcup_{i=1}^{k} \mathcal{B}_{\Delta}(\tilde{x}_i)\right\} \\ &\stackrel{\text{(b)}}{\geq} \sum_{i=1}^{k} P_{\tilde{X}}(\tilde{x}_i) - \Pr\left\{X \in \bigcup_{i=1}^{k} \mathcal{B}_{\Delta}(\hat{x}_i)\right\} \\ &= \sum_{i=1}^{k} P_{\tilde{X}}(\tilde{x}_i) - \sum_{i=1}^{k} \Pr\{X \in \mathcal{X}_i\} \\ &= \sum_{i=1}^{k} P_{\tilde{X}}(\tilde{x}_i) - \sum_{i=1}^{k} P_{\hat{X}}(\hat{x}_i) \\ &> 0, \end{aligned}$$

where 1 denotes the indicator function, (a) follows from $\sum_{i=1}^{k} P_{\tilde{X}|X}(\tilde{x}_i|x)\mathbf{1}[x \in \mathcal{B}_{\Delta}(\tilde{x}_i)] \leq \mathbf{1}[x \in \bigcup_{i=1}^{k} \mathcal{B}_{\Delta}(\tilde{x}_i)]$ for all $x \in \mathcal{X}$, and (b) follows from the definition of the order $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m$. This contradicts the fact $P_{\tilde{X}|X} \in \mathcal{W}_{\Delta}$.

APPENDIX B PROOF OF PROPOSITION 2

The proposition can be proved in the same manner as [4, Lemma 2]. We give a proof for the completeness.

Proof of Proposition 2: Letting $p_x \triangleq P_{\hat{X}}(\mathcal{A}_{\Delta}(x))$, we have

$$V_{\rho}(x|P_{\hat{X}}) = \sum_{m=1}^{\infty} {m+\rho-1 \choose \rho} \Pr\{G(x|P_{\hat{X}}) = m\}$$

= $p_x \sum_{m=1}^{\infty} {m+\rho-1 \choose \rho} (1-p_x)^{m-1}$
 $\stackrel{(a)}{=} p_x \sum_{m=1}^{\infty} {-\rho-1 \choose m-1} [-(1-p_x)]^{m-1}$
= $p_x \sum_{k=0}^{\infty} {-\rho-1 \choose k} [-(1-p_x)]^k$
 $\stackrel{(b)}{=} p_x [1-(1-p_x)]^{-\rho-1}$
= $(p_x)^{-\rho}$

where (a) follows from the relationship

$$\binom{m+\rho-1}{\rho} = (-1)^{m-1} \binom{-\rho-1}{m-1},$$

which is proved in the proof of Lemma 2 in [4], and (b) follows from the binominal formula. \Box

APPENDIX C

Asymptotics for non-integer ρ

We show that Theorem 3 also holds for non-integer $\rho > 0$. From (8), for all $x \in \mathcal{X}^n$ and $Q_{\hat{X}}$,

$$q_{\boldsymbol{x}} \triangleq Q_{\hat{X}}^{n}(\mathcal{A}_{\Delta}(\boldsymbol{x})) \doteq \exp\left\{-nR(P_{\boldsymbol{x}}, Q_{\hat{X}}|\Delta)\right\}.$$

Assume that $R(P_x, Q_{\hat{X}}|\Delta) > 0$. Then, since $q_x < 1/2$ for large n, (20) of [15] gives

$$G_{\rho}(\boldsymbol{x}|Q_{\bar{X}}^{n}) = \sum_{k=1}^{\infty} k^{\rho} (1-q_{\boldsymbol{x}})^{k-1} q_{\boldsymbol{x}}$$
$$\geq \left(\frac{1-q_{\boldsymbol{x}}}{q_{\boldsymbol{x}}}\right)^{\rho} \exp\left\{-\frac{1}{1-q_{\boldsymbol{x}}}\right\}$$
$$\geq \frac{2^{-\rho}}{e^{2}} q_{\boldsymbol{x}}^{-\rho}.$$

Thus, we have

$$\liminf_{n \to \infty} \frac{1}{n} \log \left| \frac{G_{\rho}(\boldsymbol{x} | Q_{\hat{X}}^n)}{\exp\{\rho n R(P_{\boldsymbol{x}}, Q_{\hat{X}} | \Delta)\}} \right| \ge 0.$$
(10)

On the other hand, by using Lemma 1 of [13] with $a = R(P_x, Q_{\hat{X}} | \Delta)$, we can show that

$$\limsup_{n \to \infty} \frac{1}{n} \log \left| \frac{G_{\rho}(\boldsymbol{x} | Q_{\hat{X}}^n)}{\exp\{\rho n R(P_{\boldsymbol{x}}, Q_{\hat{X}} | \Delta)\}} \right| \le 0.$$
(11)

Combining (10) and (11), we have

$$G_{\rho}(\boldsymbol{x}|Q_{\hat{X}}^{n}) \doteq \exp\left\{\rho n R(P_{\boldsymbol{x}}, Q_{\hat{X}}|\Delta)\right\}$$

and thus,

$$\sum_{\boldsymbol{x}\in\mathcal{X}^n} P_{X^n}(\boldsymbol{x}) G_{\rho}(\boldsymbol{x}|Q_{\hat{X}}^n)$$

$$\doteq \sum_{Q_X\in\mathcal{P}_n} \exp\left\{n\left[\rho R(Q_X, Q_{\hat{X}}|\Delta) - D(Q_X||P_X)\right]\right\}$$

$$\doteq \exp\left\{n\max_{Q_X} \left[\rho R(Q_X, Q_{\hat{X}}|\Delta) - D(Q_X||P_X)\right]\right\}.$$

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