# Novel one-shot inner bounds for unassisted fully quantum channels via rate splitting* 

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#### Abstract

We prove the first non-trivial one-shot inner bounds for sending quantum information over an entanglement unassisted two-sender quantum multiple access channel (QMAC) and an unassisted two-sender two-receiver quantum interference channel (QIC). Previous works only studied the unassisted QMAC in the limit of many independent and identical uses of the channel also known as the asymptotic iid limit, and did not study the unassisted QIC at all. We employ two techniques, rate splitting and successive cancellation, in order to obtain our inner bound. Rate splitting was earlier used to obtain inner bounds, avoiding time sharing, for classical channels in the asymptotic iid setting. Our main technical contribution is to extend rate splitting from the classical asymptotic iid setting to the quantum one-shot setting. In the asymptotic iid limit our one-shot inner bound for QMAC approaches the rate region of Yard et al. [2]. For the QIC we get novel non-trivial rate regions in the asymptotic iid setting. All our results also extend to the case where limited entanglement assistance is provided, in both one-shot and asymptotic iid settings. The limited entanglement results for one-setting for both QMAC and QIC are new. For the QIC the limited entanglement results are new even in the asymptotic iid setting.


## 1 Introduction

The multiple access channel (MAC), where two independent senders Alice (A) and Bob (B) have to send their respective messages to a single receiver Charlie (C) via a communication channel with two inputs and one output, is arguably the simplest multiterminal channel. Yet, it abstracts out important practical situations like several independent users transmitting their respective messages to a base station. Ahlswede [3], and independently Liao [4], obtained the first optimal rate region for the classical MAC in the asymptotic iid setting, using a powerful method called simultaneous decoding. Their region looks like the one in Figure 1 where $I(A: B):=H(A)+H(B)-H(A B)$ denotes the mutual information between two jointly distributed random variables $A, B$. Simultaneous decoding means that Charlie is able to decode any point in the rate region, e.g. point P in Figure 1 by a one-step procedure. Later on, other authors obtained the same rate region in a computationally less intensive fashion by using successive cancellation and time sharing. In successive cancellation decoding Charlie first decodes Alice's message and then uses it as an additional channel output in order to next decode Bob's message, or vice versa. In other words, Charlie can either decide to decode point S or point T in Figure [1] In order to decode another point in the rate region, e.g. point P in Figure 1. Charlie first figures out the convex combination $(\alpha, 1-\alpha)$ of points S and T that would give point P . Out of $n$ iid channel uses, Charlie then decodes the first $\alpha n$ uses according to point S's decoding strategy and the remaining $(1-\alpha) n$ channel uses according to point T's decoding strategy. This idea is called time sharing.

The interference channel is another important channel where sender Alice wants to send her message to receiver Charlie and sender Bob, whose message is independent of that of Alice, wants to send his message to receiver Damru

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Figure 1: Achievable rate region per channel use for the classical MAC in the asymptotic iid setting.
via a communication channel with two inputs and two outputs. It abstracts out the important practical situation where independent sender-receiver pairs are communicating simultaneously via a noisy medium. Han and Kobayashi [5] (see also [6]) obtained the best known inner bound for this channel in the classical asymptotic iid setting.

The multiple access and interference channels can be defined in the context of quantum information theory also. Early work studied the sending of classical information over a quantum MAC, without [7] or with [8] entanglement assistance, in the asymptotic iid setting. These works obtained the natural quantum analogues of the optimal classical rate regions using successive cancellation and time sharing. Later, Fawzi et al. [9] and Sen [10] studied the sending of classical information over a quantum interference channel in the asymptotic iid setting by first obtaining a simultaneous decoder for the quantum MAC. The latter paper managed to obtain the natural quantum analogue of the Han-Kobayashi inner bound.

For a variety of reasons recent research in Shannon theory has studied in depth the one-shot setting where the channel can be used only once. This is the most general setting and subsumes the asymptotic iid, asymptotic noniid aka information spectrum, and finite block length settings. Ideally, the one-shot inner bounds should match or supersede the best inner bounds for the respective channels in the asymptotic iid setting. Sen [11] obtained the natural one-shot quantum analogues of best known classical rate regions for sending classical information over entanglement unassisted and assisted quantum MACs and quantum interference channels. His one-shot inner bounds, obtained by simultaneous decoding, approach the optimal inner bounds known earlier for the classical and quantum asymptotic iid settings.

Note that presence of shared randomness does not affect the rates of sending classical or quantum information over channels. Also the rates of sending quantum information and classical information over entanglement assisted quantum channels are related by a factor of two because of quantum teleportation. So the main setting left unstudied in the above works is the setting of sending quantum information over an entanglement unassisted quantum channel i.e. the senders and the receivers do not share any entanglement prior to the beginning of the protocol. The first works to address this setting looked at a point-to-point quantum channel in the asymptotic iid setting [12, 13], culminating in the work of Devetak [14] which showed with full rigour that the regularised coherent information from sender $A$ to receiver $B$ defined by $I^{*}(A>B):=\lim _{k \rightarrow \infty} I\left(A^{k}>B^{k}\right) / k, I\left(A^{k}>B^{k}\right):=H\left(B^{k}\right)-H\left(A^{k} B^{k}\right)$ where $A^{k} B^{k}$ is defined by the channel action $\left(\mathcal{N}^{A^{\prime} \rightarrow B}\right)^{\otimes k}$ on an arbitrary (in general entangled) pure state $|\sigma\rangle^{A^{k}\left(A^{\prime}\right)^{k}}$, is the capacity of an unassisted quantum channel in the asymptotic iid limit.

Comment 1.1. Devetak [14] provided the first fully rigorous proof of this statement by proving an elegant connection between the quantum capacity of an unassisted quantum channel and its classical private capacity.

Hayden et al. [15]. showed that one can recover Devetak's result using a technique called decoupling

Comment 1.2. Devetak's private coding technique was geared towards a specific problem called entanglement transmission where the sender has to transmit the quantum state of a system which might be entangled with a second reference system, and the receiver must be able to decode the channel output so that at the end of the protocol the joint quantum state of the system plus reference is approximately preserved. Hayden et al.'s decoupling technique was geared towards another problem called entanglement generation where the protocol aims to create EPR pairs shared between the sender and the receiver. Once such EPR pairs are created, it is easy to solve entanglement transmission by quantum teleportation when an additional noiseless classical channel is provided. Though entanglement generation looks to be weaker than entanglement transmission and moreover requires an additional noiseless classical channel, it was shown in [16] that nevertheless both these as well as several variant tasks are essentially equivalent.

These works naturally lead one to consider unassisted multiterminal quantum channels. To the best of our knowledge, the only inner bound known for the unassisted QIC is what one would obtain by treating the channel as two independent unassisted point to point channels. For the unassisted QMAC more is known. Yard et al. [2] showed that the natural quantum analogue of the classical rate region, with mutual information replaced by regularised coherent information as in Figure 2, is an inner bound for the unassisted quantum MAC (QMAC) in the asymptotic iid setting. They proved their inner bound by time sharing and a suitable adaptation of successive cancellation.


Figure 2: Achievable rate region for the unassisted quantum MAC per channel use in the asymptotic iid setting.

The above works behoove one to consider the problem of sending quantum information over an unassisted quantum channel in the one-shot setting. Buscemi and Datta [17] proved the first one-shot achievability result for the unassisted point-to-point channel in terms of smooth modified Rényi entropies. Their result was generalised by Dupuis [18] to the case where the receiver has some side information about the sender's message. In the asymptotic iid limit, these one-shot results approach the regularised coherent information obtained in earlier works.

It is thus natural to study inner bounds for the unassisted QMAC in the one-shot setting. In this paper we take the first steps towards this problem. Observe that successive cancellation can only give the two endpoints $S$ and $T$ of the dominant line of the pentagonal rate region in Figure 2. Since time sharing cannot be used in the one-shot setting, it is not clear how to obtain other rate tuples like the point $P$. An alternative would be to develop a simultaneous decoder for the QMAC which can obtain a point like $P$ directly, but that is a major open problem with connections to the notorious simultaneous smoothing open problem [19].

Instead in this paper, we take inspiration from another powerful classical channel coding technique called rate splitting. Grant et al. [20] showed that it is possible to 'split' Alice into two senders Alice ${ }_{0}$ and Alice $_{1}$, each sending disjoint parts of Alice's original message, such that any point in the pentagonal rate region of Figure 2 like P can be obtained without time sharing by a successive cancellation process where Charlie first decodes Alice ${ }_{0}$ 's message, then Bob's message using Alice ${ }_{0}$ 's message as side information and finally Alice ${ }_{1}$ 's message using Bob's and Alice ${ }_{0}$ 's messages as side information. Though Grant et al.'s rate splitting technique was developed for the classical MAC in
the asymptotic iid setting, in this paper we show how it can be adapted to the one-shot quantum setting. This is a non-trivial task, which we tackle in two steps. In the first step we use ideas from Yard et al. [2] and Dupuis [18] and suitably adapt successive cancellation to the one-shot unassisted quantum setting. In the second step, we adapt the rate splitting function of Grant et al. [20] to the one-shot quantum setting. Our one-shot rates are in terms of the smooth coherent Rényi-2 information defined in Section 3. Since the smooth coherent Rényi-2 information is not known to possess a chain rule with equality, we get an achievable rate region of the form in Figure 3. Our achievable rate region is a subset of the 'ideal' pentagonal rate region shown by the dashed line. Nevertheless, using a quantum asymptotic equipartition result of Tomamichel et al. [21], we show that this 'subpentagonal' achievable rate region approaches the 'pentagonal' region of Yard et al. [2] (equal to the region demarcated by the dashed line) in the iid limit. The


Figure 3: One-shot achievable rate region for the unassisted QMAC (for single channel use only), contained inside the 'ideal' pentagonal region demarcated by the dashed line, and approaching it in the asymptotic iid limit. $O(\log \varepsilon)$ additive factors have been ignored in the figure.
reason why splitting of Alice into Alice $_{0}$ and Alice $_{1}$ allows one to obtain a 'middle' rate point like $P$, in addition to the 'corner' points $S$ and $T$, is as follows. The rate point $P$ is the projection onto the (Alice, Bob) plane of the 'corner' rate point $P^{\prime}$ in the $\left(\right.$ Alice $_{0}$, Bob, Alice $\left.{ }_{1}\right)$ space where the rates of Alice $_{0}$ and Alice ${ }_{1}$ are summed to obtain Alice's rate. The point $P^{\prime}$ can be obtained by a 3-step successive cancellation decoding. Note that the split of Alice depends on the rate point $P$ to be attained.


Figure 4: The 'corner' point $P^{\prime}$ can be obtained by successive cancellation following the order Alice ${ }_{0} \rightarrow$ Bob $\rightarrow$ Alice $_{1}$ with splitting of Alice followed by one use of the unassisted QMAC. Point $P^{\prime}$ projects down to point $P$ in Figure 3 , Only the 'dominant face' of the rate region is shown. Successive cancellation can only obtain the corner points of the dominant face and all 'sub-points' by 'resource wasting'. It cannot obtain 'middle' points of the 'dominant' face. $O(\log \varepsilon)$ additive factors have been ignored in the figure.

In fact, it turns out that our techniques are more general; they allow us to a obtain non-trivial achievable rate region
for sending quantum information over a QMAC with rate limited entanglement assistance. In the case of rate limited entanglement assistance, the amount of prior shared entanglement between the sender and the receiver is limited by a certain upper bound. If this upper bound is set to 0 , the situation reduces exactly to the unassisted case. As the upper bound tends to infinity, the situation becomes the same as the QMAC with unlimited entanglement assistance.

We now state our result for the unassisted QIC. The trivial inner bound treats the QIC as two independent unassisted point to point channels from Alice to Charlie and Bob to Damru. Rate splitting and successive cancellation can be similarly used to obtain non-trivial rate regions for the unassisted QIC where one party, say Alice, sacrifices her rate in order to boost Bob's rate with respect to the trivial inner bound. The situation is summarised in Figure 5 , Though the discussion above only involved unassisted QMAC and QIC, our actual results also hold for the QMAC and


Figure 5: One-shot achievable rate region (for single channel use only) for the unassisted QIC. The trivial region is shown dotted. Alice can sacrifice her rate in order to boost Bob's rate with respect to the trivial region, as shown by the solid rectangle. The dashed rectangle can be similarly obtained by Bob sacrificing his rate in order to boost Alice's. $O(\log \varepsilon)$ additive factors have been ignored in the figure.

QIC with limited entanglement assistance. However they seem to be inferior to the known results when entanglement assistance is unlimited [22].

Subsequent Works: After the arXiv and conference versions of this work were published [1], Saus and Winter [23] obtained a partially smooth one shot simultaneous coding strategy for sending quantum information across the QMAC. They proved their nice result by proving a partially smoothed generalisation of the (non smooth) multi sender decoupling theorem given by Chakraborty et al. [19]. As a result they obtain the natural smooth one shot analog of Figure 2. Hence they have an alternate derivation of the asymptotic iid rate region shown in Figure 2 without appealing to rate splitting. However, their methods don't seem to be generalisable to the case of the QMAC with limited entanglement assistance. This is because their methods cannot smooth over the Choi state of the channel, which seems crucial for obtaining any non-trivial inner bounds in the limited entanglement assisted setting. Thus to the best of our knowledge, the present work is the only one providing a non-trivial smooth one shot achievable rate region for the QMAC with limited entanglement assistance. Besides, rate splitting has proved to be a powerful technique in classical network information theory, and so its generalisation to the most general one shot quantum setting should be of independent interest.

## 2 Organisation of the Paper

The paper is organised as follows. In Section 3 we present the definitions and facts regarding one-shot entropic quantities and other necessary mathematical tools that we will need throughout the paper. In Section 5 we introduce the concept of quantum rate splitting and demonstrate it in the case of entanglement transmission across the point
to point channel. We also develop the technique of successive cancellation decoding for entanglement transmission codes in this section. In Section 6 we use the ideas introduced in Section 5 to derive inner bounds for entanglement transmission over the QMAC and the QIC. Finally, in Section 7]we present the asymptotic IID versions of the one-shot inner bounds presented in paper.

## 3 Preliminaries

### 3.1 Notation

We will use the following conventions throughout the rest of the paper :

1. We use the shorthand $M \cdot N:=M N M^{\dagger}$ for operators $M$ and $N$.
2. Suppose that $|\omega(U)\rangle^{X A^{\prime} B^{\prime}}$ be a generic intermediary state (defined in Section 5.2), where $X$ is a placeholder for other systems involved in the protocol. Suppose we are given a channel $\mathcal{N}^{A^{\prime} B^{\prime} \rightarrow C}$ and its corresponding Stinespring dilation $\mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C E}$. Then, we denote the state $\mathcal{U}_{\mathcal{N}}|\omega(U)\rangle^{X A^{\prime} B^{\prime}}$ by the symbol $|\omega(U)\rangle^{X C E}$. Although the two states are denoted using the same greek letter, we differentiate them by the systems on which they are defined. These systems will always be explicitly mentioned whenever we make use of this convention.
3. We will use the same rule for control states, For example, suppose $|\sigma\rangle^{A^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}$ is a control state for some channel coding protocol. Suppose we are given the channel $\mathcal{N}^{A^{\prime} B^{\prime} \rightarrow C}$ Then we use the following convention

$$
\sigma^{A^{\prime \prime} B^{\prime \prime} C}:=\mathcal{N} \cdot \sigma^{A^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}
$$

We will use this convention while specifying entropic quantities. It will be clear from context which state we refer to. For example, consider the expressions $H_{\min }^{\varepsilon}(A ")_{\sigma}$ and $I_{\min }^{\varepsilon}(A " \mid C)_{\sigma}$. It is clear from the arguments of the entropic expressions that in the first case $\sigma=\sigma^{A^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}$ and in the second case $\sigma=\sigma^{A^{\prime \prime} B^{\prime} C}$.
4. We will, on several occasions use the operator op ${ }^{X \rightarrow Y A^{\prime} B^{\prime}}\left(|\omega(U)\rangle^{X Y A^{\prime} B^{\prime}}\right)$. To lessen the burden on notation, whenever we use this operator, we will not mention the systems on which the argument of the op operator is defined. It will however always mention the domain and range of the op operator in these cases to avoid any confusion.

### 3.2 Smooth Entropies

For a pair of subnormalised density matrices $\rho$ and $\sigma$ in the same Hilbert space their purified distance is denoted by $P(\rho, \sigma):=\sqrt{1-F(\rho, \sigma)^{2}}$ where $F(\rho, \sigma):=\|\sqrt{\rho} \sqrt{\sigma}\|_{1}+\sqrt{(1-\operatorname{Tr}[\rho]) \cdot(1-\operatorname{Tr}[\sigma])}$ is the generalised fidelity and $\|\cdot\|_{1}$ is the Schatten 1-norm. We use $\sigma \approx_{\varepsilon} \rho$ as a shorthand for $P(\sigma, \rho) \leq \varepsilon$. See [24] for more details.

The von Neumann entropy for a normalised quantum state $\rho^{A}$ is defined by $H(A)_{\rho}:=-\operatorname{Tr}[\rho \log \rho]$. For a bipartite quantum state $\rho^{A B}$, the Coherent Information $I(A>B)$ is defined as $I(A>B):=H(A \mid E)$ where the conditional entropy $H(A \mid E)$ is computed with respect to the purification $|\rho\rangle^{A B E}$ of the state $\rho^{A B}$.

Definition 3.1. [ $\varepsilon$-smooth sandwiched Rényi-2 conditional entropy] Given a bipartite state $\rho^{A B}$, the $\varepsilon$-smooth sandwiched Rényi-2 conditional entropy is defined as

$$
\begin{aligned}
& H_{2}^{\varepsilon}(A \mid E)_{\rho}:= \\
& -2 \log \min _{\left(\rho^{\prime}\right)^{A E} \approx \rho^{A E}} \min _{\sigma^{E}}\left\|\left(\mathbb{1}^{A} \otimes\left(\sigma^{E}\right)^{-1 / 4}\right) \cdot\left(\rho^{\prime}\right)^{A E}\right\|_{2},
\end{aligned}
$$

where $\sigma^{E}$ ranges over non-singular normalised states over $E$.

## Definition 3.2. [ $\varepsilon$-smooth conditional min-entropy]

The $\varepsilon$-smooth conditional min-entropy is given by

$$
\begin{aligned}
& H_{\min }^{\varepsilon}(A \mid E)_{\rho}:= \\
& -\log \min _{\left(\rho^{\prime}\right)^{A E} \approx \varepsilon^{A E}} \min _{\sigma^{E}:\left(\rho^{\prime}\right)^{A E} \leq \mathbb{1}^{A} \otimes \sigma^{E}} \operatorname{Tr}\left[\sigma^{E}\right],
\end{aligned}
$$

where $\sigma^{E}$ ranges over positive semidefinite operators on $E$.
The unconditional smooth entropies are now defined from the conditional ones by taking the conditioning system to be one dimensional.

Definition 3.3. [ $\varepsilon$-smooth coherent min-information] Then the $\varepsilon$-smooth coherent min-information aka the negative of the $\varepsilon$-smooth conditional max-entropy is given by

$$
\left.I_{\min }^{\varepsilon}(A\rangle B\right)_{\rho}:=-H_{\max }^{\varepsilon}(A \mid B)_{\rho}:=H_{\min }^{\varepsilon}(A \mid E)_{\rho},
$$

where again $|\rho\rangle^{A B E}$ is a purification of $\rho^{A B}$.
As shown in [18], the smooth sandwiched Rényi-2 conditional entropy upper bounds the smooth conditional minentropy. The smooth conditional min-entropy is further lower bounded by the familiar conditional Shannon entropy in the amortised sense in the asymptotic iid limit [21], a result that is sometimes referred to as the fully quantum asymptotic equipartition property. To summarise, the smooth sandwiched Rényi- 2 coherent information upper bounds the Shannon coherent information in the amortised sense in the asymptotic iid limit.

We will now state some properties on the smooth conditional min entropy that we will use throughout the rest of the paper.

Fact 3.4 (Chaining for Smooth min-entropy [25, 26]). Let $\varepsilon>0$ and $\varepsilon^{\prime}, \varepsilon^{\prime \prime} \geq 0$ and let $\rho^{A B C}$ be a quantum state. Then

$$
H_{\min }^{\varepsilon+2 \varepsilon^{\prime}+\varepsilon^{\prime \prime}}(A B \mid C)_{\rho} \geq H_{\min }^{\varepsilon^{\prime}}(A \mid B C)_{\rho}+H_{\min }^{\varepsilon^{\prime \prime}}(B \mid C)_{\rho}-\log \frac{2}{\varepsilon^{2}}
$$

Fact 3.5 (Unitary Invariance of Smooth min-entropy). Given $\varepsilon \geq 0$, a quantum state $\rho^{A B}$ and isometries $U: \mathcal{H}_{A} \rightarrow$ $\mathcal{H}_{C}$ and $V: \mathcal{H}_{B} \rightarrow \mathcal{H}_{D}$, define the state $\sigma^{C D}:=(U \otimes V) \rho^{A B}\left(U^{\dagger} \otimes V^{\dagger}\right)$. Then

$$
H_{\min }^{\varepsilon}(A \mid B)_{\rho}=H_{\min }^{\varepsilon}(C \mid D)_{\sigma}
$$

Fact 3.6 (Continuity of Smooth min-entropy). Given two quantum states $\rho^{A B}$ and $\sigma^{A B}$ such that $P\left(\rho^{A B}, \sigma^{A B}\right) \leq \delta$ and $\varepsilon>0$, then

$$
\left|H_{\min }^{\varepsilon}(A \mid B)_{\rho}-H_{\min }^{\varepsilon}(A \mid B)_{\sigma}\right| \leq c \cdot \delta^{\prime}
$$

where $c$ is an absolute constant and depends on the dimensions of system $A$ and $B$ and $\delta^{\prime}=\sqrt{\delta^{2}+2 \varepsilon \delta}$
The proofs of both Fact 3.5 and Fact 3.6 can be found in [24].
Fact 3.7 (Quantum Asymptotic Equipartition Property [21]). Given a bipartite quantum state $\rho^{A B}$ on the system $\mathcal{H}_{A} \otimes \mathcal{H}_{B}, \varepsilon>0$, an integer $n \in \mathbb{N}$ and the iid extension of the state $\rho_{A B}^{n}$ it holds that

$$
\lim _{\varepsilon \rightarrow 0} \lim _{n \rightarrow \infty} \frac{1}{n} H_{\min }^{\varepsilon}\left(A^{n} \mid B^{n}\right)_{\rho^{n}}=H(A \mid B)_{\rho}
$$

### 3.3 The op Operator

One of the main technical tools we use in this paper, which is a workhorse in most of our proofs, is the notion of mapping a vector into an operator. This operation is denoted simply by 'op' and we compile some of its properties in this section for completeness. The interested reader is referred to [18] for further details.

Definition 3.8. [The op operator] Given the systems $A$ and $B$, fix the standard bases $\left|a_{i}\right\rangle^{A}$ and $\left|b_{j}\right\rangle^{B}$. Then we define $\mathrm{op}^{A \rightarrow B}: A \otimes B \rightarrow L(A, B)$ as

$$
\mathrm{op}^{A \rightarrow B}\left(\left|a_{i}\right\rangle\left|b_{j}\right\rangle\right):=\left|b_{j}\right\rangle\left\langle a_{i}\right| \quad \forall i, j
$$

Notice that this definition is basis dependant and hence whenever we use this operator a choice of bases is implied, although not always explicitly mentioned.

Fact 3.9. Let $|\psi\rangle^{A B}$ and $|\varphi\rangle^{A C}$ be vectors on the systems $A B$ and $A C$ respectively. Then

$$
\mathrm{op}^{A \rightarrow C}\left(|\varphi\rangle^{A C}\right)|\psi\rangle^{A B}=\mathrm{op}^{A \rightarrow B}\left(|\psi\rangle^{A B}\right)|\varphi\rangle^{A C}
$$

Fact 3.10. Given a vector $|\psi\rangle^{A B}$, let $|\Phi\rangle^{A A^{\prime}}$ be an EPR state, where $A \cong A^{\prime}$. Then,

$$
\sqrt{|A|} \mathrm{op}^{A \rightarrow B}\left(|\psi\rangle^{A B}\right)|\Psi\rangle^{A A^{\prime}}=|\psi\rangle^{A^{\prime} B}
$$

Fact 3.11. For all vectors $|\psi\rangle^{A B}$ and any $M^{A \rightarrow C}$,

$$
\mathrm{op}^{C \rightarrow B}(M|\psi\rangle)=\mathrm{op}^{A \rightarrow B}(|\psi\rangle) M^{T}
$$

Fact 3.12. For all $|\psi\rangle^{A B}$,

$$
\operatorname{Tr}_{B}\left[\psi^{A B}\right]=\mathrm{op}^{B \rightarrow A}(|\psi\rangle) \mathrm{op}^{B \rightarrow A}(|\psi\rangle)^{\dagger}
$$

### 3.4 The Smooth Single Sender Decoupling Theorem

Fact 3.13. Smooth Decoupling Theorem [27] Given $\varepsilon>0$ a density matrix $\rho^{A E}$ and any completely positive operator $\mathcal{T}^{A \rightarrow R}$, define $\omega^{A^{\prime} R}:=\left(\mathcal{T} \otimes \mathbb{I}^{A^{\prime}}\right) \Phi^{A A^{\prime}}$ such that $\operatorname{Tr}\left[\omega^{A^{\prime} R}\right]=1$. Then

$$
\int_{\mathbf{U}(\mathbf{A})}\left\|\mathcal{T}(U \cdot \rho)-\omega^{E} \otimes \rho^{R}\right\|_{1} \leq 2^{-\frac{1}{2} H_{2}^{\varepsilon}\left(A^{\prime} \mid R\right)_{\omega}-\frac{1}{2} H_{2}^{\varepsilon}(A \mid E)_{\rho}}+8 \varepsilon
$$

where the integration is over the Haar measure on the set of all unitaries on the system $A$, denoted by $\mathbb{U}(\mathbb{A})$.
The single sender decoupling theorem implies the following channel coding theorem.
Fact 3.14. [18] Theorem 3.14] Let $|\psi\rangle^{A B R}$ be a pure state, $\mathcal{N}^{A^{\prime} \rightarrow C}$ be any CPTP superoperator with Stinespring dilation $U_{\mathcal{N}}^{A^{\prime} \rightarrow C E}, N$ and complementary channel $\overline{\mathcal{N}}^{A^{\prime} \rightarrow E}$, let $\omega^{A^{\prime \prime} C E}:=U_{\mathcal{N}} \cdot \sigma^{A^{\prime \prime} A^{\prime}}$, where $\sigma^{A^{\prime \prime} A^{\prime}}$ is any pure state and $A " \cong A^{\prime}$, and let $\varepsilon>0$. Then, there exists an encoding partial isometry $V^{A \rightarrow A^{\prime}}$ and a decoding superoperator $\mathcal{D}^{C B \rightarrow A B}$ such that:

$$
\left\|\overline{\mathcal{N}}\left(V \cdot \psi^{A R}\right)-\omega^{E} \otimes \psi^{R}\right\|_{1} \leq 2 \sqrt{2 \delta_{1}}+\delta_{2}
$$

and

$$
\left\|(\mathcal{D} \circ \mathcal{N} \circ \mathcal{E}) \psi^{A B R}-\psi^{A B R}\right\|_{1} \leq 2 \sqrt{\left(4 \sqrt{2 \delta_{1}}+2 \delta_{2}\right)}
$$

where $\delta_{1}:=3 \times 2^{\frac{1}{2} H_{\text {max }}^{\varepsilon}(A)_{\psi}-\frac{1}{2} H_{2}^{\varepsilon}\left(A^{"}\right)_{\omega}}+24 \varepsilon, \delta_{2}:=3 \cdot 2^{-\frac{1}{2} H_{2}^{\varepsilon}\left(A^{"} \mid E\right)_{\omega}-\frac{1}{2} H_{2}^{\varepsilon}(A \mid R)_{\psi}}+24 \varepsilon$

### 3.5 Miscellaneous Useful Facts

Fact 3.15. Given states $\rho^{A B C}, \sigma^{A}, \eta^{C}, \sigma^{A B}, \omega^{B C}$ such that

$$
\begin{aligned}
& \left\|\rho^{A B C}-\sigma^{A} \otimes \omega^{B C}\right\|_{1} \leq \varepsilon_{1} \\
& \left\|\rho^{A B C}-\sigma^{A B} \otimes \eta^{C}\right\|_{1} \leq \varepsilon_{2}
\end{aligned}
$$

it holds that

$$
\left\|\rho^{A B C}-\sigma^{A} \otimes \sigma^{B} \otimes \eta^{C}\right\|_{1} \leq 2 \varepsilon_{1}+\varepsilon_{2}
$$

Fact 3.16. For any two density matrices $\rho$ and $\sigma$ and any real $c \in \mathbb{R}$, the following holds true:

$$
\|\rho-\sigma\|_{1} \leq 2\|c \rho-\sigma\|_{1}
$$

## 4 Quantum Channel Capacities: Definitions and Previous Work

The capacity of a quantum channel can have many different and distinct interpretations, based on the information processing task being considered. The various definitions of the quantum capacity arise from considerations such as whether the information being sent through the channel is classical data or whether it consists of arbitrary quantum states. Further, the definition of capacity changes whether the sender and the receiver can make use of pre-shared entanglement or EPR pairs, that they prepared before the protocol began. In this section, we will introduce the entanglement unassisted and entanglement assisted quantum capacities of a quantum channel.

We will first define the capacities assuming that the sender Alice and the receiver Bob can utilise only one copy of the channel i.e. the one-shot regime. We will then generalise to the case when many copies of the channel are available for use i.e. the iid regime.

The definition of the quantum capacity of a quantum channel stems from the following intuition: given a quantum channel $\mathcal{N}^{A^{\prime} \rightarrow B}$, we want to exhibit a subspace $A_{\text {Good }} \subset A^{\prime}$ such that channel acts approximately like the identity channel on this subspace. To make this precise, consider that the sender Alice has some system $A$ which holds her quantum message and an encoder $\mathcal{E}^{A \rightarrow A^{\prime}}$. After receiving the quantum system $B$, Bob produces a guess for the contents of Alice's $A$ system, by using the decoding map $\mathcal{D}^{B \rightarrow A}$. Note that the state on system $A$ can be arbitrarily entangled with systems that are not accessible to the protocol. To capture this notion, we consider the purification $|\psi\rangle^{A R}$ of the state on the system $A$. Thus, the goal of the protocol is to fulfil the condition:

$$
\left\|\psi^{A R}-\mathcal{D} \circ \mathcal{N} \circ \mathcal{E}(\psi)\right\|_{1} \leq \varepsilon
$$

for all pure states $|\psi\rangle^{A R}$. This is precisely equivalent to the condition that

$$
\left\|\mathbb{I}^{A}-\mathcal{D} \circ \mathcal{N} \circ \mathcal{E}\right\|_{\diamond} \leq \varepsilon
$$

It is difficult to show the existence of coding schemes using this definition of the quantum capacity, due to the maximisation over all pure states. However, in 2003, Werner and Kretschmann [16] showed that there exist several other equivalent definitions of the quantum capacity that are operationally more useful. One such definition is the entanglement transmission capacity of a quantum channel:

Definition 4.1. [Entanglement Transmission Capacity] $A(Q, \varepsilon)$ entanglement transmission code consists of an encoder $\mathcal{E}^{A \rightarrow A^{\prime}}$ and a decoding CPTP $\mathcal{D}^{B \rightarrow A}$ such that

$$
\begin{array}{r}
|\Phi\rangle^{R A}=\frac{1}{2^{Q}} \sum_{i=1}^{2^{Q}}|i\rangle^{R}|i\rangle^{A}, \\
F\left(|\Phi\rangle^{R A}, \mathbb{I}^{R} \otimes(\mathcal{D} \circ \mathcal{N} \circ \mathcal{E})\left(\Phi^{R A}\right)\right) \geq 1-\varepsilon .
\end{array}
$$

$Q$ is said to be an achievable rate for entanglement transmission if there exists a $(Q, \varepsilon)$ entanglement transmission code. The supremum of the set of all achievable rates $Q$, where the supremum is taken over all encoding and decoding maps, is defined to be the entanglement transmission capacity of the channel.

Werner and Kretschmann showed that given a $(Q, \varepsilon)$ entanglement transmission code with the encoder decoder pair $(\mathcal{E}, \mathcal{D})$, there exists another encoder decoder pair $\left(\mathcal{E}^{\prime}, \mathcal{D}^{\prime}\right)$ such that, for all pure states $|\psi\rangle^{A R}$

$$
\left\|\psi^{A R}-\mathcal{D}^{\prime} \circ \mathcal{N} \circ \mathcal{E}^{\prime}(\psi)\right\|_{1} \leq \varepsilon,
$$

where

$$
\log |A| \geq Q-1
$$

Refer to [16, 17] and [28] for details. In this paper, we will only prove the existence of codes for entanglement transmission.

We will now consider the case when Alice and Bob share EPR pairs to potentially boost the rate of entanglement transmission. In this setting, Alice and Bob share the EPR state $|\Phi\rangle^{A B}$ where $\tilde{A}$ is with Alice and $B$ lies with Bob. The two parties are allowed to use this state during the protocol, which aims to transmit the $M$ system of the maximally entangled state $|\Phi\rangle^{R M}$ from Alice to Bob. Thus Alice needs to possess an encoder $\mathcal{E}^{M \tilde{A} \rightarrow A^{\prime}}$ and Bob a decoder $\mathcal{D}^{B \rightarrow M}$ such that

$$
F\left(|\Phi\rangle^{R M}, \mathbb{I}^{R} \otimes(\mathcal{D} \circ \mathcal{N} \circ \mathcal{E})\left(\Phi^{R M} \otimes \Phi^{\tilde{A} B}\right)\right) \geq 1-\varepsilon
$$

This is known as entanglement transmission with entanglement assistance. As before, let $Q$ denote the rank of the EPR state to be transmitted (in this case $\log |M|$ ) and $E$ denote the rate at which pre-shared entanglement is available for use during the protocol (in this case $\log |\tilde{A}|$ ). Then, the rate $(Q, E)$ is said to be $\varepsilon$-achievable for entanglement transmission with entanglement assistance if there exists an encoder and decoder pair for which the above fidelity condition holds. $Q$ is said to be achievable for unassisted transmission if no pre-shared entanglement is used during the protocol i.e. if $(Q, 0)$ is $\varepsilon$-achievable.

Now suppose that instead of constraining Alice and Bob to code for only one copy of the channel, we allow them to code for $n$ tensor copies i.e. the channel $\mathcal{N}^{\otimes n}$, where $n$ can be arbitrarily large. In this case, we define the capacity of the channel as the maximum rate at which qubits can be transmitted across the channel per channel use. We state the formal definition below:

Definition 4.2. [Quantum Capacity in the iid Regime] $\operatorname{An}(n, Q)$ code for a quantum $\mathcal{N}^{A^{\prime} \rightarrow B}$ consists of an encod-
ing map $\mathcal{E}_{n}:=\mathcal{E}^{A \rightarrow A^{\prime \prime n}}$ and a decoding map $\mathcal{D}_{n}:=\mathcal{D}^{C^{n} \rightarrow A}$ such that

$$
\left\|\mathbb{I}^{A}-\mathcal{D}_{n} \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}_{n}\right\|_{\diamond} \leq \varepsilon .
$$

The rate $Q=\frac{1}{n} \log |A|$ is said to be an achievable rate for a the channel $\mathcal{N}^{A^{\prime} \rightarrow B}$ if there exists a sequence of $(n, Q)$ codes $\left(\mathcal{E}_{n}, \mathcal{D}_{n}\right)$ such that

$$
\lim _{n \rightarrow \infty}\left\|\mathbb{I}^{A}-\mathcal{D}_{n} \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}_{n}\right\|_{\diamond}=0
$$

The quantum capacity of $\mathcal{N}^{A^{\prime} \rightarrow B}$ is the supremum of all achievable rates for this channel.
One can similarly generalise the above definition to the case when entanglement assistance is available.

### 4.1 Entanglement Transmission over the QMAC

Definition 4.3. [One-Shot Entanglement Transmission over the QMAC] Given the QMAC $\mathcal{N}^{A^{\prime} B^{\prime} \rightarrow C}$, with senders Alice and Bob and receiver Charlie, suppose that Alice and Bob are given the $A$ and $B$ parts of the maximally entangled states $\left|\Phi_{1}\right\rangle^{R_{1} A}$ and $\left|\Phi_{2}\right\rangle^{R_{2} B}$. Alice and Bob want to send the systems $A$ and $B$ to Charlie via the QMAC with high fidelity. An entanglement transmission code for the QMAC then consists of the encoding maps $\mathcal{E}_{1}^{A \rightarrow A^{\prime}}$ and $\mathcal{E}_{2}^{B \rightarrow B^{\prime}}$ belonging to Alice and Bob respectively, and the decoding map $\mathcal{D}^{C \rightarrow A B}$ such that

$$
F\left(\left|\Phi_{1}\right\rangle\left|\Phi_{2}\right\rangle, \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}_{1} \otimes \mathcal{E}_{2}\left(\Phi_{1} \otimes \Phi_{2}\right)\right) \geq 1-\varepsilon .
$$

The rate of the code is defined as

$$
\begin{aligned}
R_{A} & :=\log |A| \\
R_{B} & :=\log |B| .
\end{aligned}
$$

Comment 4.4. Note that the above definition is easily generalized to the asymptotic iid case, as well as the case when Alice and Bob have access to pre-shared entanglement with Charlie.

Definition 4.5. [Entanglement Transmission Capacity Region of the QMAC] Any rate pair $\left(R_{A}, R_{B}\right)$ for which there exists a corresponding entanglement transmission code is called $\varepsilon$-achievable. The union of all $\varepsilon$-achievable rate pairs is defined as the achievable rate region for entanglement transmission over the QMAC.

A natural question is whether we can strengthen the definition of the achievable region for the QMAC to include all states that lie in the spaces corresponding to the systems $A$ and $B$. To that end, we define the task of strong subspace transmission [2]:

Definition 4.6. [Strong Subspace Transmission] Suppose Alice and Bob posses some pure quantum states $|\psi\rangle^{R_{1} A}$ and $|\varphi\rangle^{R_{2} B}$, where we place no restrictions on the systems $R_{1}$ and $R_{2}$ other than that they be finite dimensional. $A$ strong subspace transmission code then consists of encoding maps $\left(\mathcal{E}_{1}^{A \rightarrow A^{\prime}}, \mathcal{E}_{2}^{B \rightarrow B^{\prime}}\right)$ and a decoding map $\mathcal{D}^{C \rightarrow A B}$ such that, for all $|\psi\rangle^{R_{1} A}$ and $|\varphi\rangle^{R_{2} B}$

$$
F\left(|\psi\rangle|\varphi\rangle, \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}_{1} \otimes \mathcal{E}_{2}(\psi \otimes \varphi)\right) \geq 1-\varepsilon .
$$

The rate pair

$$
\left(R_{A}, R_{B}\right):=(\log |A|, \log |B|)
$$

are said to be achievable for strong subspace transmission if there exists a corresponding strong subspace transmission code. The union of all achievable rates gives the achievable region for this task.

In [2, Section 5], the authors showed that given that Alice and Bob have access to independent public coins with Charlie, the rate regions for entanglement transmission and strong subspace transmission over the QMAC are equivalent. Thus, in this paper, we will design all our protocols for entanglement transmission.

The authors of the paper [2] also provide the best known achievable bounds for this task in the asymptotic iid setting. We state their result below:

Theorem 4.7. Given the $Q M A C \mathcal{N} A^{A^{\prime} B^{\prime} \rightarrow C}$ its capacity region is given by the closure of

$$
\bigcup_{k=1}^{\infty} \frac{1}{k} \mathcal{Q}\left(\mathcal{N}^{\otimes k}\right)
$$

where the region $\mathcal{Q}(\mathcal{M})$ equals the pairs of non-negative rates $\left(R_{A}, R_{B}\right)$ satisfying

$$
\begin{aligned}
\left.R_{A}<I(A\rangle B C\right)_{\sigma} \\
\left.R_{B}<I(B\rangle A C\right)_{\sigma} \\
\left.R_{A}+R_{B}<I(A B\rangle C\right)_{\sigma},
\end{aligned}
$$

where all the entropic quantities are computed with respect to the control state

$$
\sigma^{A B C}:=\left(\mathbb{I}^{A B} \otimes \mathcal{M}\right)\left(\Omega^{A A^{\prime}} \otimes \Delta^{B B^{\prime}}\right)
$$

for a pair of pure states $|\Omega\rangle^{A A^{\prime}}$ and $|\Delta\rangle^{B B^{\prime}}$.
The achievable region shown in the theorem can be picturized by the rate region given in Fig. 6. In the figure, we use the shorthand

$$
\left.\left.I^{*}(A\rangle B\right):=\lim _{k \rightarrow \infty} \frac{1}{k} I\left(A^{k}\right\rangle B^{k}\right)
$$

Please note that the above shorthand is informal since the quantity on the right hand side is computed with respect to a state on the systems $A^{k} B^{k}$. Thus, the precise description of the rate region actually requires a union over all such states, over all values of $k$. We use this informal notation to emphasise the shape of the rate region and the fact that the rate expressions are regularised.

### 4.2 Unassisted vs. Rate Limited Assistance

As mentioned previously, a rate pair $(Q, E)$ is said to be $\varepsilon$-achievable for entanglement assisted entanglement transmission across a point-to-point quantum channel $\mathcal{N}^{A^{\prime} \rightarrow B}$ if there exists an encoder and decoder pair which consume pre-shared entanglement at a rate $E$ to faithfully transmit one half of a maximally entangled state at rate $Q$. The two extreme cases are when $E=0$ (the unassisted case) and when $E$ can be arbitrarily large. Recall that Lloyd, Shor and Devetak [12, 13, 14] showed that an achievable rate for the unassisted transmission of entanglement across the point-to-point channel is given by the maximum of the coherent information

$$
I(A\rangle B)
$$



Figure 6: Achievable rate region for the unassisted quantum MAC per channel use in the asymptotic iid setting for a fixed control state. The full region is the convex closure of all such pentagonal regions corresponding to all bipartite input control states.
over all control states of the form $\Omega^{A B}$. For the case of entanglement assisted transmission, Bennet, Shor, Smolin and Thapliyal [29] showed that the rate

$$
\frac{1}{2} I(A: B)
$$

is achievable, whenever entanglement assistance is available at the rate $\frac{1}{2}(H(A)+H(A \mid B))$. We mention that the one-shot analogue of this result was proved by Anshu, Jain and Warsi [30]. In this paper, we will be interested in proving theorems which interpolate between these two cases. To be precise, we will prove achievable bounds for entanglement transmission of the following sort:

$$
\begin{aligned}
& Q+E<I_{1} \\
& Q-E<I_{2} .
\end{aligned}
$$

Note that to recover the unassisted achievable bounds, one simply sets $E=0$. On the other hand, to recover the entanglement assisted bounds, can simply saturate the first condition. Thus this situation is more general, where we can limit the rate of entanglement assistance. We therefore call this case entanglement transmission with rate limited entanglement assistance .

It is not hard to prove a coding theorem in the case of rate limited entanglement assistance in the asymptotic iid setting, simply by time sharing between the two types of protocols. However, the situation is not so easy in the one-shot setting, since in that regime time sharing is impossible. To the best of our knowledge, bounds of this kind appeared first in the work of Dupuis [18]. All the theorems in this paper are written with this more general setting in mind.

## 5 Quantum Rate Splitting I

In this section we introduce the tools required to do quantum rate splitting. We demonstrate the technique for the point-to-point quantum channel. We will apply the tools introduced in this section to the problem of one-shot entanglement transmission over the QMAC and QIC in later sections. Another key element in our proof, which we discuss in this section, is a way to do successive cancellation decoding for entanglement transmission codes when the receiver has some side information. This allows us to generalise our bounds to the case when the sender and the receiver may share some limited number of EPR states before the protocol starts.

### 5.1 Rate Splitting for point-to-point Channels

### 5.1.1 Rate Splitting in the Classical Regime

In this section, we briefly review the idea of rate splitting, as detailed in [20]. Consider the classical point-to-point channel $\left(\mathcal{A}, P_{B \mid A}, \mathcal{B}\right)$ between Alice and Bob and let $P_{A}$ be the input distribution that maximises $I(A: B)$. The idea is to split Alice into two independent senders, Alice $_{0}$ and Alice ${ }_{1}$ and then have Bob decode their messages via a successive cancellation strategy. To do this, we create two new distributions

$$
P_{U}^{\theta} \text { and } P_{V}^{\theta}
$$

with respect to some parameter $\theta \in[0,1]$, where the random variables $U$ and $V$ both range over the same classical alphabet $\mathcal{A}$. These distributions are meant to be the input distributions of Alice $_{0}$ and Alice $_{1}$ respectively. However, we must maintain the invariant that the distribution at the input to the channel must be $P_{A}$. To do this, we define a deterministic function $f$ which has the following properties:

$$
\begin{aligned}
& f: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \\
& f(U, V) \sim P_{A}, \text { where } U \sim P_{U}^{\theta} \text { and } V \sim P_{V}^{\theta} .
\end{aligned}
$$

Moreover, when $\theta=0, V$ is distributed exactly like $A$ and $U$ is a single point distribution, and when $\theta=1, U$ is distributed exactly like $A$ and $V$ is a single point distribution. Furthermore, appealing to the properties of the mutual information one can show that

$$
I(A: B)=I\left(U^{\theta} V^{\theta}: B\right)=I\left(U^{\theta}: B\right)+I\left(V^{\theta}: B U^{\theta}\right) .
$$

From the above discussion it is clear that a simple encoding-decoding strategy is as follows:

1. Alice is split into Alice ${ }_{0}$ and Alice $_{1}$.
2. Alice $e_{0}$ uses a code of rate $I\left(U^{\theta}: B\right)$ regarding Alice ${ }_{1}$ as noise and Alice ${ }_{1}$ uses a code of rate $I\left(V^{\theta}: B U^{\theta}\right)$ regarding Alice $_{0}$ as side information at the receiver.
3. Bob decodes via successive cancellation.
4. Finally, one can show that $\left(I\left(U^{\theta}: B\right), I\left(V^{\theta}: B U^{\theta}\right)\right)$ is a continuous function in $\theta \in[0,1]$ and so the ordered pair traces out the straight line joining the points $(0, I(A: B))$ and $(I(A: B), 0)$ due to the chain rule of mutual information with equality.

With this construction in hand, one can design an encoding and decoding scheme for the classical MAC without appealing to time sharing or jointly typical simultaneous decoding. Firstly, split Alice into the two users Alice ${ }_{0}$ and Alice $_{1}$ by the construction above. Then Charlie does a successive cancellation decoding for this 3 sender MAC with senders Alice ${ }_{0}$, Bob and Alice ${ }_{1}$ : first decode Alice ${ }_{0}$ 's message treating the other senders as noise, then decode Bob's message regarding Alice, ${ }_{0}$ 's message as side information and Alice ${ }_{1}$ as noise, and finally, decode Alice ${ }_{1}$ 's message regarding Bob's and Alice ${ }_{0}$ 's message as side information. Thus three point-to-point channel decodings are done by Charlie in order to decode the sent messages at the rate triple $\left(I\left(U^{\theta}: C\right), I\left(C: B U^{\theta}\right), I\left(V^{\theta}: C B U^{\theta}\right)\right)$. Notice that, all points in the dominant face of the achievable region in Figure 1 can be achieved in this way due to continuity as $\theta$ varies from 0 to 1 . Also, observe that the split of Alice depends on $\theta$.

The triple $\left(f, P_{U}^{\theta}, P_{V}^{\theta}\right)$ with respect to the distribution $P_{A}$ is called a split of $P_{A}$. That such a triple exists is given by the following fact:

Fact 5.1. Given a distribution $P_{A}$ on the set $\mathcal{A}$, there exist two distributions $P_{U}^{\theta}$ and $P_{V}^{\theta}$ (both defined on $\mathcal{A}$ ), parameter $\theta \in[0,1]$ and a function $f: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ such that the following hold true:

1. $f(U, V) \sim P_{A}$
2. For fixed values of $x$ and $u, P_{f(U, V) \mid U}^{\theta}(a \mid u)$ is a continuous function of $\theta$.
3. $\operatorname{For} \theta=0, P_{f(U, V) \mid U}^{\theta}(a \mid u)=P_{A}(a)$.
4. For $\theta=1$, and all $u \in \mathcal{A}, P_{f(U, V) \mid U}^{\theta}(a \mid u)$ puts all its mass on one element.

Proof. We demonstrate an explicit construction, as shown in [20]. Assume that $\mathcal{A}$ is an ordered set. We describe the distribution in terms of distribution functions, for which we use the letter $F$ along with the appropriate subscript. Then, define, for all $i \in \mathcal{A}$ :

$$
\begin{aligned}
F_{U}^{\theta}(i) & :=\theta F_{A}(i)+1-\theta \\
F_{V}^{\theta}(i) & :=\frac{F_{A}(i)}{F_{U}^{\theta}(i)} \\
f(u, v) & :=\max \{u, v\} \quad \forall u, v \in \mathcal{A} .
\end{aligned}
$$

It is easy to check the triple defined above satisfies all the properties in Fact 5.1. The interested reader may look at [20] for details.

### 5.1.2 Rate Splitting in the Quantum Case

To describe rate splitting in the entanglement transmission scenario, we will define an abstract splitting scheme with some properties of interest:

Definition 5.2. Splitting Scheme Given a control state $|\Omega\rangle^{A^{\prime \prime} A^{\prime}}$ and systems $A_{0}^{\prime \prime}$ and $A_{1}^{\prime \prime}$ such that $A^{\prime \prime} \cong A_{1}^{\prime \prime} \cong A_{0}^{\prime \prime}$, we define a splitting scheme to be a family of isometric embeddings $\left\{U_{\theta}^{A^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A_{1}^{\prime \prime}}\right\}$ parametrized by a variable $\theta \in[0,1]$, such that:

1. For all $\theta, \theta^{\prime} \in[0,1]$ and $\varepsilon>0$ there exists $\delta>0$ such that whenever $\left|\theta-\theta^{\prime}\right| \leq \delta,\left\|U_{\theta} \cdot \Omega-U_{\theta^{\prime}} \cdot \Omega\right\|_{1} \leq \varepsilon$.
2. Given any channel $\mathcal{N}^{A^{\prime} \rightarrow C}$ and its Stinespring dilation $\mathcal{U}_{\mathcal{N}}^{A^{\prime} \rightarrow C E}$,

$$
\left.\left.\left.I\left(A_{0}^{\prime \prime}\right\rangle C\right)_{\mathcal{U}_{\mathcal{N}} \cdot \Omega_{0}}=I\left(A_{1}^{\prime \prime}\right\rangle C\right)_{\mathcal{U}_{\mathcal{N}} \cdot \Omega_{1}}=I\left(A^{\prime \prime}\right\rangle C\right)_{\mathcal{U}_{\mathcal{N}} \cdot \Omega}
$$

where $\Omega_{0}:=U_{0} \cdot \Omega$ and $\Omega_{1}:=U_{1} \cdot \Omega$.
The splitting scheme defined above can be defined with respect to the more general control state

$$
|\sigma\rangle^{A^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}:=|\Omega\rangle^{A^{\prime \prime} A^{\prime}}|\Delta\rangle^{B^{\prime \prime} B^{\prime}}
$$

This will be useful when we describe the splitting protocol for more general multi-terminal channels, viz. $\mathcal{N}^{A^{\prime} B^{\prime} \rightarrow C}$. In that case, the second condition in Definition 5.2 can be stated as

$$
\left.\left.\left.I\left(A_{0}^{\prime \prime} B^{\prime \prime}\right\rangle C\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma_{0}}=I\left(A_{1}^{\prime \prime} B^{\prime \prime}\right\rangle C\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma_{1}}=I\left(A^{\prime \prime} B^{\prime \prime}\right\rangle C\right)_{\sigma}
$$

where $\sigma_{0}:=U_{0} \cdot \sigma$ and $\sigma_{1}:=U_{1} \cdot \sigma$. For the purposes of this section, where we only demonstrate splitting for the point-to-point channel, one may simply ignore the state $|\Delta\rangle$. Also, note that the invariants in the splitting scheme are specified in terms of the coherent information. A more general definition would be to specify the invariants in terms of the smooth min-entropy. We work with this more general definition.

We will first give an overview of the strategy for the unassisted case. We will then state and prove the main technical lemma of this section, Proposition 5.3. The ideas used in proving this lemma will generalise easily to the setting of the multi-terminal channels such as the QMAC and the QIC.

We will emulate the strategy outlined in Section 5.1.1 for a bipartite pure quantum state

$$
|\Omega\rangle^{A^{\prime \prime} A^{\prime}}:=\sum_{a^{\prime \prime} \in \mathcal{A}^{\prime \prime}} \sqrt{P_{A^{\prime \prime}}(a)}|a\rangle^{A^{\prime \prime}}\left|\zeta_{a}\right\rangle^{A^{\prime}},
$$

where $\left|a^{\prime \prime}\right\rangle$ runs over the computational basis of $A^{\prime \prime}$ and $P_{A^{\prime \prime}}$ is a probability distribution on the basis set $\mathcal{A}^{\prime \prime}$. We split the system $A^{\prime \prime}$ into two registers $A_{0}^{\prime \prime}$ and $A_{1}^{\prime \prime}$ corresponding to the two senders Alice ${ }_{0}$ and Alice ${ }_{1}$. Let the split $\left(P_{U}^{\theta}, P_{V}^{\theta}, f\right)$ be as in the previous subsection. Define the isometric embedding $U_{\text {SPLit }}(\theta)^{A^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A_{1}^{\prime \prime}}$ as follows:

$$
\sqrt{P_{A}(a)}|a\rangle^{A^{\prime \prime}} \xrightarrow{U_{\mathrm{splur}}(\theta)} \sum_{(u, v) \in f^{-1}\left(a^{\prime \prime}\right)} \sqrt{P_{U}^{\theta}(u) P_{V}^{\theta}(v)}|u\rangle^{A_{0}^{\prime \prime}}|v\rangle^{A_{1}^{\prime \prime}},
$$

and $|\Omega(\theta)\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}:=U_{\text {SpLit }}(\theta)|\Omega\rangle^{A^{\prime \prime} A^{\prime}}$.
We now pass the system $A^{\prime}$ through a point-to-point channel $\mathcal{N}^{A^{\prime} \rightarrow B}$ and obtain the quantum state $|\Omega(\theta)\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B}$. By unitary invariance, $\left.\left.I_{\min }^{\varepsilon}\left(A^{\prime \prime}\right\rangle B\right)_{\Omega}=I_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} A_{1}^{\prime \prime}\right\rangle B\right)_{\Omega(\theta)}$. From the works of [18, 27] applied to transmission of quantum information over one-shot unassisted point to point quantum channels, we first realise that Bob can decode Alice ${ }_{0}$ 's quantum message at the rate of $\left.I_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime}\right\rangle B\right)_{\Omega(\theta)}-O\left(\log \varepsilon^{-1}\right)$ with error at most $O(\sqrt{\varepsilon})$. Then, employing the successive cancellation methods of Yard et al. [2] Bob can decode Alice ${ }_{2}$ 's quantum message at the rate of $\left.I_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{1}^{\prime \prime}\right\rangle B A_{0}^{\prime \prime}\right)_{\Omega(\theta)}-O\left(\log \varepsilon^{-1}\right)$ with error at most $O(\sqrt{\varepsilon})$.

Doing both the steps above requires us to overcome a few technical challenges. We do this by defining a notion of almost CPTP maps (see Section 5.2) and combining it with another proof technique by Dupuis for the unassisted quantum broadcast channel [18]. We believe that the notion of almost CPTP maps should be useful in other situations as well.

We have thus operationally shown the chain rule inequality $\left.\left.\left.I_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{1}^{\prime \prime}\right\rangle B A_{0}^{\prime \prime}\right)_{\Omega(\theta)}+I_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime}\right\rangle B\right)_{\Omega(\theta)} \leq I_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} A_{1}^{\prime \prime}\right\rangle B\right)_{\Omega(\theta)}$ (suppressing the log factors). One can prove this fact independently using the chain rule for the smooth min-entropy Fact 3.4. We now see that as $\theta$ varies from 0 to 1 , the point $\left.\left.\left(R_{0}(\theta), R_{1}(\theta)\right)=\left(I_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime}\right\rangle B\right)_{\Omega(\theta)}, I_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{1}^{\prime \prime}\right\rangle B A_{0}^{\prime \prime}\right)_{\Omega(\theta)}\right)$ traces out a continuous curve that lies on or below the line segment joining the point $\left.\left(I_{\min }^{\varepsilon}(A\rangle B\right), 0\right)$ to the point $\left.\left(0, I_{\text {min }}^{\varepsilon}(A\rangle B\right)\right)$ and meets it at its endpoints. The continuity of the curve follows from the continuity of the states and the functionals involved. Continuity of the functionals is implied by Fact 3.6 whereas continuity of the states is implied by Lemma A.1. This rate splitting and successive cancellation idea can now be easily generalised to the case of entanglement transmission over QMAC with limited entanglement assistance.

We will now consider the general case when Bob has side information available at the decoder. Recall that the
users Alice $_{0}$ and Alice $_{1}$ obtained from splitting the sender Alice are treated as independent senders. Hence, suppose Alice $_{0}$ and Alice ${ }_{1}$ wish to transmit the systems $A_{0}$ and $A_{1}$ of the states $|\eta\rangle^{A_{0} B_{0} R_{0}}$ and $|\psi\rangle^{A_{1} B_{1} R_{1}}$ to Bob. We wish to prove there exists an encoder $\mathcal{E}^{A_{0} A_{1} \rightarrow A^{\prime}}$ and a decoder $\mathcal{C}^{B C_{0} C_{1} \rightarrow A_{0} A_{1}}$ such that

$$
F(\mathcal{C} \circ \mathcal{N} \circ \mathcal{E}(\eta \otimes \psi), \eta \otimes \psi) \geq 1-\varepsilon
$$

Given that such an encoder decoder pair exist, set $|\eta\rangle^{A_{0} B_{0} R_{0}} \leftarrow|\Phi\rangle^{M_{0} R_{0}}|\Phi\rangle^{\tilde{A}_{0} B_{0}}$ and $|\psi\rangle^{A_{1} B_{1} R_{1}} \leftarrow|\Phi\rangle^{A_{1} M_{1}}|\Phi\rangle^{\tilde{A}_{1} B_{1}}$. Let $Q_{A_{0}}=\log \left|M_{0}\right|, Q_{A_{1}}=\log \left|M_{2}\right|$, and $E_{A_{0}}=\log \left|B_{0}\right|, E_{A_{1}}=\log \left|B_{1}\right|$. The rates $Q_{A_{0}}, Q_{A_{1}}$ are the entanglement transmission rates of Alice ${ }_{0}$ and Alice ${ }_{1}$ and $E_{A_{0}}$ and $E_{A_{1}}$ quantify the amount of pre-shared entanglement available to them before the protocol begins.

We will consider the simpler case, when Alice ${ }_{0}$ does not share any entanglement with Bob, but Alice ${ }_{1}$ does, i.e. the register $C_{0}$ is trivial. We quantify the rates in the following lemmas:

Proposition 5.3. Given the control state $|\Omega\rangle^{A^{\prime \prime} A^{\prime}}$, the point-to-point quantum channel $\mathcal{N}^{A^{\prime} \rightarrow C}$ and the splitting scheme $\left\{U_{\theta}^{A^{\prime \prime}}\right\}$, suppose Alice has to send states $|\eta\rangle^{A_{0} R_{0}} \otimes|\psi\rangle^{A_{1} B R_{1}}$ to Bob, where $A_{0}$ and $A_{1}$ are the message registers and $B$ models the side information Bob has about the $A_{1} . R_{0}$ and $R_{1}$ are reference systems. We define $\left|\Omega^{\prime}(\theta)\right\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}:=U_{\theta}^{A^{\prime \prime}}|\Omega\rangle^{A^{\prime \prime} A^{\prime}}$ and

$$
|\Omega(\theta)\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} C E}:=\mathcal{U}_{\mathcal{N}}^{A^{\prime} \rightarrow C E}|\Omega(\theta)\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}} .
$$

Then there exist an encoder $\mathcal{E}^{A_{0} A_{1} \rightarrow A^{\prime}}$ and a decoder $\mathcal{C}^{B C \rightarrow A_{0} A_{1}}$ such that

$$
\left\|\mathcal{C} \circ \mathcal{N} \circ \mathcal{E}\left(\eta^{A_{0} R_{0}} \otimes \psi^{A_{1} B_{1} R_{1}}\right)-\eta^{A_{0} R_{0}} \otimes \psi^{A_{1} B_{1} R_{1}}\right\|_{1} \leq \delta
$$

where $\delta=4 \sqrt{2 \delta_{\operatorname{dec}}(0)}+2 \sqrt{2 \delta_{\operatorname{dec}}(1)}+2 \sqrt{2 \delta_{\text {enc }(0)}+2 \delta_{\mathrm{enc}}(1)}$
and

$$
\begin{aligned}
& \delta_{\mathrm{dec}}(0)=20 \cdot 2^{-\frac{1}{2} H_{2}^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{\varepsilon_{0}}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{\Omega(\theta)}+160 \varepsilon} \\
& \delta_{\operatorname{dec}}(1)=20 \cdot 2^{-\frac{1}{2} H_{2}^{\varepsilon}\left(A_{1} \mid R_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{\varepsilon_{0}}\left(A_{1}^{\prime \prime} \mid E\right)_{\Omega(\theta)}+160 \varepsilon} \\
& \delta_{\text {enc }}(0)=20 \cdot 2^{\frac{1}{2} H_{\max }^{\varepsilon}\left(A_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{\varepsilon_{0}}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega(\theta)}+160 \varepsilon} \\
& \left.\delta_{\text {enc }}(1)=2^{\frac{1}{2} H_{\max }^{\varepsilon}\left(A_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime}\right)_{\Omega(\theta)}+12 \varepsilon,} \begin{array}{l}
\end{array}\right)
\end{aligned}
$$

where $\varepsilon_{0}=O\left(\varepsilon^{2}\right)$ for some positive $\varepsilon$.
An easy corollary of Proposition 5.3 is the following:
Corollary 5.4. Given the control state $|\Omega\rangle^{A^{\prime \prime} A^{\prime}}$, the parameter $\theta \in[0,1]$ and the point-to-point channel $\mathcal{N}^{A^{\prime} \rightarrow B}$, and the splitting scheme $\left\{U_{\theta}^{A^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A_{1}^{\prime \prime}}\right\}$, Alice can transmit EPR states to Bob at the rate $Q_{A_{0}}+Q_{A_{1}}$ given $E_{A_{1}}$ bits of

$$
\begin{gathered}
Q_{A_{0}}<H_{\max }^{\varepsilon_{0}}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega^{\prime}(\theta)}+\log 4 \varepsilon^{2} \\
\left.Q_{A_{0}}<I_{\min }^{\varepsilon_{0}}\left(A_{0}^{\prime \prime}\right\rangle B\right)_{\mathcal{U}_{\mathcal{N}} \cdot \Omega^{\prime}(\theta)}+\log 4 \varepsilon^{2} \\
Q_{A_{1}}+E_{A_{1}}<H_{\max }^{\varepsilon}\left(A_{1}^{\prime \prime}\right)_{\Omega^{\prime}(\theta)}+\log 4 \varepsilon^{2} \\
\left.Q_{A_{1}}-E_{A_{1}}<I_{\min }^{\varepsilon_{0}}\left(A_{1}^{\prime \prime}\right\rangle A_{0}^{\prime \prime} B\right)_{\mathcal{U}_{\mathcal{N}} \cdot \Omega^{\prime}(\theta)}+\log 4 \varepsilon^{2},
\end{gathered}
$$

where $\varepsilon_{0}=O\left(\varepsilon^{2}\right)$ and $\left|\Omega^{\prime}(\theta)\right\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}=U_{\theta}^{A^{\prime \prime}}|\Omega\rangle^{A^{\prime \prime} A^{\prime}}$.
Proof. We initialise the states $|\eta\rangle^{A_{0} R_{0}}$ and $|\psi\rangle^{A_{1} R_{1} B_{1}}$ as follows

$$
\begin{aligned}
& |\eta\rangle^{A_{0} R_{0}} \leftarrow|\Phi\rangle^{A_{0} R_{0}} \\
& |\psi\rangle^{A_{1} R_{1} B_{1}} \leftarrow|\Phi\rangle^{R_{1} M_{1}}|\Phi\rangle^{\tilde{A}_{1} B_{1}} .
\end{aligned}
$$

Here, the registers $M_{1} \tilde{A}_{1}$ play the roles of $A_{1}$, and the notation $\Phi$ is used generically to mean an EPR state. Let

$$
\begin{aligned}
& \left|R_{0}\right|=2^{Q_{A_{0}}} \\
& \left|R_{1}\right|=2^{Q_{A_{1}}} \text { and }\left|B_{1}\right|=2^{E_{A_{1}}} .
\end{aligned}
$$

Note that Alice's actual rate $Q_{A}$ is $Q_{A_{0}}+Q_{A_{1}}$. The following relations are easy to check:

$$
\begin{aligned}
& H_{\max }\left(A_{0}\right)_{\eta}=Q_{A_{0}} \Longrightarrow H_{\max }^{\varepsilon}\left(A_{0}\right)_{\eta} \leq Q_{A_{0}} \\
& H_{\max }\left(M_{1} \tilde{A}_{1}\right)_{\psi}=Q_{A_{1}}+E_{A_{1}} \Longrightarrow H_{\max }^{\varepsilon}\left(M_{1} \tilde{A}_{1}\right)_{\psi} \leq Q_{A_{1}}+E_{A_{1}} \\
& H_{\min }\left(A_{0} \mid R_{0}\right)_{\eta}=Q_{A_{0}} \Longrightarrow H_{\min }^{\varepsilon}\left(A_{0}\right)_{\eta} \geq Q_{A_{0}} \\
& H_{\min }\left(M_{1} \tilde{A}_{1} \mid R_{1}\right)_{\psi}=E_{A_{1}}-Q_{A_{1}} \Longrightarrow H_{\min }^{\varepsilon}\left(M_{1} \tilde{A}_{1} \mid R_{1}\right)_{\psi} \geq E_{A_{1}}-Q_{A_{1}} .
\end{aligned}
$$

Then, from Proposition 5.3, we set

$$
\begin{aligned}
\delta_{\mathrm{dec}}(0) & <200 \varepsilon \\
\delta_{\mathrm{dec}}(1) & <200 \varepsilon \\
\delta_{\mathrm{enc}}(0) & <200 \varepsilon \\
\delta_{\text {enc }}(1) & <16 \varepsilon .
\end{aligned}
$$

Plugging in these numbers in the bounds shown in Proposition 5.3 completes the proof.

### 5.2 Tools for Successive Cancellation: Intermediate States and Almost CPTP Maps

As mentioned in section Section 5.1.2, we will require the notion of almost CPTP maps to be able to successive cancellation decoding for entanglement transmission codes. An upshot of this technique is that it allows the decoder to use side information that the receiver may have, to boost the sender's entanglement transmission rate. This is what essentially allows us to provide the bounds in the general case when only limited entanglement is available.

The problem that we consider is as follows: we are given the channel $\mathcal{N}^{A^{\prime} \rightarrow C}$ and the split control state $|\Omega\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}$.

The two split senders Alice ${ }_{0}$ and Alice ${ }_{1}$ wish to send the $A_{0}$ and $A_{1}$ parts of the states $|\eta\rangle^{R_{0} A_{0}}$ and $|\psi\rangle^{A_{1} B R_{1}}$ to the receiver Charlie. Additionally, Charlie also holds the system $B$ as side information, which he can potentially use to boost Alice ${ }_{1}$ 's rate. As a first step, we embed the systems $A_{0}$ and $A_{1}$ into the systems $A_{0}^{\prime \prime}$ and $A_{1}^{\prime \prime}$ via the action of the isometries $W_{0}^{A_{0} \rightarrow A_{0}^{\prime \prime}}$ and $W_{1}^{A_{1} \rightarrow A_{1}^{\prime \prime}}$ :

$$
\begin{aligned}
|\eta\rangle^{R_{0} A_{0}^{\prime \prime}} & :=W_{0}|\eta\rangle^{R_{0} A_{0}} \\
|\psi\rangle^{R_{1} B A_{1}^{\prime \prime}} & :=W_{1}|\psi\rangle^{R_{1} B A_{1}^{\prime \prime}} .
\end{aligned}
$$

Our encoder will be of the form

$$
\mathcal{E}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} \rightarrow A^{\prime}}(\cdot) \equiv\left|A_{0}^{\prime \prime} A_{1}^{\prime \prime}\right| \mathrm{op}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} \rightarrow A^{\prime}}\left(\Omega^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right) U^{A_{0}^{\prime \prime}} \otimes U^{A_{1}^{\prime \prime}}(\cdot),
$$

where $U^{A_{0}^{\prime \prime}}$ and $U^{A_{1}^{\prime \prime}}$ are random unitaries, picked independently from the Haar measure. The above map is not trace preserving in general, and is only CPTP on average over the choices of the two random unitaries. One of our main aims will be to show that, with respect to the states $\eta$ and $\psi$, there exist fixed instantiations of $U^{A_{0}^{\prime \prime}}$ and $U^{A_{1}^{\prime \prime}}$ such that

$$
\mathcal{E}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} \rightarrow A^{\prime}} \circ W_{0} \otimes W_{1} \cdot(\eta \otimes \psi) \equiv V^{A_{0} A_{1} \rightarrow A^{\prime}} \cdot(\eta \otimes \psi)
$$

where $V^{A_{0} A_{1} \rightarrow A^{\prime}}$ is an isometry. This isometry should also have a corresponding decoding map $\mathcal{D}^{B B \rightarrow A_{0} A_{1}}$ such that

$$
F\left(|\eta\rangle^{A_{0} R_{0}}|\psi\rangle^{A_{1} B R_{1}}, \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}\left(W_{0} \cdot \eta \otimes W_{1} \cdot \psi\right)\right) \geq 1-\varepsilon .
$$

To show the existence of the encoder $V$ and its corresponding decoder $\mathcal{D}$, we first need a good way to manipulate the quantity $\mathcal{E}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} \rightarrow A^{\prime}} \circ W_{0} \otimes W_{1} \cdot(\eta \otimes \psi)$, which we henceforth abbreviate as $\mathcal{E}(\eta \otimes \psi)$. To that end, we define intermediate states.

### 5.2.1 Intermediate States

Definition 5.5. Intermediate State Given the control state $|\Omega\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}$ and the state $|\psi\rangle^{A_{1}^{\prime \prime} B R_{1}}$, we define the intermediate state

$$
|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} A^{\prime}}:=\sqrt{\left|A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}\left(|\Omega\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right) U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} B R_{1}} .
$$

The following lemma will enable us to write the encoded state in terms of the intermediate state.
Lemma 5.6. Intermediate State Lemma Given the intermediate state $|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} A^{\prime}}$, the following holds

$$
\mathcal{E}(\eta \otimes \psi)=\left|A_{0}^{\prime \prime}\right| \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow B R_{1} A^{\prime}}\left(\omega^{A_{0}^{\prime \prime} B R_{1} A^{\prime}}\right) \circ U^{A_{0}^{\prime \prime}} \cdot\left(\eta^{A_{0}^{\prime \prime} R_{0}}\right)
$$

Proof. Consider the following series of equalities:

$$
\begin{aligned}
& \sqrt{A_{0}^{\prime \prime} A_{1}^{\prime \prime}} \text { op }^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} \rightarrow A^{\prime}}\left(|\Omega\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right)\left(U^{A_{0}^{\prime \prime}} \otimes U^{A_{1}^{\prime \prime}}\right)|\eta\rangle^{A_{0}^{\prime \prime} R_{0}}|\psi\rangle^{A_{1}^{\prime \prime} B R_{1}} \\
& =\sqrt{A_{0}^{\prime \prime} A_{1}^{\prime \prime}} \text { op }^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} \rightarrow R_{0} B R_{1}}\left(\left(U^{A_{0}^{\prime \prime}} \otimes U^{A_{1}^{\prime \prime}}\right)|\eta\rangle^{A_{0}^{\prime \prime} R_{0}}|\psi\rangle^{A_{1}^{\prime \prime} B R_{1}}\right)|\Omega\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}} \\
& =\left(\sqrt{A_{0}^{\prime \prime}} \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow R_{0}}\left(U^{A_{0}^{\prime \prime}}|\eta\rangle^{A_{0}^{\prime \prime} R_{0}}\right) \otimes \sqrt{A_{1}^{\prime \prime}} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow B R_{1}}\left(U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} B R_{1}}\right)\right)|\Omega\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}} \\
& =\sqrt{A_{0}^{\prime \prime}} \text { op }^{A_{0}^{\prime \prime} \rightarrow R_{0}}\left(U^{A_{0}^{\prime \prime}}|\eta\rangle^{A_{0}^{\prime \prime} R_{0}}\right)\left(\sqrt{A_{1}^{\prime \prime}} \operatorname{op}^{A_{1}^{\prime \prime} \rightarrow B R_{1}}\left(U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} B R_{1}}\right)|\Omega\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right) \\
& =\sqrt{A_{0}^{\prime \prime}} \text { op }^{A_{0}^{\prime \prime} \rightarrow B R_{1} A^{\prime}}\left(\sqrt{A_{1}^{\prime \prime}} \text { op }^{A_{1}^{\prime \prime} \rightarrow B R_{1}}\left(U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} B R_{1}}\right)|\Omega\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right) U^{A_{0}^{\prime \prime}}|\eta\rangle^{A_{0}^{\prime \prime} R_{0}}
\end{aligned}
$$

The above derivation uses the properties of the op operator proved in Section 3.3 of Section 3. Writing the first and the last terms in the state notation gives us the required result.

Note that the intermediate states may not be quantum states in the sense that they may not have trace 1 . In the following lemma, we prove that intermediate states have trace 1 on average over the choice of random unitaries, assuming some entropic inequalities are satisfied.

Lemma 5.7. Trace of Intermediary States Given the intermediary state

$$
|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} A^{\prime}}=\sqrt{\mid A_{1}^{\prime \prime}} \mid \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}\left(\left|\Omega^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right\rangle\right) U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} R_{1} B}
$$

where $U^{A_{1}^{\prime \prime}}$ is a random unitary sampled from the Haar measure and given that

$$
H_{\max }^{\varepsilon}(A)_{\psi} \leq H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime}\right)_{\Omega}+O(\log \varepsilon)
$$

the following holds

$$
\underset{U^{A_{1}^{\prime \prime}}}{\mathbb{E}}[|\operatorname{Tr}[\omega]-1|] \leq O(\varepsilon) .
$$

Proof. Using the single sender decoupling theorem 3.13, we see that

$$
\begin{aligned}
& \underset{U^{A_{1}^{\prime \prime}}}{\mathbb{E}} \|\left|A_{1}^{\prime \prime}\right| \operatorname{Tr}_{A_{0}^{\prime \prime} A^{\prime}} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}\left(\left|\Omega^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right\rangle\right) U^{A_{1}^{\prime \prime}} \cdot \psi^{A_{1}^{\prime \prime} R_{1} B}-\psi^{R_{1} B} \|_{1} \\
\leq & 2^{-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1} \mid R_{1} B\right)_{\psi}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime}\right)_{\Omega}}+12 \varepsilon \\
= & 2^{\frac{1}{2} H_{\max }^{\varepsilon}\left(A_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime}\right)_{\Omega}}+12 \varepsilon .
\end{aligned}
$$

We can replace $A_{1}^{\prime \prime}$ with the system $A_{1}$ in the entropic quantity corresponding to $|\psi\rangle^{A_{1}^{\prime \prime} B R_{1}}$ since $|\psi\rangle^{A_{1}^{\prime \prime} B R_{1}}$ is an isometric embedding of $|\psi\rangle^{A_{1} R_{1} B}$. The last equality follows from the duality of the smooth min- and max- entropies for pure states. This concludes the proof.

We will now show the following approximate data processing type inequality: Suppose we are given the intermediary state

$$
|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} A^{\prime}}=\sqrt{\mid A_{1}^{\prime \prime}} \mid \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}\left(\left|\Omega^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right\rangle\right) U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} R_{1} B}
$$

where $U^{A_{1}^{\prime \prime}}$ is a Haar random unitary. Then, with constant probability over the choice of $U^{A_{1}^{\prime \prime}}$

$$
H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1}\right)_{\omega} \geq H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega}-O(1),
$$

where $f(\varepsilon)$ is some function of $\varepsilon$. There are several technical issues that one should note here. For example, the expression $H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1}\right)_{\omega}$ is, strictly speaking, not defined, since $\omega$ is not really a normalised state. Hence, what we actually want to show is that the above data processing like inequality holds for the quantity $H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1}\right)_{\tilde{\omega}}$, which is defined with respect to he normalised version of $\omega$. To that end, we first define almost CPTP maps in the following section.

### 5.2.2 Almost CPTP Maps

Definition 5.8 (Almost CPTP). We call a linear map $\mathcal{T}^{A_{1}^{\prime \prime} \rightarrow B R_{1}}$ as an almost CPTP if $\mathcal{T}$ has the following properties:

1. $\mathcal{T}$ is $C P$.
2. $\operatorname{Tr}\left[\mathcal{T}\left(\pi^{A_{1}^{\prime \prime}}\right)\right] \in[1-\delta, 1+\delta]$ for some small $\delta \geq 0$.
3. $\int \mathcal{T}\left(U^{A_{1}^{\prime \prime}} \cdot \xi\right) d \mu=\operatorname{Tr}[\xi] \mathcal{T}\left(\pi^{A_{1}^{\prime \prime}}\right)$.

Lemma 5.9. [Approximate Data Processing Inequality for Almost CPTP Maps] When the measure $\mu$ is set to be the Haar measure on the unitary group on $A_{1}^{\prime \prime}$, and given the condition that

$$
H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi} \leq H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\Omega}+O(\log f(\varepsilon))
$$

then the following holds with constant probability over the choice of $U^{A_{1}^{\prime \prime}}$,

$$
H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1}\right)_{\tilde{\omega}} \geq H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega}-O(1)
$$

where $\tilde{\omega}:=\frac{\omega}{\operatorname{Tr}[\omega]}$ and $f(\varepsilon)=O\left(\varepsilon^{2}\right)$.
Proof. First, given the condition that $H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi} \leq H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)+2 \log f(\varepsilon)$, from Lemma 5.7 we see that

$$
\underset{U^{A_{1}^{\prime \prime}}}{\mathbb{E}}|\operatorname{Tr}[\omega]-1| \leq 13 f(\varepsilon) .
$$

Next, define

$$
\mathcal{T}^{A_{1}^{\prime \prime} \rightarrow B R_{1}}(\xi):=\left|A_{1}^{\prime \prime}\right|\left(\mathrm{op}^{A_{1}^{\prime \prime} \rightarrow B R_{1}}(\psi) \cdot \xi\right) .
$$

## Checking $\mathcal{T}$ is almost CPTP

Firstly, it is clear that $\mathcal{T}$ is CP. Next, we see that

$$
\begin{aligned}
\operatorname{Tr}\left[\mathcal{T}\left(\pi^{A_{1}^{\prime \prime}}\right)\right] & =\operatorname{Tr}\left[\mathrm{op}^{A_{1}^{\prime \prime} \rightarrow B R_{1}}(\psi) \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow B R_{1}}(\psi)^{\dagger}\right] \\
& =\operatorname{Tr}\left[\operatorname{Tr}_{A_{1}^{\prime \prime}}(\psi)\right] \\
& =1
\end{aligned}
$$

It is another easy verification, using the properties of Haar integrals, that

$$
\int \mathcal{T}\left(U^{A_{1}^{\prime \prime}} \cdot \xi\right) d \mu=\operatorname{Tr}[\xi] \mathcal{T}\left(\pi^{A_{1}^{\prime \prime}}\right)
$$

This shows that $\mathcal{T}$ is indeed an almost CPTP.

## Applying $\mathcal{T}$ to the operator inequality

Again, using the properties of the op operator we see that

$$
\begin{aligned}
\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \Omega\right) & =\left|A_{1}^{\prime \prime}\right|\left(\operatorname{op}^{A_{1}^{\prime \prime} \rightarrow B R_{1}}(\psi) \cdot\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \Omega^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right)\right) \\
& =\left|A_{1}^{\prime \prime}\right|\left(\operatorname{op}^{A_{1}^{\prime \prime} \rightarrow B R_{1}}\left(U^{A_{1}^{\prime \prime}} \psi\right) \cdot \Omega^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right) \\
& =\left|A_{1}^{\prime \prime}\right|\left(\operatorname{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}(\Omega) \cdot\left(U^{A_{1}^{\prime \prime}} \cdot \psi^{A_{1}^{\prime \prime} B R_{1}}\right)\right) \\
& =\omega .
\end{aligned}
$$

Now, suppose that $\tilde{\Omega}$ is the optimiser in the definition of $H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega}$ and that $\|\tilde{\Omega}-\Omega\|_{1} \leq 2 f(\varepsilon)$. Suppose also that $\lambda^{A_{1}^{\prime \prime}}$ be a positive semidefinite matrix such that $\operatorname{Tr}\left[\lambda^{A_{1}^{\prime \prime}}\right]=2^{-H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega}}$ and

$$
\tilde{\Omega}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime}} \leq \mathbb{I}^{A_{0}^{\prime \prime}} \otimes \lambda^{A_{1}^{\prime \prime}}
$$

Then, using the fact that $\mathcal{T}$ is a CP map, we see that

$$
\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime}}\right) \leq \mathbb{I}_{0}^{A_{0}^{\prime \prime}} \otimes \mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \lambda^{A_{1}^{\prime \prime}}\right)^{B R_{1}}
$$

First notice that, by properties 2 and 3 of almost CPTP maps (Definition 5.8),

$$
\begin{aligned}
\int \mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \lambda^{A_{1}^{\prime \prime}}\right) d U^{A_{1}^{\prime \prime}} & =\int \mathcal{T}\left(U^{A_{1}^{\prime \prime}} \cdot \lambda^{A_{1}^{\prime \prime}}\right) d U^{A_{1}^{\prime \prime}} \\
& =\operatorname{Tr}\left[\lambda_{1}^{A_{1}^{\prime \prime}}\right] \mathcal{T}\left(\pi^{A_{1}^{\prime \prime}}\right)
\end{aligned}
$$

Taking trace on both sides

$$
\operatorname{Tr}\left[\int \mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \lambda^{A_{1}^{\prime \prime}}\right) d U^{A_{1}^{\prime \prime}}\right]=2^{-H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right) \Omega}
$$

where the last equality stems from the fact that for $\mathcal{T}$, property 2 holds with $\delta=0$. Next, from the fact that $\tilde{\Omega}-\Omega$ is Hermitian, we can write $\tilde{\Omega}-\Omega=\Delta_{+}-\Delta_{-}$where $\Delta_{ \pm}$are positive semidefinite matrices with disjoint support. This implies that

$$
\begin{aligned}
\|\tilde{\Omega}-\Omega\|_{1} & =\operatorname{Tr}\left[\Delta_{+}\right]+\operatorname{Tr}\left[\Delta_{-}\right] \\
& \leq 2 f(\varepsilon)
\end{aligned}
$$

then

$$
\begin{aligned}
\int\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)-\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \Omega\right)\right\|_{1} d U^{A_{1}^{\prime \prime}} & =\int\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot\left(\Delta_{+}-\Delta_{-}\right)\right)\right\|_{1} d U^{A_{1}^{\prime \prime}} \\
& \leq \int \operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot\left(\Delta_{+}\right)\right] d U^{A_{1}^{\prime \prime}}+\int \operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot\left(\Delta_{-}\right)\right)\right] d U^{A_{1}^{\prime \prime}}\right. \\
& =\left(\operatorname{Tr}\left[\Delta_{+}\right]+\operatorname{Tr}\left[\Delta_{-}\right]\right) \operatorname{Tr}\left[\mathcal{T}\left(\pi^{A_{1}^{\prime \prime}}\right)\right] \\
& \leq 4 f(\varepsilon) .
\end{aligned}
$$

## Derandomization

Consider the following random variables:

1. $X_{1}:=|\operatorname{Tr}[\omega]-1|$.
2. $X_{2}:=\operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \lambda^{A_{1}^{\prime \prime}}\right)\right]$.
3. $X_{3}:=\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)-\omega\right\|_{1}$.

We know from the previous arguments that

1. $\mathbb{E}\left[X_{1}\right] \leq 13 f(\varepsilon)=: \mu_{1}$.
2. $\mathbb{E}\left[X_{2}\right]=2^{-H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega}}=: \mu_{2}$.
3. $\mathbb{E}\left[X_{3}\right] \leq 4 f(\varepsilon)=: \mu_{3}$.

Then, by Markov's inequality and a union bound, for some integer $k \geq 4$, we see that

$$
\operatorname{Pr}\left[\prod_{i \in[3]}\left\{X_{i} \leq k \cdot \mu_{i}\right\}\right] \geq 1-\frac{3}{k}
$$

This implies that there exists, with at least constant probability, a fixed value of $U^{A_{1}^{\prime \prime}}$ such that

1. $\|\operatorname{Tr}[\omega]-1\|_{1} \leq k \cdot 13 f(\varepsilon)$.
2. $\operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \lambda^{A_{1}^{\prime \prime}}\right)\right] \leq k \cdot 2^{-H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega}}$.
3. $\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)-\omega\right\|_{1} \leq k \cdot 4 f(\varepsilon)$.

## Consider now

$$
\begin{aligned}
\left\|\frac{\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)}{\operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)\right]}-\frac{\omega}{\operatorname{Tr}[\omega]}\right\|_{1} & \leq\left\|\frac{\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)}{\operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)\right]}-\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)\right\|_{1} \\
& +\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)-\omega\right\|_{1} \\
& +\left\|\omega-\frac{\omega}{\operatorname{Tr}[\omega]}\right\|_{1} \\
& =\left|\operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)\right]-1\right| \\
& +\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)-\omega\right\|_{1}+|\operatorname{Tr}[\omega]-1| \\
& \leq k \cdot 17 f(\varepsilon)+k \cdot 4 f(\varepsilon)+k \cdot 13 f(\varepsilon)=k \cdot 34 f(\varepsilon):=k^{\prime} f(\varepsilon) .
\end{aligned}
$$

Notice that $\frac{\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)}{\operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)\right]}$ is a normalized state in the $\sqrt{k^{\prime} f(\varepsilon)}$ ball (w.r.t the purified distance) around the state $\tilde{\omega}$. This means that $\frac{\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)}{\operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)\right]}$ is a candidate optimiser for $H_{\min }^{\sqrt{k^{\prime} \cdot f(\varepsilon)}}\left(A_{0}^{\prime \prime} \mid B R_{1}\right) \tilde{\omega}$. To be precise, using Item 2 we see that

$$
H_{\min }^{\sqrt{k^{\prime} \cdot f(\varepsilon)}}\left(A_{0}^{\prime \prime} \mid B R_{1}\right)_{\tilde{\omega}} \geq H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega}-\log k+\log (1-17 k f(\varepsilon))
$$

We now set the function $f(\varepsilon)$ as $\frac{\varepsilon^{2}}{k^{\prime}}$. Then, substituting we get

$$
H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1}\right)_{\tilde{\omega}} \geq H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega}-O(1)
$$

which concludes the proof.
The following lemma demonstrates the use of almost CPTP maps with the decoupling theorem in a channel coding scenario.

Lemma 5.10. Given the intermediary state $|\omega\rangle^{A_{0}^{\prime \prime} A^{\prime} B R_{1}}$ and the channel $\mathcal{N}^{A^{\prime} \rightarrow C}$ with Stinespring dilation $\mathcal{U}^{A^{\prime} \rightarrow C E}$, let the measure $\mu$ to be the Haar measure over the unitary group corresponding to the system $A_{1}^{\prime \prime}$. Let $|\tilde{\omega}\rangle$ we the normalised unit vector obtained from $|\omega\rangle$. Suppose that we are given the condition

$$
H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi} \leq H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\Omega}+O(\log f(\varepsilon))
$$

Then following holds true with constant probability over the choices of $U^{A_{0}^{\prime \prime}}$ and $U^{A_{1}^{\prime \prime}}$

$$
\begin{aligned}
& \left\|\left(\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{C} \mathcal{U}_{\mathcal{N}} \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\tilde{\omega}) U^{A_{0}^{\prime \prime}} \cdot \eta^{A_{0}^{\prime \prime} R_{0}}\right)^{R_{0} B R_{1} E}-\eta^{R_{0}} \otimes \tilde{\omega}^{E B R_{1}}\right\|_{1} \\
& \leq 2^{-\frac{1}{2} H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \Omega}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}+2 \log k}+12 k \varepsilon,
\end{aligned}
$$

where $k$ is a constant positive integer and $f(\varepsilon)=O\left(\varepsilon^{2}\right)$.
Proof. First, we define the intermediary state

$$
|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} C E}:=\mathcal{U}_{\mathcal{N}}^{A^{\prime} \rightarrow C E}|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} A^{\prime}} .
$$

It is not hard to see from the properties of the op operator that

$$
|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} C E}=\sqrt{\left|A_{1}^{\prime \prime}\right|}\left(\mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} C E}\left(\mathcal{U}_{\mathcal{N}}|\Omega\rangle\right) U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} B R_{1}}\right) .
$$

Note that since $\mathcal{U}_{\mathcal{N}}$ is trace preserving, the traces of $\omega^{A_{0}^{\prime \prime} B R_{1} C E}$ and $\omega^{A_{0}^{\prime \prime} B R_{1} A^{\prime}}$ are the same. We refer to this trace quantity as $\operatorname{Tr}[\omega]$ throughout the proof.

Recall that, the condition $H_{\max }^{f(\varepsilon)}(A)_{\psi} \leq H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\Omega}+2 \log f(\varepsilon)$ along with Lemma 5.7 implies that

$$
\underset{A_{1}^{\prime \prime}}{\mathbb{E}}|\operatorname{Tr}[\omega]-1| \leq 13 f(\varepsilon) .
$$

We work with the same almost CPTP map $\mathcal{T}^{A_{1}^{\prime \prime} \rightarrow B R_{1}}$ as in Lemma 5.9 Suppose $\tilde{\Omega}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} E}$ is the optimiser in the definition of $H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \Omega}$ and $\left\|\mathcal{U}_{\mathcal{N}} \cdot \Omega-\tilde{\Omega}\right\| \leq 2 f(\varepsilon)$. Let $\lambda^{A_{1}^{\prime \prime} E}$ be a positive semidefinite matrix such that

$$
\operatorname{Tr}\left[\lambda^{A_{1}^{\prime \prime} E}\right]=2^{-H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} E\right) u_{\mathcal{N}} \cdot \Omega}
$$

and

$$
\begin{aligned}
& \tilde{\Omega} \leq \mathbb{I}_{0}^{A_{0}^{\prime \prime}} \otimes \lambda^{A_{1}^{\prime \prime} E} \\
\Longrightarrow & \mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right) \leq \mathbb{I}_{0}^{A_{0}^{\prime \prime}} \otimes \mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \lambda_{1}^{A_{1}^{\prime \prime} E}\right) .
\end{aligned}
$$

As before, we note that the action of the random map $\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T}(\cdot)\right)$ does not change the trace of $\lambda^{A_{1}^{\prime \prime} E}$ on average:

$$
\begin{aligned}
\int \mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \lambda^{A_{1}^{\prime \prime} E}\right) d U^{A_{1}^{\prime \prime}} & =\int \mathcal{T}\left(U^{A_{1}^{\prime \prime}} \cdot \lambda^{A_{1}^{\prime \prime} E}\right) d U^{A_{1}^{\prime \prime}} \\
& =\mathcal{T}\left(\pi^{A_{1}^{\prime \prime}}\right) \otimes \lambda^{E}
\end{aligned}
$$

Taking trace on both sides

$$
\begin{aligned}
\operatorname{Tr}\left[\int \mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \lambda^{A_{1}^{\prime \prime} E}\right) d U^{A_{1}^{\prime \prime}}\right] & =\operatorname{Tr}\left[\lambda^{E}\right] \\
& =\operatorname{Tr}\left[\lambda^{A_{1}^{\prime \prime} E}\right] \\
& =2^{-H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} E\right) u_{\mathcal{N}^{\Omega} \Omega}} .
\end{aligned}
$$

It is also not hard to see via the definition of $|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} C E}$ that

$$
\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \mathcal{U}_{\mathcal{N}} \cdot \Omega\right)^{A_{0}^{\prime \prime} B R_{1} C E}=\omega^{A_{0}^{\prime \prime} B R_{1} C E}
$$

We will now apply the smooth single sender decoupling theorem to the quantity on the left in the theorem statement, after appropriate normalisation:

$$
\begin{aligned}
& \int \frac{1}{\operatorname{Tr}[\omega]}\left\|\left(\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{C} \mathcal{U}_{\mathcal{N}} \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\omega) U^{A_{0}^{\prime \prime}} \cdot \eta^{A_{0}^{\prime \prime} R_{0}}\right)^{R_{0} B R_{1} E}-\eta^{R_{0}} \otimes \omega^{B R_{1} E}\right\|_{1} d U^{A_{0}^{\prime \prime}} \\
& \leq 2^{-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1} E\right)_{\tilde{\omega}-\frac{1}{2}} H_{\min }^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}}+12 \varepsilon .
\end{aligned}
$$

## Derandomization

Next, define the random variables

1. $X_{1}:=\|\operatorname{Tr}[\omega]-1\|_{1}$
2. $X_{2}:=\operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \lambda^{A_{1}^{\prime \prime} E}\right)\right]$
3. $X_{3}:=\|\left(\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}}\right)^{T} \cdot \tilde{\Omega}\right)^{A_{0}^{\prime \prime} B R_{1} E}-\omega^{A_{0}^{\prime \prime} B R_{1} E} \|_{1}\right.$
4. $X_{4}:=\frac{1}{\operatorname{Tr}[\omega]} \|\left(\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{C} \mathcal{U}_{\mathcal{N}} \text { op }^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\omega) U^{A_{0}^{\prime \prime}} \cdot \eta^{A_{0}^{\prime \prime} R_{0}}\right)^{R_{0} B R_{1} E}-\eta^{R_{0}} \otimes \omega^{B R_{1} E} \|_{1}$
5. $\mu_{4}\left(U^{A_{1}^{\prime \prime}}\right):=2^{-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1} E\right)_{\tilde{\omega}}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}+12 \varepsilon}$

We define $\mu_{1}, \mu_{2}$ and $\mu_{3}$ analogously as in Lemma 5.9. We already know from the single sender decoupling theorem that

$$
\underset{U^{A_{0}^{\prime \prime}}}{\mathbb{E}}\left[X_{4} \mid U^{A_{1}^{\prime \prime}}\right] \leq \mu_{4}\left(U^{A_{1}^{\prime \prime}}\right) .
$$

Let $k \in \mathbb{N}$ be some positive integer $\geq 5$. Then, via the conditional Markov inequality we see that

$$
\begin{aligned}
\operatorname{Pr}_{U^{A_{0}^{\prime \prime}}, U^{A_{1}^{\prime \prime}}}\left[X_{4} \geq k \cdot \mu_{4}\left(A_{1}^{\prime \prime}\right)\right] & =\sum_{U^{A_{1}^{\prime \prime}}} \operatorname{Pr}_{A_{0}^{A^{\prime \prime}}}\left[X_{4} \geq k \cdot \mu_{4}\left(A_{1}^{\prime \prime}\right) \mid U^{A_{1}^{\prime \prime}}\right] \cdot \operatorname{Pr}\left[U^{A_{1}^{\prime \prime}}\right] \\
& \leq \sum_{U^{A_{1}^{\prime \prime}}} \frac{1}{k} \operatorname{Pr}\left[U^{A_{1}^{\prime \prime}}\right] \\
& =\frac{1}{k} .
\end{aligned}
$$

Since $X_{1}, X_{2}$ and $X_{3}$ are only functions of $U^{A_{1}^{\prime \prime}}$, Markov's inequality along with a union bound imply that

$$
\operatorname{Pr}_{U^{A_{0}^{\prime \prime}, U^{A_{1}^{\prime \prime}}}}^{\operatorname{Pr}}\left[\prod_{i \in[4]}\left\{X_{i} \leq k \cdot \mu_{i}\right\}\right] \geq 1-\frac{4}{k}
$$

Then, repeating the arguments in Lemma 5.9, one can see that there exists, with probability at least $1-\frac{4}{k}$, fixed unitaries $U^{A_{0}^{\prime \prime}}$ and $U^{A_{1}^{\prime \prime}}$ such that the following holds:

$$
H_{\min }^{\sqrt{34 k f(\varepsilon)}}\left(A_{0}^{\prime \prime} \mid B R_{1} E\right)_{\tilde{\omega}} \geq H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \Omega}-\log k+\log (1-17 k f(\varepsilon))
$$

and

$$
\begin{aligned}
& \frac{1}{\operatorname{Tr}[\omega]}\left\|\left(\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{C} \mathcal{U}_{\mathcal{N}} \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\omega) U^{A_{0}^{\prime \prime}} \cdot \eta^{A_{0}^{\prime \prime} R_{0}}\right)^{R_{0} B R_{1} E}-\eta^{R_{0}} \otimes \omega^{B R_{1} E}\right\|_{1} \\
& \leq k \cdot 2^{-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1} E\right)_{\tilde{\omega}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}}+12 k \varepsilon .}
\end{aligned}
$$

Setting $f(\varepsilon)=\frac{\varepsilon^{2}}{34 k}$ and making the appropriate substitutions, we see that

$$
\begin{aligned}
& \frac{1}{\operatorname{Tr}[\omega]} \|\left(\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{C} \mathcal{U}_{\mathcal{N}} \text { op }^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\omega) U^{A_{0}^{\prime \prime}} \cdot \eta^{A_{0}^{\prime \prime} R_{0}}\right)^{B R_{0} R_{1} E}-\eta^{R_{0}} \otimes \omega^{B R_{1} E} \|_{1} \\
& \leq k \cdot \frac{1}{\left(1-\frac{\varepsilon^{2}}{2}\right)} \cdot 2^{-\frac{1}{2} H_{\min }^{\frac{\varepsilon^{2}}{34 k}}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} E\right) \mathcal{u}_{\mathcal{N}} \cdot \Omega-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}+\log k}+12 k \varepsilon \\
& \leq 2^{-\frac{1}{2} H_{\min }^{\frac{\varepsilon^{2}}{34 k}}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \Omega}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}+O(\log k)}+12 k \varepsilon .
\end{aligned}
$$

This concludes the proof.

### 5.3 Proof of Proposition 5.3

We are now ready to prove our main theorem in this section, which is Proposition 5.3.

Proof. At the outset we assume that

$$
H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi} \leq H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\Omega}+O(\log f(\varepsilon))
$$

where $f(\varepsilon)=O\left(\varepsilon^{2}\right)$. Consider the randomised encoder

$$
\mathcal{E}_{\mathrm{RAND}}^{A_{0} A_{1} \rightarrow A^{\prime}} \equiv \sqrt{\left|A_{0}^{\prime \prime}\right|\left|A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} \rightarrow A^{\prime}}\left(\Omega^{\prime}(\theta)\right)\left(U^{A_{0}^{\prime \prime}} W_{0}^{A_{1} \rightarrow A_{0}^{\prime \prime}} \otimes U_{1}^{A_{1}^{\prime \prime}} W_{1}^{A_{1} \rightarrow A_{1}^{\prime \prime}}\right)
$$

From the Lemma 5.6 we know that

$$
\mathcal{E}(\eta \otimes \psi)=\left|A_{0}^{\prime \prime}\right| \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow B R_{1} A^{\prime}}\left(|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} A^{\prime}}\right) U^{A_{0}^{\prime \prime}} \cdot \eta^{A_{0}^{\prime \prime} R_{0}}
$$

where $|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} A^{\prime}}$ is the intermediate state defined as

$$
|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} A^{\prime}}=\sqrt{\left|A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}\left(\left|\Omega^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right\rangle\right) U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} R_{1} B} .
$$

We also use the convention that $\tilde{\omega}$ is the normalised version of $\omega$.

## The Decoupling Step

We consider the four decoupling equations corresponding to the encoding and decoding steps for Alice ${ }_{0}$ and Alice ${ }_{1}$.

## The Encoding Equations

$$
\begin{aligned}
& \underset{U^{A_{0}^{\prime \prime}}}{\mathbb{E}}\left[\|\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{A^{\prime}} \text { op }^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\tilde{\omega}) U^{A_{0}^{\prime \prime}} W_{0} \cdot \eta^{R_{0} A_{0}}-\eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1}} \|_{1}\right] \\
& \leq 2^{\frac{1}{2} H_{\max }^{\varepsilon}\left(A_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B_{1} R_{1}\right)_{\tilde{\omega}}}+12 \varepsilon . \quad \text { (enc_Alice }{ }_{0} \text { ) } \\
& \underset{U^{A_{1}^{\prime \prime}}}{\mathbb{E}}\left[\|\left|A_{1}^{\prime \prime}\right| \operatorname{Tr}_{A_{0}^{\prime \prime} A^{\prime}} \text { op }^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}(\Omega(\theta)) U^{A_{1}^{\prime \prime}} W_{1} \cdot \psi^{R_{1} B A_{1}}-\psi^{R_{1} B} \|_{1}\right] \\
&\left.\leq 2^{\frac{1}{2} H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\Omega}}+12 f(\varepsilon) . \quad \text { (enc_Alice }{ }_{1}\right)
\end{aligned}
$$

## The Decoding Equations

$$
\begin{aligned}
\underset{U^{A_{0}^{\prime \prime}}}{\mathbb{E}}\left[\|\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{C} \mathcal{U}_{\mathcal{N}} \text { op }^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\tilde{\omega}) U^{A_{0}^{\prime \prime}} W_{0} \cdot \eta^{R_{0} A_{0}}-\right. & \left.\eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1} E} \|_{1}\right] \\
& \left.\leq 2^{-\frac{1}{2} H_{2}^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1} E\right)_{\tilde{\omega}}}+12 \varepsilon . \quad \text { (dec_Alice }{ }_{0}\right)
\end{aligned}
$$

$$
\begin{aligned}
\underset{U^{A_{1}^{\prime \prime}}}{\mathbb{E}}\left[\|\left|A_{1}^{\prime \prime}\right| \operatorname{Tr}_{C A_{0}^{\prime \prime}} \mathcal{U}_{\mathcal{N}} \text { op }^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}(\Omega) U^{A_{1}^{\prime \prime}} W_{1} \cdot \psi^{R_{1} A_{1}}-\right. & \left.\psi^{R_{1}} \otimes \Omega^{E} \|_{1}\right] \\
& \leq 2^{-\frac{1}{2} H_{2}^{\varepsilon}\left(A_{1} \mid R_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime} \mid E\right)_{\mathcal{N}_{\mathcal{N}}} \cdot \Omega}+12 \varepsilon .
\end{aligned}
$$

(dec_Alice ${ }_{1}$ )

## Derandomisation

Note that under the assumption that

$$
H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi} \leq H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\Omega}+O(\log f(\varepsilon)),
$$

the upper bound in Eq. enc_Alice $_{1}$ ) is at most $13 f(\varepsilon)$. Then, following steps that are similar to the arguments in Lemma 5.10, we see that for a large enough but constant integer $k$, there exist unitaries $U^{A_{0}^{\prime \prime}}$ and $U^{A_{1}^{\prime \prime}}$ such that

$$
\begin{aligned}
& \left\|\left|\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{A^{\prime}} \operatorname{op}^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\tilde{\omega}) U^{A_{0}^{\prime \prime}} W_{0} \cdot \eta^{R_{0} A_{0}}-\eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1}} \|_{1}\right.\right. \\
& \leq k \cdot 2^{\frac{1}{2} H_{\text {max }}^{\varepsilon}\left(A_{0}\right)_{\eta}-\frac{1}{2} H_{\text {min }}^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B_{1} R_{1}\right)_{\bar{\omega}}}+12 k \varepsilon, \\
& \|\left|A_{1}^{\prime \prime}\right| \operatorname{Tr}_{A_{0}^{\prime \prime} A^{\prime}} \text { op }^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}(\Omega(\theta)) U^{A_{1}^{\prime \prime}} W_{1} \cdot \psi^{R_{1} B A_{1}}-\psi^{R_{1} B} \|_{1} \\
& \leq 13 k f(\varepsilon), \\
& \left\|\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{C} \mathcal{U}_{\mathcal{N}} \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\tilde{\omega}) U^{A_{0}^{\prime \prime}} W_{0} \cdot \eta^{R_{0} A_{0}}-\eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1} E}\right\|_{1} \\
& \leq k \cdot 2^{-\frac{1}{2} H_{2}^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1} E\right)_{\tilde{\omega}}}+12 k \varepsilon, \\
& \left\|\left|A_{1}^{\prime \prime}\right| \operatorname{Tr}_{C A_{0}^{\prime \prime}} \mathcal{U}_{\mathcal{N}} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}(\Omega) U^{A_{1}^{\prime \prime}} W_{1} \cdot \psi^{R_{1} A_{1}}-\psi^{R_{1}} \otimes \Omega^{E}\right\|_{1} \\
& \leq k \cdot 2^{-\frac{1}{2} H_{2}^{\varepsilon}\left(A_{1} \mid R_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime} \mid E\right) u_{\mathcal{N}} \cdot \Omega}+12 k \varepsilon, \\
& H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1}\right)_{\tilde{\omega}} \geq H_{\min }^{\frac{\varepsilon^{2}}{34 k}}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega}-\log k+\log \left(1-\frac{\varepsilon^{2}}{34}\right), \\
& H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B R_{1} E\right)_{\tilde{\omega}} \geq H_{\min }^{\frac{\varepsilon^{2}}{34 k}}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \Omega}-\log k+\log \left(1-\frac{\varepsilon^{2}}{34}\right),
\end{aligned}
$$

where we get the data processing inequalities by setting $f(\varepsilon)=\frac{\varepsilon^{2}}{34 k}$. Simplifying the above and using the definition of $\omega$ we see that the above inequalities imply that

$$
\begin{aligned}
& \|\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{A^{\prime}} \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\tilde{\omega}) U^{A_{0}^{\prime \prime}} W_{0} \cdot \eta^{R_{0} A_{0}}- \eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1}} \|_{1} \\
& \leq 2^{\frac{1}{2} H_{\max }^{\varepsilon}\left(A_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\Omega}+O(\log k)}+12 k \varepsilon \\
&:=\delta_{\mathrm{enc}}(1), \\
&\left\|\omega^{R_{1} B}-\psi^{R_{1} B}\right\|_{1} \leq 13 k f(\varepsilon):=\delta_{\mathrm{enc}}(2), \\
& \|\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{C} \mathcal{U}_{\mathcal{N}} \operatorname{op}^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\tilde{\omega}) U^{A_{0}^{\prime \prime}} W_{0} \cdot \eta^{R_{0} A_{0}}- \eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1} E} \|_{1} \\
& \leq 2^{-\frac{1}{2} H_{2}^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{u_{\mathcal{N}} \cdot \Omega}+O(\log k)}+12 k \varepsilon \\
&:=\delta_{\operatorname{dec}(1)}, \\
&\left\|\omega^{R_{1} E}-\psi^{R_{1}} \otimes \Omega^{E}\right\|_{1} \leq 2^{-\frac{1}{2} H_{2}^{\varepsilon}\left(A_{1} \mid R_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime} \mid E\right)_{u_{\mathcal{N}} \cdot \Omega}+O(\log k)}+12 k \varepsilon:=\delta_{\operatorname{dec}}(2) .
\end{aligned}
$$

## Normalisation and Uhlmann's Theorem

Note that, the matrices on the left inside each 1-norm expression is unnormalised. We use Lemma A. 2 to replace each of these with their normalised counterparts, which increases each of the upper bounds by a multiplicative factor of 2 . Also note that, by Corollary A.3,

$$
\left\|\frac{\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{A^{\prime}} \text { op }^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\omega) U^{A_{0}^{\prime \prime}} W_{0} \cdot \eta^{R_{0} A_{0}}}{\operatorname{Tr}\left[\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{A^{\prime}} \operatorname{op}^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\omega) U^{A_{0}^{\prime \prime}} W_{0} \cdot \eta^{R_{0} A_{0}}\right]}-\eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1}}\right\|_{1} \leq 2 \delta_{\mathrm{enc}}(1)
$$

which, by the definition of $\mathcal{E}(\eta \otimes \psi)$ is equivalent to

$$
\left\|\operatorname{Tr}_{A^{\prime}} \frac{\mathcal{E}(\eta \otimes \psi)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}-\eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1}}\right\|_{1} \leq 2 \delta_{\mathrm{enc}}(1) .
$$

Then, combining the first two inequalities (after appropriately appending $\eta^{R_{0}}$ to the second inequality) we see that

$$
\left\|\operatorname{Tr}_{A^{\prime}} \frac{\mathcal{E}(\eta \otimes \psi)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}-\eta^{R_{0}} \otimes \psi^{R_{1} B}\right\|_{1} \leq 2 \delta_{\mathrm{enc}}(1)+2 \delta_{\mathrm{enc}}(2) .
$$

Thus, applying Uhlmann's theorem, we see that there exists an encoding isometry $V_{\text {enc }}^{A_{0} A_{1} \rightarrow A^{\prime}}$ such that

$$
\left\|\frac{\mathcal{E}(\eta \otimes \psi)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}-V_{\mathrm{enc}}^{A_{0} A_{1} \rightarrow A^{\prime}} \cdot \eta^{A_{0} R_{0}} \otimes \psi^{A_{1} B R_{1}}\right\|_{1} \leq 2 \sqrt{2 \delta_{\mathrm{enc}}(1)+2 \delta_{\mathrm{enc}}(2)} .
$$

Next, note that $\frac{1}{\sqrt{\operatorname{Tr}[\omega]}}|\omega\rangle^{A_{0}^{\prime \prime} B R_{1} C E}$ is a valid purification of the state $\tilde{\omega}^{B R_{1} E}$ appearing in the inequality corresponding to $\delta_{\operatorname{dec}}(1)$ and also of the state $\tilde{\omega}^{R_{1} E}$ appearing in the normalised version of the inequality corresponding to $\delta_{\operatorname{dec}}(2)$. Then, via Uhlmann's theorem we see that there exist isometries

$$
\begin{aligned}
& V_{1}^{C \rightarrow A_{0} \stackrel{\circ}{C} A^{\prime \prime}{ }_{0}} \\
& V_{2}^{\AA_{0}^{\prime \prime} \stackrel{\circ}{C} B \rightarrow A_{1} B F},
\end{aligned}
$$

such that

$$
\left\|V_{1}^{C \rightarrow A_{0} \stackrel{\circ}{C} \stackrel{\circ}{A}^{\prime \prime}} \cdot \mathcal{U}_{\mathcal{N}}\left(\frac{\mathcal{E}(\eta \otimes \psi)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}\right)-\eta^{A_{0} R_{0}} \otimes \tilde{\omega}^{\stackrel{\circ}{A}_{0}^{\prime \prime} \stackrel{\circ}{C} R_{1} B E}\right\|_{1} \leq 2 \sqrt{2 \delta_{\operatorname{dec}}(1)}
$$

and

$$
\left\|V_{2}^{\stackrel{\circ}{A}_{0}^{\prime \prime} \stackrel{\circ}{C} B \rightarrow A_{1} B F} \cdot \tilde{\omega}^{\stackrel{\circ}{A}_{0}^{\prime \prime} \stackrel{\circ}{C} R_{1} B E}-\psi^{A_{1} B R_{1}} \otimes \Omega^{E F}\right\|_{1} \leq 2 \sqrt{2 \delta_{\operatorname{dec}}(2)} .
$$

Then, by using the triangle inequality after appending the state $\eta^{A_{0} R_{0}}$ to the second inequality and applying the isometry $V_{2}$ to the first inequality, we see that

$$
\begin{aligned}
\left\|V_{2} \circ V_{1} \circ \mathcal{U}_{\mathcal{N}}\left(\frac{\mathcal{E}(\eta \otimes \psi)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}\right)-\eta^{A_{0} R_{0}} \otimes \psi^{A_{1} B R_{1}} \otimes \Omega^{E F}\right\|_{1} & \leq 2 \sqrt{2 \delta_{\operatorname{dec}}(1)}+2 \sqrt{2 \delta_{\operatorname{dec}}(2)} \\
& :=\delta^{\prime}
\end{aligned}
$$

Defining

$$
\mathcal{C}^{B C \rightarrow A_{0} A_{1}}:=\operatorname{Tr}_{F} V_{2} \circ V_{1}
$$

and discarding the $E$ system, we see that

$$
\left\|\mathcal{C} \circ \mathcal{N}\left(\frac{\mathcal{E}(\eta \otimes \psi)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}\right)-\eta^{A_{0} R_{0}} \otimes \psi^{A_{1} B R_{1}}\right\|_{1} \leq \delta^{\prime}
$$

A further application of the triangle inequality with the expression which bounds the encoding error (after acting the operator $\mathcal{C} \circ \mathcal{N}$ on it) shows that

$$
\begin{aligned}
\left\|\mathcal{C} \circ \mathcal{N} \circ V_{\mathrm{enc}}(\eta \otimes \psi)-\eta^{A_{0} R_{0}} \otimes \psi^{A_{1} B R_{1}}\right\|_{1} & \leq \delta^{\prime}+2 \sqrt{2 \delta_{\text {enc }}(1)+2 \delta_{\mathrm{enc}}(2)} \\
& :=\delta .
\end{aligned}
$$

## Successive Cancellation

The decoding algorithm is now clear.

1. Alice creates a state close to $\left(\frac{\mathcal{E}(\eta \otimes \psi)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}\right)$ by using the encoding isometry $V_{\text {enc }}$ on $|\eta\rangle^{A_{0} R_{0}}|\psi\rangle^{A_{1} B R_{1}}$.
2. Charlie first recovers the $A_{0}$ system of $|\eta\rangle^{A_{0} R_{0}}$ by applying the isometry $V_{1}$. This isometry also does the job of locally simulating the system $A_{0}^{\prime \prime}$ of the state $\omega$ at Charlie's end.
3. Using the locally created $A_{0}^{\prime \prime}$ system and the pre-shared $B$ system as 'side information', Bob decodes the $A_{1}$ register of the state $|\psi\rangle^{A_{1} B R_{1}}$.
4. The entire decoding procedure, after discarding the purifying system $F$ is encapsulated by the operator $\mathcal{C}$.

This completes the proof of the theorem.

## 6 Quantum Rate Splitting II

In this section, we apply the quantum rate splitting and successive cancellation decoding techniques developed in Section 5 to the problem of entanglement transmission across the QMAC and the QIC in the one-shot setting. This allows us to prove non-trivial achievability results in the one-shot setting in terms of smoothed one-shot entropic quantities, without appealing to a simultaneous decoder. We have already mentioned that Yard et al. [2] showed that the natural quantum analogue of the pentagonal rate region, with the mutual information replaced by the regularised coherent information, is achievable for the QMAC. To the best of our knowledge, the only inner bound known for the QIC (for both the unassisted and entanglement assisted regimes) is what one would obtain by treating the channel as two independent unassisted point-to-point channels.

Recall that the idea in rate splitting is to 'split' Alice into two senders Alice ${ }_{0}$ and Alice $_{1}$, each sending disjoint parts of Alice's original message, such that any point in the pentagonal rate region of Figure 7 like P can be obtained without time sharing by a successive cancellation process where Charlie first decodes Alice ${ }_{0}$ 's message, then Bob's message using Alice ${ }_{0}$ 's message as side information and finally Alice ${ }_{1}$ 's message using Bob's and Alice ${ }_{0}$ 's messages as side information.


Figure 7: Achievable rate region for the unassisted quantum MAC per channel use in the asymptotic iid setting with respect to a fixed bipartite input control state. The actual rate region is the convex closure of all such pentagonal regions.

Our one-shot rates are in terms of the smooth coherent min-information defined in Definition 3.3. Since the smooth coherent min-information is not known to possess a chain rule with equality, we get an achievable rate region of the form in Figure 8 Our achievable rate region is a subset of the 'ideal' pentagonal rate region shown by the dashed line. Nevertheless, using a quantum asymptotic equipartition result of Tomamichel et al. [21], we show that this 'subpentagonal' achievable rate region approaches the 'pentagonal' region of Yard et al. [2] (equal to the region demarcated by the dashed line) in the iid limit.

Figure 8: One-shot achievable rate region for the unassisted QMAC (for single channel use only), contained inside the 'ideal' pentagonal region demarcated by the dashed line, and approaching it in the asymptotic iid limit. $O(\log \varepsilon)$ additive factors have been ignored in the figure. The dotted curve shows the situation for some large finite $n$, intuitively indicating that the region is approaching the dashed line when $n \rightarrow \infty$. Note that the above region corresponds to a fixed bipartite input control state. The actual region is a union over all such regions.

### 6.1 The QMAC

In this section, we will use the techniques developed till now to show the existence of encoders and a decoder for entanglement transmission over the QMAC $\mathcal{N}^{A^{\prime} B^{\prime} \rightarrow C}$ with senders Alice and Bob and receiver Charlie. As usual, we consider the Stinespring dilation of this operator $\mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C E}$. We will first specify the control state

$$
|\sigma\rangle^{A^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}:=|\Omega\rangle^{A^{\prime \prime} A^{\prime}}|\Delta\rangle^{B^{\prime \prime} B^{\prime}} .
$$

The state $|\Omega\rangle^{A^{\prime \prime} A^{\prime}}$ is associated with Alice and $|\Delta\rangle^{B^{\prime \prime} B^{\prime}}$ is associated with Bob. As before we consider the action of a splitting scheme on this state. In particular, we will again split Alice into the two senders Alice ${ }_{0}$ and Alice $_{1}$, by acting a unitary $U_{\theta}$ on the system $A^{\prime \prime}$. Thus we will deal with the split control state

$$
|\sigma(\theta)\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}:=|\Omega(\theta)\rangle_{0}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}|\Delta\rangle^{B^{\prime \prime} B^{\prime}} .
$$

As before we will often omit the $\theta$ in the notation for the state. The states to be transmitted are denoted as $|\eta\rangle^{A_{0} C_{0} R_{0}},|\psi\rangle^{A_{1} C_{1} R_{1}}$ for Alice $|\varphi\rangle^{B D S}$ for Bob, and where $C_{0}, C_{1}$ and $D$ represent side information held by the receiver Charlie. The systems $R_{0}, R_{1}$ and $S$ represent reference systems which remain untouched by the protocol.

We will prove the following lemma:
Proposition 6.1. Consider the quantum multiple access channel $\mathcal{N}{ }^{A^{\prime} B^{\prime} \rightarrow C}$. Consider a pure 'control state' $|\sigma\rangle^{A^{\prime \prime} B^{\prime \prime} A^{\prime} B^{\prime}}:=$ $|\Omega\rangle^{A^{\prime \prime} A^{\prime}}|\Delta\rangle^{B^{\prime \prime} B^{\prime}}$. Let $|\psi\rangle^{A_{1} C_{1} R_{1}} \otimes|\eta\rangle^{A_{0} C_{0} R_{0}}$ and $|\phi\rangle^{B D S}$ be the states that are to be sent to Charlie through the channel by Alice and Bob respectively, where $C_{0}, C_{1}, D$ model the side information about the respective messages $A_{0}, A_{1}, B$ that Charlie possesses and $R_{0}, R_{1}, S$ are reference systems that are untouched by channel and coding operators. Let $\mathbb{I}$ denote the identity superoperator. For $\theta \in[0,1]$, let $\left\{U_{\theta}\right\}^{A^{\prime \prime}}$ be a splitting scheme. We define $|\sigma(\theta)\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}:=U_{\theta}|\Omega\rangle^{A^{\prime \prime} A^{\prime}}|\Delta\rangle^{B^{\prime \prime} B^{\prime}}$ and

$$
\sigma(\theta)^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B^{\prime \prime} C}:=\left(\mathcal{N}^{A^{\prime} B^{\prime} \rightarrow C} \otimes \mathbb{I}_{0}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B^{\prime \prime}}\right)\left(\sigma(\theta)^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}\right)
$$

Then there exist encoding maps $\mathcal{A}^{A_{0} A_{1} \rightarrow A^{\prime}}, \mathcal{B}^{B \rightarrow B^{\prime}}$ and a decoding map $\mathcal{C}^{C C_{0} C_{1} D \rightarrow A_{0} C_{0} A_{1} C_{1} B D}$ such that

$$
\left\|\left(\mathcal{C} \otimes \mathbb{I}^{R_{0} R_{1} S}\right)\left(\left(\mathcal{N} \otimes \mathbb{I}^{C_{0} R_{0} C_{1} R_{1} D S}\right)\left(\left(\mathcal{A} \otimes \mathcal{B} \otimes \mathbb{I}^{C_{0} R_{0} C_{1} R_{1} D S}\right)((\eta \otimes \psi) \otimes \phi)\right)\right)-\eta \otimes \psi \otimes \phi\right\|_{1} \leq \delta,
$$

whenever

$$
\begin{aligned}
& H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi} \leq H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\sigma}+2 \log f(\varepsilon), \\
& H_{\max }^{f(\varepsilon)}(B)_{\phi} \leq H_{\min }^{f(\varepsilon)}\left(B^{\prime \prime}\right)_{\sigma}+2 \log f(\varepsilon),
\end{aligned}
$$

where $\delta:=\delta_{\text {enc }}+\delta_{\text {dec }}$ and

$$
\begin{aligned}
2 \sqrt{2 \delta_{\mathrm{enc}}(3)}+2 \sqrt{2 \delta_{\mathrm{enc}}(2)+2 \delta_{\mathrm{enc}}(1)} & :=\delta_{\mathrm{enc}}, \\
2 \sqrt{2 \delta_{\mathrm{dec}}(1)}+2 \sqrt{2 \delta_{\operatorname{dec}}(2)}+2 \sqrt{2 \delta_{\operatorname{dec}}(3)} & :=\delta_{\mathrm{dec}}, \\
2 \cdot\left(2^{-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1} \mid R_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{1}^{\prime \prime} \mid E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}+O(\log k)}+12 k \varepsilon\right) & :=2 \delta_{\mathrm{dec}}(3), \\
2 \cdot\left(2^{-\frac{1}{2} H_{\min }^{\varepsilon}(B \mid S)_{\phi}-\frac{1}{2} H_{\min }^{O\left(\varepsilon^{2}\right)}\left(B^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{u_{\mathcal{N}} \cdot \sigma}+O(\log k)}+12 k \varepsilon\right) & :=2 \delta_{\operatorname{dec}}(2), \\
2 \cdot\left(2^{-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B E\right)_{u_{\mathcal{N}} \cdot \sigma}+O(\log k)}+12 k \varepsilon\right) & :=2 \delta_{\operatorname{dec}}(1), \\
2 \cdot\left(k \cdot 2^{\frac{1}{2} H_{\max }^{f(\varepsilon)}(B)_{\phi}-\frac{1}{2} H_{\min }^{f(\varepsilon)}\left(B^{\prime \prime}\right)_{\Delta}}+12 k f(\varepsilon)\right) & :=2 \delta_{\mathrm{enc}}(3), \\
2 \cdot\left(k \cdot 2^{\frac{1}{2} H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\Omega}}+12 k f(\varepsilon)\right) & :=2 \delta_{\mathrm{enc}}(2), \\
2 \cdot\left(2^{\frac{1}{2} H_{\max }^{\varepsilon}\left(A_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\sigma}+O(\log k)}+12 k \varepsilon\right) & :=2 \delta_{\mathrm{enc}}(1),
\end{aligned}
$$

where $k$ is a constant integer and $f(x)=O\left(x^{2}\right)$.
Proposition 6.1 immediately implies the following theorem, by the arguments presented in Corollary 5.4.
Theorem 6.2. Consider the setting of Proposition 6.1 Let $Q_{A}, E_{A}, Q_{B}, E_{B}$ be the number of message qubits and number of available ebits of Alice and Bob respectively. Let $\theta, \varepsilon \in[0,1]$ and $\varepsilon_{0}:=O\left(\varepsilon^{2}\right)$, where the order hides some multiplicative constant. Then there exist encoding and decoding maps such that any message cum ebit rate 4 -tuple satisfying either the following set of constraints or the set obtained by interchanging $A_{0}^{\prime \prime}$ with $A_{1}^{\prime \prime}, Q_{A}(0)$ with $Q_{A}(1)$ and $E_{A}(0)$ with $E_{A}(1)$ in the right-hand sides of the first two inequalities, is achievable with error at most $O(\sqrt{\varepsilon})$ :

$$
\begin{aligned}
Q_{A} & =Q_{A}(0)+Q_{A}(1), E_{A}=E_{A}(0)+E_{A}(1), \\
Q_{A}(0)+E_{A}(0) & <H_{\min }^{\varepsilon_{0}}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\sigma(\theta)}+O(\log \varepsilon)-O(1), \\
Q_{A}(1)+E_{A}(1) & <H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\sigma(\theta)}+O(\log \varepsilon), \\
Q_{A}(0)-E_{A}(0) & \left.<I_{\min }^{\varepsilon_{0}}\left(A_{0}^{\prime \prime}\right\rangle C\right)_{\sigma(\theta)}+O(\log \varepsilon)-O(1), \\
Q_{A}(1)-E_{A}(1) & \left.<I_{\min }^{\varepsilon_{0}}\left(A_{\rangle}^{\prime \prime}\right\rangle C A_{0}^{\prime \prime} B^{\prime \prime}\right)_{\sigma(\theta)}+O(\log \varepsilon)-O(1), \\
Q_{B}+E_{B} & <H_{\min }^{f(\varepsilon)}\left(B^{\prime \prime}\right)_{\sigma(\theta)}+O(\log \varepsilon), \\
Q_{B}-E_{B} & \left.<I_{\min }^{\varepsilon_{0}}\left(B^{\prime \prime}\right\rangle C A_{0}^{\prime \prime}\right)_{\sigma(\theta)}+O(\log \varepsilon)-O(1) .
\end{aligned}
$$

The $O(1)$ in the region above hides the $O(\log k)$ factors from Proposition 6.1

### 6.1.1 Intermediate States and Almost CPTP Maps for the QMAC

The development in this section closely follows the layout and logical flow of Section 5.2,

We first define isometric embeddings $W_{0}^{A_{0} \rightarrow A_{0}^{\prime \prime}}, W_{1}^{A_{1} \rightarrow A_{1}^{\prime \prime}}$ and $W_{3}^{B \rightarrow B^{\prime \prime}}$ which map

$$
\begin{aligned}
& |\eta\rangle_{0}^{A_{0}^{\prime \prime} C_{0} R_{0}}:=W_{0}|\eta\rangle^{A_{0} C_{0} R_{0}}, \\
& |\psi\rangle_{1}^{A_{1}^{\prime \prime} C_{1} R_{1}}:=W_{1}|\psi\rangle_{1}^{A_{1} C_{1} R_{1}}, \\
& |\varphi\rangle^{B^{\prime \prime} D S}:=W_{3}|\varphi\rangle^{B D S} .
\end{aligned}
$$

We define an intermediate state as follows:
Definition 6.3. Intermediate State for the QMAC We define

$$
\left|\omega_{12}\right\rangle^{A_{0}^{\prime \prime} C_{1} R_{1} D S A^{\prime} B^{\prime}}:=\sqrt{\left|B^{\prime \prime} A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle)\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}}|\phi\rangle^{B^{\prime \prime} D S} .
$$

Lemma 6.4. Trace of Intermediate State Given the conditions

$$
\begin{aligned}
& H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi} \leq H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\sigma}+2 \log f(\varepsilon), \\
& H_{\max }^{f(\varepsilon)}(B)_{\phi} \leq H_{\min }^{f(\varepsilon)}\left(B^{\prime \prime}\right)_{\sigma}+2 \log f(\varepsilon)
\end{aligned}
$$

we have that

$$
\underset{U^{A_{1}^{\prime \prime}, U^{B^{\prime \prime}}}}{\mathbb{E}}\left[\left|\operatorname{Tr}\left[\omega_{12}\right]-1\right|\right] \leq 26 f(\varepsilon) .
$$

Proof. First, notice that

$$
\begin{aligned}
\left|\omega_{12}\right\rangle & =\sqrt{\mid B^{\prime \prime} A_{1}^{\prime \prime}} \mid \mathrm{op}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle)\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}}|\phi\rangle^{B^{\prime \prime} D S} \\
& =\left(\sqrt{\left|B^{\prime \prime}\right|} \mathrm{op}^{B^{\prime \prime} \rightarrow B^{\prime}}(|\Delta\rangle) U^{A_{1}^{\prime \prime}}|\phi\rangle^{B^{\prime \prime} D S}\right) \otimes\left(\sqrt{\left|A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}(|\Omega\rangle) U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}}\right) \\
& :=\left|\omega_{1}\right\rangle^{B^{\prime} D S}\left|\omega_{2}\right\rangle_{0}^{A_{0}^{\prime \prime} C_{1} R_{1} A^{\prime}} .
\end{aligned}
$$

Notice that the state $\omega_{2}$ is similar to the state $\omega$ in the last section. We essentially repeat the analysis of Lemma 5.7 for $\left|\omega_{1}\right\rangle$ and $\left|\omega_{2}\right\rangle$. Notice that this implies that if the conditions given in the hypothesis of the lemma are satisfied, then

$$
\begin{aligned}
& \underset{U^{B^{\prime \prime}}}{\mathbb{E}}\left[\left|\operatorname{Tr}\left[\omega_{1}\right]-1\right|\right] \leq 13 f(\varepsilon), \\
& \underset{U^{A_{1}^{\prime \prime}}}{\mathbb{E}}\left[\left|\operatorname{Tr}\left[\omega_{2}\right]-1\right|\right] \leq 13 f(\varepsilon) .
\end{aligned}
$$

Then notice that

$$
\begin{aligned}
\left|\operatorname{Tr}\left[\omega_{12}\right]-1\right| & =\left|\operatorname{Tr}\left[\omega_{1}\right] \cdot \operatorname{Tr}\left[\omega_{2}\right]-1\right| \\
& \leq \operatorname{Tr}\left[\omega_{1}\right] \cdot\left|\operatorname{Tr}\left[\omega_{2}\right]-1\right|+\left|\operatorname{Tr}\left[\omega_{1}\right]-1\right|
\end{aligned}
$$

Since $U^{A_{1}^{\prime \prime}}$ and $U^{B^{\prime \prime}}$ are sampled independently,

$$
\underset{U^{A_{1}^{\prime \prime}, U^{B^{\prime \prime}}}}{\mathbb{E}}\left[\left|\operatorname{Tr}\left[\omega_{12}\right]-1\right|\right] \leq \underset{U^{B^{\prime \prime}}}{\mathbb{E}}\left[\operatorname{Tr}\left[\omega_{1}\right]\right] \cdot \underset{U^{A_{1}^{\prime \prime}}}{\mathbb{E}}\left[\left|\operatorname{Tr}\left[\omega_{2}\right]-1\right|\right]+\underset{U^{B^{\prime \prime}}}{\mathbb{E}}\left[\left|\operatorname{Tr}\left[\omega_{1}\right]-1\right|\right] .
$$

It is not hard to see that

$$
\underset{U^{B^{\prime \prime}}}{\mathbb{E}}\left[\operatorname{Tr}\left[\omega_{1}\right]\right]=1
$$

Therefore

$$
\underset{U^{A_{1}^{\prime \prime}, U^{B^{\prime \prime}}}}{\mathbb{E}}\left[\left|\operatorname{Tr}\left[\omega_{12}\right]-1\right|\right] \leq 26 f(\varepsilon)
$$

This concludes the proof.
Lemma 6.5. Rewriting the Intermediate State for Coding The intermediate state $\left|\omega_{12}\right\rangle_{9}^{A_{9}^{\prime \prime} C_{1} R_{1} D S A^{\prime} B^{\prime}}$ can be rewritten as

$$
\left|\omega_{12}\right\rangle^{A_{0}^{\prime \prime} C_{1} R_{1} D S A^{\prime} B^{\prime}}:=\sqrt{B^{\prime \prime}} \mathrm{op}^{B^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime} B^{\prime} C_{1} R_{1}}\left(|\omega\rangle_{3}\right) U^{B^{\prime \prime}}|\phi\rangle^{B^{\prime \prime} D S}
$$

where

$$
\left|\omega_{3}\right\rangle^{B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime} C_{1} R_{1}}:=\sqrt{\left|A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle) U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}}
$$

Proof.

$$
\begin{aligned}
\left|\omega_{12}\right\rangle^{A_{0}^{\prime \prime} C_{1} R_{1} D S A^{\prime} B^{\prime}} & =\sqrt{\left|B^{\prime \prime} A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle)\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)|\psi\rangle_{1}^{A_{1}^{\prime \prime} C_{1} R_{1}}|\phi\rangle^{B^{\prime \prime} D S} \\
& =\sqrt{\mid B^{\prime \prime} A_{1}^{\prime \prime}} \mathrm{op}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow C_{1} R_{1} D S}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)|\psi\rangle_{1}^{A_{1}^{\prime \prime} C_{1} R_{1}}|\phi\rangle^{B^{\prime \prime} D S}\right)|\sigma\rangle_{0}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B^{\prime \prime} A^{\prime} B^{\prime}} \\
& =\left(\sqrt{\left.\left|B^{\prime \prime}\right| \mathrm{op}^{B^{\prime \prime} \rightarrow D S}\left(U^{B^{\prime \prime}}|\phi\rangle^{B^{\prime \prime} D S}\right) \otimes \sqrt{\mid A_{1}^{\prime \prime}} \mid \mathrm{op}_{1}^{A_{1}^{\prime \prime} \rightarrow C_{1} R_{1}}\left(U^{A_{1}^{\prime \prime}}|\psi\rangle_{1}^{A_{1}^{\prime \prime} C_{1} R_{1}}\right)\right)|\sigma\rangle}\right. \\
& =\sqrt{\left|B^{\prime \prime}\right|} \mathrm{op}^{B^{\prime \prime} \rightarrow D S}\left(U^{B^{\prime \prime}}|\phi\rangle^{B^{\prime \prime} D S}\right)\left(\sqrt{\left|A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow C_{1} R_{1}}\left(U^{A_{1}^{\prime \prime}}|\psi\rangle_{1}^{A_{1}^{\prime \prime} C_{1} R_{1}}\right)|\sigma\rangle\right) \\
& =\sqrt{\left|B^{\prime \prime}\right|} \mathrm{op}^{B^{\prime \prime} \rightarrow D S}\left(U^{B^{\prime \prime}}|\phi\rangle^{B^{\prime \prime} D S}\right)\left(\sqrt{\left|A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle) U^{A_{1}^{\prime \prime}}|\psi\rangle_{1}^{A_{1}^{\prime \prime} C_{1} R_{1}}\right) \\
& =\sqrt{\left|B^{\prime \prime}\right|} \mathrm{op}^{B^{\prime \prime} \rightarrow D S}\left(U^{B^{\prime \prime}}|\phi\rangle^{B^{\prime \prime} D S}\right)\left|\omega_{3}\right\rangle^{B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime} C_{1} R_{1}} \\
& =\sqrt{\left|B^{\prime \prime}\right|} \mathrm{op}^{B^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime} B^{\prime} C_{1} R_{1}}\left(\left|\omega_{3}\right\rangle\right) U^{B^{\prime \prime}}|\phi\rangle^{B^{\prime \prime} D S}
\end{aligned}
$$

where e have used the properties of the op operator as proved in Section 3.3.
Lemma 6.6. Trace of $\left|\omega_{3}\right\rangle$.

$$
\operatorname{Tr}\left[\omega_{3}\right]=\operatorname{Tr}\left[\omega_{2}\right]
$$

Proof. Recall that, from Lemma 6.4,

$$
\left|\omega_{2}\right\rangle^{A_{0}^{\prime \prime} C_{1} R_{1} A^{\prime}}=\sqrt{\left|A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}(|\Omega\rangle) U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}}
$$

Now

$$
\begin{aligned}
\left|\omega_{3}\right\rangle^{B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime} C_{1} R_{1}} & =\sqrt{\mid A_{1}^{\prime \prime}} \mid \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle) U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}} \\
& \left.=\sqrt{\mid A_{1}^{\prime \prime}}\left|\mathrm{op}^{A_{1}^{\prime \prime} \rightarrow B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime}}\left(|\Omega\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}|\Delta\rangle^{B^{\prime \prime} B^{\prime}}\right) U^{A_{1}^{\prime \prime}}\right| \psi\right\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}} \\
& =|\Delta\rangle^{B^{\prime \prime} B^{\prime}} \otimes \sqrt{\mid A_{1}^{\prime \prime}} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}\left(|\Omega\rangle_{0}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\right) U^{A_{1}^{\prime \prime}}|\psi\rangle_{1}^{A_{1}^{\prime \prime} C_{1} R_{1}} \\
& =|\Delta\rangle^{B^{\prime \prime} B^{\prime}}\left|\omega_{2}\right\rangle_{0}^{A_{0}^{\prime \prime} C_{1} R_{1} A^{\prime}}
\end{aligned}
$$

Since by definition, $\operatorname{Tr}[\Delta]=1$, we have that

$$
\operatorname{Tr}\left[\omega_{3}\right]=\operatorname{Tr}\left[\omega_{2}\right] .
$$

This concludes the proof.
Lemma 6.7. Approximate DPI with $\left|\omega_{12}\right\rangle$ Given the intermediate state

$$
\left|\omega_{12}\right\rangle^{A_{0}^{\prime \prime} C_{1} R_{1} D S A^{\prime} B^{\prime}}:=\sqrt{\left|B^{\prime \prime} A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle)\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}}|\phi\rangle^{B^{\prime \prime} D S},
$$

and the relations

$$
\begin{aligned}
& H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi} \leq H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\sigma}+2 \log f(\varepsilon), \\
& H_{\max }^{f(\varepsilon)}(B)_{\phi} \leq H_{\min }^{f(\varepsilon)}\left(B^{\prime \prime}\right)_{\sigma}+2 \log f(\varepsilon)
\end{aligned}
$$

there exist unitaries $U^{A_{1}^{\prime \prime}}$ and $U^{B^{\prime \prime}}$, with constant probability, such that

$$
H_{\min }^{\sqrt{O(f(\varepsilon))}}\left(A_{0}^{\prime \prime} \mid C_{1} R_{1} D S E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \omega_{12}} \geq H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}-O(1)
$$

Proof. Define the map

$$
\mathcal{T}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow C_{1} R_{1} D S}(\xi):=\left|A_{1}^{\prime \prime} B^{\prime \prime}\right|\left(\mathrm{op}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow C_{1} R_{1} D S}(|\psi\rangle|\varphi\rangle) \cdot \xi\right)
$$

First, recall the following properties of Haar integration

1. $\int U_{1}^{A} \otimes U_{2}^{B} \cdot \rho^{A B} d U_{1} d U_{2}=\operatorname{Tr}\left[\rho^{A B}\right] \pi^{A B}$,
2. $\int U_{1}^{A} \otimes U_{2}^{B} \otimes I^{C} \cdot \rho^{A B C} d U_{1} d U_{2}=\pi^{A B} \otimes \rho^{C}$.

It is now easy to verify that $\mathcal{T}$ is indeed an almost CPTP. The first two properties can be shown to be true using reasoning similar to that used in Lemma 5.9. Finally, using property 1 of double Haar integration above, one can immediately see that

$$
\int \mathcal{T}\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}} \cdot \xi^{A_{1}^{\prime \prime} B^{\prime \prime}}\right) d U_{1} d U_{2}=\operatorname{Tr}[\xi] \mathcal{T}\left(\pi^{A_{1}^{\prime \prime} B^{\prime \prime}}\right)
$$

Next suppose $\tilde{\sigma}^{A_{1}^{\prime \prime} A_{0}^{\prime \prime} B^{\prime \prime} C E}$ be a state such that $H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}=H_{\min }\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B^{\prime \prime} E\right)_{\tilde{\sigma}}$ where $\left\|\tilde{\sigma}-\mathcal{U}_{\mathcal{N}} \cdot \sigma\right\|_{1} \leq$ $2 f(\varepsilon)$. Let $\lambda^{A_{1}^{\prime \prime} B^{\prime \prime} E}$ be a positive semidefinite matrix such that

$$
\operatorname{Tr}[\lambda]=2^{-H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B^{\prime \prime} E\right)_{\mathcal{N}_{\mathcal{N}} \cdot \sigma}}
$$

and

$$
\tilde{\sigma}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B^{\prime \prime} E} \leq \mathbb{I}_{0}^{A_{0}^{\prime \prime}} \otimes \lambda_{1}^{A_{1}^{\prime \prime} B^{\prime \prime} E} .
$$

Then, since $\mathcal{T}$ is CP ,

$$
\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \tilde{\sigma}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B^{\prime \prime} E}\right) \leq \mathbb{I}_{0}^{A_{0}^{\prime \prime}} \otimes \mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \lambda^{A_{1}^{\prime \prime} B^{\prime \prime} E}\right)
$$

Then, it holds that

$$
\begin{aligned}
\operatorname{Tr}\left[\int \mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \cdot \lambda^{A_{1}^{\prime \prime} B^{\prime \prime} E} d U^{A_{1}^{\prime \prime}} d U^{B^{\prime \prime}}\right]\right. & =\operatorname{Tr}\left[\mathcal{T}\left(\pi^{A_{1}^{\prime \prime} B^{\prime \prime}}\right) \otimes \lambda^{E}\right] \\
& =2^{-H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}}
\end{aligned}
$$

Note that

$$
\begin{aligned}
\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \mathcal{U}_{\mathcal{N}} \cdot|\sigma\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B^{\prime \prime} A^{\prime} B^{\prime}}\right) & =\left|A_{1}^{\prime \prime} B^{\prime \prime}\right|\left(\operatorname{op}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow C_{1} R_{1} D S}(|\psi\rangle|\phi\rangle)\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C E} \cdot \sigma\right) \\
& =\left|A_{1}^{\prime \prime} B^{\prime \prime}\right|\left(\operatorname{op}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow C_{1} R_{1} D S}\left(U^{A_{1}^{\prime \prime}}|\psi\rangle \otimes U^{B^{\prime \prime}}|\phi\rangle\right) \mathcal{U}_{\mathcal{N}} \cdot \sigma\right) \\
& =\left|A_{1}^{\prime \prime} B^{\prime \prime}\right|\left(\operatorname{op}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow A_{0}^{\prime \prime} C E}\left(\mathcal{U}_{\mathcal{N}}|\sigma\rangle\right)\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right) \cdot(\psi \otimes \phi)\right) \\
& =\left(\mathcal{U}_{\mathcal{N}} \cdot \omega_{12}\right)^{A_{0}^{\prime \prime} C_{1} R_{1} D S C E},
\end{aligned}
$$

which implies that
$\operatorname{Tr}_{C} \mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \mathcal{U}_{\mathcal{N}} \cdot|\sigma\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B^{\prime \prime} A^{\prime} B^{\prime}}\right)=\left(\mathcal{U}_{\mathcal{N}} \cdot \omega_{12}\right)^{A_{0}^{\prime \prime} C_{1} R_{1} D S E}$.
Also, from the arguments used in Lemma 5.9, we see that there exist positive matrices $\Delta^{+}$and $\Delta^{-}$such that

$$
\begin{aligned}
& \underset{U^{A_{1}^{\prime \prime}, U^{B^{\prime \prime}}}}{\mathbb{E}}\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \tilde{\sigma}\right)-\mathcal{U}_{\mathcal{N}} \cdot \omega_{12}\right\|_{1} \\
& =\underset{U^{A_{1}^{\prime \prime}, U^{B^{\prime \prime}}}}{\mathbb{E}}\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \tilde{\sigma}\right)-\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \mathcal{U}_{\mathcal{N}} \cdot \sigma\right)\right\|_{1} \\
& =\underset{U^{A_{1}^{\prime \prime}, U^{B^{\prime \prime}}}}{\mathbb{E}}\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \cdot\left(\tilde{\sigma}-\mathcal{U}_{\mathcal{N}} \cdot \sigma\right)\right)\right\|_{1} \\
& =\underset{U^{A_{1}^{\prime \prime}, U^{B^{\prime \prime}}}}{\mathbb{E}}\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \cdot\left(\Delta^{+}-\Delta^{-}\right)\right)\right\|_{1} \\
& \leq \underset{U^{A_{1}^{\prime \prime}, U^{B^{\prime \prime}}}}{\mathbb{E}}\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \cdot \Delta^{+}\right)\right\|_{1}+\underset{U^{A_{1}^{\prime \prime}, U^{B^{\prime \prime}}}}{\mathbb{E}}\left\|\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \cdot \Delta^{-}\right)\right\|_{1} \\
& =\underset{U^{A_{1}^{\prime \prime}, U^{B^{\prime \prime}}}}{\mathbb{E}} \operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \cdot \Delta^{+}\right)\right]+\underset{U^{A_{1}^{\prime}, U^{B^{\prime \prime}}}}{\mathbb{E}} \operatorname{Tr}\left[\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \cdot \Delta^{-}\right)\right] \\
& =\left(\operatorname{Tr}\left[\Delta^{+}\right]+\operatorname{Tr}\left[\Delta^{-}\right]\right) \cdot \operatorname{Tr}\left[\mathcal{T}\left(\pi^{A_{1}^{\prime \prime} B^{\prime \prime}}\right)\right] \\
& \leq 4 f(\varepsilon) \text {. }
\end{aligned}
$$

Note also that given the entropic conditions in the hypothesis of the lemma, we see via Lemma 6.4 that

$$
\underset{U^{A_{1}^{\prime \prime}, U^{B^{\prime \prime}}}}{\mathbb{E}}\left[\left|\operatorname{Tr}\left[\omega_{12}\right]-1\right|\right] \leq 26 f(\varepsilon)
$$

Then, via the derandomisation arguments used in Lemma 5.9, we see that there exists, with probability at least
$1-\frac{3}{k}$, for some constant integer $k$, fixed unitaries $U^{A_{1}^{\prime \prime}}$ and $U^{B^{\prime \prime}}$ such that

$$
\left\|\frac{\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \tilde{\sigma}\right)}{\operatorname{Tr}\left(\mathcal{T}\left(\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)^{T} \tilde{\sigma}\right)\right)}-\frac{\mathcal{U}_{\mathcal{N}} \cdot \omega_{12}}{\operatorname{Tr}\left[\omega_{12}\right]}\right\|_{1} \leq k \cdot 60 \cdot f(\varepsilon),
$$

and

$$
H_{\min }^{\sqrt{60 k f(\varepsilon)}}\left(A_{0}^{\prime \prime} \mid C_{1} R_{1} D S E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \omega_{12}} \geq H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}-\log k+\log (1-30 k f(\varepsilon))
$$

Setting $f(\varepsilon):=\frac{\varepsilon^{2}}{60 k}$ and plugging this into the above inequality we get that

$$
H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid C_{1} R_{1} D S E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \omega_{12}} \geq H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}-\log k+\log \left(1-O\left(\varepsilon^{2}\right)\right)
$$

This concludes the proof.
We will now prove the main theorem for the QMAC.

### 6.1.2 Proof of Proposition 6.1

Proof. We will begin with the assumptions that

$$
\begin{aligned}
& H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi} \leq H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\sigma}+2 \log f(\varepsilon) \\
& H_{\max }^{f(\varepsilon)}(B)_{\phi} \leq H_{\min }^{f(\varepsilon)}\left(B^{\prime \prime}\right)_{\sigma}+2 \log f(\varepsilon)
\end{aligned}
$$

We will use the intermediate states

$$
\begin{aligned}
& \left|\omega_{12}\right\rangle_{A_{0}^{\prime \prime} C_{1} R_{1} D S A^{\prime} B^{\prime}}:=\sqrt{\left|B^{\prime \prime} A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle)\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right)|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}}|\phi\rangle^{B^{\prime \prime} D S}, \\
& \left|\omega_{3}\right\rangle^{B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime} C_{1} R_{1}}:=\sqrt{\left|A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle) U^{A_{1}^{\prime \prime}}|\psi\rangle_{1}^{A_{1}^{\prime \prime} C_{1} R_{1}}
\end{aligned}
$$

and

$$
\left|\omega_{2}\right\rangle^{A_{0}^{\prime \prime} C_{1} R_{1} A^{\prime}}=\sqrt{\left|A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}(|\Omega\rangle) U^{A_{1}^{\prime \prime}}|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}} .
$$

Recall from Lemma 6.5 we have that

$$
\left|\omega_{12}\right\rangle^{A_{0}^{\prime \prime} C_{1} R_{1} D S A^{\prime} B^{\prime}}:=\sqrt{\left|B^{\prime \prime}\right|} \mathrm{op}^{B^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime} B^{\prime} C_{1} R_{1}}\left(\left|\omega_{3}\right\rangle\right) U^{B^{\prime \prime}}|\phi\rangle^{B^{\prime \prime} D S} .
$$

We define the maps

$$
\begin{aligned}
& \mathcal{E}^{A_{0} A_{1} \rightarrow A^{\prime}}(|\xi\rangle):=\sqrt{\mid A_{0}^{\prime \prime} A_{1}^{\prime \prime}} \mid \mathrm{op} A_{0}^{\prime \prime} A_{1}^{\prime \prime} \rightarrow A^{\prime}(|\Omega\rangle)\left(U^{A_{0}^{\prime \prime}} \otimes U^{A_{1}^{\prime \prime}}\right)\left(W_{0}^{A_{0} \rightarrow A_{0}^{\prime \prime}} \otimes W_{1}^{A_{1} \rightarrow A_{1}^{\prime \prime}}\right)|\xi\rangle, \\
& \mathcal{F}^{B \rightarrow B^{\prime}}(|\zeta\rangle):=\sqrt{\left|B^{\prime \prime}\right|} \mid \mathrm{op}^{B^{\prime \prime} \rightarrow B^{\prime}}(|\Delta\rangle) U^{B^{\prime \prime}} W^{B \rightarrow B^{\prime \prime}}|\zeta\rangle .
\end{aligned}
$$

We start with the vector

$$
\begin{aligned}
& (\mathcal{E} \otimes \mathcal{F})\left(|\eta\rangle^{A_{0} C_{0} R_{0}}|\psi\rangle^{A_{1} C_{1} R_{1}}|\phi\rangle^{B D S}\right) \\
= & \sqrt{\left|A_{0}^{\prime \prime} B^{\prime \prime} A_{1}^{\prime \prime}\right|} \mathrm{op}^{A_{0}^{\prime \prime} B^{\prime \prime} A_{1}^{\prime \prime} \rightarrow A^{\prime} B^{\prime}}(|\sigma\rangle) U^{A_{0}^{\prime \prime}} \otimes U^{B^{\prime \prime}} \otimes U^{A_{1}^{\prime \prime}}|\eta\rangle^{A_{0}^{\prime \prime} C_{0} R_{0}}|\phi\rangle^{B^{\prime \prime} D S}|\psi\rangle_{1}^{A_{1}^{\prime \prime} C_{1} R_{1}} \\
= & \left.\sqrt{\left|A_{0}^{\prime \prime} B^{\prime \prime} A_{1}^{\prime \prime}\right|}\left|\mathrm{op}^{A_{0}^{\prime \prime} B^{\prime \prime} A_{1}^{\prime \prime} \rightarrow C_{0} R_{0} D S C_{1} R_{1}}\left(U^{A_{0}^{\prime \prime}} \otimes U^{B^{\prime \prime}} \otimes U^{A_{1}^{\prime \prime}}|\eta\rangle^{A_{0}^{\prime \prime} C_{0} R_{0}}|\phi\rangle^{B^{\prime \prime} D S}|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}}\right)\right| \sigma\right\rangle_{0}^{A_{0}^{\prime \prime} B^{\prime \prime} A_{1}^{\prime \prime} A^{\prime} B^{\prime}} \\
= & \left(\sqrt{\mid A_{0}^{\prime \prime}} \mid \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow C_{0} R_{0}}\left(U^{A_{0}^{\prime \prime}}|\eta\rangle_{0}^{A_{0}^{\prime \prime} C_{0} R_{0}}\right) \otimes \sqrt{\mid B^{\prime \prime} A_{1}^{\prime \prime}}\left|\mathrm{op}^{B^{\prime \prime} A_{1}^{\prime \prime} \rightarrow D S C_{1} R_{1}}\left(U^{B^{\prime \prime}} \otimes U^{A_{1}^{\prime \prime}}|\phi\rangle^{B^{\prime \prime} D S}|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}}\right)\right)|\sigma\rangle\right. \\
= & \left.\sqrt{\left|A_{0}^{\prime \prime}\right|} \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow C_{0} R_{0}}\left(U^{A_{0}^{\prime \prime}}|\eta\rangle_{0}^{A_{0}^{\prime \prime} C_{0} R_{0}}\right)\left(\sqrt{\mid B^{\prime \prime} A_{1}^{\prime \prime}}\left|\mathrm{op}^{B^{\prime \prime} A_{1}^{\prime \prime} \rightarrow D S C_{1} R_{1}}\left(U^{B^{\prime \prime}} \otimes U^{A_{1}^{\prime \prime}}|\phi\rangle^{B^{\prime \prime} D S}|\psi\rangle_{1}^{A_{1}^{\prime \prime} C_{1} R_{1}}\right)\right| \sigma\right\rangle\right) \\
= & \sqrt{\mid A_{0}^{\prime \prime}} \mid \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow C_{0} R_{0}}\left(U^{A_{0}^{\prime \prime}}|\eta\rangle_{0}^{A_{0}^{\prime \prime} C_{0} R_{0}}\right)\left(\sqrt{\mid B^{\prime \prime} A_{1}^{\prime \prime}} \mid \mathrm{op}^{B^{\prime \prime} A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle) U^{B^{\prime \prime}} \otimes U^{A_{1}^{\prime \prime}}|\phi\rangle^{B^{\prime \prime} D S}|\psi\rangle_{1}^{A_{1}^{\prime \prime} C_{1} R_{1}}\right) \\
= & \sqrt{\left|A_{0}^{\prime \prime}\right|} \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow C_{0} R_{0}}\left(U^{A_{0}^{\prime \prime}}|\eta\rangle_{0}^{A_{0}^{\prime \prime} C_{0} R_{0}}\right)\left|\omega_{12}\right\rangle_{0}^{A_{0}^{\prime \prime} D S C_{1} R_{1} A^{\prime} B^{\prime}} \\
= & \sqrt{\mid A_{0}^{\prime \prime}} \mid \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow D S C_{1} R_{1} A^{\prime} B^{\prime}}\left(\left|\omega_{12}\right\rangle\right) U^{A_{0}^{\prime \prime}}|\eta\rangle_{0}^{A_{0}^{\prime} C_{0} R_{0}} .
\end{aligned}
$$

Using similar reasoning one can also see that

$$
\begin{aligned}
& \sqrt{\mid A_{0}^{\prime \prime} A_{1}^{\prime \prime}} \mid \mathrm{op}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} \rightarrow A^{\prime}}(|\Omega\rangle) U^{A_{0}^{\prime \prime}} \otimes U^{A_{1}^{\prime \prime}}|\eta\rangle^{A_{0}^{\prime \prime} C_{0} R_{0}}|\psi\rangle^{A_{1}^{\prime \prime} C_{1} R_{1}} \\
= & \sqrt{\mid A_{0}^{\prime \prime}} \mid \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow C_{1} R_{1} A^{\prime}}\left(\left|\omega_{2}\right\rangle_{0}^{A_{0}^{\prime \prime} C_{1} R_{1} A^{\prime}}\right) U^{A_{0}^{\prime \prime}}|\eta\rangle_{0}^{A_{0}^{\prime \prime} C_{0} R_{0}} .
\end{aligned}
$$

## The Decoupling Step

As before, we will first consider the relevant decoupling conditions that will ensure the existence of our encoders and decoders.

## The Encoding Equations

$$
\begin{aligned}
& \underset{U^{A_{0}^{\prime \prime}}}{\mathbb{E}} \|\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{A^{\prime}} \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow C_{1} R_{1} A^{\prime}}\left(\left|\tilde{\omega}_{2}\right\rangle\right) U^{A_{0}^{\prime \prime}} \cdot \eta^{A_{0}^{\prime \prime} C_{0} R_{0}}-\eta^{R_{0} C_{0}} \otimes \tilde{\omega}_{2}^{C_{1} R_{1}} \|_{1} \\
& \leq 2^{-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0} \mid C_{0} R_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid C_{1} R_{1}\right)_{\tilde{\omega}_{2}}}+12 \varepsilon, \\
& \underset{U^{A_{1}^{\prime \prime}}}{\mathbb{E}}\left|\left|\left|A_{1}^{\prime \prime}\right| \operatorname{Tr}_{A_{0}^{\prime \prime} A^{\prime}} \mathrm{op}^{A_{1}^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime}}(|\Omega\rangle) U^{A_{1}^{\prime \prime}} \cdot \psi^{A_{1}^{\prime \prime} C_{1} R_{1}}-\psi^{C_{1} R_{1}} \|_{1}\right.\right. \\
& \leq 2^{-\frac{1}{2} H_{\text {min }}^{f(\varepsilon)}\left(A_{1} \mid C_{1} R_{1}\right)_{\psi}-\frac{1}{2} H_{\text {min }}^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\Omega}}+12 f(\varepsilon), \\
& \underset{U^{B^{\prime \prime}}}{\mathbb{E}} \|\left|B^{\prime \prime}\right| \operatorname{Tr}_{B^{\prime}} \mathrm{op}^{B^{\prime \prime} \rightarrow B^{\prime}}(|\Delta\rangle) U^{B^{\prime \prime}} \cdot \phi^{B^{\prime \prime} D S}-\phi^{D S} \|_{1} \\
& \leq 2^{-\frac{1}{2} H_{\min }^{f(\varepsilon)}(B \mid D S)_{\phi}-\frac{1}{2} H_{\min }^{f(\varepsilon)}\left(B^{\prime \prime}\right)_{\Delta}}+12 f(\varepsilon) .
\end{aligned}
$$

## The Decoding Equations

$$
\begin{gathered}
\underset{U^{A_{0}^{\prime \prime}}}{\mathbb{E}} \|\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{C} \mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C E} \mathrm{op}^{A_{0}^{\prime \prime} \rightarrow D S C_{1} R_{1} A^{\prime} B^{\prime}}\left(\left|\tilde{\omega}_{12}\right\rangle\right) U^{A_{0}^{\prime \prime}} \cdot \eta^{A_{0}^{\prime \prime} R_{0}}-\eta^{R_{0}} \otimes \tilde{\omega}_{12}^{D S C_{1} R_{1} E} \|_{1} \\
\leq 2^{-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid D S C_{1} R_{1} E\right) u_{\mathcal{N}} \cdot \tilde{\omega}_{12}}+12 \varepsilon, \\
\underset{U^{B^{\prime \prime}}}{\mathbb{E}} \|\left|B^{\prime \prime}\right| \operatorname{Tr}_{A_{0}^{\prime \prime} C} \mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C E} \mathrm{op}^{B^{\prime \prime} \rightarrow A_{0}^{\prime \prime} C_{1} R_{1} A^{\prime} B^{\prime}}\left(\left|\tilde{\omega}_{3}\right\rangle\right) U^{B^{\prime \prime}} \cdot \phi^{B^{\prime \prime} S}-\phi^{S} \otimes \tilde{\omega}_{3}^{C_{1} R_{1} E} \|_{1} \\
\leq 2^{-\frac{1}{2} H_{\min }^{\varepsilon}(B \mid S)_{\phi}-\frac{1}{2} H_{\min }^{\varepsilon}\left(B^{\prime \prime} \mid C_{1} R_{1} E\right) u_{\mathcal{N}} \cdot \tilde{\omega}_{3}}+12 \varepsilon, \\
\underset{U^{A_{1}^{\prime \prime}}}{\mathbb{E}} \|\left|A_{1}^{\prime \prime}\right| \operatorname{Tr}_{A_{0}^{\prime \prime} B^{\prime \prime} C} \mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C E}{ }^{\mathrm{op}^{A_{1}^{\prime \prime} \rightarrow B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle) U^{A_{1}^{\prime \prime}} \cdot \psi^{A_{1}^{\prime \prime} R_{1}}-\psi^{R_{1}} \otimes \sigma^{E} \|_{1}} \\
\leq 2^{-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1} \mid R_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime} \mid E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}+12 \varepsilon .}
\end{gathered}
$$

## Derandomisation

Using the derandomisation arguments used previously, we see that there exists a constant integer $k$ and constant integers $n_{1}, n_{2}, n_{2}$ such that, there exist fixed unitaries $U^{A_{0}^{\prime \prime}}, U^{B^{\prime \prime}}$ and $U^{A_{1}^{\prime \prime}}$ with probability at least $1-\frac{m}{k}$ (where $m$ is the number of events in the intersection and $k$ is chosen to be larger than $m$ ) such that the encoding and decoding conditions are satisfied along with the three data processing inequalities

$$
\begin{aligned}
& H_{\min }^{\sqrt{n_{1} \cdot k \cdot f(\varepsilon)}}\left(A_{0}^{\prime \prime} \mid C_{1} R_{1}\right)_{\tilde{\omega}_{2}} \geq H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)-\log k+\log \left(1-n_{1} \cdot k \cdot f(\varepsilon)\right), \\
& H_{\min }^{\sqrt{n_{2} \cdot k \cdot f(\varepsilon)}}\left(B^{\prime \prime} \mid C_{1} R_{1} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \tilde{\omega}_{3}} \geq H_{\min }^{f(\varepsilon)}\left(B^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}-\log k+\log \left(1-n_{2} \cdot k \cdot f(\varepsilon)\right), \\
& H_{\min }^{\sqrt{n_{3} \cdot k \cdot f(\varepsilon)}}\left(A_{0}^{\prime \prime} \mid D S C_{1} R_{1} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \tilde{\omega}_{12}} \geq H_{\min }^{f(\varepsilon)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}-\log k+\log \left(1-n_{3} \cdot k \cdot f(\varepsilon)\right) .
\end{aligned}
$$

We choose $f(\varepsilon):=\frac{\varepsilon^{2}}{k \cdot \max \left\{n_{1}, n_{2}, n_{3}\right\}}$. Then using the fact that the smooth min-entropy increases with increasing $\varepsilon$, we see that

$$
\begin{aligned}
& H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid C_{1} R_{1}\right)_{\tilde{\omega}_{2}} \geq H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\sigma}-\log k+\log \left(1-O\left(\varepsilon^{2}\right)\right) \\
& H_{\min }^{\varepsilon}\left(B^{\prime \prime} \mid C_{1} R_{1} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \tilde{\omega}_{3}} \geq H_{\min }^{O\left(\varepsilon^{2}\right)}\left(B^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}-\log k+\log \left(1-O\left(\varepsilon^{2}\right)\right) \\
& H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid D S C_{1} R_{1} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \tilde{\omega}_{12}} \geq H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}-\log k+\log \left(1-O\left(\varepsilon^{2}\right)\right) .
\end{aligned}
$$

Then, plugging in the above bounds into the encoding and decoding equations, using the definitions of $\omega_{12}, \omega_{2}$ and $\omega_{3}$ and using Lemma A. 2 we see that the derandomised encoding and decoding equations are equivalent to

## Encoding

$$
\begin{aligned}
& \left\|\frac{\operatorname{Tr}_{A^{\prime}} \mathcal{E}^{A_{0} A_{1} \rightarrow A^{\prime}}\left(\eta^{A_{0} R_{0} C_{0}} \otimes \psi^{A_{1} R_{1} C_{1}}\right)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}-\eta^{R_{0} C_{0}} \otimes \tilde{\omega}_{2}^{C_{1} R_{1}}\right\|_{1} \\
& \leq 2 \cdot\left(2^{\frac{1}{2} H_{\text {max }}^{\varepsilon}\left(A_{0}\right)_{\eta}-\frac{1}{2} H_{\text {min }}^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\sigma}+O(\log k)}+12 k \varepsilon\right) \\
& \quad:=2 \delta_{\text {enc }}(1), \\
& \| \begin{aligned}
\left\|\tilde{\omega}_{2}^{C_{1} R_{1}}-\psi^{C_{1} R_{1}}\right\|_{1} \leq 2 \cdot\left(k \cdot 2^{\frac{1}{2} H_{\text {max }}^{f(\varepsilon)}\left(A_{1}\right)_{\psi}-\frac{1}{2} H_{\text {min }}^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\Omega}}+12 k f(\varepsilon)\right) \\
:=2 \delta_{\text {enc }}(2),
\end{aligned} \\
& \left\|\frac{\operatorname{Tr}_{B^{\prime}} \mathcal{F}^{B \rightarrow B^{\prime}}\left(\phi^{B D S}\right)}{\operatorname{Tr}[\mathcal{F}(\phi)]}-\phi^{D S}\right\|_{1} \\
& \leq 2 \cdot\left(k \cdot 2^{\frac{1}{2} H_{\text {max }}^{(f \varepsilon)}(B)_{\phi}-\frac{1}{2} H_{\min }^{f(\varepsilon)}\left(B^{\prime \prime}\right)_{\Delta}}+12 k f(\varepsilon)\right) \\
& :=2 \delta_{\text {enc }}(3) .
\end{aligned}
$$

## Decoding

$$
\begin{aligned}
& \left\|\frac{\operatorname{Tr}_{C C_{0}} \mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C E}(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \psi \otimes \phi)}{\operatorname{Tr}[(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \psi \otimes \phi)]}-\eta^{R_{0}} \otimes \tilde{\omega}_{12}^{D S C_{1} R_{1} E}\right\|_{1} \\
& \quad \leq 2 \cdot\left(2^{-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B E\right) \mathcal{U}_{\mathcal{N}} \cdot \sigma+O(\log k)}+12 k \varepsilon\right) \\
& :=2 \delta_{\operatorname{dec}}(1), \\
& \leq 2 \cdot\left(2^{-\frac{1}{2} H_{\min }^{\varepsilon}(B \mid S)_{\phi}-\frac{1}{2} H_{\min }^{\varepsilon}\left(B^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{u_{\mathcal{N}} \cdot \sigma}+O(\log k)}+12 k \varepsilon\right) \\
& :=2 \delta_{\operatorname{dec}}(2), \\
& \operatorname{Tr}_{A_{0}^{\prime \prime} C D}\left[\tilde{\omega}_{12}^{D S C_{1} R_{1} A_{0}^{\prime \prime} C E}\right]-\phi^{S} \otimes \tilde{\omega}_{3}^{C_{1} R_{1} E} \|_{1} \\
& \quad \leq 2 \cdot\left(2^{\left.-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1} \mid R_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime} \mid E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma}+12 k \varepsilon\right)}\right. \\
& :=2 \delta_{\operatorname{dec}}(3) .
\end{aligned}
$$

## Uhlmann's Theorem

From the first two inequalities in the encoding part, by using the triangle inequality we see that

$$
\left\|\frac{\operatorname{Tr}_{A^{\prime}} \mathcal{E}^{A_{0} A_{1} \rightarrow A^{\prime}}\left(\eta^{A_{0} R_{0} C_{0}} \otimes \psi^{A_{1} R_{1} C_{1}}\right)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}-\eta^{R_{0} C_{0}} \otimes \psi^{R_{1} C_{1}}\right\|_{1} \leq 2 \delta_{\mathrm{enc}}(1)+2 \delta_{\mathrm{enc}}(2) .
$$

The third equation in the encoding part gives

$$
\left\|\frac{\operatorname{Tr}_{B^{\prime}} \mathcal{F}^{B \rightarrow B^{\prime}}\left(\phi^{B D S}\right)}{\operatorname{Tr}[\mathcal{F}(\phi)]}-\phi^{D S}\right\|_{1} \leq 2 \delta_{\mathrm{enc}}(3) .
$$

Uhlmann's Theorem then implies that there exist isometries $V_{\text {Alice }}^{A_{0} A_{1} \rightarrow A^{\prime}}$ and $V_{\mathrm{Bob}}^{B \rightarrow B^{\prime}}$ such that

$$
\begin{aligned}
& \left\|\frac{\mathcal{E}^{A_{0} A_{1} \rightarrow A^{\prime}}\left(\eta^{A_{0} R_{0} C_{0}} \otimes \psi^{A_{1} R_{1} C_{1}}\right)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}-V_{\text {Alice }}^{A_{0} A_{1} \rightarrow A^{\prime}} \cdot\left(\eta^{A_{0} R_{0} C_{0}} \otimes \psi^{A_{1} R_{1} C_{1}}\right)\right\|_{1} \leq 2 \sqrt{2 \delta_{\mathrm{enc}}(1)+2 \delta_{\mathrm{enc}}(2)}, \\
& \left\|\frac{\mathcal{F}^{B \rightarrow B^{\prime}}\left(\phi^{B D S}\right)}{\operatorname{Tr}[\mathcal{F}(\phi)]}-V_{\mathrm{Bob}}^{B \rightarrow B^{\prime}} \cdot \phi^{B D S}\right\|_{1} \leq 2 \sqrt{2 \delta_{\mathrm{enc}}(3)} .
\end{aligned}
$$

Then

$$
\begin{aligned}
& \left\|\frac{\mathcal{E} \otimes \mathcal{F}(\eta \otimes \psi \otimes \phi)}{\operatorname{Tr}[\mathcal{E} \otimes \mathcal{F}(\eta \otimes \psi \otimes \phi)]}-V_{\text {Alice }} \otimes V_{\text {Bob }} \cdot(\eta \otimes \psi \otimes \phi)\right\|_{1} \\
\leq & \left\|\frac{\mathcal{E} \otimes \mathcal{F}(\eta \otimes \psi \otimes \phi)}{\operatorname{Tr}[\mathcal{E} \otimes \mathcal{F}(\eta \otimes \psi \otimes \phi)]}-\frac{\mathcal{F}(\phi)}{\operatorname{Tr}[\mathcal{F}(\phi)]} \otimes V_{\text {Alice }} \cdot(\eta \otimes \psi)\right\|_{1} \\
+ & \left\|\frac{\mathcal{F}(\phi)}{\operatorname{Tr}[\mathcal{F}(\phi)]} \otimes V_{\text {Alice }} \cdot(\eta \otimes \psi)-V_{\text {Alice }} \otimes V_{\text {Bob }} \cdot(\eta \otimes \psi \otimes \phi)\right\|_{1} \\
= & \left\|\frac{\mathcal{E}(\eta \otimes \psi)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}-V_{\text {Alice }} \cdot(\eta \otimes \psi)\right\|_{1}+\left\|\frac{\mathcal{F}(\phi)}{\operatorname{Tr}[\mathcal{F}(\phi)]}-V_{\text {Bob }} \cdot \phi\right\|_{1} \\
\leq & 2 \sqrt{2 \delta_{\text {enc }}(3)}+2 \sqrt{2 \delta_{\text {enc }}(1)+2 \delta_{\text {enc }}(1)} \\
: & \delta_{\text {enc }}
\end{aligned}
$$

Again, from the decoding inequalities, by applying Uhlmann's theorem we see that there exist isometries

$$
\begin{aligned}
& V_{\mathrm{dec} \_1}^{C C_{0} \rightarrow A_{0}^{\prime \prime} C C_{0} A_{0}}, \\
& V_{\text {dec_2 }}^{A_{0}^{\prime \prime} C D \rightarrow A_{0}^{\prime \prime} B^{\prime \prime} C B D}
\end{aligned}
$$

and

$$
V_{\text {dec_3 }}^{A_{0}^{\prime \prime} B^{\prime \prime} C C_{1} \rightarrow A_{1} C_{1} F}
$$

such that

$$
\begin{aligned}
& \| V_{\operatorname{dec}-1}^{C C_{0} \rightarrow A_{0}^{\prime \prime} C C_{0} A_{0}} \cdot \frac{\mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C E}(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \psi \otimes \phi)}{\operatorname{Tr}[(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \psi \otimes \phi)]}-\eta^{A_{0} R_{0} C_{0} \otimes \tilde{\omega}_{12}^{D S C_{1} R_{1} A_{0}^{\prime \prime} C E} \|_{1}} \begin{array}{l}
\leq 2 \sqrt{2 \delta_{\operatorname{dec}}(1)}, \\
\left\|V_{\operatorname{dec}-2}^{A_{0}^{\prime \prime} C D \rightarrow A_{0}^{\prime \prime} B^{\prime \prime} C B D} \cdot \tilde{\omega}_{12}^{D S C_{1} R_{1} A_{0}^{\prime \prime} C E}-\phi^{B D S} \otimes \tilde{\omega}_{3}^{B^{\prime \prime} A_{0}^{\prime \prime} C_{1} R_{1} C E}\right\|_{1} \leq 2 \sqrt{2 \delta_{\operatorname{dec}}(2)}, \\
\left\|V_{\operatorname{dec} \_3}^{A_{0}^{\prime \prime} B^{\prime \prime} C C_{1} \rightarrow A_{1} C_{1} F} \cdot \tilde{\omega}_{3}^{B^{\prime \prime} A_{0}^{\prime \prime} C_{1} R_{1} C E}-\psi^{A_{1} R_{1} C_{1}} \otimes \sigma^{E F}\right\|_{1} \leq 2 \sqrt{2 \delta_{\operatorname{dec}}(3)}
\end{array}
\end{aligned}
$$

Finally, defining

$$
V_{\mathrm{dec}}^{C C_{0} D C_{1} \rightarrow A_{0} C_{0} B D A_{1} C_{1} F}:=V_{\mathrm{dec}_{2} 3} \circ V_{\mathrm{dec}_{-} 2} \circ V_{\mathrm{dec}-1}
$$

and using the triangle inequality shows that

$$
\begin{aligned}
& \left\|V_{\mathrm{dec}} \cdot \frac{\mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C E}(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \psi \otimes \phi)}{\operatorname{Tr}[(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \psi \otimes \phi)]}-\eta^{A_{0} R_{0} C_{0}} \otimes \phi^{B D S} \otimes \psi^{A_{1} R_{1} C_{1}} \otimes \sigma^{E F}\right\|_{1} \\
\leq & 2 \sqrt{2 \delta_{\operatorname{dec}}(1)}+2 \sqrt{2 \delta_{\operatorname{dec}}(2)}+2 \sqrt{2 \delta_{\operatorname{dec}}(3)} \\
: & \delta_{\mathrm{dec}} .
\end{aligned}
$$

Tracing out the systems $E F$ from the above inequality gives us the promised decoding map

$$
\mathcal{C}^{C C_{0} D C_{1} \rightarrow A_{0} C_{0} B D A_{1} C_{1}}
$$

A further triangle inequality with the encoding condition shows that

$$
\left\|\mathcal{C} \circ \mathcal{N}^{A^{\prime} B^{\prime} \rightarrow C} \circ\left(V_{\mathrm{Alice}} \otimes V_{\mathrm{Bob}}\right) \cdot(\eta \otimes \psi \otimes \phi)-\eta \otimes \psi \otimes \phi\right\|_{1} \leq \delta_{\mathrm{enc}}+\delta_{\mathrm{dec}}
$$

## Successive Cancellation

The decoding algorithm is now clear:

1. Alice creates a state close to $\frac{\mathcal{E}(\eta \otimes \psi)}{\operatorname{Tr}[\mathcal{E}(\eta \otimes \psi)]}$ by using the encoding isometry $V_{\text {Alice }}$.
2. Bob creates a state close to $\frac{\mathcal{F}(\phi)}{\operatorname{Tr}[\mathcal{F}(\phi)]}$ by using the isometric encoder $V_{\mathrm{Bob}}$.
3. They then enter the $A^{\prime}$ and $B^{\prime}$ parts of their respective encoded states into the channel.
4. Charlie first decodes for $|\eta\rangle^{A_{0} R_{0} C_{0}}$ by using the map $V_{\text {dec_1 }}$ on the systems $C C_{0}$, and also locally prepares the system $A_{0}^{\prime \prime}$ and a copy of $C$.
5. He then decodes for $|\phi\rangle^{B D S}$ by using the decoder $V_{\text {dec_2 }}$ on the systems $A_{0}^{\prime \prime} C D$ which also locally prepares the systems $A_{0}^{\prime \prime} B^{\prime \prime}$ and also another copy of $C$.
6. Finally, Charlie decodes for the state $|\psi\rangle^{A_{1} R_{1} C_{1}}$ by using the map $V_{\text {dec_3 }}$ on the systems $A_{0}^{\prime \prime} B^{\prime \prime} C_{1}$.
7. The composition of all three decoding maps and disregarding the environment $E$ and the junk system $F$ gives us the decoder $\mathcal{C}$.

This concludes the proof of the theorem.

### 6.2 The QIC

In this section, we prove inner bounds for rate-limited entanglement assisted entanglement transmission through the Quantum Interference Channel (QIC) $\mathcal{N}^{A^{\prime} B^{\prime} \rightarrow C D}$. We wish for Alice to send EPR pairs to Charlie and for Bob to send EPR pairs to Damru. Note that, for a fixed control state $|\sigma\rangle^{A^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}:=|\Omega\rangle^{A^{\prime \prime} A^{\prime}}|\Delta\rangle^{B^{\prime \prime} B^{\prime}}$, one can consider this situation as two point-to-point channels, one from Alice to Charlie and one from Bob to Damru. In that case, the achievable region becomes a rectangle of all non negative rate pairs less than $\left.\left.\left.\left(I_{\text {min }}^{\varepsilon}\left(A^{\prime \prime}\right\rangle C\right)_{\sigma}, I_{\text {min }}^{\varepsilon}\left(B^{\prime \prime}\right\rangle D\right)\right)_{\sigma}\right)$ (suppressing the additive $\log$ terms).


Figure 9: One-shot achievable rate region (for single channel use only) for the unassisted QIC. The trivial region is shown dotted. Alice can sacrifice her rate in order to boost Bob's rate with respect to the trivial region, as shown by the solid rectangle. The dashed rectangle can be similarly obtained by Bob sacrificing his rate in order to boost Alice's. $O(\log \varepsilon)$ additive factors have been ignored in the figure. The above region is with respect to a fixed control state. The actual achievable rate region is a union over all such regions.

The trivial inner bound treats the QIC as two independent unassisted point-to-point channels from Alice to Charlie and Bob to Damru. Rate splitting and successive cancellation can be similarly used to obtain non-trivial rate regions for the unassisted QIC where one party, say Alice, sacrifices her rate in order to boost Bob's rate with respect to the trivial inner bound. The situation is summarised in Figure 9 .

In order to show that a larger region is achievable, we use splitting schemes and successive cancellation. Essentially, we split Alice into two senders, Alice ${ }_{0}$ and Alice $_{1}$, and we require Alice ${ }_{0}$ 's input to be decoded by Damru instead of Charlie. This allows Damru to treat Alice $0_{0}$ 's input as side information while decoding Bob's input, which allows us to boost Bob's rate. Alice's rate to Charlie, however, takes a hit because of this. Using a splitting scheme to do this allows us to adjust the amount of resources that Alice dedicates towards boosting Bob's rate, with the extreme cases $\theta \in\{0,1\}$ corresponding to situations when either Alice does not help Bob at all (the case of the two point-to-point channels) to when Alice dedicates all her resources to help Bob while her own rates drops to 0 .

The precise statements can be found in Proposition 6.8 and Theorem 6.10,
Proposition 6.8. Consider the quantum interference channel $\mathcal{N} A^{A^{\prime} B^{\prime} \rightarrow C D}$. Consider a pure 'control state' $|\sigma\rangle^{A^{\prime \prime} B^{\prime \prime} A^{\prime} B^{\prime}}:=$ $|\Omega\rangle^{A^{\prime \prime} A^{\prime}}|\Delta\rangle^{B^{\prime \prime} B^{\prime}}$. Let $|\psi\rangle^{A_{1} C_{1} R_{1}}$ and $|\eta\rangle^{A_{0} R_{0}}$ be the states that are to be sent by Alice to Charlie and Damru respectively and let $|\phi\rangle^{B D_{0} S}$ be the state to be sent from Bob to Damru, where $C_{1}, D_{0}$ model the side information about the respective messages $A_{1}, B$ that Charlie and Damru possess and $R_{0}, R_{1}, S$ are reference systems that are untouched by channel and coding operators. Let $\mathbb{I}$ denote the identity superoperator. For $\theta \in[0,1]$, let $\left\{U_{\theta}^{A^{\prime \prime}}\right\}$ be a splitting scheme. We define $|\sigma(\theta)\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}:=U_{\theta}|\Omega\rangle^{A^{\prime \prime} A^{\prime}}|\Delta\rangle^{B^{\prime \prime} B^{\prime}}$ and

$$
\sigma(\theta)^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B^{\prime \prime} C D}:=\left(\mathcal{N}^{A^{\prime} B^{\prime} \rightarrow C D} \otimes \mathbb{I}_{0}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B^{\prime \prime}}\right)\left(\sigma(\theta)^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}\right) .
$$

Then there exist encoding maps $\mathcal{A}^{A_{0} A_{1} \rightarrow A^{\prime}}$ and $\mathcal{B}^{B \rightarrow B^{\prime}}$ and decoding maps $\mathcal{C}^{C C_{1} \rightarrow A_{1} C_{1}}$ and $\mathcal{D}^{D D_{0} \rightarrow A_{0} B D_{0}}$ such that

$$
\|(\mathcal{C} \otimes \mathcal{D}) \circ \mathcal{N} \circ(\mathcal{A} \otimes \mathcal{B}) \cdot(\psi \otimes \eta \otimes \varphi)-\psi \otimes \eta \otimes \varphi\|_{1} \leq \delta .
$$

Here, $\delta=\delta_{\text {enc }}+\delta_{\text {dec }}$ where,

$$
\begin{aligned}
& \delta_{\mathrm{enc}}=2 \sqrt{2 \delta_{\mathrm{enc}}(3)}+2 \sqrt{2 \delta_{\mathrm{enc}}(2)+2 \delta_{\mathrm{enc}}(1)}, \\
& \delta_{\mathrm{dec}}=4 \sqrt{2 \delta_{\mathrm{dec}}(0)}+4 \sqrt{2 \delta_{\mathrm{dec}}(1)}+2 \sqrt{2 \delta_{\mathrm{dec}}(2)},
\end{aligned}
$$

and

$$
\begin{aligned}
& \delta_{\mathrm{enc}}(0)=2 \cdot 2^{\frac{1}{2} H_{\max }^{\varepsilon}\left(A_{0}\right)_{\eta}-\frac{1}{2} H_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\sigma(\theta)}+O(1)}+O(\varepsilon) \\
& \delta_{\mathrm{enc}}(1)=2 \cdot 2^{\frac{1}{2} H_{\max }^{f(\varepsilon)}\left(A_{1}\right)_{\psi}-\frac{1}{2} H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\sigma(\theta)}+O(1)}+O(\varepsilon), \\
& \delta_{\mathrm{enc}}(2)=2 \cdot 2^{\frac{1}{2} H_{\max }^{f(\varepsilon)}(B)_{\varphi}-\frac{1}{2} H_{\min }^{f(\varepsilon)}\left(B^{\prime \prime}\right)_{\sigma(\theta)}+O(1)}+O(\varepsilon), \\
& \delta_{\operatorname{dec}}(0)=2 \cdot 2^{\left.-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}-\frac{1}{2} I_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{0}^{\prime \prime}\right\rangle D\right)_{\sigma(\theta)}+O(1)}+O(\varepsilon), \\
& \delta_{\operatorname{dec}}(1)=2 \cdot 2^{\left.-\frac{1}{2} H_{2}^{\varepsilon}(B \mid S)_{\varphi}-\frac{1}{2} I_{\min }^{O\left(\varepsilon^{2}\right)}\left(B^{\prime \prime}\right\rangle D A_{0}^{\prime \prime}\right)_{\sigma(\theta)}+O(1)}+O(\varepsilon), \\
& \delta_{\operatorname{dec}}(2)=2 \cdot 2^{\left.-\frac{1}{2} H_{2}^{\varepsilon}\left(A_{1} \mid R_{1}\right)_{\psi}-\frac{1}{2} I_{\min }^{O\left(\varepsilon^{2}\right)}\left(A_{1}^{\prime \prime}\right\rangle C\right)_{\sigma(\theta)}+O(1)}+O(\varepsilon),
\end{aligned}
$$

where $f(\varepsilon)=O\left(\varepsilon^{2}\right)$.
Remark 6.9. The $O(1)$ in the bounds in Proposition 6.8 hide the $\log k$ terms, as in Proposition 6.1, where $k$ is some constant integer.

We are now ready to state our main one-shot coding theorem. In this case, we denote by $Q_{0}$ the number of qubits available to Alice for sending to Damru, to use as side information to boost Bob's rate. The quantities of interest however are $\left(Q_{A}, E_{A}, Q_{B}, E_{B}\right)$ which denote, in order, the number of message qubits and ebits available to Alice, and the analogous quantities for Bob.

Theorem 6.10. Consider the setting of Proposition 6.8 Let $Q_{A}, E_{A}, Q_{B}, E_{B}$ be the number of message qubits and number of available ebits of Alice and Bob respectively. Additionally, let $Q_{0}$ denote the number of message qubits available to Alice for transmission to Damru. Let $\theta, \varepsilon \in[0,1]$ and $\varepsilon_{0}:=O\left(\varepsilon^{2}\right)$. Then there exist encoding and decoding maps such that any message cum ebit rate 4-tuple satisfying the following inequalities, is achievable with error at most $O(\sqrt{\varepsilon})$ achievable for partial entanglement assisted entanglement transmission

$$
\begin{aligned}
& \left.Q_{0}<I_{\min }^{\varepsilon_{0}}\left(A_{0}^{\prime \prime}\right\rangle D\right)_{\sigma(\theta)}+O(\log \varepsilon)-O(1) \\
& Q_{0}<H_{\min }^{\varepsilon_{0}}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime}\right)_{\sigma(\theta)}+O(\log \varepsilon)-O(1) \\
& Q_{A}+E_{A}<H_{\min }^{f(\varepsilon)}\left(A_{1}^{\prime \prime}\right)_{\sigma(\theta)}+O(\log \varepsilon) \\
& \left.Q_{A}-E_{A}<I_{\min }^{\varepsilon_{0}}\left(A_{1}^{\prime \prime}\right\rangle C\right)_{\sigma(\theta)}+O(\log \varepsilon)-O(1) \\
& Q_{B}+E_{B}<H_{\min }^{f(\varepsilon)}\left(B^{\prime \prime}\right)_{\sigma(\theta)}+O(\log \varepsilon) \\
& \left.Q_{B}-E_{B}<I_{\min }^{\varepsilon_{0}}\left(B^{\prime \prime}\right\rangle A_{0}^{\prime \prime} D\right)_{\sigma(\theta)}+O(\log \varepsilon)-O(1)
\end{aligned}
$$

The proof of Theorem 6.10 follows from Proposition 6.8 using the arguments in Section 5, We present the proof of Proposition 6.8 below.

Proof. As mentioned before, the idea is for Alice to use some part of her input to boost Bob's rate to Damru. to do this we split Alice into Alice $_{0}$ and Alice $_{1}$. We can then treat the interference channel as a QMAC from Alice ${ }_{0}$ and Bob to Damru, and as a point-to-point quantum channel from Alice ${ }_{1}$ to Charlie. We will use the techniques used to prove Proposition 6.1 to derive achievable rates for entanglement transmission from Bob to Charlie. Note that in this case, we assume that Alice $_{0}$ shares no pre-shared entanglement with Damru. The inner bound for entanglement
transmission from Alice ${ }_{1}$ to Charlie can be derived by considering the coding scheme for entanglement transmission over a point-to-point channel.

Note that the above analysis will give us two separate 1-norm expressions, one for the QMAC among Alice ${ }_{0}$, Bob and Damru and the other for the point-to-point channel from Alice ${ }_{1}$ to Charlie. To combine these two expressions we will need the following fact, whose proof can be found in the appendix. This fact appears in [18, Lemma 5.1]:

Fact 6.11. Given density operators $\rho^{A B C}, \sigma^{A}, \omega^{B C}, \tau^{A B}, \eta^{C}$ such that

$$
\begin{aligned}
& \left\|\rho^{A B C}-\sigma^{A} \otimes \omega^{B C}\right\|_{1} \leq \varepsilon_{1}, \\
& \left\|\rho^{A B C}-\tau^{A B} \otimes \eta^{C}\right\|_{1} \leq \varepsilon_{2},
\end{aligned}
$$

then

$$
\left\|\rho^{A B C}-\sigma^{A} \otimes \tau^{B} \otimes \eta^{C}\right\|_{1} \leq 2 \varepsilon_{1}+\varepsilon_{2}
$$

We denote by $\mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C D E}$ the Stinespring dilation of the interference channel $\mathcal{N}^{A^{\prime} B^{\prime} \rightarrow C D}$. We define our encoding maps $\mathcal{E}^{A_{0} A_{1} \rightarrow A^{\prime}}$ and $\mathcal{F}^{B \rightarrow B^{\prime}}$ as in the proof of Proposition 6.1. First, we repeat the decoding protocol for the QMAC, where Damru decodes Alice ${ }_{0}$ first and then Bob. The intermediate states used for this part of the protocol are as follows:

$$
\begin{aligned}
& \left|\omega_{12}\right\rangle^{A_{0}^{\prime \prime} A^{\prime} B^{\prime} C_{1} R_{1} D_{0} S}:=\sqrt{\mid B^{\prime \prime} A_{1}^{\prime \prime}} \mathrm{op}^{A_{1}^{\prime \prime} B^{\prime \prime} \rightarrow A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle)\left(U^{A_{1}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right) W_{1}^{A_{1} \rightarrow A_{1}^{\prime \prime}}|\psi\rangle^{A_{1} R_{1} C_{1}} W_{2}^{B \rightarrow B^{\prime \prime}}|\varphi\rangle^{B_{1}^{\prime \prime} D_{0} S}, \\
& \left|\omega_{3}\right\rangle^{B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime} C_{1} R_{1}}:=\sqrt{A_{1}^{\prime \prime}} \mathrm{p}^{A_{1}^{\prime \prime} \rightarrow B^{\prime \prime} A_{0}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle) U^{A_{1}^{\prime \prime}} W_{1}^{A_{1} \rightarrow A_{1}^{\prime \prime}}|\psi\rangle^{A_{1} R_{1} C_{1}},
\end{aligned}
$$

where $W_{1}$ and $W_{2}$ are isometric embeddings, as before. Damru decodes Alice ${ }_{0}$ 's message and then Bob's. Damru is not required to decode Alice 1 's message. In fact, for the protocol to work, we will treat Damru as part of the environment so that Charlie can decode Alice, ${ }_{1}$ 's message, effectively making it impossible for Damru to decode Alice $_{1}$ 's message. We can then show the existence of a decoding isometry

$$
V_{\mathrm{Boв}}^{D D_{0} \rightarrow A_{0}^{\prime \prime} B^{\prime \prime} D D_{0} A_{0} B}
$$

such that

$$
\begin{aligned}
& \| V_{\mathrm{BOB}}^{D D_{0} \rightarrow A_{0}^{\prime \prime} B^{\prime \prime} D D_{0} A_{0} B \frac{\left.\mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C D E}(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \phi \otimes \psi)\right)}{\operatorname{Tr}[(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \phi \otimes \psi)]}-\eta^{A_{0} R_{0}} \otimes \phi^{B D_{0} S} \otimes \tilde{\omega}_{3}^{B^{\prime \prime} A_{0}^{\prime \prime} C_{1} R_{1} D C E} \|_{1}} \\
& \leq 2 \sqrt{2 \delta_{\operatorname{dec}(0)}(0)}+2 \sqrt{2 \delta_{\operatorname{dec}}(1)} .
\end{aligned}
$$

We now consider the channel from Alice ${ }_{1}$ to Charlie. We will need the following intermediate state:

$$
\left|\omega_{4}\right\rangle^{A_{1}^{\prime \prime} A^{\prime} B^{\prime} R_{0} D_{0} S}:=\sqrt{\mid A_{0}^{\prime \prime} B^{\prime \prime}} \mid \mathrm{op}^{A_{0}^{\prime \prime} B^{\prime \prime} \rightarrow A_{1}^{\prime \prime} A^{\prime} B^{\prime}}(|\sigma\rangle)\left(U^{A_{0}^{\prime \prime}} \otimes U^{B^{\prime \prime}}\right) W_{0}^{A_{0} \rightarrow A_{0}^{\prime \prime}}|\eta\rangle^{A_{0} R_{0}} W_{2}^{B \rightarrow B^{\prime \prime}}|\varphi\rangle^{B_{1}^{\prime \prime} D_{0} S} .
$$

We can then show the existence of an isometric decoder

$$
V_{\mathrm{AlICE}}^{C C_{1} \rightarrow C C_{1} A_{1}^{\prime \prime} A_{1}}
$$

such that

$$
\begin{aligned}
& \left\|V_{\mathrm{ALICE}}^{C C_{1} \rightarrow C C_{1} A_{1}^{\prime \prime} A_{1}} \frac{\left.\mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C D E}(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \phi \otimes \psi)\right)}{\operatorname{Tr}[(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \phi \otimes \psi)]}-\psi^{A_{1} C_{1} R_{1}} \otimes \tilde{\omega}_{4}^{D D_{0} E S R_{0} A_{1}^{\prime \prime} C}\right\|_{1} \\
& \leq 2 \sqrt{2 \delta(2)} .
\end{aligned}
$$

Next, through some standard algebraic manipulation, we see that the two inequalities above are equivalent to

$$
\begin{aligned}
& \left\|V_{\mathrm{ALICE}} \otimes V_{\mathrm{BoB}} \frac{\left.\mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C D E}(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \phi \otimes \psi)\right)}{\operatorname{Tr}[(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \phi \otimes \psi)]}-\eta^{A_{0} R_{0}} \otimes \phi^{B D_{0} S} \otimes \zeta_{1}^{B^{\prime \prime} A_{0}^{\prime \prime} A_{1}^{\prime \prime} A_{1} C C_{1} R_{1} D E}\right\|_{1} \\
& \leq 2 \sqrt{2 \delta_{\mathrm{dec}}(0)}+2 \sqrt{2 \delta_{\mathrm{dec}}(1)}, \\
& \left\|V_{\mathrm{Bob}} \otimes V_{\mathrm{AlicE}} \frac{\left.\mathcal{U}_{\mathcal{N}}^{A^{\prime} B^{\prime} \rightarrow C D E}(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \phi \otimes \psi)\right)}{\operatorname{Tr}[(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \phi \otimes \psi)]}-\psi^{A_{1} C_{1} R_{1}} \otimes \zeta_{2}^{D D_{0} A_{0}^{\prime \prime} B^{\prime \prime} A_{0} B E S R_{0} A_{1}^{\prime \prime} C}\right\|_{1} \\
& \leq 2 \sqrt{2 \delta(2)},
\end{aligned}
$$

where

$$
\begin{aligned}
& \left|\zeta_{1}\right\rangle^{B^{\prime \prime} A_{0}^{\prime \prime} A_{1}^{\prime \prime} A_{1} C C_{1} R_{1} D E}:=V_{\mathrm{AlicE}}\left|\tilde{\omega}_{3}\right\rangle^{B^{\prime \prime} A_{0}^{\prime \prime} C_{1} R_{1} D C E}, \\
& \left|\zeta_{2}\right\rangle^{D D_{0} A_{0}^{\prime \prime} B^{\prime \prime} A_{0} B E S R_{0} A_{1}^{\prime \prime} C}:=V_{\mathrm{BoB}}\left|\tilde{\omega}_{4}\right\rangle_{1}^{A_{1}^{\prime \prime} D C E R_{0} D_{0} S} .
\end{aligned}
$$

We can now use Fact 6.11 to conclude that:

$$
\begin{aligned}
& \left\|V_{\mathrm{ALICE}} \otimes V_{\mathrm{BOB}} \frac{\left.\mathcal{U}_{\mathcal{N}}^{\mathrm{A}^{\prime} B^{\prime} \rightarrow C D E}(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \phi \otimes \psi)\right)}{\operatorname{Tr}[(\mathcal{E} \otimes \mathcal{F}) \cdot(\eta \otimes \phi \otimes \psi)]}-\eta^{A_{0} R_{0}} \otimes \phi^{B D_{0} S} \otimes \zeta_{2}^{A_{0}^{\prime \prime} B^{\prime \prime} A_{1}^{\prime \prime} D C E} \otimes \psi^{A_{1} R_{1} C_{1}}\right\|_{1} \\
& \leq 4 \sqrt{2 \delta_{\operatorname{dec}}(0)}+4 \sqrt{2 \delta_{\operatorname{dec}}(1)}+2 \sqrt{2 \delta_{\operatorname{dec}}(2)}
\end{aligned}
$$

The rest of the proof is identical to the analysis of the encoding error in the proof of Proposition 6.1. This concludes the proof.

## 7 Asymptotic IID Analysis

In this section we present the asymptotic iid versions of the one-shot achievability results presented in the previous section. Based on the discussion so far, we have seen that we can achieve the following rate point for Alice ${ }_{0}$, Bob and Alice $_{1}$ :

$$
\left.\left(H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid B A_{1}^{\prime \prime} E\right), H_{\min }^{\varepsilon}\left(B^{\prime \prime} \mid A_{1}^{\prime \prime} E\right), H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime} \mid E\right)\right)\right)
$$

for all values of the parameter $\theta$. It is tempting to conclude using the Quantum Asymptotic Equipartition Property [21] to the above rate point and conclude the achievability of the following rate point in the asymptotic iid setting:

$$
\left.\left(H\left(A_{0}^{\prime \prime} \mid B A_{1}^{\prime \prime} E\right), H\left(B^{\prime \prime} \mid A_{1}^{\prime \prime} E\right), H\left(A_{1}^{\prime \prime} \mid E\right)\right),\right) .
$$

Things are not so simple however, due to the fact that after rate splitting, one of the terms in this rate point could be negative. For example, if Alice $0_{0}$ 's rate is negative, then the protocol no longer works for decoding Bob and Alice ${ }_{1}$.

First, note that we only have to worry about Alice $_{0}$ 's being negative. This is because Bob's rate could never be
negative due to the following data processing inequality:

$$
H_{\min }^{\varepsilon}\left(B^{\prime \prime} \mid A^{\prime \prime} E\right)=H_{\min }^{\varepsilon}\left(B^{\prime \prime} \mid A_{0}^{\prime \prime} A_{1}^{\prime \prime} E\right) \leq H_{\min }^{\varepsilon}\left(B^{\prime \prime} \mid A_{1}^{\prime \prime} E\right) \leq H_{\min }^{\varepsilon}\left(B^{\prime \prime} \mid E\right) .
$$

This also implies that Alice $_{0}$ and Alice ${ }_{1}$ 's combined rate cannot be negative, since the sum rate of Alice and Bob is invariant. Thus, if Alice ${ }_{1}$ 's rate is negative, that implies that Alice ${ }_{0}$ 's rate is more than the combined rate given of Alice $_{0}$ and Alice $_{1}$. This is a good situation since we could then simply perform the protocol for only Alice ${ }_{0}$ and Bob.

Thus, the difficulty lies in the case when Alice ${ }_{0}$ 's rate is negative. We will show that we can achieve the desired rate region in the asymptotic iid limit, with a small amount of pre-shared entanglement between Alice ${ }_{0}$ and Charlie. These pre-shared EPR pairs will be used catalytically with the added advantage that the rate at which we require these pre-shared EPR pairs go to 0 in the asymptotic iid limit.

We will divide the $n$ channels into $\sqrt{n}$ blocks, where each block is of size $\sqrt{n}$. To avoid cumbersome notation, we use the following convention:

$$
H_{\min }^{\varepsilon}(A \mid B)_{\rho}(n):=H_{\min }^{\varepsilon}\left(A^{\otimes n} \mid B^{\otimes n}\right)_{\rho^{A B} \otimes n} .
$$

We will consider two situations:

1. $H_{\text {min }}^{\varepsilon}\left(A_{0} \mid A_{1}\right)(\sqrt{n}) \geq 0$,
2. $H_{\text {min }}^{\varepsilon}\left(A_{0} \mid A_{1}\right)(\sqrt{n})<0$.

Case I: $H_{\text {min }}^{\varepsilon}\left(A_{0} \mid A_{1}\right)(\sqrt{n}) \geq 0$.

For each block of size $\sqrt{n}$ set:

$$
\begin{aligned}
& Q_{A_{0}^{\sqrt{n}}}=0, \\
& E_{A_{0}^{\sqrt{n}}}=\left|H_{\min }^{\varepsilon}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B E\right)(\sqrt{n})\right| .
\end{aligned}
$$

Since $H_{\text {min }}^{\varepsilon}\left(A_{0} \mid A_{1}\right)(\sqrt{n})$ is positive, this implies that there exists an isometric encoder for Alice and the protocol can start. Note that at the end of the protocol, Alice ${ }_{1}$ shares a maximally entangled state with Charlie of rank

$$
2^{H_{\min }^{€}}\left(A_{1}^{\prime \prime} \mid E\right)(\sqrt{n}) .
$$

Alice can now keep aside $2^{\left|H\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B E\right)(\sqrt{n})\right|}$ EPR pairs and use them as the seed pre-shared EPR states for the next block of $\sqrt{n}$ channels. Repeating this argument for each block, we see that Alice's rate for entanglement transmission is

$$
\left.\frac{1}{n} \sqrt{n}\left(H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime} \mid E\right)(\sqrt{n})+H\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B E\right)(\sqrt{n})\right)\right) .
$$

In the asymptotic iid limit, the above quantity is equal to

$$
H\left(A_{1}^{\prime \prime} \mid E\right)+H\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B^{\prime \prime} E\right)
$$

which is Alice's desired rate. Note that we only needed to use $2{ }^{H_{\text {min }}^{\varepsilon}}\left(A_{1}^{\prime \prime} \mid E\right)(\sqrt{n})$ EPR pairs for the very first block, and
thereafter these EPR pairs were regenerated by the protocol. This implies that the rate of seed EPR pairs is given by

$$
\frac{1}{n} H_{\min }^{\varepsilon}\left(A_{1}^{\prime \prime} \mid E\right)(\sqrt{n}),
$$

which is 0 in the asymptotic iid limit. Thus in this case, we can prove the following theorem:
Theorem 7.1. Given a quantum multiple access channel $\mathcal{N}^{A^{\prime} B^{\prime} \rightarrow C}$ all rate points in the closure of the following region are achievable for partial entanglement assisted entanglement generation:

$$
\bigcup_{k=1}^{\infty} \frac{1}{k} \mathcal{Q}\left(\mathcal{N}^{\otimes k}\right)
$$

where $\mathcal{Q}\left(\mathcal{N}^{\otimes k}\right)$ is the set of non negative rate tuples $\left(Q_{A}, E_{A}, Q_{B}, E_{B}\right)$ in the set

$$
\begin{gathered}
Q_{A}+E_{A}<H\left(A^{\prime \prime k}\right)_{\sigma_{k}}, \\
\left.Q_{A}-E_{A}<I\left(A^{\prime \prime k}\right\rangle B^{\prime \prime k} C^{k}\right) \mathcal{U}_{\mathcal{N}} \cdot \sigma_{k}, \\
Q_{B}+E_{B}<H\left(B^{\prime \prime k}\right)_{\sigma_{k}}, \\
\left.Q_{B}-E_{B}<I\left(B^{\prime \prime k}\right\rangle A^{\prime \prime k} C^{k}\right) \mathcal{U}_{\mathcal{N}} \cdot \sigma_{k}, \\
\left.Q_{A}-E_{A}+Q_{B}-E_{B}<I\left(A^{\prime \prime k} B^{\prime \prime k}\right\rangle C^{k}\right) \mathcal{U}_{\mathcal{N}} \cdot \sigma_{k},
\end{gathered}
$$

where $\left|\sigma_{k}\right\rangle^{A^{\prime \prime} k} B^{\prime \prime k} A^{\prime k} B^{\prime k}:=|\Omega\rangle^{A^{\prime \prime}{ }^{\prime} A^{\prime k}}|\Delta\rangle^{B^{\prime \prime} k^{\prime} B^{k}}$.
Case II: $H_{\text {min }}^{\varepsilon}\left(A_{0} \mid A_{1}\right)(\sqrt{n})<0$.

In this case we cannot prove the existence of an isometric encoder for Alice. We get around this by using the state merging protocol. To be precise, for the value for $\theta$ for which Case II occurs, Alice and Bob simply send the bipartite pure state $\left(|\Omega\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}|\Delta\rangle^{B^{\prime \prime} B^{\prime}}\right)^{\otimes n}$ through $n$ copies of the channel. This results in the following state:

$$
\left(|\sigma\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B^{\prime \prime} C E}\right)^{\otimes n}
$$

We divide this state into $\sqrt{n}$ blocks, where each block corresponds to the state

$$
\left(|\sigma\rangle^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B^{\prime \prime} C E}\right)^{\otimes \sqrt{n}} .
$$

The parties then do a multi-party state merging protocol for this state [31], with parties Alice ${ }_{0}$, Bob and Alice ${ }_{1}$. Since the expression $H\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B E\right)(\sqrt{n})$ is assumed to be negative, Alice ${ }_{0}$ must use $2^{H\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B E\right)(\sqrt{n})}$ pre-shared EPR pairs with Charlie to merge her share of the state. The EPR pairs are regenerated when the protocol ends by Alice ${ }_{1}$ merging her state with Charlie. Then, the same arguments as in Case I show that, with a vanishing amount of pre-shared seed, the following rate point is achievable for unassisted state merging, in the asymptotic iid limit

$$
\begin{aligned}
& Q_{A}<H\left(A_{1}^{\prime \prime} \mid E\right)+H\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} B^{\prime \prime} E\right), \\
& Q_{B}<H\left(B^{\prime \prime} \mid A_{1}^{\prime \prime} E\right) .
\end{aligned}
$$

It is known that entanglement generation and entanglement transmission are equivalent [16]. This implies that there exists an unassisted entanglement transmission protocol which also achieves the above rates. Finally, one can obtain the rates for the case when rate-limited entanglement assistance is available by simply time sharing between the unassisted protocol above and the completely assisted protocol of Bennet et al. [29]. This implies that Theorem 7.1]is true for all cases.

The arguments above can now be used to prove a similar theorem for the QIC as well:
Theorem 7.2. Given a quantum interference channel $\mathcal{N}^{\otimes k}$, the control state $|\sigma\rangle^{A^{\prime \prime} A^{\prime} B^{\prime \prime} B^{\prime}}$ the following regularised rate region is achievable for partial entanglement assisted entanglement transmission:

$$
\bigcup_{k=1}^{\infty} \frac{1}{k} \mathcal{Q}\left(\mathcal{N}^{\otimes k}\right)
$$

For each $k \in \mathbb{N}$,

$$
\mathcal{Q}\left(\mathcal{N}^{\otimes k}\right)=\bigcup \mathcal{A}_{\theta}^{k} \bigcup \bigcup \mathcal{B}_{\theta}^{k},
$$

where, for a fixed $\theta \in[0,1], \mathcal{A}_{\theta}^{k}$ is the set of all non-negative tuples $\left(Q_{A}, E_{A}, Q_{B}, E_{B}\right)$ such that

$$
\begin{aligned}
& Q_{A}+E_{A}<H\left(A_{1}^{\prime \prime k}\right)_{\sigma_{k}(\theta)}, \\
& \left.Q_{A}-E_{A}<I\left(A_{1}^{\prime \prime k}\right\rangle C_{1}^{k}\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma_{k}(\theta)}, \\
& Q_{B}+E_{B}<H\left(B^{\prime \prime k}\right)_{\sigma_{k}(\theta)}, \\
& \left.Q_{B}-E_{B}<I\left(B^{\prime \prime k}\right\rangle A_{0}^{\prime \prime k} C_{2}^{k}\right)_{\mathcal{U}_{\mathcal{N}} \cdot \sigma_{k}(\theta)},
\end{aligned}
$$

where $\left|\sigma_{k}\right\rangle^{A^{\prime \prime k} B^{\prime \prime k} A^{\prime k} B^{\prime k}}:=|\Omega\rangle^{A^{\prime \prime k} A^{\prime} k}|\Delta\rangle^{B^{\prime \prime} k^{\prime} k}$ and $\left|\sigma_{k}\right\rangle:=U^{A^{\prime \prime k} \rightarrow A_{0}^{\prime \prime k} A_{1}^{\prime \prime k}}\left|\sigma_{k}\right\rangle$. We assume that $\left\{U_{\theta}\right\}$ is a splitting scheme. Analogously, $\mathcal{B}_{\theta}^{k}$ is the set of those points which are obtained when the splitting isometry acts on the system $B^{\prime \prime k}$.

## 8 Conclusion

In this paper we use the technique of quantum rate splitting and successive cancellation decoding of entanglement transmission codes to design entanglement transmission codes for the QMAC and QIC. We recover a non-trivial rate region for the QMAC in the one-shot setting. Suitable adaptations of our techniques also achieve the ideal pentagonal rate region in the asymptotic iid setting.

For the QIC, we show the existence of a non-trivial rate region both in the asymptotic iid and one-shot setting, which is larger than the region one would obtain by considering the QIC as two point-to-point channels.

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## A Appendix

Lemma A.1. Given $\theta, \theta^{\prime} \in[0,1]$ such that $\left|\theta-\theta^{\prime}\right| \leq \delta$, we have that

$$
P\left(\Omega^{\prime}(\theta)^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B E}, \Omega^{\prime}\left(\theta^{\prime}\right)^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} B E}\right) \leq O(\sqrt{\delta}) .
$$

Proof of Lemma A.1 For the course of the proof we will neglect to mention the registers in the superscript to ease the notation, unless necessary. Since both $\Omega^{\prime}(\theta)$ and $\Omega^{\prime}\left(\theta^{\prime}\right)$ are pure, we will use the identity :

$$
P\left(\Omega^{\prime}(\theta), \Omega^{\prime}\left(\theta^{\prime}\right)\right)=\sqrt{1-\left|\left\langle\Omega^{\prime}(\theta) \mid \Omega^{\prime}\left(\theta^{\prime}\right)\right\rangle\right|^{2}}
$$

Recall that since

$$
\left|\Omega^{\prime}\right\rangle^{A_{0}^{\prime \prime} A_{0}^{\prime \prime} B E}(\theta)=U_{\mathcal{N}}^{A^{\prime}} \circ U_{f}^{A_{0}^{\prime \prime} A_{1}^{\prime \prime} A^{\prime}}\left(\sum_{u \in \mathcal{A}} \sqrt{P_{U}^{\theta}(u)}|u\rangle^{A_{0}^{\prime \prime}}\right) \otimes\left(\sum_{v \in \mathcal{A}} \sqrt{P_{V}^{\theta}(v)}|v\rangle^{A_{0}^{\prime \prime}}\right)|0\rangle^{A^{\prime}}
$$

and similarly for $\left|\Omega^{\prime}\right\rangle\left(\theta^{\prime}\right)$,

$$
\left\langle\Omega^{\prime}(\theta) \mid \Omega^{\prime}\left(\theta^{\prime}\right)\right\rangle=F\left(P_{U}^{\theta}, P_{U}^{\theta^{\prime}}\right) F\left(P_{V}^{\theta}, P_{V}^{\theta^{\prime}}\right)
$$

It is thus sufficient to show that the distributions $P_{U}^{\theta}$ and $P_{V}^{\theta}$ are close to $P_{U}^{\theta^{\prime}}$ and $P_{V}^{\theta^{\prime}}$ respectively. Then, recalling the explicit form of $P_{U}^{\theta}$ observe that :

$$
\begin{aligned}
\left\|P_{U}^{\theta^{\prime}}-P_{U}^{\theta}\right\|_{1} & =\left|\left(1-\theta+\theta P_{A}(0)\right)-\left(1-\theta^{\prime}+\theta^{\prime} P_{A}(0)\right)\right|+\sum_{i \neq 0}\left|\theta P_{A}(i)-\theta^{\prime} P_{A}(i)\right| \\
& \leq\left|\theta-\theta^{\prime}\right|+\left|\theta-\theta^{\prime}\right| \sum_{i \in \mathcal{A}} P_{A}(i) \\
& \leq 2 \delta
\end{aligned}
$$

Next, observe that, for any $i \in \mathcal{A}$

$$
P_{V}^{\theta}(i)=\frac{F_{A}(i)}{F_{U}^{\theta}(i)}-\frac{F_{A}(i-1)}{F_{U}^{\theta}(i-1)}
$$

It holds that

$$
\begin{aligned}
F_{A}(i) F_{U}^{\theta}(i-1)-F_{A}(i-1) F_{U}^{\theta}(i) & =\left(P_{A}(i)+F_{A}(i-1)\right) F_{U}^{\theta}(i-1)-F_{A}(i-1)\left(\theta F_{A}(i)+1-\theta\right) \\
& =\left(P_{A}(i)+F_{A}(i-1)\right) F_{U}^{\theta}(i-1)-F_{A}(i-1)\left(F_{U}^{\theta}(i-1)+\theta P_{A}(i)\right) \\
& =P_{A}(i)\left(\theta F_{A}(i-1)+1-\theta\right)-\theta F_{A}(i-1) P_{A}(i) \\
& =(1-\theta) P_{A}(i)
\end{aligned}
$$

Denote $F_{U}^{\theta}(i) F_{U}^{\theta}(i-1):=g(\theta)$. Then,

$$
\begin{aligned}
& \left|g(\theta)-g\left(\theta^{\prime}\right)\right| \\
& =\left|F_{U}^{\theta}(i) F_{U}^{\theta}(i-1)-F_{U}^{\theta^{\prime}}(i) F_{U}^{\theta^{\prime}}(i-1)\right| \\
& \leq\left|F_{U}^{\theta}(i) F_{U}^{\theta}(i-1)-F_{U}^{\theta^{\prime}}(i) F_{U}^{\theta}(i-1)\right|+\left|F_{U}^{\theta^{\prime}}(i) F_{U}^{\theta}(i-1)-F_{U}^{\theta^{\prime}}(i) F_{U}^{\theta^{\prime}}(i-1)\right| \\
& \leq 4 \delta
\end{aligned}
$$

Let $p^{*}=\min _{i \in \mathcal{A}} P_{A}(i)$. Then,

$$
\begin{aligned}
g(\theta) & \geq\left(1-\theta+\theta p^{*}\right)^{2} \\
& \geq p^{* 2} \\
\left|P_{V}^{\theta}(i)-P_{V}^{\theta^{\prime}}(i)\right|= & P_{A}(i)\left|\frac{1-\theta}{g(\theta)}-\frac{1-\theta^{\prime}}{g\left(\theta^{\prime}\right)}\right| \\
& =\frac{P_{A}(i)}{\left|g(\theta) g\left(\theta^{\prime}\right)\right|} \cdot\left|(1-\theta) g\left(\theta^{\prime}\right)-\left(1-\theta^{\prime}\right) g(\theta)\right| \\
& \stackrel{(a)}{\leq} \frac{P_{A}(i)}{p^{* 4}} \cdot c \cdot \delta,
\end{aligned}
$$

where $c$ is some constant and we have used the triangle inequality and the lower bound for $g(\theta)$ in $(a)$.
Then, the above bound implies that:

$$
\left\|P_{V}^{\theta}-P_{V}^{\theta^{\prime}}\right\|_{1} \leq O(\delta) .
$$

Using the property that $F(P, Q) \geq 1-\|P-Q\|_{1}$ for any two distributions $P$ and $Q$, we see that

$$
\left|\left\langle\Omega^{\prime}(\theta) \mid \Omega^{\prime}\left(\theta^{\prime}\right)\right\rangle\right|^{2} \geq 1-O(\delta) .
$$

This concludes the proof.
Lemma A.2. [Normalisation Lemma] Given a state $\rho$ and any positive matrix $\sigma$, not necessarily of trace 1 , given

$$
\|\sigma-\rho\|_{1} \leq \delta
$$

then

$$
\left\|\frac{\sigma}{\operatorname{Tr}[\sigma]}-\rho\right\|_{1} \leq 2 \delta .
$$

Proof. First note that for any positive matrix $\Omega$,

$$
\|\sigma\|_{1}=\operatorname{Tr}[\sigma] .
$$

Then

$$
\begin{aligned}
\left\|\frac{\sigma}{\operatorname{Tr}[\sigma]}-\rho\right\|_{1} & \leq\left\|\frac{\sigma}{\operatorname{Tr}[\sigma]}-\sigma\right\|_{1}+\|\sigma-\rho\|_{1} \\
& =\left|\frac{1}{\operatorname{Tr}[\sigma]}-1\right| \cdot\|\sigma\|_{1}+\|\sigma-\rho\|_{1} \\
& =|\operatorname{Tr}[\sigma]-1|+\|\sigma-\rho\|_{1} .
\end{aligned}
$$

Now, by the given condition,

$$
\begin{gathered}
\|\sigma-\rho\|_{1} \leq \delta \\
\Longrightarrow|\operatorname{Tr}[\sigma]-1| \leq \delta
\end{gathered}
$$

by the monotonicity of 1-norm under trace. Thus, we see that

$$
\left\|\frac{\sigma}{\operatorname{Tr}[\sigma]}-\rho\right\|_{1} \leq 2 \delta
$$

This completes the proof.
Corollary A.3. [Purification Lemma] Given the setting of Lemma 5.10] define

$$
\kappa^{R_{0} B R_{1} E}:=\left|A_{0}^{\prime \prime}\right| \operatorname{Tr}_{C} \mathcal{U}_{\mathcal{N}} \text { op }^{A_{0}^{\prime \prime} \rightarrow A^{\prime} B R_{1}}(\omega) U^{A_{0}^{\prime \prime}} \cdot \eta^{A_{0}^{\prime \prime} R_{0}}
$$

and

$$
\delta:=2^{-\frac{1}{2} H_{\min }^{\frac{\varepsilon^{2}}{26 k}}\left(A_{0}^{\prime \prime} \mid A_{1}^{\prime \prime} E\right)_{\mathcal{U}_{\mathcal{N}} \cdot \Omega}-\frac{1}{2} H_{\min }^{\varepsilon}\left(A_{0} \mid R_{0}\right)_{\eta}+O(\log k)}+12 k \varepsilon
$$

Then

$$
\left\|\frac{\kappa^{R_{0} B R_{1} E}}{\operatorname{Tr}[\kappa]}-\eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1} E}\right\|_{1} \leq 2 \delta
$$

Proof. Lemma 5.10 tells us that

$$
\left\|\frac{\kappa^{R_{0} B R_{1} E}}{\operatorname{Tr}[\omega]}-\eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1} E}\right\|_{1} \leq \delta
$$

Then by Lemma A.2, we see that

$$
\left\|\frac{1}{\operatorname{Tr}\left[\frac{\kappa}{\operatorname{Tr}[\omega]}\right]} \frac{\kappa^{R_{0} B R_{1} E}}{\operatorname{Tr}[\omega]}-\eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1} E}\right\| \leq 2 \delta
$$

which implies that

$$
\left\|\frac{\kappa^{R_{0} B R_{1} E}}{\operatorname{Tr}[\kappa]}-\eta^{R_{0}} \otimes \tilde{\omega}^{B R_{1} E}\right\|_{1} \leq 2 \delta
$$

This concludes the proof.


[^0]:    *A preliminary version of this work appeared in the proceedings of ISIT 2021 [1].
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