List Decoding of Quaternary Codes in the Lee Metric

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Abstract—We present a list decoding algorithm for quaternary negacyclic codes over the Lee metric. To achieve this result, we use a Sudan-Guruswami type list decoding algorithm for Reed-Solomon codes over certain ring alphabets. Our decoding strategy for negacyclic codes over the ring \mathbb{Z}_4 combines the list decoding algorithm by Wu with the Gröbner basis approach for solving a key equation due to Byrne and Fitzpatrick.

Index Terms—Codes over rings, negacyclic codes, list decoding, polynomial factorization, interpolation, lifting.

I. INTRODUCTION

It has been observed in the literature, that cyclic codes over the alphabet \mathbb{Z}_4 equipped with the Lee distance often have a larger minimum distance and a better decoding capability than predicted by their "designed distance". For this reason a list decoding approach is suggested, although technical difficulties due to zero divisors in rings are to be expected.

Codes over integer residue rings equipped with the Lee metric have currently received increasing attention in the community. This stems on one hand from the fact, that McEliece type cryptosystems based on the Lee metric may offer increased security but lack so far the availability of good codes. On the other hand, connections between lattice-based cryptography and coding theory are gaining attention, a connection which is established by the Lee metric as it approximates the Euclidean distance on lattices.

In this paper, we present a list decoding algorithm for quaternary negacyclic codes with Lee distance. To arrive there, we employ a Sudan-Guruswami type list decoding algorithm for Reed-Solomon codes over ring alphabets. Note that this part is related to work by Armand [1], [2] (see also [10]), while our setup and factorisation algorithm slightly differs. For negacyclic codes over the ring \mathbb{Z}_4 (cf. [12], [5]), our decoding strategy combines the list decoding algorithm by Wu [13] with the Gröbner basis approach for solving a key equation due to Byrne and Fitzpatrick [4].

II. PRELIMINARIES

Let p be a prime and let m and r be positive integers. We denote by \mathbb{F}_{p^m} the finite field with p^m elements and let $\mathrm{GR}(p^r,m)$ be the Galois ring of characteristic p^r and degree m. The latter can be constructed as the quotient ring $\mathbb{Z}_{p^r}[X]/(f)$ with monic polynomial $f \in \mathbb{Z}_{p^r}[X]$ of degree m

such that $f \mod p$ in $\mathbb{Z}_p[X]$ is irreducible. The Galois ring $\mathrm{GR}(p^r,m)$ is a local ring with maximal ideal (p) such that all its ideals form a chain $\{(p^i) \mid 0 \le i \le r\}$, and one has the canonical homomorphism onto the residue field

$$\mu \colon \operatorname{GR}(p^r, m) \longrightarrow \mathbb{F}_{p^m}$$
.

A. Hensel lifting

Hensel lifting is important for both the construction of Galois rings and the factorisation of polynomials over such rings. The setup actually applies to a quite general situation, following [8, Ch. 15]. By a ring we mean a commutative ring with identity.

Elements g,h in a ring R are called *Bézout-coprime* if sg+th=1 for certain $s,t\in R$. Note that for principal ideal domains this definition amounts to the usual notion of coprimeness of having no common factor. But in general it is stronger; in fact, coprime elements need not be Bézout-coprime, consider e.g. $2,X\in \mathbb{Z}_4[X]$ or $X,Y\in \mathbb{F}_2[X,Y]$.

Now let R be a ring and $a \in R$. The basic Hensel step allows to lift a polynomial factorisation over the quotient ring R/(a) to a factorisation over the quotient ring $R/(a^2)$. More precisely, suppose that we have $f^* \in (R/(a^2))[X]$ and $g, h \in (R/(a))[X]$, with h monic, being Bézout-coprime such that

$$f^* \mod a = g \cdot h$$
,

then we find $g^*, h^* \in (R/(a^2))[X]$, with h^* monic, still Bézout-coprime such that

$$f^* = g^* \cdot h^*.$$

Furthermore, given $s,t \in (R/(a))[X]$ such that sg+th=1 we can also compute elements $s^*,t^*\in (R/(a^2))[X]$ satisfying $s^*g^*+t^*h^*=1$. The details are given in Algorithm 1 (cf. [8, Alg. 15.10]).

Applying the algorithm repeatedly, we can lift this way factorisations modulo a to factorisations modulo a^2, a^4, a^8 , etc.

B. Bivariate polynomial factorisation

Due to zero divisors in general rings one cannot expect their polynomials to have as nice factorisation properties and algorithms as in the field case. Simple examples like

Algorithm 1: Hensel step

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Input : f^* \in (R/(a^2))[X] and g,h,s,t \in (R/(a))[X],h monic such that f^* \mod a = gh and sg+th=1

Output: g^*,h^*,s^*,t^* \in (R/(a^2))[X],h^* monic such that f^* = g^*h^* and s^*g^* + t^*h^* = 1

1 coerce g,h,s,t into (R/(a^2))[X]

2 e = f^* - gh

3 q,r = \text{quo\_rem}(se,h) (note that a \mid e,q,r)

4 g^* = g + te + qg; h^* = h + r

5 b = sg^* + th^* - 1

6 c,d = \text{quo\_rem}(sb,h^*) (note that a \mid b,c,d)

7 s^* = s - d; t^* = t - tb - cg^*

8 return g^*,h^*,s^*,t^*
```

 $X \cdot X = (X+2) \cdot (X+2) \in \mathbb{Z}_4[X]$ already show a non-unique factorisation behaviour. However, provided that the factors can be mapped into square-free Bézout-coprime factors, say in the univariate polynomial ring over a field, we are able to obtain a factorisation by Hensel lifting.

For the list decoding at hand we are interested in the factorisation of a bivariate polynomial $Q \in R[X,Y]$ over a Galois ring $R := GR(p^r,m)$. Note that for Hensel lifting one cannot simply use a factorisation as a bivariate polynomial over its residue field $F := \mathbb{F}_{p^m}$, as the prime factors in F[X,Y] will usually not be Bézout-coprime.

In the following we describe an adaption of the Zassenhaus factorisation method (cf. [8, Sec. 15.6]) to work over Galois rings. The idea is to use Hensel lifting on two levels, first for lifting a univariate polynomial factorisation in F[X] to R[X], and then for lifting the factorisation in $(R[Y]/(Y-u))[X] \cong R[X]$ to one in $(R[Y]/(Y-u)^{\ell})[X]$, from which we may deduce the factorisation in R[X,Y]. Details follow.

Algorithm 2: Bivariate polynomial factorisation

Input: $Q \in R[X,Y]$ Output: factorisation $Q = Q_1 \cdot ... \cdot Q_t$ into irreducibles 1 choose $u \in R$ such that $\mu Q(X,u) \in F[X]$ is square-free

- 2 factorise $\mu Q(X,u) = dg_1 \cdot ... \cdot g_s$ over F
- 3 use Hensel lifting to obtain $Q(X, u) = cf_1 \cdot ... \cdot f_s$ over R
- 4 use Hensel lifting to obtain $\overline{Q} = CF_1 \cdot ... \cdot F_s$ over $R[Y]/(Y-u)^\ell$
- 5 combine factors to obtain $Q = Q_1 \cdot ... \cdot Q_t$ over R[Y]
 - 1) Given $Q \in R[X,Y]$, we compute a univariate polynomial $Q_u := Q(X,u) \in R[X]$ for some $u \in R$ and consider its reduction $\overline{Q}_u := \mu(Q_u) \in F[X]$, which we can factorise by classical methods. If the polynomial \overline{Q}_u is square-free, so that there are no repeated factors, we proceed; otherwise we try a different choice of u.

- 2) In the prime factorisation $\overline{Q}_u = dg_1 \cdot ... \cdot g_s$ over F (with $d \in F$ and the g_i monic), products of distinct factors are coprime and thus Bézout-coprime, since F[X] is a principal ideal domain. Thus we may apply multifactor Hensel lifting (cf. [8, Alg. 15.17]) using the basic Hensel step (Algorithm 1) to obtain a factorisation of $Q_u = cf_1 \cdot ... \cdot f_s \in R[X]$ into irreducibles, where $\mu f_i = g_i$ for all i.
- 3) Given a factorisation of $Q_u = cf_1 \cdot ... \cdot f_s$ in R[X] into distinct Bézout-coprime irreducibles (with $c \in R$ and the f_i monic), we apply Hensel lifting over the polynomial ring R[Y] using the modulus a := Y u to arrive at a factorisation $\overline{Q} = CF_1 \cdot ... \cdot F_s$ in $(R[Y]/(Y-u)^{\ell})[X]$ for some large enough ℓ .
- 4) We combine the factors C, F_1, \ldots, F_s into products Q_1, \ldots, Q_t (with $t \leq s$) such that $Q = Q_1 \cdot \ldots \cdot Q_t$ holds in R[X,Y], a step which is necessary as the quotient ring may introduce additional factors. Since there is a bound in the Y-degree of the coefficients of the actual factors Q_i , we can find them by computing products of $1,2,3,\ldots$ factors until the bound is satisfied.

We summarise our bivariate polynomial factorisation method in Algorithm 2. Notice that all steps are polynomial time except possibly for the last combine-factors step, which however seems to be very efficient in practice. We leave a more thorough study of the factoring algorithm for future work.

III. LIST DECODING OF REED-SOLOMON CODES OVER RINGS

List decoding of Reed-Solomon and related codes over Galois rings has been considered by Armand [1], [2], while our setup and factorisation algorithm is slightly different. See also [10] for list decoding of Reed-Solomon codes over more general rings. We briefly present here the main concepts for Galois rings, as required subsequently.

Let $R:=\operatorname{GR}(p^r,m)$ be a Galois ring of characteristic p^r and degree m, and let $\vartheta\in R$ be an element of multiplicative order p^m-1 . Such an element can be obtained by taking the defining polynomial $f\in\mathbb{Z}_{p^r}[X]$ of R to be the Hensel lift of a primitive polynomial over \mathbb{Z}_p of degree m and then letting ϑ be the class of X modulo f.

The set $\mathcal{T} := \{\vartheta^i \mid 0 \leq i < p^m - 1\} \cup \{0\} \subseteq R$ then maps bijectively onto the residue field \mathbb{F}_{p^m} under the canonical map μ , and is called *Teichmüller set*.

Definition 1. Given $n \leq p^m$ and $1 \leq k \leq n$ as well as $\alpha_1, \ldots, \alpha_n \in \mathcal{T}$ distinct, we define the [n, k] Reed-Solomon code over R as the evaluation code

$$\mathcal{C} := \{ \operatorname{ev}(f) := (f(\alpha_1), \dots, f(\alpha_n)) \mid f \in R[X], \operatorname{deg} f < k \}.$$

Lemma 2. The (Hamming) minimum distance of C equals d := n - k + 1, thus the code is maximum distance separable.

Proof. Suppose that $c = \text{ev}(f) \in C$ has weight < d, so that $f \in R[X]$ has at least k zeros, say (w.l.o.g.) $\alpha_1, \ldots, \alpha_k$.

Writing $f = \sum_{i=0}^{k-1} f_i X^i$ it follows that $(f_0, \dots, f_{k-1})V = 0$ for the Vandermonde matrix $V := (\alpha_j^{\ i})_{ij}$, with

$$\det V = \prod_{i < j} (\alpha_j - \alpha_i)$$

a unit in R, since $\mu \det V = \prod_{i < j} (\mu \alpha_j - \mu \alpha_i) \neq 0$. Therefore, f = 0 and thus c = 0.

We can in fact correct error weights beyond half the minimum distance by adapting the list decoding approach by Sudan [11], as described next. It consists of an interpolation step which produces a bivariate polynomial $Q \in R[X,Y]$, and a factorisation step using Algorithm 2 by which the codewords within the list decoding radius are obtained.

For the interpolation step we fix a finite set S of indices (i,j) describing terms X^iY^j , and require that S has more than n elements. Given a received word $y \in R^n$ we consider the interpolation problem

$$Q(\alpha_i, y_i) = 0$$
 for $1 \le i \le n$,

where $Q = \sum_{(i,j) \in S} c_{ij} X^i Y^j \in R[X,Y]$ with the coefficients $c_{ij} \in R$ to be determined. This is a linear system with more variables than equations and thus contains a nonzero solution. Such a solution can be obtained using Smith normal form, which is available over Galois rings as these are chain rings (cf. [6, Sec. 2-D]).

Concretely, for a list error radius t we let

$$S := \{(i, j) \mid i + (k-1)j \le n - t\}.$$

The next result shows that the interpolation polynomial Q carries information on all codewords within distance $\leq t$.

Lemma 3. Suppose that y = c + e with $c = \operatorname{ev}(f) \in \mathcal{C}$ for $f \in R[X]$, $\deg f < k$, and e an error vector of weight $\leq t$. Then Q(X, f) = 0.

Proof. Considering $h := Q(X,f) = \sum_{(i,j) \in S} c_{ij} X^i f^j \in R[X]$, then since $\deg f < k$ and by the definition of S, we see that $\deg h \le n-t$. On the other hand, we have $y_i = c_i = f(\alpha_i)$ and thus $h(\alpha_i) = Q(\alpha_i, f(\alpha_i)) = Q(\alpha_i, y_i) = 0$ whenever $e_i = 0$, so for at least n-t values α_i . As in the proof of Lemma 2 we can use the Vandermonde determinant and the fact that the $\mu\alpha_i$ are distinct to deduce that h = 0. \square

Example 4. Consider the [64,6] Reed-Solomon code over $R := \mathrm{GR}(4,6)$ defined by the full Teichmüller set. While the minimum distance is d=59 by Lemma 2 and thus the unique decoding radius is 29, we can list decode up to radius t=41. Indeed, the set $S := \{(i,j) \mid i+5j \leq 23\}$ is of cardinality 65, so we can conpute an interpolation polynomial Q and in light of Lemma 3 find the list of codewords by factorising this bivariate polynomial using Algorithm 2.

A. Multiplicities

We can also apply the Guruswami-Sudan list decoding approach [7] incorporating multiplicities to the present situation. For this we alter the interpolation step such that every (α_i, y_i) should be a zero $Q \in R[X,Y]$ with multiplicity e, which

means that for $Q(X+\alpha_i,Y+y_i)$ every coefficient of X^iY^j with i+j< e vanishes. This amounts to $\frac{1}{2}e(e+1)$ linear conditions for each point, so that we require the set S to have more than $\frac{1}{2}e(e+1)n$ elements. But now, if there are t errors, the polynomials f for every codeword within this radius the polynomial $h:=Q(X,f)\in R[X]$ has n-t roots with multiplicity at least e, i.e., $(X-\alpha)^e\mid h$. This forces h to be zero, provided that we take $S:=\{(i,j)\mid i+(k-1)j\leq e(n-t)\}$, as the next result shows.

Lemma 5. Let $h \in R[X]$ be a polynomial of degree < ke such that $(X-\alpha_i)^e \mid h$ for at least k distinct $\alpha_i \in \mathcal{T}$. Then h equals zero.

Proof. Over the residue field we have $(X-\mu\alpha_i)^e \mid \mu h$, where the $\mu\alpha_i \in F$ are distinct, hence we can argue by degrees to deduce $\mu h = 0$. Therefore, $h = p\tilde{h}$ and we may view the polynomial \tilde{h} over $\mathrm{GR}(p^{r-1},m)$ with $\deg \tilde{h} = \deg h$ and still have $(X-\mu\alpha_i)^e \mid \mu \tilde{h}$ over the corresponding residue field. Continuing this way, we see that h = 0.

Example 6. Consider the [64,6] Reed-Solomon code over $R := \mathrm{GR}(4,6)$ from Example 4. Using multiplicity e=2, i.e., double roots, we can now list decode up to radius t=43. We take a set $S:=\{(i,j)\mid i+5j\leq 42\}$ of cardinality $198>3\cdot 64$, which guarantees the existence of an interpolation polynomial Q. Then every codeword polynomial f within the decoding radius satisfies Q(X,f)=0, so we can find these again by factoring Q with Algorithm 2.

IV. A JUMP INTO THE BYRNE-FITZPATRICK ALGORITHM

The Byrne-Fitzpatrick algorithm [3], [4] can be used to solve key equations over Galois rings R. It considers for $U \in R[Z]$ the solution modules

$$\mathcal{M}_k := \{ (f, g) \in R[Z]^2 \mid Uf \equiv g \bmod Z^k \},$$

and given a Gröbner basis for \mathcal{M}_k refines it to a Gröbner basis for \mathcal{M}_{k+1} . The method is dubbed "solution by approximations" and is reminiscent of the Berlekamp-Massey algorithm. Recently, the algorithm was adapted to work also over skew polynomials over Galois rings [9].

More specifically, one considers a term order on the set of terms $(Z^j,0)$ and $(0,Z^j)$, $j=0,1,2,\ldots$, so that leading term and leading monomial of a nonzero pair $(f,g)\in R[Z]^2$ are defined. Given a module $\mathcal{M}\subseteq R[Z]^2$, a set $\mathcal{B}\subseteq \mathcal{M}$ is called *Gröbner basis* for \mathcal{M} if for each $(f,g)\in \mathcal{M}$ there exists a Gröbner basis element such that its leading monomial divides the leading monomial of (f,g).

Now given a Gröbner basis \mathcal{B}_k for \mathcal{M}_k , to construct a Gröbner basis \mathcal{B}_{k+1} for \mathcal{M}_{k+1} one computes for each $(f_i, g_i) \in \mathcal{B}_k$ the *discrepancy*

$$\zeta_i := (Uf_i - g_i)_k \in R$$
,

where the subscript denotes the k-th coefficient. Then if $\zeta_i=0$ we put (f_i,g_i) into \mathcal{B}_{k+1} . Otherwise, we look for some $(f_j,g_j)\in\mathcal{B}_k$ with smaller leading term such that $\zeta_j\mid\zeta_i$, say $\zeta_i=q\zeta_j$ for some $q\in R$, in which case we put

 $(f_i, g_i) - q(f_j, g_j)$ into \mathcal{B}_{k+1} ; if there is none, we put $Z(f_i, g_i)$ into \mathcal{B}_{k+1} .

In order to adapt the list decoding approach by Wu [13], we are interested in the following question: Given a Gröbner basis for \mathcal{M}_k , how can we construct a Gröbner basis for $\mathcal{M}_{k+\ell}$ with $\ell > 1$? Thus we deal with a "jump" in the solution-by-approximations method. At this point we are unable to fully solve this problem, but we outline a method to construct elements in $\mathcal{M}_{k+\ell}$ satisfying certain degree constraints, which will be sufficient for the list decoding approach.

Lemma 7. Given Gröbner basis elements (f_i, g_i) and (f_j, g_j) in \mathcal{B}_k with leading coefficient a unit in R, there exist polynomials $a, b \in R[Z]$ with unit leading coefficient and $\deg a + \deg b \leq \ell$ such that $a(f_i, g_i) - b(f_j, g_j) \in \mathcal{M}_{k+\ell}$.

Proof. To address the problem at hand, we introduce for $(f_i, g_i) \in \mathcal{B}_k$ the discrepancy polynomials

$$h_i := \sum_{\lambda=0}^{\ell-1} (Uf_i - g_i)_{k+\lambda} \in R[Z].$$

Then we look for an expression $ah_i - bh_j = 0 \mod Z^\ell$ for some $a,b \in R[Z]$ of low degree, in which case we have $a(f_i,g_i)-b(f_j,g_j) \in \mathcal{M}_{k+\ell}$. Such a pair (a,b) can in turn be found by computing a Gröbner basis for the solution module

$$\mathcal{N} := \{ (a, b) \in R[Z]^2 \mid ah_i - bh_i \equiv 0 \bmod Z^{\ell} \}$$

by a (slight adaption) of the Byrne-Fitzpatrick algorithm.

In the decoding scenario described in the next section we do not know the discrepancy polynomials and thus cannot compute the polynomials a and b directly, but we are able to deduce these by a list decoding approach.

V. LIST DECODING OF QUATERNARY NEGACYLIC CODES

Let n>1 be an odd integer. By a quaternary negacyclic code of length n we mean an ideal in the ring $\mathbb{Z}_4[X]/(X^n-1)$. Such codes have been investigated by Wolfman [12], who examined their structure. We equip the ring \mathbb{Z}_4 with the Lee weight $w(x) := \min(x, 4-x)$ and build upon the algebraic decoding of Lee errors [5]. Notice that the map induced by $X\mapsto -X$ sends any negacyclic code isometrically onto a cyclic code, though the negacyclic representation offers some advantage regarding decoding.

As in the case of BCH codes we can specify negacyclic codes in terms of roots. For this we choose a Galois ring $R:=\mathrm{GR}(4,m)$ such that $n\mid 2^m-1$ together with an element ϑ of order 2^m-1 (see Section III). Then there exists an element β of order n and we fix a root $\alpha:=-\beta$ of order 2n, satisfying $\alpha^n=-1$.

Definition 8. The quaternary negacyclic code with t roots $\alpha, \alpha^3, \dots, \alpha^{2t-1}$ is given by

$$\mathcal{C} := \left\{ c \in \mathbb{Z}_4[X]/(X^n + 1) \mid c(\alpha^{2i - 1}) = 0 \text{ for } 1 \leq i \leq t \right\}.$$

It is shown [5, Thm. 1] that the code \mathcal{C} has minimum Hamming distance $\geq 2t+1$, so this clearly holds for the minimum Lee distance, too. An algebraic decoding method for

TABLE I PARAMETERS OF NEGACYCLIC CODES OF LENGTH n, DESIGNED ERROR-CORRECTING CAPABILITY t, AND RANK k (i.e., size 4^k).

n	t	k	2t+1	d_{Lee}
15	1	11	3	3
	2	7	5	5
	3	5	7	10
31	1	26	3	4
	2	21	5	7
	3	16	7	12
	5	11	11	16
	7	6	15	26

errors up to Lee weight t was devised [5], based on a Gröbner basis algorithm by Byrne and Fitzpatrick [4]. However, it has been observed that many such codes have a larger minimum Lee distance than 2t+1 (see Table I), which motiviates a list decoding approach.

A. The key equation

Here we adjust the list decoding method of Wu [13] to the present situation. The central idea of this algorithm is to start with a Berlekamp-Massey solution to the key equation, and to "refine" it afterwards by formulating a list decoding problem. We recall therefore the key equation for negacyclic codes and outline its algebraic decoding.

For an error vector $e \in \mathbb{Z}_4[X]/(X^n+1)$ we define the error locator polynomial

$$\sigma := \prod_{i=0}^{n-1} (1 - X_i Z)^{w(e_i)} \in R[Z],$$

with $X_i := \alpha^{-i}$ if $e_i = 1, 2$ and $X_i := -\alpha^{-i}$ if $e_i = 3$, so that $(1-X_iZ)^{w(e_i)}$ equals $1-\alpha^iZ$ if $e_i = 1$, $(1\pm\alpha^iZ)^2$ if $e_i = 2$ and $1+\alpha^iZ$ if $e_i = 3$. Then the error pattern is completely determined by the roots α^i for $0 \le i < 2n$ of σ .

We also let the syndrome polynomial be $s:=\sum_{i=1}^t y(\alpha^{2i-1})Z^{2i-1}=\sum_{i=1}^t e(\alpha^{2i-1})Z^{2i-1}\in R[Z],$ which is known to the decoder. This polynomial determines an odd polynomial $u:=\sum_{i=1}^t u_{2i-1}Z^{2i-1}$ by the equation $s(u^2-1)=Zu',$ which in turn defines a polynomial $T:=\sum_{i=1}^t T_iZ^i$ by the relation $(1+T(Z^2))(1+Zu)\equiv 1\pmod{Z^{2t+2}}.$ We arrive at the key equation

$$(1+T)\varphi \equiv \omega \pmod{Z^{t+1}},$$

from which we recover the even and the odd part of the error locator polynomial σ by $\omega(Z^2) = \sigma_e, \ \varphi(Z^2) = \sigma_e + Z\sigma_o$.

This key equation can be solved by considering the solution module \mathcal{M}_{t+1} as in Section IV with U := 1+T and employing the Gröbner basis approach [5].

B. List decoding

Even though we only know the syndrome polynomial up to degree 2t-1 and thus the polynomial T in the key equation up to degree t, we would like to correct more than t errors. For this we pretend that we actually have access to more

syndromes and presume that we can set up a key equation modulo $Z^{t+1+\ell}.$

Suppose that (φ_i, ω_i) and (φ_j, ω_j) have been computed as Gröbner basis elements for \mathcal{M}_k , then according to Lemma 7 there is a solution (Φ, Ω) for $\mathcal{M}_{k+\ell}$ such that $a\varphi_i - b\varphi_j = \Phi$ and $a\omega_i - b\omega_j = \Omega$. Hence, for the error locator polynomial there holds

$$\Sigma = \Sigma_e + \Sigma_o = \Omega(Z^2) + \frac{1}{Z}(\Phi(Z^2) - \Omega(Z^2))$$

$$= a(\omega_i(Z^2) + \frac{1}{Z}(\varphi_i(Z^2) - \omega_i(Z^2))$$

$$- b(\omega_j(Z^2) + \frac{1}{Z}(\varphi_j(Z^2) - \omega_j(Z^2))$$

$$= a\sigma_i - b\sigma_j.$$

Therefore, whenever $\Sigma(\gamma) = 0$ then

$$\frac{\sigma_j}{\sigma_i}(\gamma) = \frac{a}{b}(\gamma) \,,$$

which holds for the $\tau > t$ roots γ of Σ . This is a rational approximation problem: We look for a rational function of small degree that interpolates at least τ out of 2n given values. Such kind of problem has been addressed by the list decoding algorithm by Wu [13], which we adapt by our list decoding algorithm of Section III.

More precisely, in the list decoding setup we have the 2n evaluation points α^i for $0 \le i < 2n$, and we have τ error positions γ for which

$$(a\sigma_i - b\sigma_i)(\gamma) = 0,$$

with $a,b\in R[Z]$ unknown of degree $\leq \frac{\ell}{2}$, where $\ell:=\tau-t$. In the context of Section III this corresponds to an evaluation code of length 2n, rank $\ell+1$ and with $2n-\tau$ "errors". Taking into account the particular form of the factors, a suitable set of indices is

$$S := \left\{ (i, j) \mid \max\{i, \lfloor \frac{\ell}{2} \rfloor j\} \le \frac{\tau}{2} \right\},\,$$

which we require to have more than 2n elements in order to solve the interpolation step (in the single-multiplicity e=1 case). One technical difficulty arising is that $\frac{\sigma_j}{\sigma_i}(\gamma)$ might be infinity, in which case, following Wu [13], rather than $Q(\gamma,\infty)=0$ we impose the linear condition $\tilde{Q}(\gamma,0)=0$ with $\tilde{Q}:=Q(X,\frac{1}{\nabla})Y^d$ the Y-reverse polynomial of Q.

The bivariate polynomial $Q \in R[X,Y]$ has then the property that the degree of the numerator of $Q(X,\frac{a}{b})$ is at most τ . Suppose for now that the $\mu\gamma_j$ are distinct for the τ error positions $\gamma_i \in \langle \alpha \rangle$ (no "double error" occurs). In that case we infer that $Q(X,\frac{a}{b})=0$ by a similar proof as Lemma 3, and hence we have a factor $bY-a\mid Q$.

In the general case, following the strategy in [5, Sec. 7], we consider the reduction modulo the residue field F and have τ error locations (possibly with multiplicity) $\mu\gamma$ such that

$$\frac{\mu \sigma_j}{\mu \sigma_i}(\mu \gamma) = \frac{\mu a}{\mu b}(\mu \gamma).$$

Using the list decoding algorithm over fields by Wu [13], we can find μa and μb . Thus we compute $\mu \Sigma$, by which we deduce all double errors $e_i=2$ by its double roots. Then we subtract a vector consisting of only those double roots, by which we are

able to reduce the decoding problem to the first case without double errors.

Observe that the cardinality of the set S equals

$$(\lfloor \frac{\tau}{2} \rfloor + 1)(\lfloor \frac{\tau}{2|\ell/2|} \rfloor + 1)$$
,

which exceeds 2n provided that $\tau^2 > 4n\ell = 4n(\tau-t)$. Incorporating sufficient large multiplicities e, as in Section III-A, it suffices to require $\tau^2 > 2n(\tau-t)$, or $\tau < n-\sqrt{n(n-2t)}$. Therefore, we arrive at the following result.

Theorem 9. For a quaternary negacyclic code of length n with t roots and designed distance d=2t+1, the proposed list decoding algorithm corrects all codewords within radius τ from the received word, provided that

$$\tau < n - \sqrt{n(n-d)}$$
.

Example 10. Let n=63 and consider a quaternary negacyclic code with t=16 roots and designed distance 2t+1=33. The algebraic decoding method [5] is thus able to correct up to 16 Lee weight errors. With our list decoding approach (with multiplicity e=2) we can correct however up to $\tau=19$ Lee errors. For this we let $S:=\{(i,j)\mid i,j\leq 19\}$ of size $400>3\cdot 2n$, so we can construct a bivariate interpolation polynomial $Q\in R[X,Y]$ of max-degree 19. Provided that no double error occured, by factorising Q using Algorithm 2 and looking for factors bY-a with $\deg a, \deg b\leq 1$, we are able to solve the list decoding problem. Otherwise, we employ the strategy outlined above.

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