# Gaussian Data Privacy Under Linear Function Recoverability 

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#### Abstract

A user's data is represented by a Gaussian random variable. Given a linear function of the data, a querier is required to recover, with at least a prescribed accuracy level, the function value based on a query response provided by the user. The user devises the query response, subject to the recoverability requirement, so as to maximize privacy of the data from the querier. Recoverability and privacy are both measured by $\ell_{2}$-distance criteria. An exact characterization is provided of maximum user data privacy under the recoverability condition. An explicit optimal achievability scheme for the user is given whose privacy is shown to match a converse upper bound.


## Index Terms

Gaussian data privacy, linear function computation, query response, recoverability

## I. Introduction

A (legitimate) user's data is represented by a Gaussian random variable (rv) and a querier wishes to compute a given linear function of the data from a query response provided by the user. The query response is a suitably randomized version of the data which the user constructs so as to enable the querier to recover the function value with a prescribed accuracy. Under this recoverability requirement, the user wishes to maximize privacy of the data from the querier. Both recoverability and privacy are measured by $\ell_{2}$-distance criteria.

The contributions of this paper are as follows. In our formulation, the user-provided query response should be such that its expected $\ell_{2}$-distance from the function value is no greater than $\rho, \rho \geq 0$. Under this recoverability constraint, we consider a notion of privacy measured by the expected $\ell_{2}$-distance between the user data and the querier's best estimate of it based on the query response, i.e., the corresponding minimum mean-square estimation error (MMSE). We provide an exact characterization of maximum privacy under the $\rho$-recoverablity requirement as a function of $\rho$, and specify an explicit query response that attains it. This maximum privacy is shown to be a nondecreasing piecewise affine function of $\rho$, and depends on the linear mapping only through its rank and singular values. The user implements the optimal query response by attenuating the function value and adding to it a suitable independent Gaussian noise. This query response constitutes a multidimensional extension of a scheme in [22] in the separate context of maximizing the MMSE in estimating a one-dimensional Gaussian rv on the basis of a one-dimensional randomized version of it under a constraint on the expected $\ell_{2}$-distance between the input and the randomized output, where the maximization is over all possible randomization mechanisms satisfying the constraint.

This work is motivated by applications involving analog user data in a database which then must release functional information to a querying consumer. An important goal is to preserve user data privacy while ensuring high accuracy of the released information. For example, when analog biometric data, such as fingerprint or voice patterns, are collected and certain attributes are released, an objective is to safeguard the privacy of individual user data while ensuring high utility of the released information.

Our approach is in the spirit of prior works [21], [13], [4], [12], [16], [17], [18] that deal with maximizing data privacy for a given level of utility of a randomized function of the data. Specifically, for example, in [21], [13], [4], a setting where a user possesses private finite-valued data with associated nonprivate correlated data is considered. A randomized version of the nonprivate data is released publicly. The utility constraint is that the expected distortion

[^0]between the nonprivate and public data should be no more than a specified level. The public data is designed by the user in such a way that privacy measured in terms of the mutual information between the private and public data is maximized. Next, in [16], [17], user data is represented by a finite-valued rv and a querier which wishes to compute a given function of user data. Privacy is gauged in terms of probability of error incurred by the querier in estimating the user data based on a query response that is a user-provided randomized version of the data. Privacy is maximized under a constraint on the utility of the query response as measured by a conditional probability of error criterion. In all these works, the data that is considered is of the digital type. Our work is motivated by applications involving analog data. This work is a preliminary Gaussian counterpart of our earlier works on data privacy under function recoverability for a finite-valued rv [16], [17]; notions of privacy and recoverability are different, as is the technical approach.

Related works in [1], [2] deal with private and nonprivate correlated data with a given joint distribution. In [1], a Gaussian noise independent of the private and nonprivate data is added to the latter and released publicly. The parameters of the Gaussian noise are obtained as a result of minimizing the MMSE in estimating the nonprivate data from the public data. This minimization is done under a constraint on the MMSE in estimating the private data from the public data. In [2], first a Gaussian noise independent of the private and nonprivate data is added to the latter, and the sum is quantized and released publicly. In this case, the parameters of the Gaussian noise are obtained as a result of maximizing the mutual information between the public and nonprivate data under a constraint on the mutual information between the public and private data. In another related work in [5], motivated by MMSE as a measure of information leakage, a neural network-based estimator of MMSE is characterized. These works involve maximizing recoverability under a privacy constraint. In contrast, our goal is to maximize privacy under a recoverability constraint.

An important movement in data privacy that has dominated attention over the years is differential privacy, introduced in [6], [7] and explored further in [14], [3], [11], [15], among others. Consider a data vector that represents multiple users' data. The notion of differential privacy requires that altering a data vector slightly leads only to a near-indistinguishable change in the corresponding probability distribution of the output of the data release mechanism, which is a randomized function of the data vector. A large body of work exists that seeks to maximize function recoverability under a differential privacy constraint, by minimizing a discrepancy cost between function value and randomized output; cf. e.g., [10], [8], [9]. Our alternative approach, i.e., maximizing privacy under a recoverability constraint, can be viewed as a complement to this body of work.

Our model for $\rho$-recoverable linear function computation with associated privacy is described in Section $\Pi$ and the derivation of $\rho$-privacy is given in Section [III. A closing discussion is contained in Section IV]

## II. Preliminaries

Let a user's data be represented by a $\mathbb{R}^{n}$-valued redundant Gaussian rv $X \sim \mathcal{N}\left(\mathbf{0}, I_{n}\right)$ with covariance matrix $I_{n}, n \geq 1$, the identity matrix of size $n$. A querier - who does not know $X$ - wishes to compute a given linear function of the user data $A X$, where $A \in \mathbb{R}^{m \times n}$ and has rank $1 \leq r \leq \min \{m, n\}$. The querier obtains from the user a query response $(\mathrm{QR}) Z$ that is a $\mathbb{R}^{m}$-valued rv generated by a conditional distribution $P_{Z \mid X}$. A $\mathrm{QR} Z$ must satisfy the following recoverability condition.

Definition 1. Given $\rho \geq 0$, a QR $Z$ is $\rho$-recoverable if

$$
\begin{equation*}
\mathbb{E}\left[\|A X-Z\|^{2}\right] \leq \rho \tag{1}
\end{equation*}
$$

where the expectation is with respect to the distribution of $(X, Z)$. Such a $\rho$-recoverable $Z$ will be termed $\rho$-QR. We note that the distribution of $Z$ will depend, in general, on $\rho$.

Definition 2. Given $\rho \geq 0$, the privacy of a $\rho$ - $\mathrm{QR} Z$ satisfying (1) is

$$
\begin{equation*}
\pi_{\rho}(Z) \triangleq \inf _{g} \mathbb{E}\left[\|X-g(Z)\|^{2}\right]=\operatorname{mmse}(X \mid Z) \tag{2}
\end{equation*}
$$

where the infimum is taken over all (Borel) measurable estimators $g: \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ of $X$ on the basis of $Z$. Clearly, the infimum in (2) is attained by an MMSE estimator so that $\pi_{\rho}(Z)$ is mmse $(X \mid Z)$, where $\mathrm{mmse}(X \mid Z)$ denotes the MMSE in estimating $X$ on the basis of $Z$.

Definition 3. Given $\rho \geq 0$, the maximum privacy that can be attained by a $\rho$ - $\mathrm{QR} Z$ is termed $\rho$-privacy and denoted by $\pi(\rho)$, i.e.,

$$
\begin{equation*}
\pi(\rho) \triangleq \sup _{P_{Z \mid X}: \mathbb{E}\left[\|A X-Z\|^{2}\right] \leq \rho} \pi_{\rho}(Z) \tag{3}
\end{equation*}
$$

Our objective is to characterize $\rho$-privacy $\pi(\rho), \rho \geq 0$, and identify a $\rho$-QR that achieves it. This objective is addressed in Section III.

## III. $\rho$-Privacy

Theorem 1 below provides an exact characterization of $\rho$-privacy in (3). This is done by obtaining first an upper bound (converse) for $\pi(\rho), \rho \geq 0$, and then identifying a $\rho$ - QR whose privacy meets the bound (achievability).

Throughout the rest of the paper, we consider a particular singular value decomposition of ${ }^{11}$

$$
\begin{equation*}
A=U S V^{T} \tag{4}
\end{equation*}
$$

where $U$ and $V$ are, respectively, $m \times m$ - and $n \times n$-orthonormal matrices. In (4), the $m \times n$ matrix $S$ containing the singular values of $A$ and represented by

$$
S_{i j}=\left\{\begin{array}{ll}
s_{k}, & i=j=k,  \tag{5}\\
0, & \text { otherwise },
\end{array} \quad i \in\{1, \ldots, r, ~(1, \ldots, m\}, j \in\{1, \ldots, n\},\right.
$$

is such that

$$
\begin{equation*}
0<s_{1} \leq \cdots \leq s_{r} \tag{6}
\end{equation*}
$$

where $s_{1}, \ldots, s_{r}$ are the nonzero singular values of $A$. Let $\tilde{S}$ be the $r \times n$-matrix consisting of the first $r$ rows of $S$; the remaining $m-r$ rows of $S$ are all-zero rows.

Recalling that $U$ and $V$ are orthonormal matrices and using (4), (5), standard calculations, repeated in Appendix A for the sake of completeness, show that $\operatorname{var}(A X)=\operatorname{tr}\left(A A^{T}\right)=\sum_{i=1}^{r} s_{i}^{2}$ and

$$
\begin{equation*}
\operatorname{mmse}(X \mid A X)=n-r . \tag{7}
\end{equation*}
$$

Theorem 1. $\rho$-privacy equals

$$
\begin{equation*}
\pi(\rho)=n-r+\min \left\{\frac{\rho}{s_{1}^{2}}, 1+\frac{\rho-s_{1}^{2}}{s_{2}^{2}}, \ldots, r-1+\frac{\rho-\sum_{i=1}^{r-1} s_{i}^{2}}{s_{r}^{2}}, r\right\}, \quad \rho \geq 0 \tag{8}
\end{equation*}
$$

## Remarks:

(i) By Theorem 1 and (6),

$$
\pi(\rho)= \begin{cases}n-r+\frac{\rho}{s_{1}^{2}}, & 0 \leq \rho \leq s_{1}^{2}  \tag{9}\\ n-r+1+\frac{\rho-s_{1}^{2}}{s_{2}^{2}}, & s_{1}^{2} \leq \rho \leq s_{1}^{2}+s_{2}^{2} \\ \vdots & \vdots \\ n-r+r-1+\frac{\rho-\sum_{i=1} s_{i}^{2}}{s_{r}^{2}}, & \sum_{i=1}^{r-1} s_{i}^{2} \leq \rho \leq \sum_{i=1}^{r} s_{i}^{2} \\ n, & \rho \geq \sum_{i=1}^{r} s_{i}^{2},\end{cases}
$$

is piecewise affine in $\rho$. For example, a plot of $\pi(\rho)$ vs. $\rho$ is given in Fig. 1 for $n=5, m=r=3, s_{1}=$ $2, s_{2}=3$ and $s_{3}=4$.

[^1](ii) In particular, for $\rho=0, \pi(\rho)=n-r$ which, from (7), is the error of an MMSE estimator of $X$ on the basis of $A X$. For $\rho \geq \sum_{i=1}^{r} s_{i}^{2}=\operatorname{tr}\left(A A^{T}\right)=\operatorname{var}(A X), \pi(\rho)=n=\operatorname{var}(X)$ is the error of a MMSE estimator of $X$ without any observation.
(iii) $\rho$-privacy $\pi(\rho), \rho \geq 0$, is a nonincreasing function of each individual singular value when the remaining $r-1$ singular values are fixed.


Fig. 1: $\pi(\rho)$ vs. $\rho$.

The following Lemmas 2 and 3 are pertinent to the proof of Theorem 1 Their proofs are relegated to Appendix B
Lemma 2. For $\rho \geq 0$,

$$
\begin{equation*}
\sup _{P_{Z \mid X}: \mathbb{E}\left[\|A X-Z\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(Z)\|^{2}\right]=\sup _{P_{\bar{Z} \mid X}: \mathbb{E}\left[\left\|S V^{T} X-\bar{Z}\right\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(\bar{Z})\|^{2}\right], \tag{10}
\end{equation*}
$$

where the $\mathbb{R}^{m}$-valued $r v \bar{Z}$ represents a generic stand-in for a $\rho-Q R$ under recoverability of $S V^{T} X$.
Remark: Since $V$ is orthonormal and $X \sim \mathcal{N}\left(\mathbf{0}, I_{n}\right), \bar{X}=V^{T} X$ has the same distribution as $X$. Hence, the right-side of (10) can be interpreted as $\rho$-privacy under recoverability of $S \bar{X}, \bar{X} \sim \mathcal{N}\left(\mathbf{0}, I_{n}\right)$. For reasons of ease of notation, we do not use this observation for the proof of Theorem 1 .

The significance of Lemma 2 is that $\rho$-privacy under recoverability of $A X$ from $Z$ is equivalent to $\rho$-privacy under recoverability of $S V^{T} X$ from $\bar{Z}$. Recalling that $\tilde{S}$ is the $r \times n$-matrix consisting of the $r$ nonzero rows of $S$, the covariance matrix of the $\mathbb{R}^{r}$-valued rv $\tilde{S} V^{T} X$ is a diagonal matrix $\tilde{S} \tilde{S}^{T}$, an observation that facilitates establishing the converse for the proof of Theorem 1 .

The following notation is relevant for the rest of the paper. For given $\mathbb{R}^{m}$-valued rvs $\Phi$ and $\phi$, let the components of $\Phi-\phi$ be denoted by $\left[(\Phi-\phi)_{1}, \ldots,(\Phi-\phi)_{m}\right]^{T}$.
Lemma 3. For $\rho \geq 0$,

$$
\begin{equation*}
\sup _{P_{\bar{Z} \mid X}: \mathbb{E}\left[\left\|S V^{T} X-\bar{Z}\right\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(\bar{Z})\|^{2}\right]=\sup _{\substack{P_{\tilde{Z} \mid X}: \mathbb{E}\left[\left\|\tilde{S} V^{T} X-\tilde{Z}\right\|^{2}\right] \leq \rho \\ \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right] \leq s_{i}^{2}, i=1, \ldots, r}} \inf _{g} \mathbb{E}\left[\|X-g(\tilde{Z})\|^{2}\right], \tag{11}
\end{equation*}
$$

where the $\mathbb{R}^{r}$-valued rv $\tilde{Z}$ represents a generic stand-in for a $\rho-Q R$ under recoverability of $\tilde{S} V^{T} X$ and satisfies the additional constraints given under the supremum in the right-side of (11).

By Lemma 3, we can restrict the class of $\rho$-QRs $\bar{Z}$ for recovering $S V^{T} X$ in Lemma 2 to those specified $\tilde{Z}$ for the recoverability of $\tilde{S} V^{T} X$, as detailed under the supremum in the right-side of (11) without any loss in $\rho$-privacy.

Proof of Theorem [1) Using Lemmas 2 and 3, we get in (3) that

$$
\begin{align*}
\pi(\rho) & =\sup _{P_{Z \mid X}: \mathbb{E}\left[\|A X-Z\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(Z)\|^{2}\right] \\
& =\sup _{P_{\tilde{Z} \mid X}: \mathbb{E}\left[\left\|S V^{T} X-\bar{Z}\right\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(\bar{Z})\|^{2}\right] \\
& =\sup _{\substack{\left.P_{\tilde{Z} \mid X}: \mathbb{E}\left[\left\|\tilde{S} V^{T} X-\tilde{Z}\right\|^{2}\right] \leq \rho \\
\mathbb{E}\left[\mid\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right]^{2}\right] \leq s_{i}^{2}, i=1, \ldots, r}} \inf _{g} \mathbb{E}\left[\|X-g(\tilde{Z})\|^{2}\right] . \tag{12}
\end{align*}
$$

We establish (8), with (12) serving as the springboard. First, we prove a converse proof showing that $\pi(\rho)$ cannot exceed the right-side of (12). Then an achievability proof shows the reverse inequality by identifying an explicit $\rho$ - QR that attains the right-side of (12) as $\rho$-privacy.

Starting with the converse, we have that for every $\mathbb{R}^{r}$-valued rv $\tilde{Z}$, generated according to a $P_{\tilde{Z} \mid X}$ satisfying the constraints in the right-side of (12),

$$
\begin{equation*}
\mathbb{E}\left[\left\|\tilde{S} V^{T} X-\tilde{Z}\right\|^{2}\right] \leq \rho, \quad \mathbb{E}\left[\|\left.\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right] \leq s_{i}^{2}, \quad i=1, \ldots, r, \tag{13}
\end{equation*}
$$

so that

$$
\begin{align*}
& \inf _{g} \mathbb{E}\left[\|X-g(\tilde{Z})\|^{2}\right] \\
& \leq \mathbb{E}\left[\left\|X-V \tilde{S}^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1} \tilde{Z}\right\|^{2}\right]  \tag{14}\\
& \left.=\mathbb{E}\left[\left\|X-V \tilde{S}^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1} \tilde{S} V^{T} X+V \tilde{S}^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1} \tilde{S} V^{T} X-V \tilde{S}^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1} \tilde{Z}\right\|^{2}\right]\right] \\
& =\mathbb{E}\left[\left\|X-V \tilde{S}^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1} \tilde{S} V^{T} X\right\|^{2}\right] \\
& \left.+\mathbb{E}\left[\left\|V \tilde{S}^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1} \tilde{S} V^{T} X-V \tilde{S}^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1} \tilde{Z}\right\|^{2}\right]\right] \\
& +2 \mathbb{E}\left[X^{T}\left(I_{n}-V \tilde{S}^{T}\left(\tilde{S} \tilde{S} \tilde{S}^{T}\right)^{-1} \tilde{S} V^{T}\right)^{T} V \tilde{S}^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1}\left(\tilde{S} V^{T} X-\tilde{Z}\right)\right] \\
& =\operatorname{mmse}\left(X \mid \tilde{S} V^{T} X\right)+\mathbb{E}\left[\left\|V \tilde{S}^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1}\left(\tilde{S} V^{T} X-\tilde{Z}\right)\right\|^{2}\right]  \tag{15}\\
& +2 \mathbb{E}\left[X^{T}\left(\tilde{S} V^{T}-\tilde{S} V^{T} V \tilde{S}^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1} \tilde{S} V^{T}\right)^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1}\left(\tilde{S} V^{T} X-\tilde{Z}\right)\right] \\
& =\operatorname{mmse}\left(X \mid U^{T} A X\right)+\mathbb{E}\left[\left\|\tilde{S}^{T}\left(\tilde{S} \tilde{S}^{T}\right)^{-1}\left(\tilde{S} V^{T} X-\tilde{Z}\right)\right\|^{2}\right]  \tag{16}\\
& =\operatorname{mmse}(X \mid A X)+\sum_{i=1}^{r} \frac{1}{s_{i}^{2}} \mathbb{E}\left[\|\left.\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right]  \tag{17}\\
& =n-r+\sum_{i=1}^{r} \frac{1}{s_{i}^{2}} \mathbb{E}\left[\|\left.\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right], \tag{18}
\end{align*}
$$

where: (14) uses the MMSE estimator of $X$ on the basis of $\tilde{S} V^{T} X$, leading to the first term in (15); the first term in (16) is obtained since $\tilde{S}$ is the matrix consisting of the nonzero rows of $S$ and (4); the second terms in (16)
and (17), and the first term in (18), are obtained by the orthonormality of $V$, (5) and (7), respectively. Considering the second term in the right-side of (18), for $t=0,1, \ldots, r$,

$$
\begin{align*}
& \sum_{i=1}^{r} \frac{1}{s_{i}^{2}} \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right] \\
& =\sum_{i=1}^{t} \frac{1}{s_{i}^{2}} \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right]+\sum_{i=t+1}^{r} \frac{1}{s_{i}^{2}} \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right]  \tag{19}\\
& \leq \sum_{i=1}^{t} \frac{1}{s_{i}^{2}} \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right]+\frac{1}{s_{t+1}^{2}} \sum_{i=t+1}^{r} \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right], \quad \text { using (6) } \\
& =\sum_{i=1}^{t}\left(\frac{1}{s_{i}^{2}}-\frac{1}{s_{t+1}^{2}}\right) \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right]+\frac{1}{s_{t+1}^{2}} \sum_{i=1}^{r} \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right] \\
& \leq \sum_{i=1}^{t}\left(\frac{1}{s_{i}^{2}}-\frac{1}{s_{t+1}^{2}}\right) s_{i}^{2}+\frac{\rho}{s_{t+1}^{2}}, \\
& =t+\frac{\rho-\sum_{i=1}^{t} s_{i}^{2}}{s_{t+1}^{2}} \tag{20}
\end{align*}
$$

from 3 which we get

$$
\begin{equation*}
\sum_{i=1}^{r} \frac{1}{s_{i}^{2}} \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right] \leq \min \left\{\frac{\rho}{s_{1}^{2}}, 1+\frac{\rho-s_{1}^{2}}{s_{2}^{2}}, \ldots, r-1+\frac{\rho-\sum_{i=1}^{r-1} s_{i}^{2}}{s_{r}^{2}}, r\right\} \tag{21}
\end{equation*}
$$

Using (21) in (18), the right-side of (12), and therefore $\pi(\rho)$, is bounded above as

$$
\begin{equation*}
\pi(\rho) \leq n-r+\min \left\{\frac{\rho}{s_{1}^{2}}, 1+\frac{\rho-s_{1}^{2}}{s_{2}^{2}}, \ldots, r-1+\frac{\rho-\sum_{i=1}^{r-1} s_{i}^{2}}{s_{r}^{2}}, r\right\} \tag{22}
\end{equation*}
$$

Next, we demonstrate achievability of the privacy in the right-side of (12) by describing explicitly a $\rho$ - QR for the purpose. This $\rho$-QR represents an extension of the scheme in [22, Theorem 13] in the separate context of maximizing (i.e., under worst-case noise) the MMSE of estimating a one-dimensional Gaussian rv on the basis of a one-dimensional noisy version of it under a constraint on the expected $\ell_{2}$-distance between the input and the noisy output. The scheme in [22, Theorem 13] has the structure of attenuation of the input followed by additive independent Gaussian noise which will be the case for our achievability scheme, too, and is therefore an extension. To this end, by Lemmas 2, 3, it suffices to show a $\rho-\mathrm{QR} \tilde{Z}=\tilde{Z}_{o}$ for the recoverability of $\tilde{S} V^{T} X$ and satisfies the constraints in (13). Our $\rho$ - QR is the $\mathbb{R}^{r}$-valued rv given by

$$
\begin{equation*}
\tilde{Z}_{o}=D_{a} \tilde{S} V^{T} X+D_{n o} N \tag{23}
\end{equation*}
$$

where

$$
\begin{gather*}
D_{a}=\operatorname{diag}\left(1-\frac{\rho_{1}}{s_{1}^{2}}, \ldots, 1-\frac{\rho_{r}}{s_{r}^{2}}\right), \\
D_{n o}=\operatorname{diag}\left(\sqrt{\rho_{1}-\frac{\rho_{1}^{2}}{s_{1}^{2}}}, \ldots, \sqrt{\rho_{r}-\frac{\rho_{r}^{2}}{s_{r}^{2}}}\right) \tag{24}
\end{gather*}
$$

[^2]where $\operatorname{diag}\left(d_{1}, \ldots, d_{r}\right)$ denotes a diagonal matrix with (diagonal) elements $\left(d_{1}, \ldots, d_{r}\right)$, and $N \sim \mathcal{N}\left(\mathbf{0}, I_{r}\right)$ is a $\mathbb{R}^{r}$-valued zero-mean Gaussian rv independent of $X$. Thus, $\tilde{Z}_{o}$ entails attenuating $\tilde{S} V^{T} X$ by $D_{a}$ and contaminating it with an additive independent Gaussian noise $D_{n o} N$. The values of $\rho_{1}, \ldots, \rho_{r}$ in (24) are chosen for various ranges of values of $\rho$ as follows.

- $0 \leq \rho \leq s_{1}^{2}: \quad \rho_{1}=\rho, \rho_{2}=\cdots=\rho_{r}=0 ;$
- $s_{1}^{2} \leq \rho \leq s_{1}^{2}+s_{2}^{2}: \quad \rho_{1}=s_{1}^{2}, \rho_{2}=\rho-s_{1}^{2}, \rho_{3}=\cdots=\rho_{r}=0$;
- $s_{1}^{2}+s_{2}^{2} \leq \rho \leq s_{1}^{2}+s_{2}^{2}+s_{3}^{2}: \quad \rho_{1}=s_{1}^{2}, \rho_{2}=s_{2}^{2}, \rho_{3}=\rho-s_{1}^{2}-s_{2}^{2}, \rho_{4}=\cdots=\rho_{r}=0$;
$\vdots$
- $\sum_{i=1}^{r-1} s_{i}^{2} \leq \rho \leq \sum_{i=1}^{r} s_{i}^{2}: \quad \rho_{1}=s_{1}^{2}, \ldots, \rho_{r-1}=s_{r-1}^{2}, \quad \rho_{r}=\rho-\sum_{i=1}^{r-1} s_{i}^{2} ;$
- $\rho \geq \sum_{i=1}^{r} s_{i}^{2}: \quad \rho_{1}=s_{1}^{2}, \ldots, \rho_{r}=s_{r}^{2}$.

We show in Appendix $C$ that $\tilde{Z}_{o}$ as in (23), (24), (25) satisfies the constraints in (13) (with $\tilde{Z}=\tilde{Z}_{o}$ ). Observing that $X$ and $\tilde{Z}_{o}$ are jointly Gaussian, we have that the right-side of (12) is bounded below by

$$
\begin{align*}
\inf _{g} \mathbb{E}\left[\left\|X-g\left(\tilde{Z}_{o}\right)\right\|^{2}\right] & =\operatorname{mmse}\left(X \mid \tilde{Z}_{o}\right) \\
& =\mathbb{E}\left[\left\|X-\mathbb{E}\left[X \tilde{Z}_{o}^{T}\right]\left(\mathbb{E}\left[\tilde{Z}_{o} \tilde{Z}_{o}^{T}\right]\right)^{-1} \tilde{Z}_{o}\right\|^{2}\right] \\
& =\mathbb{E}\left[\left(X-\mathbb{E}\left[X \tilde{Z}_{o}^{T}\right]\left(\mathbb{E}\left[\tilde{Z}_{o} \tilde{Z}_{o}^{T}\right]\right)^{-1} \tilde{Z}_{o}\right)^{T} X\right] \\
& =\operatorname{tr}\left(\mathbb{E}\left[X X^{T}\right]\right)-\operatorname{tr}\left(\mathbb{E}\left[X \tilde{Z}_{o}^{T}\right]\left(\mathbb{E}\left[\tilde{Z}_{o} \tilde{Z}_{o}^{T}\right]\right)^{-1} \mathbb{E}\left[\tilde{Z}_{o} X^{T}\right]\right) \\
& =\operatorname{tr}\left(I_{n}\right)-\operatorname{tr}\left(I_{n} V \tilde{S}^{T} D_{a}^{T}\left(D_{a} \tilde{S} V^{T} I_{n} V \tilde{S}^{T} D_{a}^{T}+D_{n o} I_{r} D_{n o}^{T}\right)^{-1} D_{a} \tilde{S} V^{T} I_{n}\right) \\
& =n-\operatorname{tr}\left(\tilde{S}^{T} D_{a}^{T}\left(D_{a} \tilde{S} \tilde{S}^{T} D_{a}^{T}+D_{n o} D_{n o}^{T}\right)^{-1} D_{a} \tilde{S}\right), \quad \text { since } V \text { is orthonormal }  \tag{26}\\
& =n-r+\sum_{i=1}^{r} \frac{\rho_{i}}{s_{i}^{2}}, \tag{27}
\end{align*}
$$

where the second term in the right-side of (26) is calculated using

$$
\begin{aligned}
D_{a} \tilde{S} \tilde{S}^{T} D_{a}^{T} & =\operatorname{diag}\left(s_{1}^{2}+\frac{\rho_{1}^{2}}{s_{1}^{2}}-2 \rho_{1}, \ldots, s_{r}^{2}+\frac{\rho_{r}^{2}}{s_{r}^{2}}-2 \rho_{r}\right) \\
D_{n o} D_{n o}^{T} & =\operatorname{diag}\left(\rho_{1}-\frac{\rho_{1}^{2}}{s_{1}^{2}}, \ldots, \rho_{r}-\frac{\rho_{r}^{2}}{s_{r}^{2}}\right)
\end{aligned}
$$

and for $i \in\{1, \ldots, r\}, j \in\{1, \ldots, m\}$,

$$
\left(D_{a} \tilde{S}\right)_{i j}= \begin{cases}s_{k}-\frac{\rho_{k}}{s_{k}}, & i=j=k, k=1, \ldots, r \\ 0, & \text { otherwise }\end{cases}
$$

and

$$
\tilde{S}^{T} D_{a}^{T}\left(D_{a} \tilde{S} \tilde{S}^{T} D_{a}^{T}+D_{n o} D_{n o}^{T}\right)^{-1} D_{a} \tilde{S}=\operatorname{diag}\left(1-\frac{\rho_{1}}{s_{1}^{2}}, \ldots, 1-\frac{\rho_{r}}{s_{r}^{2}}\right)
$$

By recalling the equivalence of the right-sides of (8) and (9) and substituting (25) in (27), we get that the right-side
of (12), and therefore $\pi(\rho)$, is bounded below as

$$
\begin{equation*}
\pi(\rho) \geq n-r+\min \left\{\frac{\rho}{s_{1}^{2}}, 1+\frac{\rho-s_{1}^{2}}{s_{2}^{2}}, \ldots, r-1+\frac{\rho-\sum_{i=1}^{r-1} s_{i}^{2}}{s_{r}^{2}}, r\right\} \tag{28}
\end{equation*}
$$

The theorem follows from (22) and (28).
Remark: Recalling (4), let $\tilde{U}$ be the $m \times r$-matrix containing the first $r$ columns of $U$. The achievability scheme $\tilde{Z}_{o}=D_{a} \tilde{S} V^{T} X+D_{n o} N(23)$ is for maximizing privacy under recoverability of $\tilde{S} V^{T} X$. The corresponding achievability scheme or $\rho$-QR for the recoverability of $A X$, denoted by $Z_{o}$, is $Z_{o}=\tilde{U} D_{a} \tilde{U}^{T} A X+\tilde{U} D_{n o} N$ which can be shown readily from the proof of Lemma 2, and also has the same features of attenuation and independent additive Gaussian noise.

We conclude this section by extending $\rho$-privacy to the case when the querier wishes to compute an affine function $A X+b$ of the Gaussian user data $X \sim \mathcal{N}\left(\mathbf{0}, I_{n}\right), n \geq 1$, for a given $A$ as in Section $\Pi$, and $b \in \mathbb{R}^{m}$. In Definition 1, (3) becomes

$$
\begin{align*}
\pi(\rho) & =\sup _{P_{Z \mid X}: \mathbb{E}\left[\|A X+b-Z\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(Z)\|^{2}\right]  \tag{29}\\
& =\sup _{P_{Z \mid X}: \mathbb{E}\left[\|A X-(Z-b)\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(Z-b)\|^{2}\right] \\
& =\sup _{P_{\hat{Z} \mid X}: \mathbb{E}\left[\|A X-\hat{Z}\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(\hat{Z})\|^{2}\right]
\end{align*}
$$

from which we conclude that $\pi(\rho)$ as in (29) is equal to the right-side of (8). Observe that $\pi(\rho)$ does not depend on $b$, as is to be expected.

## IV. DISCUSSION

We give a heuristic explanation of the form of $\pi(\rho)$ in (8). By Lemma 2, note that the recoverability of $A X$ is equivalent to the recoverability of $S V^{T} X$, and therefore $\tilde{S} V^{T} X$, which consists of $r$ components. We recall that $\tilde{S}$ is the matrix composed of the $r$ nonzero rows of $S$, due to which $A X$ consists effectively of $r$ components corresponding to the $r$ singular values of $A$. At $\rho=0$, the querier is provided the exact value of $A X$. From (23), (24), (25), observe that as $\rho$ increases to $s_{1}^{2}$, the component of $A X$ corresponding to the smallest singular value of $A$ is concealed from the querier. As $\rho$ increases further, more components of $A X$ are hidden from the querier, with each subsequent component corresponding to a larger singular value of $A$. This explains the piecewise affine form of $\pi(\rho), \rho \geq 0$, in (8). Therefore, for lower values of $\rho$, i.e., in the high recoverability regime, only those components that correspond to smaller singular values of $A$, can be concealed from the querier.

This work is an initial foray into tackling the larger objective of characterizing data privacy under function recoverability, where the data is of the analog type. Therefore, several open questions remain some of which are stated next.

For reasons of mathematical tractability, we have assumed that the covariance matrix of the Gaussian user data $X$ is $I_{n}$. The problem of computing $\rho$-privacy for an arbitrary (positive-definite) covariance is open. We conjecture that the form of $\pi(\rho)$ (piecewise affine in $\rho$ ) and the structure of the achievability scheme (attenuation and independent additive Gaussian noise) in Theorem 1, will hold.

A natural extension of this work involves the querier obtaining from the user multiple QRs each satisfying the $\rho$-recoverabilty condition. Specifically, given user data $X \sim \mathcal{N}\left(\mathbf{0}, I_{n}\right)$, a querier receives multiple $\rho$-QRs $Z_{1}, \ldots, Z_{t}, t \geq 1$, each satisfying (1) and generated by conditional distributions $P_{Z_{1} \mid X}, \ldots, P_{Z_{t} \mid X}$. The $\rho$-QRs are taken to be conditionally mutually independent, conditioned on $X$, but not necessarily identically distributed. Correspondingly, for each $\rho \geq 0$ and $t \geq 1$, the $\rho$-privacy $\pi_{t}(\rho)$ is defined as

$$
\pi_{t}(\rho)=\sup _{\substack{P_{Z_{1} \mid X}, \ldots, P_{Z_{t} \mid X}:}} \inf _{g_{t}} \mathbb{E}\left[\left\|X-g_{t}\left(Z_{1}, \ldots, Z_{t}\right)\right\|^{2}\right]
$$

where the infimum is taken over all estimators $g_{t}: \mathbb{R}^{m \times t} \rightarrow \mathbb{R}^{n}$ of $X$ on the basis of $Z_{1}, \ldots, Z_{t}$. The task is to characterize $\pi_{t}(\rho)$ and obtain the rate of decay of $\pi_{t}(\rho)$ with $t$. A candidate for the for $\rho$-QRs is (23) with mutually independent Gaussian noise rvs added to them. Will this be optimal in attaining $\pi_{t}(\rho)$ ?

Another broader extension of this work entails recoverability and privacy being measured by $\ell_{p}$-distance and $\ell_{q}$-distance criteria, $p, q \geq 1$, respectively. We seek a characterization of $\pi_{p, q}(\rho)$ given by

$$
\pi_{p, q}(\rho)=\sup _{\left.P_{Z \mid X}: \mathbb{E}\|A X-Z\|^{p}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(Z)\|^{q}\right] .
$$

Does the structure of the solution in Theorem 1 change?
Finally, it is also of interest to examine $\rho$-privacy under recoverability of an arbitrary but given measurable function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}, m \geq 1$, not limited to being linear. This problem, of a more demanding nature due to the (possible) nonlinearity of the mapping $f$, requires a new approach.

## Appendix A

CALCULATION OF $\operatorname{Var}(A X)$ And mmse $(X \mid A X)$
We have

$$
\operatorname{var}(A X)=\operatorname{tr}\left(A A^{T}\right)=\operatorname{tr}\left(U S V^{T} V S^{T} U^{T}\right)=\operatorname{tr}\left(U S S^{T} U^{T}\right)=\operatorname{tr}\left(S S^{T}\right)=\sum_{i=1}^{r} s_{i}^{2}
$$

Next, noting that $X$ and $A X=U S V^{T} X$ are jointly Gaussian, we have

$$
\begin{aligned}
& \operatorname{mmse}(X \mid A X)=\operatorname{mmse}\left(X \mid U S V^{T} X\right) \\
& =\mathbb{E}\left[\left\|X-\mathbb{E}\left[X\left(U S V^{T} X\right)^{T}\right]\left(\mathbb{E}\left[U S V^{T} X\left(U S V^{T} X\right)^{T}\right]\right)^{-1} U S V^{T} X\right\|^{2}\right] \\
& =\mathbb{E}\left[\left(X-\mathbb{E}\left[X\left(U S V^{T} X\right)^{T}\right]\left(\mathbb{E}\left[U S V^{T} X\left(U S V^{T} X\right)^{T}\right]\right)^{-1} U S V^{T} X\right)^{T} X\right] \\
& =\operatorname{tr}\left(\mathbb{E}\left[X X^{T}\right]\right) \\
& \quad \quad-\operatorname{tr}\left(\mathbb{E}\left[X\left(U S V^{T} X\right)^{T}\right]\left(\mathbb{E}\left[\left(U S V^{T} X\right)\left(U S V^{T} X\right)^{T}\right]\right)^{-1} \mathbb{E}\left[U S V^{T} X X^{T}\right]\right) \\
& =\operatorname{tr}\left(I_{n}\right)-\operatorname{tr}\left(I_{n} V S^{T} U^{T}\left(U S V^{T} I_{n} V S^{T} U^{T}\right)^{-1} U S V^{T} I_{n}\right) \\
& =n-\operatorname{tr}\left(S^{T}\left(S S^{T}\right)^{-1} S\right)=n-r .
\end{aligned}
$$

## Appendix B

Proofs of Lemmas 2, 3
Proof of Lemma 2] Recalling (4), we have

$$
\mathbb{E}\left[\|A X-Z\|^{2}\right]=\mathbb{E}\left[\left\|U\left(U^{T} A X-U^{T} Z\right)\right\|^{2}\right]=\mathbb{E}\left[\left\|S V^{T} X-U^{T} Z\right\|^{2}\right]
$$

from which we get

$$
\begin{aligned}
& \sup _{P_{Z \mid X}: \mathbb{E}\left[\|A X-Z\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(Z)\|^{2}\right] \\
& =\sup _{P_{Z \mid X}: \mathbb{E}\left[\left\|S V^{T} X-U^{T} Z\right\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\left\|X-g\left(U^{T} Z\right)\right\|^{2}\right] \\
& =\sup _{P_{\bar{Z} \mid X}: \mathbb{E}\left[\left\|S V^{T} X-\bar{Z}\right\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(\bar{Z})\|^{2}\right], \quad \text { since } U \text { is orthonormal. }
\end{aligned}
$$

Proof of Lemma 3: Since the supremum in the right-side of (11) is over a restricted set compared with the left-side, it suffices to show that (11) holds with " $\leq$," i.e.,

$$
\begin{equation*}
\sup _{P_{\tilde{Z} \mid X}: \mathbb{E}\left[\left\|S V^{T} X-\bar{Z}\right\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(\bar{Z})\|^{2}\right] \leq \sup _{\substack{P_{\tilde{Z} \mid X}: \mathbb{E}\left[\left\|\tilde{S} V^{T} X-\tilde{Z}\right\|^{2}\right] \leq \rho \\ \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{n}\right|^{2}\right] \leq s_{i}^{2}, i=1, \ldots, r}} \inf _{g} \mathbb{E}\left[\|X-g(\tilde{Z})\|^{2}\right] . \tag{30}
\end{equation*}
$$

Since $S$ contains only $r$ nonzero rows, $S V^{T} X$ in the left-side of (11) is a $\mathbb{R}^{m}$-valued rv containing at most $r$ nonzero elements. Recalling that $\tilde{S}$ is the $r \times n$-matrix consisting of the nonzero rows of $S$, it is easily seen that

$$
\begin{equation*}
\sup _{P_{\tilde{Z} \mid X}: \mathbb{E}\left[\left\|S V^{T} X-\bar{Z}\right\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(\bar{Z})\|^{2}\right]=\sup _{P_{\tilde{Z} \mid X}: \mathbb{E}\left[\left\|\tilde{S} V^{T} X-\check{Z}\right\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(\check{Z})\|^{2}\right] \tag{31}
\end{equation*}
$$

where $\check{Z}$ is a $\mathbb{R}^{r}$-valued rv denoting a $\rho$ - QR under recoverability of $\tilde{S} V^{T} X$. For every $\mathbb{R}^{r}$-valued rv $\check{Z}=$ $\left[\check{Z}_{1}, \ldots, \check{Z}_{r}\right]^{T}$, consider the $\mathbb{R}^{r}$-valued rv $\tilde{Z}$ given by

$$
\begin{equation*}
\tilde{Z}=\left[\check{Z}_{1} \mathbb{1}\left(\mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\check{Z}\right)_{1}\right|^{2}\right] \leq s_{1}^{2}\right), \ldots, \check{Z}_{r} \mathbb{1}\left(\mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\check{Z}\right)_{r}\right|^{2}\right] \leq s_{r}^{2}\right)\right]^{T} \tag{32}
\end{equation*}
$$

Observe that

$$
\begin{align*}
\mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right] & = \begin{cases}\mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\check{Z}\right)_{i}\right|^{2}\right], & \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\check{Z}\right)_{i}\right|^{2}\right] \leq s_{i}^{2} \\
s_{i}^{2}, & \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\check{Z}\right)_{i}\right|^{2}\right]>s_{i}^{2}\end{cases}  \tag{33}\\
& \leq \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\check{Z}\right)_{i}\right|^{2}\right], \quad i=1, \ldots, r,
\end{align*}
$$

from which, owing to the constraint under the supremum in the right-side of (31), we get

$$
\begin{equation*}
\mathbb{E}\left[\left\|\tilde{S} V^{T} X-\tilde{Z}\right\|^{2}\right] \leq \mathbb{E}\left[\left\|\tilde{S} V^{T} X-\tilde{Z}\right\|^{2}\right] \leq \rho \tag{34}
\end{equation*}
$$

Since

$$
X-0-\check{Z}-0-\tilde{Z},
$$

and using data processing inequality for MMSE [23], [22], we get

$$
\begin{equation*}
\inf _{g} \mathbb{E}\left[\|X-g(\check{Z})\|^{2}\right] \leq \inf _{g} \mathbb{E}\left[\|X-g(\tilde{Z})\|^{2}\right] . \tag{35}
\end{equation*}
$$

We have shown that for every $\mathbb{R}^{r}$-valued rv $\check{Z}$ that satisfies the $\rho$-recoverability constraint, there exists another $\mathbb{R}^{r}$-valued rv $\tilde{Z}$ (32) that also satisfies the $\rho$-recoverability constraint (34) and additionally, due to (33), meets the constraints

$$
\begin{equation*}
\mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right] \leq s_{i}^{2}, \quad i=1, \ldots, r . \tag{36}
\end{equation*}
$$

Therefore, using (34), (35), (36), we get

$$
\sup _{P_{\tilde{Z} \mid X:}:\left[\left\|\tilde{S} V^{T} X-\check{Z}\right\|^{2}\right] \leq \rho} \inf _{g} \mathbb{E}\left[\|X-g(\check{Z})\|^{2}\right] \leq \sup _{\substack{P_{\tilde{Z} \mid X}: \mathbb{E}\left[\left\|\tilde{S} V^{T} X-\tilde{Z}\right\|^{2}\right] \leq \rho \\ \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}\right)_{i}\right|^{2}\right] \leq s_{i}^{2}, i=1, \ldots, r}} \inf _{g} \mathbb{E}\left[\|X-g(\tilde{Z})\|^{2}\right],
$$

which along with (31) gives (30).

## Appendix C <br> VERIFICATION THAT $\tilde{Z}_{o}$ SATISFIES (13)

For $i=1, \ldots, r$,

$$
\begin{align*}
\mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-\tilde{Z}_{o}\right)_{i}\right|^{2}\right] & =\mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-D_{a} \tilde{S} V^{T} X-D_{n o} N\right)_{i}\right|^{2}\right] \\
& =\mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-D_{a} \tilde{S} V^{T} X\right)_{i}\right|^{2}\right]+\rho_{i}-\frac{\rho_{i}^{2}}{s_{i}^{2}} \\
& =\frac{\rho_{i}^{2}}{s_{i}^{2}}+\rho_{i}-\frac{\rho_{i}^{2}}{s_{i}^{2}}=\rho_{i}  \tag{37}\\
& \leq s_{i}^{2} .
\end{align*}
$$

Next,

$$
\begin{aligned}
\mathbb{E}\left[\left\|\tilde{S} V^{T} X-\tilde{Z}_{o}\right\|^{2}\right] & =\sum_{i=1}^{r} \mathbb{E}\left[\left|\left(\tilde{S} V^{T} X-D_{a} \tilde{S} V^{T} X-D_{n o} N\right)_{i}\right|^{2}\right] \\
& =\sum_{i=1}^{r} \rho_{i}, \quad \text { using (37) } \\
& =\rho .
\end{aligned}
$$

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[^1]:    ${ }^{1}$ Some notation relevant for the rest of the paper: For a matrix $B$, we denote its transpose and trace by $B^{T}$ and $\operatorname{tr}(B)$, respectively. For a rv $Y, \operatorname{var}(Y)$ will denote the trace of the covariance matrix of $Y$.
    ${ }^{2}$ The columns of $U$ and columns of $V$ are the left-singular vectors and right-singular vectors of $A$, respectively.

[^2]:    ${ }^{3}$ In the right-side of 19 , the first and second terms are vacuous for $t=0$ and $t=r$, respectively. In the right-side of (20), the second term and the summation in the second term are vacuous for $t=r$ and $t=0$, respectively.

