# List-decodable Codes for Single-deletion Single-substitution with List-size Two

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Abstract—In this paper, we present an explicit construction of list-decodable codes for single-deletion and single-substitution with list size two and redundancy  $3 \log n+4$ , where n is the block length of the code. Our construction has lower redundancy than the best known explicit construction by Gabrys *et al.* (arXiv 2021), whose redundancy is  $4 \log n + O(1)$ .

### I. INTRODUCTION

Codes correcting insertion, deletion and substitution errors (collectively referred to as edit errors) have gone through a long history from the seminal work of Levenshtein [1]. It was shown in [1] that the binary Varshamov-Tenengolts (VT) code [2], which is given by

$$\mathscr{C}_n(a) = \left\{ \boldsymbol{x} \in \{0,1\}^n : \sum_{i=1}^n ix_i \equiv a \pmod{n+1} \right\},$$

can correct a single edit error and is asymptotically optimal in redundancy, given by  $\log n + 2$ . Order-optimal non-binary single-edit correcting codes were studied in [3], [4].

Constructing optimal multiple-edit error correcting codes is much more challenging, even for binary deletion codes. A generalization of the VT construction for multiple-deletion correcting codes was presented in [5], but this generalized construction has asymptotic rate strictly smaller than 1. Recently, there were many works on explicit construction of lowredundancy t-deletion correcting codes for  $t \ge 2$  (e.g., see [6]-[12]). For t = 2, Guruswami and Håstad constructed a family of 2-deletion correcting codes with length n and redundancy  $4\log n + O(\log \log n)$  [12], which matches the best known upper bound obtained via the Gilbert-Varshamovtype greedy algorithm [6]. By introducing the higher order VT syndromes and the syndrome compression technique, Sima et al. constructed a family of t-deletion correcting codes with redundancy  $8t \log n + o(\log n)$  [10]. Unfortunately, for t > 2, all existential constructions of t-deletion correcting codes have redundancy greater than the Gilbert-Varshamov-type bound.

The best known t-edit correcting codes for  $t \ge 2$  were given by Sima *et al.*, which have redundancy  $4t \log n + o(\log n)$ [11]. The method in [11] was improved by the authors in [15], which gave a construction of t-deletion s-substitution correcting codes with redundancy  $(4t+3s) \log n + o(\log n)$ . A family of single-deletion single-substitution correcting binary codes with redundancy  $6 \log n + 8$  was constructed in [13]. So far, constructing optimal (with respect to redundancy) multiple-edit correcting codes is still an open problem, even for single-deletion single-substitution correcting codes.

As a relaxation of the decoding requirement, list-decoding for insertions and deletions have been considered by several research teams, mainly focusing on list-decoding for some fraction of deletions/insertions [16]–[20]. Unlike the traditional decoding (also referred to as unique-decoding), listdecoding with list-size  $\ell$  allows to give a set of  $\ell$  codewords from each corrupted sequence. A family of explicit listdecodable codes for two deletions with length n and list-size two was constructed in [12], which has redundancy  $3 \log n$ . Note that the redundancy of the construction in [12] is lower than the Gilbert-Varshamov-type bound, which is  $4 \log n$ . The improvement in redundancy is achieved by the relaxation in the decoding requirement.

In this paper, we present an explicit construction of listdecodable codes for single-deletion and single-substitution with list-size two and redundancy  $3 \log n + 4$ . Our construction improves the recent work by Gabrys *et al.* [4], which constructed such codes with redundancy  $4 \log n + O(1)$ .

The rest of this paper is organized as follows. In Section II, the basic concepts are introduced and some preliminary properties of the errors are discussed. Our construction of list-decodable codes for single-deletion and single-substitution is presented in Section III. The auxiliary lemma used by our construction is proved in Section IV.

#### **II. PRELIMINARIES**

For any positive integers m and n such that  $m \leq n$ , denote  $[m,n] = \{m, m+1, \ldots, n\}$ . If m > n, let  $[m,n] = \emptyset$ . For simplicity, we denote [n] = [1,n] and  $\mathbb{Z}_n = [0, n-1]$ .

In this work, we consider binary codes. For any sequence (vector) x of length n, we use  $x_i$  to denote the *i*th symbol of x, and hence x can be denoted as  $x = (x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$  or simply,  $x = x_1 x_2 \ldots x_n$ . The weight of x, denoted by wt(x), is the number of non-zero symbols (the symbol 1 for binary sequence) in x. Clearly, for binary sequence x, we have wt(x) =  $\sum_{i=1}^{n} x_i$ .

Given non-negative integers t and s such that t + s < n, for any  $x \in \{0,1\}^n$ , the *error ball* of x under t-deletion ssubstitution, denoted by  $\mathcal{B}_{t,s}(x)$ , is the set of all sequences that can be obtained from x by t deletions (i.e., deleting t symbols of x) and at most s substitutions (i.e., substituting at most s symbols of x, each with a different symbol). A code

Cases of error combination	$wt({m x}) - wt({m y})$
$1 \rightarrow \epsilon$ , no substitution	1
$1 \rightarrow \epsilon, \ 1 \rightarrow 0$	2
$1 \to \epsilon, \ 0 \to 1$	0
$0 \rightarrow \epsilon$ , no substitution	0
$0 \to \epsilon, \ 1 \to 0$	1
$0 \to \epsilon, \ 0 \to 1$	-1

Table 1. The value of wt(x) - wt(y) for different cases of single-deletion single-substitution, where  $a \to \epsilon$  means a symbol  $a \in \{0, 1\}$  is deleted from x and for  $b \in \{0, 1\} \setminus \{a\}, a \to b$  means a symbol a of x is substituted by the symbol b.

 $\mathscr{C} \subseteq \{0,1\}^n$  is *list-decodable* for *t*-deletion *s*-substitution with list size  $\ell$  if any  $\boldsymbol{y} \in \{0,1\}^{n-1}$  is contained by the error ball of at most  $\ell$  codewords of  $\mathscr{C}$ . In other words, for any  $\boldsymbol{y} \in \{0,1\}^{n-1}$ , there exist at most  $\ell$  codewords of  $\mathscr{C}$  from which  $\boldsymbol{y}$  can be obtained by *t* deletions and at most *s* substitutions.

In this work, we consider list-decodable codes for single deletion and single substitution, i.e., t = s = 1. Suppose  $\boldsymbol{x} \in \{0,1\}^n$  and  $\boldsymbol{y} \in \{0,1\}^{n-1}$  such that  $\boldsymbol{y}$  can be obtained from x by deleting one symbol of x and substituting at most one symbol of x with a different symbol in  $\{0, 1\}$ . We can compute the difference between the weights of x and y for all possible cases (see Table 1). According to Table 1, we have  $wt(x) - wt(y) \in \{-1, 0, 1, 2\}$ . If  $wt(x) - wt(y) \in \{-1, 2\}$ , then the values of the deleted and substituted symbols can be determined. If wt(x) - wt(y) = 0, then y can be obtained from x by deleting a 0, or by deleting a 1 and substituting a 0 with a 1. For the case that y is obtained from x by deleting a 0, unless x is the all-zero sequence, y can also be obtained from x by deleting a 1 and substituting a 0 with a 1.<sup>1</sup> Hence, if wt(x) - wt(y) = 0, then y can always be obtained from  $\boldsymbol{x}$  by deleting a 1 and substituting a 0 with a 1. Similarly, if wt(x) - wt(y) = 1 and x is not the all-one sequence, then y can always be obtained from x by deleting a 0 and substituting a 1 with a 0. In summary, we have the following remark.

*Remark 1:* Suppose  $x \in \{0,1\}^n \setminus \{1^n,0^n\}$ , where  $1^n$  and  $0^n$  are the all-one sequence and the all-zero sequence of length *n* respectively, and  $y \in \{0,1\}^{n-1}$  such that y can be obtained from x by deleting one symbol of x and substituting *at most* one symbol of x. Then y can be obtained from x by deleting one symbol of x, and the values of the deleted and substituted symbols can be determined by wt(x) (mod 4) and wt(y).

## **III. MAIN RESULTS**

In this section, we present our construction of list-decodable codes for single-deletion and single-substitution. Our construction only uses the weight and the first two order VT syndromes for binary sequences.

We adopt the method of [10] to define the higher order VT syndromes. For each positive integer j and each  $x \in \{0, 1\}^n$ , the *j*th-order VT syndrome of x is defined as

$$f_j(\boldsymbol{x}) = \sum_{i=1}^n \left( \sum_{\ell=1}^i \ell^{j-1} \right) x_i.$$
 (1)

As in [14], we can rearrange the terms and obtain

$$f_j(\boldsymbol{x}) = \sum_{\ell=1}^n \left(\sum_{i=1}^\ell i^{j-1}\right) x_\ell = \sum_{i=1}^n \left(\sum_{\ell=i}^n x_\ell\right) i^{j-1}.$$
 (2)

The code is given by the following definition, where  $1^n$  and  $0^n$  denote the all-one sequence and the all-zero sequence of length n, respectively.

Definition 1: For any fixed values  $c_0 \in \mathbb{Z}_4$ ,  $c_1 \in \mathbb{Z}_{2n}$  and  $c_2 \in \mathbb{Z}_{2n^2}$ , let  $\mathscr{C}_n(c_0, c_1, c_2)$  be the set of all sequences  $\boldsymbol{x} \in \{0, 1\}^n \setminus \{1^n, 0^n\}$  satisfying the following three conditions:

- (C0) wt( $\boldsymbol{x}$ )  $\equiv c_0 \pmod{4}$ . (C1)  $f_1(\boldsymbol{x}) \equiv c_1 \pmod{2n}$ .
- (C2)  $f_2(\boldsymbol{x}) \equiv c_2 \pmod{2n^2}$ .

Then our main result can be stated as the following theorem. Theorem 1: There exists a  $(c_0, c_1, c_2) \in \mathbb{Z}_4 \times \mathbb{Z}_{2n} \times \mathbb{Z}_{2n^2}$ such that the code  $\mathscr{C}_n(c_0, c_1, c_2)$  in Definition 1 has redundancy at most  $3 \log n + 4$  and is list-decodable from singledeletion and single-substitution with list size 2.

In the rest of this section, we always assume that  $(c_0, c_1, c_2) \in \mathbb{Z}_4 \times \mathbb{Z}_{2n} \times \mathbb{Z}_{2n^2}$  and  $\mathscr{C}_n(c_0, c_1, c_2)$  is given by Definition 1. For any  $x \in \{0, 1\}$  and  $\{d, e\} \subseteq [n]$ , let E(x, d, e) denote the sequence obtained from x by deleting  $x_d$ and substituting  $x_e$  with  $\bar{x}_e = 1 - x_e$  (i.e.,  $\bar{x}_e = 1$  if  $x_e = 0$  and  $\bar{x}_e = 0$  if  $x_e = 1$ ). Clearly,  $E(x, d, e) \in \{0, 1\}^{n-1}$  is uniquely determined by x, d and e. We also need the following lemma, which will be proved in Section IV.

*Lemma 1:* Suppose  $\boldsymbol{x}, \, \boldsymbol{x}' \in \mathscr{C}_n(c_0, c_1, c_2)$  and  $\{d_1, e_1\}, \{d_2, e_2\} \subseteq [n]$  such that  $\boldsymbol{x} \neq \boldsymbol{x}', \, d_1 \leq d_2$  and  $E(\boldsymbol{x}, d_1, e_1) = E(\boldsymbol{x}', d_2, e_2)$ . We have  $d_1 < e_1 \leq d_2$  and  $d_1 \leq e_2 < d_2$ .

In formally speaking, if there exists a  $y \in \{0,1\}^{n-1}$  such that y can be obtained from x and x' by deleting one symbol and substituting one symbol, then the two substituted symbols are both located between the two deleted symbols.

Using Lemma 1, we can prove Theorem 1 as follows.

*Proof of Theorem 1:* By the pigeonhole principle, there exists a  $(c_0, c_1, c_2) \in \mathbb{Z}_4 \times \mathbb{Z}_{2n} \times \mathbb{Z}_{2n^2}$  such that the code  $\mathscr{C}_n(c_0, c_1, c_2)$  has size at least  $\frac{2^n - 2}{16n^3}$ , hence the redundancy of  $\mathscr{C}_n(c_0, c_1, c_2)$  is at most  $3 \log n + 4$ .

It remains to prove that  $\mathscr{C}_n(c_0, c_1, c_2)$  is list-decodable from single-deletion and single-substitution with list size 2. We need to prove that for any given  $\boldsymbol{y} \in \{0, 1\}^{n-1}$ , there exist at most two codewords in  $\mathscr{C}_n(c_0, c_1, c_2)$ , from which  $\boldsymbol{y}$  can be obtained by one deletion and at most one substitution. This can be proved by contradiction as follows.

Suppose x, x' and x'' are three distinct sequences in  $\mathscr{C}_n(c_0, c_1, c_2)$  from which y can be obtained by one deletion and at most one substitution. By Remark 1, we can assume

<sup>&</sup>lt;sup>1</sup>For example, let  $\boldsymbol{x} = 0110001$  and let  $\boldsymbol{y} = 011001$  be obtained from  $\boldsymbol{x}$  by deleting  $x_5 = 0$ . Then  $\boldsymbol{y}$  can also be obtained from  $\boldsymbol{x}$  by deleting  $x_3 = 1$  and substituting  $x_4 = 0$  with  $\bar{x}_4 = 1$ . In general, if  $\boldsymbol{x}$  is not the all-zero sequence and  $\boldsymbol{y}$  can be obtained from  $\boldsymbol{x}$  by deleting a 0, then we can always find a 0 (denoted by  $\hat{0}$ ) in the same run with the deleted 0 that is adjacent to a 1 (denoted by  $\hat{1}$ ). Then  $\boldsymbol{y}$  can always be viewed as being obtained from  $\boldsymbol{x}$  by deleting the  $\hat{1}$  and substituting the  $\hat{0}$  with 1.

 $y = E(x, d_1, e_1) = E(x', d_2, e_2) = E(x'', d_3, e_3)$ . Without loss of generality, assume  $d_1 \le d_2$ .

First, consider x and x'. By Lemma 1, we have

$$d_1 < e_1 \le d_2 \tag{3}$$

and

$$d_1 \le e_2 < d_2. \tag{4}$$

For further discussions, we have the following three cases. Case 1:  $d_3 \leq d_1$ . Considering x and x'', by Lemma 1, we have  $d_3 \leq e_1 < d_1$  and  $d_3 < e_3 \leq d_1$ . Combining with (3), we have  $e_1 < d_1 < e_1$ , a contradiction.

Case 2:  $d_1 < d_3 \leq d_2$ . Considering x and x'', by Lemma 1, we have  $d_1 < e_1 \leq d_3$  and  $d_1 \leq e_3 < d_3$ . On the other hand, considering x' and x'', by Lemma 1, we have  $d_3 < e_3 \leq d_2$  and  $d_3 \leq e_2 < d_2$ . Hence, we obtain  $e_3 < d_3 < e_3$ , a contradiction.

Case 3:  $d_2 < d_3$ . Considering x' and x'', by Lemma 1, we have  $d_2 < e_2 \le d_3$  and  $d_2 \le e_3 < d_3$ . Combining with (4), we get  $e_2 < d_2 < e_2$ , a contradiction.

From the above discussions, we can conclude that there exist at most two codewords in  $\mathscr{C}_n(c_0, c_1, c_2)$  from which  $\boldsymbol{y}$  can be obtained by one deletion and at most one substitution, which proves Theorem 1.

## IV. PROOF OF LEMMA 1

In this section, we prove Lemma 1. We always suppose that  $x, x' \in \mathcal{C}_n(c_0, c_1, c_2)$ , and  $\{d_1, e_1\}, \{d_2, e_2\} \subseteq [n]$  such that  $d_1 \leq d_2$  and  $E(x, d_1, e_1) = E(x', d_2, e_2)$ . We first enumerate all the possible cases according to the order of  $d_1, e_1, d_2, e_2$ .

*Remark 2:* Consider  $e_1, d_1$  and  $d_2$ . Since  $d_1 \le d_2$ , we have three cases:  $e_1 < d_1, d_1 < e_1 \le d_2$  and  $d_2 < e_1$ . Similarly, for  $e_2, d_1$  and  $d_2$ , we have three cases:  $e_2 < d_1, d_1 \le e_2 < d_2$  and  $d_2 < e_2$ . Combining these two scenarios we have a total of nine cases to consider. However, we can merge some cases and consider the following six cases.

(i)  $e_1 < d_1 \le d_2$  and  $e_2 < d_1 \le d_2$ . (ii)  $e_1 < d_1 \le e_2 < d_2$  or  $e_2 < d_1 < e_1 \le d_2$ . (iii)  $e_1 < d_1 \le d_2 < e_2$  or  $e_2 < d_1 \le d_2 < e_1$ . (iv)  $d_1 < e_1 \le d_2$  and  $d_1 \le e_2 < d_2$ . (v)  $d_1 < e_1 \le d_2 < e_2$  or  $d_1 \le e_2 < d_2 < e_1$ . (vi)  $d_1 \le d_2 < e_1$  and  $d_1 \le d_2 < e_2$ .

We will prove that x = x' for all cases in Remark 2 except for Case (iv). Hence, if  $x \neq x'$ , then it must fall into Case (iv), that is,  $d_1 < e_1 \le d_2$  and  $d_1 \le e_2 < d_2$ .

Denote  $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$  and  $\boldsymbol{x}' = (x_1', x_2', \dots, x_n')$ . For each  $i \in [n]$ , let

$$u_i \triangleq \sum_{\ell=i}^n x_\ell - \sum_{\ell=i}^n x'_\ell.$$
(5)

To prove x = x', it suffices to prove  $u_i = 0$  for all  $i \in [n]$ .

The following lemma will be used in our discussions. (Recall that for each positive integer j and  $x \in \{0,1\}^n$ ,  $f_j(x)$  is the *j*th-order VT syndrome of x defined by (1) or (2).) *Lemma 2:* Let m be a fixed positive integer. Suppose  $(f_1(\boldsymbol{x}), \ldots, f_{m+1}(\boldsymbol{x})) = (f_1(\boldsymbol{x}'), \ldots, f_{m+1}(\boldsymbol{x}'))$  and there exist m positive integers, say  $p_1, p_2, \ldots, p_m$ , such that  $1 \leq p_1 < p_2 < \cdots < p_m \leq n$  and for each  $j \in [m+1]$ , either  $u_i \geq 0$  for all  $i \in [p_{j-1} + 1, p_j]$  or  $u_i \leq 0$  for all  $i \in [p_{j-1} + 1, p_j]$ , where  $p_0 = 1$  and  $p_{m+1} = n$ . Then  $u_i = 0$  for all  $i \in [n]$ , and hence we have  $\boldsymbol{x} = \boldsymbol{x}'$ .

The proof of Lemma 2 is omitted because it is (implicitly) contained in the proof of [10, Proposition 2].

The following simple remark is also useful in our proof.

*Remark 3:* Since wt( $\boldsymbol{x}$ )  $\equiv$  wt( $\boldsymbol{x}'$ )  $\equiv c_0 \pmod{4}$  (because  $\boldsymbol{x}, \boldsymbol{x}' \in \mathscr{C}_n(c_0, c_1, c_2)$ ) and  $E(\boldsymbol{x}, d_1, e_1) = E(\boldsymbol{x}', d_2, e_2)$ , then by Remark 1, we have  $x_{d_1} = x'_{d_2}$  and wt( $\boldsymbol{x}$ ) = wt( $\boldsymbol{x}'$ ).

In the following five subsections, we will prove that for all cases in Remark 2 except for Case (iv), we have  $(f_1(x), f_2(x)) = (f_1(x'), f_2(x'))$ , and there exists a  $p_1 \in [n]$ such that for each  $j \in \{1, 2\}$ , either  $u_i \ge 0$  for all  $i \in [p_{j-1} + 1, p_j]$  or  $u_i \le 0$  for all  $i \in [p_{j-1} + 1, p_j]$ , where  $p_0 = 1$  and  $p_2 = n$ . Then by Lemma 2 (for the special case of m = 1), we have x = x'. Thus, if  $x \ne x'$ , then it must fall into Case (iv), that is,  $d_1 < e_1 \le d_2$  and  $d_1 \le e_2 < d_2$ .

## A. Proof of x = x' for Case (i)

For this case, we have  $e_1 < d_1 \le d_2$  and  $e_2 < d_1 \le d_2$ . If  $e_1 = e_2$ , then  $\boldsymbol{x} = \boldsymbol{x}'$ .<sup>2</sup> Therefore, we assume  $e_1 \ne e_2$ . To simplify the presentation, let  $\lambda_1 = \min\{e_1, e_2\}$  and  $\lambda_2 = \max\{e_1, e_2\}$ . Then  $1 \le \lambda_1 < \lambda_2 < d_1 \le d_2$ . Since  $E(\boldsymbol{x}, d_1, e_1) = E(\boldsymbol{x}', d_2, e_2)$ , we can obtain<sup>3</sup>

$$x_{i} = \begin{cases} x'_{i}, & \text{for } i \in [1, d_{1} - 1] \setminus \{\lambda_{1}, \lambda_{2}\}; \\ x'_{i-1}, & \text{for } i \in [d_{1} + 1, d_{2}]; \\ x'_{i}, & \text{for } i \in [d_{2} + 1, n]. \end{cases}$$
(6)

Moreover, we have  $x_{\lambda_1} \neq x'_{\lambda_1}$  and  $x_{\lambda_2} \neq x'_{\lambda_2}$  because of the substitution error. According to (6), this case can be illustrated by Fig. 1.

Fig. 1. Illustration of Case (i): The bits (symbols) of each sequence is denoted by a row of black dots, where each column corresponds to the two symbols at the same position in the respective sequences. Each pair of bits connected by a solid segment are of equal value, while those connected by a dashed segment have different values because of the substitution error.

We can use (6) or Fig. 1 to simplify  $u_i$  for each  $i \in [n]$  (In fact, Fig. 1 is more intuitive than (6).) as follows.

<sup>2</sup>If  $e_1 = e_2 < d_1 \le d_2$ , then there is a  $y' \in \{0,1\}^{n-1}$  such that y' can be obtained from x (resp. x') by a single deletion. By (C1) of Definition 1,  $\mathscr{C}_n(c_0, c_1, c_2)$  is a single-deletion correcting code, so we can obtain x = x'. <sup>3</sup>In fact, let  $y = E(x, d_1, e_1) = E(x', d_2, e_2)$ , which means that y can be obtained from x by deleting  $x_{d_1}$  and substituting  $x_{e_1}$  with  $\bar{x}_{e_1} = 1 - x_{e_1}$ , and y can also be obtained from x' by deleting  $x'_{d_2}$  and substituting  $x'_{e_2}$  with  $\bar{x}'_{e_2} = 1 - x'_{e_2}$ . Then (6) can be obtained by comparing the elements of x and x' with the elements of y:  $x_i = y_i = x'_i$  for  $i \in [1, d_1 - 1] \setminus \{\lambda_1, \lambda_2\}$ ;  $x_i = y_{i-1} = x'_{i-1}$  for  $i \in [d_1 + 1, d_2]$ ; and  $x_i = y_i = x'_i$  for  $i \in [d_2 + 1, n]$ .

First, we simplify  $u_i = \sum_{\ell=i}^n x_\ell - \sum_{\ell=i}^n x'_\ell$  for  $i \in [1, \lambda_1]$ . From Fig. 1 we can see that all terms in  $\sum_{\ell=i}^{n} x_{\ell}$  can be cancelled by their corresponding terms in  $\sum_{\ell=i}^{n} x_{\ell}'$  except for  $x_{\lambda_1}, x_{\lambda_2}$  and  $x_{d_1}$ , and all terms in  $\sum_{\ell=i}^{n} x_{\ell}'$  can be cancelled except for  $x'_{\lambda_1}, x'_{\lambda_2}$  and  $x'_{d_2}$ , so we have

$$u_{i} = \sum_{\ell=i}^{n} x_{\ell} - \sum_{\ell=i}^{n} x_{\ell}'$$
  
=  $x_{\lambda_{1}} + x_{\lambda_{2}} + x_{d_{1}} - x_{\lambda_{1}}' - x_{\lambda_{2}}' - x_{d_{2}}'$  (7)

In particular, we have  $\operatorname{wt}(\boldsymbol{x}) - \operatorname{wt}(\boldsymbol{x}') = \sum_{\ell=1}^{n} x_{\ell} - \sum_{\ell=1}^{n} x'_{\ell} = u_1 = x_{\lambda_1} + x_{\lambda_2} + x_{d_1} - x'_{\lambda_1} - x'_{\lambda_2} - x'_{d_2}$ . Note that by Remark 3,  $\operatorname{wt}(\boldsymbol{x}) = \operatorname{wt}(\boldsymbol{x}')$  and  $x_{d_1} = x'_{d_2}$ . Therefore, by (7), we have

$$0 = \mathsf{wt}(\boldsymbol{x}) - \mathsf{wt}(\boldsymbol{x}') = x_{\lambda_1} + x_{\lambda_2} + x_{d_1} - x'_{\lambda_1} - x'_{\lambda_2} - x'_{d_2} = x_{\lambda_1} + x_{\lambda_2} - x'_{\lambda_1} - x'_{\lambda_2}$$
(8)

and

$$u_i = x_{\lambda_1} + x_{\lambda_2} - x'_{\lambda_1} - x'_{\lambda_2} = 0, \ \forall i \in [1, \lambda_1].$$

Similarly, from Fig. 1, by cancelling the corresponding equivalent terms in  $\sum_{\ell=i}^{n} x_{\ell}$  and  $\sum_{\ell=i}^{n} x'_{\ell}$ , we can obtain:

- $u_i = x_{\lambda_2} + x_{d_1} x'_{\lambda_2} x'_{d_2} = x_{\lambda_2} x'_{\lambda_2}$  for each  $i \in [\lambda_1 + 1, \lambda_2]$ , where the second equality holds because  $x_{d_1} = x'_{d_2}$  (according to Remark 3).
- $u_i = x_{d_1} x'_{d_2} = 0$  for each  $i \in [\lambda_2 + 1, d_1]$ .
- $u_i = x_i x'_{d_2}$  for each  $i \in [d_1 + 1, d_2]$ .
- $u_i = 0$  for each  $i \in [d_2 + 1, n]$ .

Collectively, we have

$$u_{i} = \begin{cases} 0, & \text{for } i \in [1, \lambda_{1}]; \\ x_{\lambda_{2}} - x'_{\lambda_{2}}, & \text{for } i \in [\lambda_{1} + 1, \lambda_{2}]; \\ 0, & \text{for } i \in [\lambda_{2} + 1, d_{1}]; \\ x_{i} - x'_{d_{2}}, & \text{for } i \in [d_{1} + 1, d_{2}]; \\ 0, & \text{for } i \in [d_{2} + 1, n]. \end{cases}$$
(9)

Moreover, we have the following claim.

Claim 1: Let  $p_1 = \lambda_2$ . Then for each  $j \in \{1, 2\}$ , either  $u_i \geq 1$ 0 for all  $i \in [p_{j-1}+1, p_j]$  or  $u_i \le 0$  for all  $i \in [p_{j-1}+1, p_j]$ , where  $p_0 = 1$  and  $p_2 = n$ . Moreover,  $|u_i| \le 1$  for all  $i \in [n]$ .

*Proof of Claim 1:* For  $i \in [1, \lambda_2]$ , by (9), we have  $u_i = 0$ or  $u_i = x_{\lambda_2} - x'_{\lambda_2}$ . If  $x'_{\lambda_2} = 0$ , then  $u_i \in \{0, 1\}$  for all  $i \in [\lambda_2 + 1, n]$ ; if  $x'_{\lambda_2} = 1$ , then  $u_i \in \{-1, 0\}$  for all  $i \in [\lambda_2 + 1, n]$ .

For  $i \in [\lambda_2 + 1, n]$ , by (9), we have  $u_i = 0$  or  $u_i = x_i - x'_{d_2}$ . If  $x'_{d_2} = 0$ , then  $u_i \in \{0, 1\}$  for all  $i \in [\lambda_2 + 1, n]$ ; if  $x'_{d_2} = 1$ , then  $u_i \in \{-1, 0\}$  for all  $i \in [\lambda_2 + 1, n]$ .

Thus,  $p_1 = \lambda_2$  satisfies the desired property and  $|u_i| \leq 1$ for all  $i \in [n]$ , which proves Claim 1.

By (2), for j = 1, 2, we have

$$|f_j(\boldsymbol{x}) - f_j(\boldsymbol{x}')| = \left| \sum_{i=1}^n \left( \sum_{\ell=i}^n x_\ell \right) i^{j-1} - \sum_{i=1}^n \left( \sum_{\ell=i}^n x'_\ell \right) i^{j-1} \right|$$
$$= \left| \sum_{i=1}^n u_i i^{j-1} \right|$$
$$\leq \sum_{i=1}^n i^{j-1}$$
$$< n^j, \tag{10}$$

where the first inequality holds because by Claim 1,  $|u_i| < 1$ for all  $i \in [n]$ . Note that by (C1) and (C2) of Definition 1,  $f_j(\boldsymbol{x}) \equiv f_j(\boldsymbol{x}') \pmod{2n^j}$ , so by (10), we have  $f_j(\boldsymbol{x}) =$  $f_j(\mathbf{x}')$ . Thus, by Claim 1 and Lemma 2, we have  $\mathbf{x} = \mathbf{x}'$ .

Example 1: To help the reader to understand the proof, consider an example with

$$\begin{aligned} & \boldsymbol{x} = 1101101000101110, \\ & \boldsymbol{y} = 110111100101110, \\ & \boldsymbol{x}' = 1001111001011010, \end{aligned}$$

where n = 16. We can check that y can be obtained from x by deleting  $x_{10} = 0$  and substituting  $x_6 = 0$  with  $y_6 =$  $\bar{x}_6 = 1$ , and  $\boldsymbol{y}$  can also be obtained from  $\boldsymbol{x}'$  by deleting  $x'_{14} = 0$  and substituting  $x'_2 = 0$  with  $y_2 = \overline{x}'_2 = 1$ . Hence, y = E(x, 10, 6) = E(x', 14, 2), that is,  $d_1 = 10, e_1 = 6$ ,  $d_2 = 14$  and  $e_2 = 2$ . Since  $e_2 < e_1$ , we take  $\lambda_1 = e_2 = 2$  and  $\lambda_2 = e_1 = 6$ . For this example, x and x' can be illustrated by Fig. 2, which is an instance of Fig. 1. It is easy to check that:

- For  $i \in [\lambda_1] = \{1, 2\}, u_i = \sum_{\ell=i}^n x_\ell \sum_{\ell=i}^n x'_\ell = x_{\lambda_1} + x_{\lambda_2} + x_{d_1} x'_{\lambda_1} x'_{\lambda_2} x'_{d_2} = 0;$  For  $i \in [\lambda_1 + 1, \lambda_2] = \{3, 4, 5, 6\}, u_i = \sum_{\ell=i}^n x_\ell \sum_{\ell=i}$
- $\sum_{\ell=i}^{n} x_{\ell}' = x_{\lambda_2} + x_{d_1} x_{\lambda_2}' x_{d_2}' = x_{\lambda_2} x_{\lambda_2}' = -1;$  For  $i \in [\lambda_2 + 1, d_1] = \{7, 8, 9, 10\}, u_i = \sum_{\ell=i}^{n} x_{\ell} 1$
- $\sum_{\ell=i}^{n} x_{\ell}' = x_{d_1} x_{d_2}' = 0;$
- For  $i \in [d_1 + 1, d_2] = \{11, 12, 13, 14\}, u_i = \sum_{\ell=i}^n x_\ell \sum_{\ell=i}^n x'_\ell = x_i x'_{d_2} = x_i \in \{0, 1\};$
- For  $i \in [d_2 + 1, n] = \{15, 16\}, u_i = \sum_{\ell=i}^n x_\ell$  $\sum_{\ell=i}^{n} x_{\ell}' = 0;$

In summary, we have

$$(u_1, u_2, \cdots, u_n) = (0, 0, -1, -1, -1, -1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0).$$

We can see that  $u_i \leq 0$  for all  $i \in [1, \lambda_2] = \{1, 2, \dots, 6\}$ , and  $u_i \ge 0$  for all  $i \in [\lambda_2 + 1, n] = \{7, 8, \cdots, 16\}.$ 





## B. Proof of x = x' for Case (ii)

For this case, we have  $e_1 < d_1 \le e_2 < d_2$  or  $e_2 < d_1 < e_1 \le d_2$ . If  $e_1 < d_1 \le e_2 < d_2$ , let  $\lambda_1 = e_1$  and  $\lambda_2 = e_2 + 1$ ; If  $e_2 < d_1 < e_1 \le d_2$ , let  $\lambda_1 = e_2$  and  $\lambda_2 = e_1$ . Then for both cases, we always have  $\lambda_1 < d_1 < \lambda_2 \le d_2$ . Since  $E(\boldsymbol{x}, d_1, e_1) = E(\boldsymbol{x}', d_2, e_2)$ , analogous to (6), we can obtain

$$x_{i} = \begin{cases} x'_{i}, & \text{for } i \in [1, d_{1} - 1] \setminus \{\lambda_{1}\}, \\ x'_{i-1}, & \text{for } i \in [d_{1} + 1, d_{2}] \setminus \{\lambda_{2}\}, \\ x'_{i}, & \text{for } i \in [d_{2} + 1, n]. \end{cases}$$
(11)

Moreover, we have  $x_{\lambda_1} \neq x'_{\lambda_1}$  and  $x_{\lambda_2} \neq x'_{\lambda_2-1}$  because of the substitution error. According to (11), this case can be illustrated by Fig. 3.

Fig. 3. Illustration of Case (ii).

By Remark 3, we have wt( $\boldsymbol{x}$ ) = wt( $\boldsymbol{x}'$ ) and  $x_{d_1} = x'_{d_2}$ . Then by (11) or Fig. 3, and through a cancelling process similar to Case (i), we can obtain  $0 = \text{wt}(\boldsymbol{x}) - \text{wt}(\boldsymbol{x}') =$  $\sum_{\ell=1}^{n} x_{\ell} - \sum_{\ell=1}^{n} x'_{\ell} = x_{\lambda_1} + x_{\lambda_2} + x_{d_1} - x'_{\lambda_1} - x'_{\lambda_2-1} - x'_{d_2} =$  $x_{\lambda_1} + x_{\lambda_2} - x'_{\lambda_1} - x'_{\lambda_2-1} \text{ and}$ 

$$u_{i} = \begin{cases} 0, & \text{for } i \in [1, \lambda_{1}]; \\ x_{\lambda_{2}} - x'_{\lambda_{2}-1}, & \text{for } i \in [\lambda_{1}+1, d_{1}]; \\ x_{i} + x_{\lambda_{2}} - x'_{\lambda_{2}-1} - x'_{d_{2}}, & \text{for } i \in [d_{1}+1, \lambda_{2}-1]; \\ x_{i} - x'_{d_{2}}, & \text{for } i \in [\lambda_{2}, d_{2}]; \\ 0, & \text{for } i \in [d_{2}+1, n]. \end{cases}$$

$$(12)$$

Moreover, we have the following Claim.

Claim 2: Let  $p_1 = \lambda_2 - 1$ . Then for each  $j \in \{1, 2\}$ , either  $u_i \ge 0$  for all  $i \in [p_{j-1} + 1, p_j]$  or  $u_i \le 0$  for all  $i \in [p_{j-1} + 1, p_j]$ , where  $p_0 = 1$  and  $p_2 = n$ . Moreover, we have  $|u_i| \le 2$  for all  $i \in [n]$ .

Proof of Claim 2: First consider  $i \in [\lambda_2, n]$ . By (12),  $u_i = 0$  or  $u_i = x_i - x'_{d_2}$ . Clearly, if  $x'_{d_2} = 0$ , then  $u_i \ge 0$  for all  $i \in [\lambda_2, n]$ ; if  $x'_{d_2} = 1$ , then  $u_i \le 0$  for all  $i \in [\lambda_2, n]$ .

Now, consider  $i \in [1, \lambda_2 - 1]$ . Note that  $x_{\lambda_2} \neq x'_{\lambda_2 - 1}$ . Then we have  $x_{\lambda_2} - x'_{\lambda_2 - 1} \in \{-1, 1\}$ . We need to consider the following two subcases.

Case (ii.1):  $x_{\lambda_2} - x'_{\lambda_2 - 1} = 1$ . By (12), we have

$$u_i = \begin{cases} 0, & \text{for } i \in [1, \lambda_1]; \\ 1, & \text{for } i \in [\lambda_1 + 1, d_1]; \\ x_i + 1 - x'_{d_2}, & \text{for } i \in [d_1 + 1, \lambda_2 - 1]. \end{cases}$$

Note that  $x_i \ge 0$  and  $1 - x'_{d_2} \ge 0$  (because  $x'_{d_2} \in \{0, 1\}$ ). Then  $u_i \ge 0$  for all  $i \in [1, \lambda_2 - 1]$ .

Case (ii.2):  $x_{\lambda_2} - x'_{\lambda_2-1} = -1$ .

By (12), we have

$$u_i = \begin{cases} 0, & \text{for } i \in [1, \lambda_1]; \\ -1, & \text{for } i \in [\lambda_1 + 1, d_1]; \\ x_i - 1 - x'_{d_2}, & \text{for } i \in [d_1 + 1, \lambda_2 - 1]. \end{cases}$$

Note that  $x_i - 1 \leq 0$  (because  $x'_{d_2} \in \{0, 1\}$ ) and  $-x'_{d_2} \leq 0$ . Then  $u_i \leq 0$  for all  $i \in [1, \lambda_2 - 1]$ .

Thus,  $p_1 = \lambda_2 - 1$  satisfies the desired property.

Finally, note that  $|x_{\lambda_2} - x'_{\lambda_2-1}| \le 1$  and  $|x_i - x'_{d_2}| \le 1$ . Then it is easy to see from (12) that  $|u_i| \le 2$  for all  $i \in [n]$ , which proves Claim 2.

Similar to Case (i), by (2) and Claim 2, for j = 1, 2, we have

$$|f_j(\boldsymbol{x}) - f_j(\boldsymbol{x}')| \le \sum_{i=1}^n |u_i| i^{j-1} \le \sum_{i=1}^n 2i^{j-1} < 2n^j.$$

On the other hand, by (C1) and (C2) of Definition 1, we have  $f_j(\boldsymbol{x}) \equiv f_j(\boldsymbol{x}') \pmod{2n^j}$ , so  $f_j(\boldsymbol{x}) = f_j(\boldsymbol{x}')$ . Then by Claim 2 and Lemma 2, we have  $\boldsymbol{x} = \boldsymbol{x}'$ .

*Example 2:* Consider an example with

$$\begin{aligned} & \boldsymbol{x} = 1001011101001110, \\ & \boldsymbol{y} = 110111101001110, \\ & \boldsymbol{x}' = 1101111000011010, \end{aligned}$$

where n = 16. We can check that y can be obtained from x by deleting  $x_5 = 0$  and substituting  $x_2 = 0$  with  $y_2 = \bar{x}_2 = 1$ , and y can also be obtained from x' by deleting  $x'_{14} = 0$  and substituting  $x'_9 = 0$  with  $y_9 = \bar{x}'_9 = 1$ . Hence, we have y = E(x, 5, 2) = E(x', 14, 9), that is,  $d_1 = 5$ ,  $e_1 = 2$ ,  $d_2 = 14$  and  $e_2 = 9$ . Since  $e_1 < d_1 < e_2 < d_2$ , we take  $\lambda_1 = e_1 = 2$  and  $\lambda_2 = e_2 + 1 = 10$ . For this example, x and x' can be illustrated by Fig. 4, which is an instance of Fig. 3. It is easy to check that:

- For  $i \in [\lambda_1] = \{1, 2\}, u_i = \sum_{\ell=i}^n x_\ell \sum_{\ell=i}^n x'_\ell = x_{\lambda_1} + x_{\lambda_2} + x_{d_1} x'_{\lambda_1} x'_{\lambda_2} x'_{d_2} = 0;$
- For  $i \in [\lambda_1 + 1, d_1] = \{3, 4, 5\}, u_i = \sum_{\ell=i}^n x_\ell \sum_{\ell=i}^n x'_\ell = x_{d_1} + x_{\lambda_2} x'_{\lambda_2-1} x'_{d_2} = x_{\lambda_2} x'_{\lambda_2-1} = 1;$
- For  $i \in [d_1 + 1, \lambda_2 1] = \{6, 7, 8, 9\}, u_i = \sum_{\ell=i}^n x_\ell \sum_{\ell=i}^n x'_\ell = x_i + x_{\lambda_2} x'_{\lambda_2 1} x'_{d_2} = x_i + 1 \in \{1, 2\};$
- For  $i \in [\lambda_2, d_2] = \{10, 11, 12, 13, 14\}, u_i = \sum_{\ell=i}^n x_\ell \sum_{\ell=i}^n x'_\ell = x_i x'_{d_2} = x_i \in \{0, 1\};$
- For  $i \in [d_2 + 1, n] = \{15, 16\}, u_i = \sum_{\ell=i}^n x_\ell \sum_{\ell=i}^n x'_\ell = 0;$

In summary, we have

$$(u_1, u_2, \cdots, u_n) = (0, 0, 1, 1, 1, 2, 2, 2, 1, 1, 0, 0, 1, 1, 0, 0).$$

We can see that  $u_i \ge 0$  for all  $i \in [1, \lambda_2 - 1] = \{1, 2, \dots, 9\}$ , and  $u_i \ge 0$  for all  $i \in [\lambda_2, n] = \{10, 11, \dots, 16\}$ . Note that in this example, we have  $u_i \ge 0$  for all  $i \in [n]$ , which is stronger than Claim 2. However, this is not the case in general.

Fig. 4. An example of Case (ii).

## C. Proof of x = x' for Case (iii)

For this case, we have  $e_1 < d_1 \leq d_2 < e_2$  or  $e_2 < d_1 \leq$  $d_2 < e_1$ . Let  $\lambda_1 = \min\{e_1, e_2\}$  and  $\lambda_2 = \max\{e_1, e_2\}$ . Then we have  $\lambda_1 < d_1 \leq d_2 < \lambda_2$ . Since  $E(\boldsymbol{x}, d_1, e_1) =$  $E(\mathbf{x}', d_2, e_2)$ , analogous to (6), we can obtain

$$x_{i} = \begin{cases} x'_{i}, & \text{for } i \in [1, d_{1} - 1] \setminus \{\lambda_{1}\}, \\ x'_{i-1}, & \text{for } i \in [d_{1} + 1, d_{2}], \\ x'_{i}, & \text{for } i \in [d_{2} + 1, n] \setminus \{\lambda_{2}\}. \end{cases}$$
(13)

Moreover, we have  $x_{\lambda_1} \neq x'_{\lambda_1}$  and  $x_{\lambda_2} \neq x'_{\lambda_2}$  because of the substitution error. According to (13), this case can be illustrated by Fig. 5.



By Remark 3, we have wt(x) = wt(x') and  $x_{d_1} = x'_{d_2}$ . Then by (13) or Fig. 5, and through a cancelling process similar to Case (i), we can obtain 0 = wt(x) - wt(x') = $\sum_{\ell=1}^{n} x_{\ell} - \sum_{\ell=1}^{n} x'_{\ell} = x_{\lambda_1} + x_{\lambda_2} + x_{d_1} - x'_{\lambda_1} - x'_{\lambda_2} - x'_{d_2} = x_{\lambda_1} + x_{\lambda_2} - x'_{\lambda_1} - x'_{\lambda_2} \text{ and }$ 

$$u_{i} = \begin{cases} 0, & \text{for } i \in [1, \lambda_{1}]; \\ x_{\lambda_{2}} - x'_{\lambda_{2}}, & \text{for } i \in [\lambda_{1} + 1, d_{1}]; \\ x_{i} + x_{\lambda_{2}} - x'_{d_{2}} - x'_{\lambda_{2}}, & \text{for } i \in [d_{1} + 1, d_{2}]; \\ x_{\lambda_{2}} - x'_{\lambda_{2}}, & \text{for } i \in [d_{2} + 1, \lambda_{2}]; \\ 0, & \text{for } i \in [\lambda_{2} + 1, n]. \end{cases}$$
(14)

Then we have the following Claim.

Claim 3: Either  $u_i \ge 0$  for all  $i \in [n]$  or  $u_i \le 0$  for all  $i \in [n]$ . Moreover,  $|u_i| \leq 2$  for all  $i \in [n]$ .

*Proof of Claim 3:* Note that  $x_{\lambda_2} \neq x'_{\lambda_2}$ . Then we have  $x_{\lambda_2} - x'_{\lambda_2} \in \{-1, 1\}$ . To prove Claim 3, similar to Case (ii), we consider the following two subcases.

Case (iii.1):  $x_{\lambda_2} - x'_{\lambda_2} = 1$ .

By (14), we can obtain

$$u_{i} = \begin{cases} 0, & \text{for } i \in [1, \lambda_{1}]; \\ 1, & \text{for } i \in [\lambda_{1} + 1, d_{1}]; \\ x_{i} - x'_{d_{2}} + 1, & \text{for } i \in [d_{1} + 1, \lambda_{2} - 1]; \\ 1, & \text{for } i \in [\lambda_{2}, d_{2}]; \\ 0, & \text{for } i \in [d_{2} + 1, n]. \end{cases}$$

Note that  $1 - x'_{d_2} \ge 0$  and  $x_i \ge 0$  for all  $i \in [n]$ . Then  $u_i \ge 0$ for all  $i \in [n]$ .

Case (iii.2):  $x_{\lambda_2} - x'_{\lambda_2} = -1$ .

By (14), we can obtain

$$u_i = \begin{cases} 0, & \text{for } i \in [1, \lambda_1]; \\ -1, & \text{for } i \in [\lambda_1 + 1, d_1]; \\ x_i - x'_{d_2} - 1, & \text{for } i \in [d_1 + 1, \lambda_2 - 1]; \\ -1, & \text{for } i \in [\lambda_2, d_2]; \\ 0, & \text{for } i \in [d_2 + 1, n]. \end{cases}$$

Since  $-x'_{d_2} \leq 0$  and  $x_i - 1 \leq 0$  for all  $i \in [n]$ , then  $u_i \leq 0$ for all  $i \in [n]$ .

Thus, either  $u_i \ge 0$  for all  $i \in [n]$  or  $u_i \le 0$  for all  $i \in [n]$ . Note that  $|x_{\lambda_2} - x'_{\lambda_2}| \le 1$  and  $|x_i - x'_{d_2}| \le 1$ . It is easy to see from (14) that  $|u_i| \le 2$  for all  $i \in [n]$ , which proves Claim 3.

Similar to Case (i), by (2) and by Claim 3, for j = 1, 2,

$$|f_j(\boldsymbol{x}) - f_j(\boldsymbol{x}')| \le \sum_{i=1}^n |u_i| i^{j-1} \le \sum_{i=1}^n 2i^{j-1} < 2n^j.$$

On the other hand, by (C1) and (C2) of Definition 1, we have  $f_i(\mathbf{x}) \equiv f_i(\mathbf{x}') \pmod{2n^j}$ , so  $f_i(\mathbf{x}) = f_i(\mathbf{x}')$ . Then by Claim 3 and Lemma 2, we have x = x'.

Example 3: Consider an example with

$$x = 10010101010011111,$$
  
 $y = 100110101001101,$   
 $x' = 110110100001101,$ 

where n = 16. We can check that y can be obtained from x by deleting  $x_5 = 0$  and substituting  $x_{15} = 1$  with  $y_{14} =$  $\bar{x}_{15} = 0$ , and y can also be obtained from x' by deleting  $x'_{10} = 0$  and substituting  $x'_2 = 1$  with  $y_2 = \bar{x}'_2 = 0$ . Hence, y = E(x, 5, 15) = E(x', 10, 2), that is,  $d_1 = 5, e_1 = 15$ ,  $d_2 = 10$  and  $e_2 = 2$ . Since  $e_2 < e_1$ , we take  $\lambda_1 = e_2 = 2$  and  $\lambda_2 = e_1 = 15$ . For this example,  $\boldsymbol{x}$  and  $\boldsymbol{x}'$  can be illustrated by Fig. 6, which is an instance of Fig. 5. It is easy to check that:

• For  $i \in [\lambda_1] = \{1, 2\}, u_i = \sum_{\ell=i}^n x_\ell - \sum_{\ell=i}^n x'_\ell = x_{\lambda_1} + x_{\lambda_2} + x_{d_1} - x'_{\lambda_1} - x'_{\lambda_2} - x'_{d_2} = 0;$ 

• For 
$$i \in [\lambda_1 + 1, d_1] = \{3, 4, 5\}, u_i = \sum_{\ell=i}^n x_\ell - \sum_{\ell=i}^n x'_\ell = x_{d_1} + x_{\lambda_2} - x'_{d_2} - x'_{\lambda_2} = x_{\lambda_2} - x'_{\lambda_2} = 1;$$

- For  $i \in [d_1 + 1, d_2] = \{6, 7, 8, 9, 10\}, u_i = \sum_{\ell=i}^n x_\ell \sum_{\ell=i}^n x'_\ell = x_i + x_{\lambda_2} x'_{\lambda_2} = x_i + 1 \in \{1, 2\};$  For  $i \in [d_2 + 1, \lambda_2] = \{11, 12, 13, 14, 15\}, u_i = \sum_{\ell=i}^n x_\ell \sum_{\ell=i}^n x'_\ell = x_{\lambda_2} x'_{\lambda_2} = 1;$  For  $i \in [\lambda_2 + 1, n] = \{16\}, u_i = \sum_{\ell=i}^n x_\ell \sum_{\ell=i}^n x'_\ell = 0;$

In summary, we have

$$(u_1, u_2, \cdots, u_n) = (0, 0, 1, 1, 1, 2, 1, 2, 1, 2, 1, 2, 1, 1, 1, 1, 1, 0).$$

We can see that  $u_i \ge 0$  for all  $i \in [n] = \{1, 2, \dots, 16\}$ .

Fig. 6. An example of Case (iii).

#### D. Proof of x = x' for Case (v)

For this case, we have  $d_1 < e_1 \le d_2 < e_2$  or  $d_1 \le e_2 < d_2 < e_1$ . If  $d_1 < e_1 \le d_2 < e_2$ , let  $\lambda_1 = e_1$  and  $\lambda_2 = e_2$ ; If  $d_1 \le e_2 < d_2 < e_1$ , let  $\lambda_1 = e_2 + 1$  and  $\lambda_2 = e_1$ . Then for both cases, we always have  $d_1 < \lambda_1 \le d_2 < \lambda_2$ . Since  $E(\boldsymbol{x}, d_1, e_1) = E(\boldsymbol{x}', d_2, e_2)$ , analogous to (6), we can obtain

$$x_{i} = \begin{cases} x'_{i}, & \text{for } i \in [1, d_{1} - 1], \\ x'_{i-1}, & \text{for } i \in [d_{1} + 1, d_{2}] \setminus \{\lambda_{1}\}, \\ x'_{i}, & \text{for } i \in [d_{2} + 1, n] \setminus \{\lambda_{2}\}. \end{cases}$$
(15)

Moreover, we have  $x_{\lambda_1} \neq x'_{\lambda_1-1}$  and  $x_{\lambda_2} \neq x'_{\lambda_2}$  because of the substitution error. According to (15), this case can be illustrated by Fig. 7.

$$\begin{array}{c} x_i: & \dots & d_1 & \dots & \lambda_1 & \dots & d_2 \\ x_i': & \dots & & & & & & & & & & & & & & & & \\ \end{array}$$

Fig. 7. Illustration of Case (v).

By Remark 3, we have wt(x) = wt(x') and  $x_{d_1} = x'_{d_2}$ . Then by (15) or Fig. 7, and through a cancelling process similar to Case (i), we can obtain  $0 = \text{wt}(x) - \text{wt}(x') = \sum_{\ell=1}^{n} x_\ell - \sum_{\ell=1}^{n} x'_\ell = x_{\lambda_1} + x_{\lambda_2} + x_{d_1} - x'_{\lambda_1-1} - x'_{\lambda_2} - x'_{d_2} = x_{\lambda_1} + x_{\lambda_2} - x'_{\lambda_1-1} - x'_{\lambda_2}$  and

$$u_{i} = \begin{cases} 0, & \text{for } i \in [1, d_{1}]; \\ x_{i} - x'_{d_{2}}, & \text{for } i \in [d_{1} + 1, \lambda_{1} - 1]; \\ x_{i} + x_{\lambda_{2}} - x'_{d_{2}} - x'_{\lambda_{2}}, & \text{for } i \in [\lambda_{1}, d_{2}]; \\ x_{\lambda_{2}} - x'_{\lambda_{2}}, & \text{for } i \in [d_{2} + 1, \lambda_{2}]; \\ 0, & \text{for } i \in [\lambda_{2} + 1, n]. \end{cases}$$
(16)

Then we have the following Claim.

Claim 4: Let  $p_1 = \lambda_1 - 1$ . Then for each  $j \in \{1, 2\}$ , either  $u_i \ge 0$  for all  $i \in [p_{j-1} + 1, p_j]$  or  $u_i \le 0$  for all  $i \in [p_{j-1} + 1, p_j]$ , where  $p_0 = 1$  and  $p_2 = n$ . Moreover, we have  $|u_i| \le 2$  for all  $i \in [n]$ .

Proof of Claim 4: For  $i \in [1, \lambda_1 - 1]$ , by (16),  $u_i = 0$ or  $u_i = x_i - x'_{d_2}$ . Clearly, if  $x'_{d_2} = 0$ , then  $u_i \ge 0$  for all  $i \in [1, \lambda_1 - 1]$ ; if  $x'_{d_2} = 1$ , then  $u_i \le 0$  for all  $i \in [1, \lambda_1 - 1]$ . For  $i \in [\lambda_1, n]$ , similar to Case (ii), we need to consider the following two subcases.

Case (v.1):  $x_{\lambda_2} - x'_{\lambda_2} = 1$ . By (16), we have

$$u_i = \begin{cases} x_i - x'_{d_2} + 1, \text{ for } i \in [\lambda_1, d_2];\\ 1, & \text{ for } i \in [d_2 + 1, \lambda_2];\\ 0, & \text{ for } i \in [\lambda_2 + 1, n]. \end{cases}$$

Note that  $1 - x'_{d_2} \ge 0$  and  $x_i \ge 0$  for all  $i \in [n]$ . Then  $u_i \ge 0$  for all  $i \in [\lambda_1, n]$ .

Case (v.2):  $x_{\lambda_2} - x'_{\lambda_2} = -1$ . By (16), we have

$$u_i = \begin{cases} x_i - x'_{d_2} - 1, \text{ for } i \in [\lambda_1, d_2]; \\ -1, \text{ for } i \in [d_2 + 1, \lambda_2]; \\ 0, \text{ for } i \in [\lambda_2 + 1, n]. \end{cases}$$

Since  $-x'_{d_2} \leq 0$  and  $x_i - 1 \leq 0$  for all  $i \in [n]$ , then  $u_i \leq 0$  for all  $i \in [\lambda_1, n]$ .

Thus,  $p_1 = \lambda_1 - 1$  satisfies the desired property.

Note that  $|x_{\lambda_2} - x'_{\lambda_2}| \le 1$  and  $|x_i - x'_{d_2}| \le 1$ . It is easy to see from (16) that  $|u_i| \le 2$  for all  $i \in [n]$ , which proves Claim 4.

Similar to Case (i), by (2) and by Claim 4, for j = 1, 2,

$$|f_j(\boldsymbol{x}) - f_j(\boldsymbol{x}')| \le \sum_{i=1}^n |u_i| i^{j-1} \le \sum_{i=1}^n 2i^{j-1} < 2n^j.$$

On the other hand, by (C1) and (C2) of Definition 1, we have  $f_j(\boldsymbol{x}) \equiv f_j(\boldsymbol{x}') \pmod{2n^j}$ , so  $f_j(\boldsymbol{x}) = f_j(\boldsymbol{x}')$ . Then by Claim 4 and Lemma 2, we have  $\boldsymbol{x} = \boldsymbol{x}'$ .

# E. Proof of x = x' for Case (vi)

For this case, we have  $d_1 \leq d_2 < e_1$  and  $d_1 \leq d_2 < e_2$ . Similar to Case (i), if  $e_1 = e_2$ , then  $\boldsymbol{x} = \boldsymbol{x}'$ . Therefore, we assume  $e_1 \neq e_2$ . Let  $\lambda_1 = \min\{e_1, e_2\}$  and  $\lambda_2 = \max\{e_1, e_2\}$ . Then we have  $d_1 \leq d_2 < \lambda_1 < \lambda_2$ . Since  $E(\boldsymbol{x}, d_1, e_1) = E(\boldsymbol{x}', d_2, e_2)$ , analogous to (6), we can obtain

$$x_{i} = \begin{cases} x'_{i}, & \text{for } i \in [1, d_{1} - 1], \\ x'_{i-1}, & \text{for } i \in [d_{1} + 1, d_{2}], \\ x'_{i}, & \text{for } i \in [d_{2} + 1, n] \setminus \{\lambda_{1}, \lambda_{2}\}. \end{cases}$$
(17)

Moreover, we have  $x_{\lambda_1} \neq x'_{\lambda_1}$  and  $x_{\lambda_2} \neq x'_{\lambda_2}$  because of the substitution error. According to (17), this case can be illustrated by Fig. 8.

### Fig. 8. Illustration of Case (vi).

By Remark 3, we have wt( $\boldsymbol{x}$ ) = wt( $\boldsymbol{x}'$ ) and  $x_{d_1} = x'_{d_2}$ . Then by (17) or Fig. 8, and through a cancelling process similar to Case (i), we can obtain  $0 = \text{wt}(\boldsymbol{x}) - \text{wt}(\boldsymbol{x}') =$  $\sum_{\ell=1}^{n} x_{\ell} - \sum_{\ell=1}^{n} x'_{\ell} = x_{\lambda_1} + x_{\lambda_2} + x_{d_1} - x'_{\lambda_1} - x'_{\lambda_2} - x'_{d_2} =$  $x_{\lambda_1} + x_{\lambda_2} - x'_{\lambda_1} - x'_{\lambda_2} \text{ and}$ 

$$u_{i} = \begin{cases} 0, & \text{for } i \in [1, d_{1}]; \\ x_{i} - x'_{d_{2}}, & \text{for } i \in [d_{1} + 1, d_{2}]; \\ 0, & \text{for } i \in [d_{2} + 1, \lambda_{1}]; \\ x_{\lambda_{2}} - x'_{\lambda_{2}}, & \text{for } i \in [\lambda_{1} + 1, \lambda_{2}]; \\ 0, & \text{for } i \in [\lambda_{2} + 1, n]. \end{cases}$$
(18)

Then we have the following Claim.

Claim 5: Let  $p_1 = d_2$ . Then for each  $j \in \{1, 2\}$ , either  $u_i \geq d_2$ 0 for all  $i \in [p_{j-1}+1, p_j]$  or  $u_i \leq 0$  for all  $i \in [p_{j-1}+1, p_j]$ , where  $p_0 = 1$  and  $p_2 = n$ . Moreover,  $|u_i| \le 1$  for all  $i \in [n]$ .

*Proof of Claim 5:* For  $i \in [1, d_2]$ , by (18), we have  $u_i = 0$  or  $u_i = x_i - x'_{d_2}$ . If  $x'_{d_2} = 0$ , then  $u_i \ge 0$  for all  $i \in [1, d_2]$ ; if  $x'_{d_2} = 1$ , then  $u_i \le 0$  for all  $i \in [1, d_2]$ .

For  $i \in [d_2 + 1, n]$ , by (18), we have  $u_i = 0$  or  $u_i =$  $x_{\lambda_2} - x'_{\lambda_2}$ . If  $x'_{\lambda_2} = 0$ , then  $u_i \ge 0$  for all  $i \in [d_2 + 1, n]$ ; if  $x'_{\lambda_2} = 1$ , then  $u_i \le 0$  for all  $i \in [d_2 + 1, n]$ . Thus,  $p_1 = d_2$  satisfies the desired property.

Note that  $|x_{\lambda_2} - x'_{\lambda_2}| \leq 1$  and  $|x_i - x'_{d_2}| \leq 1$ . It is easy to see from (18) that  $|u_i| \leq 1$  for all  $i \in [n]$ , which proves Claim 5.

Similar to Case (i), by (2) and by Claim 5, for j = 1, 2,

$$|f_j(\boldsymbol{x}) - f_j(\boldsymbol{x}')| \le \sum_{i=1}^n |u_i| i^{j-1} \le \sum_{i=1}^n i^{j-1} < n^j.$$

On the other hand, by (C1) and (C2) of Definition 1, we have  $f_i(\boldsymbol{x}) \equiv f_i(\boldsymbol{x}') \pmod{2n^j}$ , so  $f_i(\boldsymbol{x}) = f_i(\boldsymbol{x}')$ . Then by Claim 5 and Lemma 2, we have x = x'.

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