# List-decodable Codes for Single-deletion Single-substitution with List-size Two 

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#### Abstract

In this paper, we present an explicit construction of list-decodable codes for single-deletion and single-substitution with list size two and redundancy $3 \log n+4$, where $n$ is the block length of the code. Our construction has lower redundancy than the best known explicit construction by Gabrys et al. (arXiv 2021), whose redundancy is $4 \log n+O(1)$.


## I. Introduction

Codes correcting insertion, deletion and substitution errors (collectively referred to as edit errors) have gone through a long history from the seminal work of Levenshtein [1]. It was shown in [1] that the binary Varshamov-Tenengolts (VT) code [2], which is given by

$$
\mathscr{C}_{n}(a)=\left\{\boldsymbol{x} \in\{0,1\}^{n}: \sum_{i=1}^{n} i x_{i} \equiv a(\bmod n+1)\right\}
$$

can correct a single edit error and is asymptotically optimal in redundancy, given by $\log n+2$. Order-optimal non-binary single-edit correcting codes were studied in [3], [4].

Constructing optimal multiple-edit error correcting codes is much more challenging, even for binary deletion codes. A generalization of the VT construction for multiple-deletion correcting codes was presented in [5], but this generalized construction has asymptotic rate strictly smaller than 1 . Recently, there were many works on explicit construction of lowredundancy $t$-deletion correcting codes for $t \geq 2$ (e.g., see [6]- [12]). For $t=2$, Guruswami and Håstad constructed a family of 2 -deletion correcting codes with length $n$ and redundancy $4 \log n+O(\log \log n)$ [12], which matches the best known upper bound obtained via the Gilbert-Varshamovtype greedy algorithm [6]. By introducing the higher order VT syndromes and the syndrome compression technique, Sima et al. constructed a family of $t$-deletion correcting codes with redundancy $8 t \log n+o(\log n)$ [10]. Unfortunately, for $t>2$, all existential constructions of $t$-deletion correcting codes have redundancy greater than the Gilbert-Varshamov-type bound.

The best known $t$-edit correcting codes for $t \geq 2$ were given by Sima et al., which have redundancy $4 t \log n+o(\log n)$ [11]. The method in [11] was improved by the authors in [15], which gave a construction of $t$-deletion $s$-substitution correcting codes with redundancy $(4 t+3 s) \log n+o(\log n)$. A family of single-deletion single-substitution correcting binary codes with redundancy $6 \log n+8$ was constructed in [13]. So far, constructing optimal (with respect to redundancy)
multiple-edit correcting codes is still an open problem, even for single-deletion single-substitution correcting codes.

As a relaxation of the decoding requirement, list-decoding for insertions and deletions have been considered by several research teams, mainly focusing on list-decoding for some fraction of deletions/insertions [16]- [20]. Unlike the traditional decoding (also referred to as unique-decoding), listdecoding with list-size $\ell$ allows to give a set of $\ell$ codewords from each corrupted sequence. A family of explicit listdecodable codes for two deletions with length $n$ and list-size two was constructed in [12], which has redundancy $3 \log n$. Note that the redundancy of the construction in [12] is lower than the Gilbert-Varshamov-type bound, which is $4 \log n$. The improvement in redundancy is achieved by the relaxation in the decoding requirement.

In this paper, we present an explicit construction of listdecodable codes for single-deletion and single-substitution with list-size two and redundancy $3 \log n+4$. Our construction improves the recent work by Gabrys et al. [4], which constructed such codes with redundancy $4 \log n+O(1)$.

The rest of this paper is organized as follows. In Section II, the basic concepts are introduced and some preliminary properties of the errors are discussed. Our construction of listdecodable codes for single-deletion and single-substitution is presented in Section III. The auxiliary lemma used by our construction is proved in Section IV.

## II. Preliminaries

For any positive integers $m$ and $n$ such that $m \leq n$, denote $[m, n]=\{m, m+1, \ldots, n\}$. If $m>n$, let $[m, n]=\emptyset$. For simplicity, we denote $[n]=[1, n]$ and $\mathbb{Z}_{n}=[0, n-1]$.

In this work, we consider binary codes. For any sequence (vector) $\boldsymbol{x}$ of length $n$, we use $x_{i}$ to denote the $i$ th symbol of $\boldsymbol{x}$, and hence $\boldsymbol{x}$ can be denoted as $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in\{0,1\}^{n}$ or simply, $\boldsymbol{x}=x_{1} x_{2} \ldots x_{n}$. The weight of $\boldsymbol{x}$, denoted by $\mathrm{wt}(\boldsymbol{x})$, is the number of non-zero symbols (the symbol 1 for binary sequence) in $\boldsymbol{x}$. Clearly, for binary sequence $\boldsymbol{x}$, we have $\mathrm{wt}(\boldsymbol{x})=\sum_{i=1}^{n} x_{i}$.

Given non-negative integers $t$ and $s$ such that $t+s<n$, for any $\boldsymbol{x} \in\{0,1\}^{n}$, the error ball of $\boldsymbol{x}$ under $t$-deletion $s$ substitution, denoted by $\mathscr{B}_{t, s}(\boldsymbol{x})$, is the set of all sequences that can be obtained from $x$ by $t$ deletions (i.e., deleting $t$ symbols of $\boldsymbol{x}$ ) and at most $s$ substitutions (i.e., substituting at most $s$ symbols of $\boldsymbol{x}$, each with a different symbol). A code

| Cases of error combination | $\mathrm{wt}(\boldsymbol{x})-\mathrm{wt}(\boldsymbol{y})$ |
| :---: | :---: |
| $1 \rightarrow \epsilon$, no substitution | 1 |
| $1 \rightarrow \epsilon, 1 \rightarrow 0$ | 2 |
| $1 \rightarrow \epsilon, 0 \rightarrow 1$ | 0 |
| $0 \rightarrow \epsilon$, no substitution | 0 |
| $0 \rightarrow \epsilon, 1 \rightarrow 0$ | 1 |
| $0 \rightarrow \epsilon, 0 \rightarrow 1$ | -1 |

Table 1. The value of $w t(\boldsymbol{x})-\mathrm{wt}(\boldsymbol{y})$ for different cases of single-deletion single-substitution, where $a \rightarrow \epsilon$ means a symbol $a \in\{0,1\}$ is deleted from $\boldsymbol{x}$ and for $b \in\{0,1\} \backslash\{a\}, a \rightarrow b$ means a symbol $a$ of $\boldsymbol{x}$ is substituted by the symbol $b$.
$\mathscr{C} \subseteq\{0,1\}^{n}$ is list-decodable for $t$-deletion $s$-substitution with list size $\ell$ if any $\boldsymbol{y} \in\{0,1\}^{n-1}$ is contained by the error ball of at most $\ell$ codewords of $\mathscr{C}$. In other words, for any $\boldsymbol{y} \in$ $\{0,1\}^{n-1}$, there exist at most $\ell$ codewords of $\mathscr{C}$ from which $\boldsymbol{y}$ can be obtained by $t$ deletions and at most $s$ substitutions.

In this work, we consider list-decodable codes for single deletion and single substitution, i.e., $t=s=1$. Suppose $\boldsymbol{x} \in\{0,1\}^{n}$ and $\boldsymbol{y} \in\{0,1\}^{n-1}$ such that $\boldsymbol{y}$ can be obtained from $\boldsymbol{x}$ by deleting one symbol of $\boldsymbol{x}$ and substituting at most one symbol of $\boldsymbol{x}$ with a different symbol in $\{0,1\}$. We can compute the difference between the weights of $\boldsymbol{x}$ and $\boldsymbol{y}$ for all possible cases (see Table 1). According to Table 1, we have $w \mathrm{wt}(\boldsymbol{x})-\mathrm{wt}(\boldsymbol{y}) \in\{-1,0,1,2\}$. If $\mathrm{wt}(\boldsymbol{x})-\mathrm{wt}(\boldsymbol{y}) \in\{-1,2\}$, then the values of the deleted and substituted symbols can be determined. If $w t(\boldsymbol{x})-\mathrm{wt}(\boldsymbol{y})=0$, then $\boldsymbol{y}$ can be obtained from $\boldsymbol{x}$ by deleting a 0 , or by deleting a 1 and substituting a 0 with a 1 . For the case that $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by deleting a 0 , unless $\boldsymbol{x}$ is the all-zero sequence, $\boldsymbol{y}$ can also be obtained from $\boldsymbol{x}$ by deleting a 1 and substituting a 0 with a $1 .{ }^{1}$ Hence, if $w t(\boldsymbol{x})-\mathrm{wt}(\boldsymbol{y})=0$, then $\boldsymbol{y}$ can always be obtained from $\boldsymbol{x}$ by deleting a 1 and substituting a 0 with a 1 . Similarly, if $\mathrm{wt}(\boldsymbol{x})-\mathrm{wt}(\boldsymbol{y})=1$ and $\boldsymbol{x}$ is not the all-one sequence, then $\boldsymbol{y}$ can always be obtained from $\boldsymbol{x}$ by deleting a 0 and substituting a 1 with a 0 . In summary, we have the following remark.

Remark 1: Suppose $\boldsymbol{x} \in\{0,1\}^{n} \backslash\left\{1^{n}, 0^{n}\right\}$, where $1^{n}$ and $0^{n}$ are the all-one sequence and the all-zero sequence of length $n$ respectively, and $\boldsymbol{y} \in\{0,1\}^{n-1}$ such that $\boldsymbol{y}$ can be obtained from $\boldsymbol{x}$ by deleting one symbol of $\boldsymbol{x}$ and substituting at most one symbol of $\boldsymbol{x}$. Then $\boldsymbol{y}$ can be obtained from $\boldsymbol{x}$ by deleting one symbol and substituting exactly one symbol of $\boldsymbol{x}$, and the values of the deleted and substituted symbols can be determined by $w t(\boldsymbol{x})(\bmod 4)$ and $\operatorname{wt}(\boldsymbol{y})$.

## III. Main Results

In this section, we present our construction of list-decodable codes for single-deletion and single-substitution. Our construction only uses the weight and the first two order VT syndromes

[^0]for binary sequences.
We adopt the method of [10] to define the higher order VT syndromes. For each positive integer $j$ and each $\boldsymbol{x} \in\{0,1\}^{n}$, the $j$ th-order VT syndrome of $\boldsymbol{x}$ is defined as
\[

$$
\begin{equation*}
f_{j}(\boldsymbol{x})=\sum_{i=1}^{n}\left(\sum_{\ell=1}^{i} \ell^{j-1}\right) x_{i} \tag{1}
\end{equation*}
$$

\]

As in [14], we can rearrange the terms and obtain

$$
\begin{equation*}
f_{j}(\boldsymbol{x})=\sum_{\ell=1}^{n}\left(\sum_{i=1}^{\ell} i^{j-1}\right) x_{\ell}=\sum_{i=1}^{n}\left(\sum_{\ell=i}^{n} x_{\ell}\right) i^{j-1} \tag{2}
\end{equation*}
$$

The code is given by the following definition, where $1^{n}$ and $0^{n}$ denote the all-one sequence and the all-zero sequence of length $n$, respectively.

Definition 1: For any fixed values $c_{0} \in \mathbb{Z}_{4}, c_{1} \in \mathbb{Z}_{2 n}$ and $c_{2} \in \mathbb{Z}_{2 n^{2}}$, let $\mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$ be the set of all sequences $\boldsymbol{x} \in$ $\{0,1\}^{n} \backslash\left\{1^{n}, 0^{n}\right\}$ satisfying the following three conditions:
(C0) $\mathrm{wt}(\boldsymbol{x}) \equiv c_{0}(\bmod 4)$.
(C1) $f_{1}(\boldsymbol{x}) \equiv c_{1}(\bmod 2 n)$.
(C2) $f_{2}(\boldsymbol{x}) \equiv c_{2}\left(\bmod 2 n^{2}\right)$.
Then our main result can be stated as the following theorem.
Theorem 1: There exists a $\left(c_{0}, c_{1}, c_{2}\right) \in \mathbb{Z}_{4} \times \mathbb{Z}_{2 n} \times \mathbb{Z}_{2 n^{2}}$ such that the code $\mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$ in Definition 1 has redundancy at most $3 \log n+4$ and is list-decodable from singledeletion and single-substitution with list size 2.

In the rest of this section, we always assume that $\left(c_{0}, c_{1}, c_{2}\right) \in \mathbb{Z}_{4} \times \mathbb{Z}_{2 n} \times \mathbb{Z}_{2 n^{2}}$ and $\mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$ is given by Definition 1. For any $\boldsymbol{x} \in\{0,1\}$ and $\{d, e\} \subseteq[n]$, let $E(\boldsymbol{x}, d, e)$ denote the sequence obtained from $\boldsymbol{x}$ by deleting $x_{d}$ and substituting $x_{e}$ with $\bar{x}_{e}=1-x_{e}$ (i.e., $\bar{x}_{e}=1$ if $x_{e}=0$ and $\bar{x}_{e}=0$ if $x_{e}=1$. Clearly, $E(\boldsymbol{x}, d, e) \in\{0,1\}^{n-1}$ is uniquely determined by $\boldsymbol{x}, d$ and $e$. We also need the following lemma, which will be proved in Section IV.

Lemma 1: Suppose $\boldsymbol{x}, \boldsymbol{x}^{\prime} \in \mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$ and $\left\{d_{1}, e_{1}\right\}$, $\left\{d_{2}, e_{2}\right\} \subseteq[n]$ such that $\boldsymbol{x} \neq \boldsymbol{x}^{\prime}, d_{1} \leq d_{2}$ and $E\left(\boldsymbol{x}, d_{1}, e_{1}\right)=$ $E\left(\boldsymbol{x}^{\prime}, d_{2}, e_{2}\right)$. We have $d_{1}<e_{1} \leq d_{2}$ and $d_{1} \leq e_{2}<d_{2}$.

In formally speaking, if there exists a $\boldsymbol{y} \in\{0,1\}^{n-1}$ such that $\boldsymbol{y}$ can be obtained from $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime}$ by deleting one symbol and substituting one symbol, then the two substituted symbols are both located between the two deleted symbols.

Using Lemma 1, we can prove Theorem 1 as follows.
Proof of Theorem 1: By the pigeonhole principle, there exists a $\left(c_{0}, c_{1}, c_{2}\right) \in \mathbb{Z}_{4} \times \mathbb{Z}_{2 n} \times \mathbb{Z}_{2 n^{2}}$ such that the code $\mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$ has size at least $\frac{2^{n}-2}{16 n^{3}}$, hence the redundancy of $\mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$ is at most $3 \log n+4$.

It remains to prove that $\mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$ is list-decodable from single-deletion and single-substitution with list size 2 . We need to prove that for any given $\boldsymbol{y} \in\{0,1\}^{n-1}$, there exist at most two codewords in $\mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$, from which $\boldsymbol{y}$ can be obtained by one deletion and at most one substitution. This can be proved by contradiction as follows.

Suppose $\boldsymbol{x}, \boldsymbol{x}^{\prime}$ and $\boldsymbol{x}^{\prime \prime}$ are three distinct sequences in $\mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$ from which $\boldsymbol{y}$ can be obtained by one deletion and at most one substitution. By Remark 1, we can assume
$\boldsymbol{y}=E\left(\boldsymbol{x}, d_{1}, e_{1}\right)=E\left(\boldsymbol{x}^{\prime}, d_{2}, e_{2}\right)=E\left(\boldsymbol{x}^{\prime \prime}, d_{3}, e_{3}\right)$. Without loss of generality, assume $d_{1} \leq d_{2}$.

First, consider $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime}$. By Lemma 1, we have

$$
\begin{equation*}
d_{1}<e_{1} \leq d_{2} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
d_{1} \leq e_{2}<d_{2} \tag{4}
\end{equation*}
$$

For further discussions, we have the following three cases.
Case 1: $d_{3} \leq d_{1}$. Considering $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime \prime}$, by Lemma 1 , we have $d_{3} \leq e_{1}<d_{1}$ and $d_{3}<e_{3} \leq d_{1}$. Combining with (3), we have $e_{1}<d_{1}<e_{1}$, a contradiction.

Case 2: $d_{1}<d_{3} \leq d_{2}$. Considering $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime \prime}$, by Lemma 1 , we have $d_{1}<e_{1} \leq d_{3}$ and $d_{1} \leq e_{3}<d_{3}$. On the other hand, considering $\boldsymbol{x}^{\prime}$ and $\boldsymbol{x}^{\prime \prime}$, by Lemma 1 , we have $d_{3}<$ $e_{3} \leq d_{2}$ and $d_{3} \leq e_{2}<d_{2}$. Hence, we obtain $e_{3}<d_{3}<e_{3}$, a contradiction.

Case 3: $d_{2}<d_{3}$. Considering $\boldsymbol{x}^{\prime}$ and $\boldsymbol{x}^{\prime \prime}$, by Lemma 1 , we have $d_{2}<e_{2} \leq d_{3}$ and $d_{2} \leq e_{3}<d_{3}$. Combining with (4), we get $e_{2}<d_{2}<e_{2}$, a contradiction.

From the above discussions, we can conclude that there exist at most two codewords in $\mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$ from which $\boldsymbol{y}$ can be obtained by one deletion and at most one substitution, which proves Theorem 1.

## IV. Proof of Lemma 1

In this section, we prove Lemma 1 . We always suppose that $\boldsymbol{x}, \boldsymbol{x}^{\prime} \in \mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$, and $\left\{d_{1}, e_{1}\right\},\left\{d_{2}, e_{2}\right\} \subseteq[n]$ such that $d_{1} \leq d_{2}$ and $E\left(\boldsymbol{x}, d_{1}, e_{1}\right)=E\left(\boldsymbol{x}^{\prime}, d_{2}, e_{2}\right)$. We first enumerate all the possible cases according to the order of $d_{1}, e_{1}, d_{2}, e_{2}$.

Remark 2: Consider $e_{1}, d_{1}$ and $d_{2}$. Since $d_{1} \leq d_{2}$, we have three cases: $e_{1}<d_{1}, d_{1}<e_{1} \leq d_{2}$ and $d_{2}<e_{1}$. Similarly, for $e_{2}, d_{1}$ and $d_{2}$, we have three cases: $e_{2}<d_{1}, d_{1} \leq e_{2}<d_{2}$ and $d_{2}<e_{2}$. Combining these two scenarios we have a total of nine cases to consider. However, we can merge some cases and consider the following six cases.
(i) $e_{1}<d_{1} \leq d_{2}$ and $e_{2}<d_{1} \leq d_{2}$.
(ii) $e_{1}<d_{1} \leq e_{2}<d_{2}$ or $e_{2}<d_{1}<e_{1} \leq d_{2}$.
(iii) $e_{1}<d_{1} \leq d_{2}<e_{2}$ or $e_{2}<d_{1} \leq d_{2}<e_{1}$.
(iv) $d_{1}<e_{1} \leq d_{2}$ and $d_{1} \leq e_{2}<d_{2}$.
(v) $d_{1}<e_{1} \leq d_{2}<e_{2}$ or $d_{1} \leq e_{2}<d_{2}<e_{1}$.
(vi) $d_{1} \leq d_{2}<e_{1}$ and $d_{1} \leq d_{2}<e_{2}$.

We will prove that $\boldsymbol{x}=\boldsymbol{x}^{\prime}$ for all cases in Remark 2 except for Case (iv). Hence, if $\boldsymbol{x} \neq \boldsymbol{x}^{\prime}$, then it must fall into Case (iv), that is, $d_{1}<e_{1} \leq d_{2}$ and $d_{1} \leq e_{2}<d_{2}$.

Denote $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ and $\boldsymbol{x}^{\prime}=\left(x_{1}^{\prime}, x_{2}^{\prime}, \ldots, x_{n}^{\prime}\right)$. For each $i \in[n]$, let

$$
\begin{equation*}
u_{i} \triangleq \sum_{\ell=i}^{n} x_{\ell}-\sum_{\ell=i}^{n} x_{\ell}^{\prime} \tag{5}
\end{equation*}
$$

To prove $\boldsymbol{x}=\boldsymbol{x}^{\prime}$, it suffices to prove $u_{i}=0$ for all $i \in[n]$.
The following lemma will be used in our discussions. (Recall that for each positive integer $j$ and $\boldsymbol{x} \in\{0,1\}^{n}, f_{j}(\boldsymbol{x})$ is the $j$ th-order VT syndrome of $\boldsymbol{x}$ defined by (1) or (2).)

Lemma 2: Let $m$ be a fixed positive integer. Suppose $\left(f_{1}(\boldsymbol{x}), \ldots, f_{m+1}(\boldsymbol{x})\right)=\left(f_{1}\left(\boldsymbol{x}^{\prime}\right), \ldots, f_{m+1}\left(\boldsymbol{x}^{\prime}\right)\right)$ and there exist $m$ positive integers, say $p_{1}, p_{2}, \ldots, p_{m}$, such that $1 \leq$ $p_{1}<p_{2}<\cdots<p_{m} \leq n$ and for each $j \in[m+1]$, either $u_{i} \geq 0$ for all $i \in\left[p_{j-1}+1, p_{j}\right]$ or $u_{i} \leq 0$ for all $i \in\left[p_{j-1}+1, p_{j}\right]$, where $p_{0}=1$ and $p_{m+1}=n$. Then $u_{i}=0$ for all $i \in[n]$, and hence we have $\boldsymbol{x}=\boldsymbol{x}^{\prime}$.

The proof of Lemma 2 is omitted because it is (implicitly) contained in the proof of [10, Proposition 2].

The following simple remark is also useful in our proof.
Remark 3: Since $\mathrm{wt}(\boldsymbol{x}) \equiv \mathrm{wt}\left(\boldsymbol{x}^{\prime}\right) \equiv c_{0}(\bmod 4)$ (because $\left.\boldsymbol{x}, \boldsymbol{x}^{\prime} \in \mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)\right)$ and $E\left(\boldsymbol{x}, d_{1}, e_{1}\right)=E\left(\boldsymbol{x}^{\prime}, d_{2}, e_{2}\right)$, then by Remark 1, we have $x_{d_{1}}=x_{d_{2}}^{\prime}$ and $w t(\boldsymbol{x})=w t\left(\boldsymbol{x}^{\prime}\right)$.

In the following five subsections, we will prove that for all cases in Remark 2 except for Case (iv), we have $\left(f_{1}(\boldsymbol{x}), f_{2}(\boldsymbol{x})\right)=\left(f_{1}\left(\boldsymbol{x}^{\prime}\right), f_{2}\left(\boldsymbol{x}^{\prime}\right)\right)$, and there exists a $p_{1} \in[n]$ such that for each $j \in\{1,2\}$, either $u_{i} \geq 0$ for all $i \in\left[p_{j-1}+1, p_{j}\right]$ or $u_{i} \leq 0$ for all $i \in\left[p_{j-1}+1, p_{j}\right]$, where $p_{0}=1$ and $p_{2}=n$. Then by Lemma 2 (for the special case of $m=1$ ), we have $\boldsymbol{x}=\boldsymbol{x}^{\prime}$. Thus, if $\boldsymbol{x} \neq \boldsymbol{x}^{\prime}$, then it must fall into Case (iv), that is, $d_{1}<e_{1} \leq d_{2}$ and $d_{1} \leq e_{2}<d_{2}$.

## A. Proof of $\boldsymbol{x}=\boldsymbol{x}^{\prime}$ for Case (i)

For this case, we have $e_{1}<d_{1} \leq d_{2}$ and $e_{2}<d_{1} \leq$ $d_{2}$. If $e_{1}=e_{2}$, then $\boldsymbol{x}=\boldsymbol{x}^{\prime} .{ }^{2}$ Therefore, we assume $e_{1} \neq$ $e_{2}$. To simplify the presentation, let $\lambda_{1}=\min \left\{e_{1}, e_{2}\right\}$ and $\lambda_{2}=\max \left\{e_{1}, e_{2}\right\}$. Then $1 \leq \lambda_{1}<\lambda_{2}<d_{1} \leq d_{2}$. Since $E\left(\boldsymbol{x}, d_{1}, e_{1}\right)=E\left(\boldsymbol{x}^{\prime}, d_{2}, e_{2}\right)$, we can obtain ${ }^{3}$

$$
x_{i}= \begin{cases}x_{i}^{\prime}, & \text { for } i \in\left[1, d_{1}-1\right] \backslash\left\{\lambda_{1}, \lambda_{2}\right\}  \tag{6}\\ x_{i-1}^{\prime}, & \text { for } i \in\left[d_{1}+1, d_{2}\right] \\ x_{i}^{\prime}, & \text { for } i \in\left[d_{2}+1, n\right]\end{cases}
$$

Moreover, we have $x_{\lambda_{1}} \neq x_{\lambda_{1}}^{\prime}$ and $x_{\lambda_{2}} \neq x_{\lambda_{2}}^{\prime}$ because of the substitution error. According to (6), this case can be illustrated by Fig. 1 .


Fig. 1. Illustration of Case (i): The bits (symbols) of each sequence is denoted by a row of black dots, where each column corresponds to the two symbols at the same position in the respective sequences. Each pair of bits connected by a solid segment are of equal value, while those connected by a dashed segment have different values because of the substitution error.

We can use (6) or Fig. 1 to simplify $u_{i}$ for each $i \in[n]$ (In fact, Fig. 1 is more intuitive than (6).) as follows.

[^1]First, we simplify $u_{i}=\sum_{\ell=i}^{n} x_{\ell}-\sum_{\ell=i}^{n} x_{\ell}^{\prime}$ for $i \in\left[1, \lambda_{1}\right]$. From Fig. 1 we can see that all terms in $\sum_{\ell=i}^{n} x_{\ell}$ can be cancelled by their corresponding terms in $\sum_{\ell=i}^{n} x_{\ell}^{\prime}$ except for $x_{\lambda_{1}}, x_{\lambda_{2}}$ and $x_{d_{1}}$, and all terms in $\sum_{\ell=i}^{n} x_{\ell}^{\prime}$ can be cancelled except for $x_{\lambda_{1}}^{\prime}, x_{\lambda_{2}}^{\prime}$ and $x_{d_{2}}^{\prime}$, so we have

$$
\begin{align*}
u_{i} & =\sum_{\ell=i}^{n} x_{\ell}-\sum_{\ell=i}^{n} x_{\ell}^{\prime} \\
& =x_{\lambda_{1}}+x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime}-x_{d_{2}}^{\prime} \tag{7}
\end{align*}
$$

In particular, we have $\mathrm{wt}(\boldsymbol{x})-\mathrm{wt}\left(\boldsymbol{x}^{\prime}\right)=\sum_{\ell=1}^{n} x_{\ell}-\sum_{\ell=1}^{n} x_{\ell}^{\prime}=$ $u_{1}=x_{\lambda_{1}}+x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime}-x_{d_{2}}^{\prime}$. Note that by Remark 3 , $\mathrm{wt}(\boldsymbol{x})=\mathrm{wt}\left(\boldsymbol{x}^{\prime}\right)$ and $x_{d_{1}}=x_{d_{2}}^{\prime}$. Therefore, by (7), we have

$$
\begin{align*}
0 & =\mathrm{wt}(\boldsymbol{x})-\mathrm{wt}\left(\boldsymbol{x}^{\prime}\right) \\
& =x_{\lambda_{1}}+x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime}-x_{d_{2}}^{\prime} \\
& =x_{\lambda_{1}}+x_{\lambda_{2}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime} \tag{8}
\end{align*}
$$

and

$$
u_{i}=x_{\lambda_{1}}+x_{\lambda_{2}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime}=0, \forall i \in\left[1, \lambda_{1}\right] .
$$

Similarly, from Fig. 1, by cancelling the corresponding equivalent terms in $\sum_{\ell=i}^{n} x_{\ell}$ and $\sum_{\ell=i}^{n} x_{\ell}^{\prime}$, we can obtain:

- $u_{i}=x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{2}}^{\prime}-x_{d_{2}}^{\prime}=x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}$ for each $i \in\left[\lambda_{1}+1, \lambda_{2}\right]$, where the second equality holds because $x_{d_{1}}=x_{d_{2}}^{\prime}$ (according to Remark 3).
- $u_{i}=x_{d_{1}}-x_{d_{2}}^{\prime}=0$ for each $i \in\left[\lambda_{2}+1, d_{1}\right]$.
- $u_{i}=x_{i}-x_{d_{2}}^{\prime}$ for each $i \in\left[d_{1}+1, d_{2}\right]$.
- $u_{i}=0$ for each $i \in\left[d_{2}+1, n\right]$.

Collectively, we have

$$
u_{i}= \begin{cases}0, & \text { for } i \in\left[1, \lambda_{1}\right]  \tag{9}\\ x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}, & \text { for } i \in\left[\lambda_{1}+1, \lambda_{2}\right] \\ 0, & \text { for } i \in\left[\lambda_{2}+1, d_{1}\right] \\ x_{i}-x_{d_{2}}^{\prime}, & \text { for } i \in\left[d_{1}+1, d_{2}\right] \\ 0, & \text { for } i \in\left[d_{2}+1, n\right]\end{cases}
$$

Moreover, we have the following claim.
Claim 1: Let $p_{1}=\lambda_{2}$. Then for each $j \in\{1,2\}$, either $u_{i} \geq$ 0 for all $i \in\left[p_{j-1}+1, p_{j}\right]$ or $u_{i} \leq 0$ for all $i \in\left[p_{j-1}+1, p_{j}\right]$, where $p_{0}=1$ and $p_{2}=n$. Moreover, $\left|u_{i}\right| \leq 1$ for all $i \in[n]$.

Proof of Claim 1: For $i \in\left[1, \lambda_{2}\right]$, by (9), we have $u_{i}=0$ or $u_{i}=x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}$. If $x_{\lambda_{2}}^{\prime}=0$, then $u_{i} \in\{0,1\}$ for all $i \in$ $\left[\lambda_{2}+1, n\right]$; if $x_{\lambda_{2}}^{\prime}=1$, then $u_{i} \in\{-1,0\}$ for all $i \in\left[\lambda_{2}+1, n\right]$.

For $i \in\left[\lambda_{2}+1, n\right]$, by (9), we have $u_{i}=0$ or $u_{i}=x_{i}-x_{d_{2}}^{\prime}$. If $x_{d_{2}}^{\prime}=0$, then $u_{i} \in\{0,1\}$ for all $i \in\left[\lambda_{2}+1, n\right]$; if $x_{d_{2}}^{\prime}=1$, then $u_{i} \in\{-1,0\}$ for all $i \in\left[\lambda_{2}+1, n\right]$.

Thus, $p_{1}=\lambda_{2}$ satisfies the desired property and $\left|u_{i}\right| \leq 1$ for all $i \in[n]$, which proves Claim 1 .

By (2), for $j=1,2$, we have

$$
\begin{align*}
\left|f_{j}(\boldsymbol{x})-f_{j}\left(\boldsymbol{x}^{\prime}\right)\right| & =\left|\sum_{i=1}^{n}\left(\sum_{\ell=i}^{n} x_{\ell}\right) i^{j-1}-\sum_{i=1}^{n}\left(\sum_{\ell=i}^{n} x_{\ell}^{\prime}\right) i^{j-1}\right| \\
& =\left|\sum_{i=1}^{n} u_{i} i^{j-1}\right| \\
& \leq \sum_{i=1}^{n} i^{j-1} \\
& <n^{j} \tag{10}
\end{align*}
$$

where the first inequality holds because by Claim $1,\left|u_{i}\right| \leq 1$ for all $i \in[n]$. Note that by (C1) and (C2) of Definition 1, $f_{j}(\boldsymbol{x}) \equiv f_{j}\left(\boldsymbol{x}^{\prime}\right)\left(\bmod 2 n^{j}\right)$, so by (10), we have $f_{j}(\boldsymbol{x})=$ $f_{j}\left(\boldsymbol{x}^{\prime}\right)$. Thus, by Claim 1 and Lemma 2, we have $\boldsymbol{x}=\boldsymbol{x}^{\prime}$.

Example 1: To help the reader to understand the proof, consider an example with

$$
\begin{aligned}
\boldsymbol{x} & =1101101000101110 \\
\boldsymbol{y} & =110111100101110 \\
\boldsymbol{x}^{\prime} & =1001111001011010
\end{aligned}
$$

where $n=16$. We can check that $\boldsymbol{y}$ can be obtained from $\boldsymbol{x}$ by deleting $x_{10}=0$ and substituting $x_{6}=0$ with $y_{6}=$ $\bar{x}_{6}=1$, and $\boldsymbol{y}$ can also be obtained from $\boldsymbol{x}^{\prime}$ by deleting $x_{14}^{\prime}=0$ and substituting $x_{2}^{\prime}=0$ with $y_{2}=\bar{x}_{2}^{\prime}=1$. Hence, $\boldsymbol{y}=E(\boldsymbol{x}, 10,6)=E\left(\boldsymbol{x}^{\prime}, 14,2\right)$, that is, $d_{1}=10, e_{1}=6$, $d_{2}=14$ and $e_{2}=2$. Since $e_{2}<e_{1}$, we take $\lambda_{1}=e_{2}=2$ and $\lambda_{2}=e_{1}=6$. For this example, $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime}$ can be illustrated by Fig. 2, which is an instance of Fig. 1. It is easy to check that:

- For $i \in\left[\lambda_{1}\right]=\{1,2\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{\lambda_{1}}+$ $x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime}-x_{d_{2}}^{\prime}=0$;
- For $i \in\left[\lambda_{1}+1, \lambda_{2}\right]=\{3,4,5,6\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-$ $\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{2}}^{\prime}-x_{d_{2}}^{\prime}=x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}=-1$;
- For $i \in\left[\lambda_{2}+1, d_{1}\right]=\{7,8,9,10\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-$ $\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{d_{1}}-x_{d_{2}}^{\prime}=0$;
- For $i \in\left[d_{1}+1, d_{2}\right]=\{11,12,13,14\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-$ $\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{i}-x_{d_{2}}^{\prime}=x_{i} \in\{0,1\} ;$
- For $i \in\left[d_{2}+1, n\right]=\{15,16\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-$ $\sum_{\ell=i}^{n} x_{\ell}^{\prime}=0$;
In summary, we have

$$
\begin{aligned}
& \left(u_{1}, u_{2}, \cdots, u_{n}\right) \\
& =(0,0,-1,-1,-1,-1,0,0,0,0,1,0,1,1,0,0)
\end{aligned}
$$

We can see that $u_{i} \leq 0$ for all $i \in\left[1, \lambda_{2}\right]=\{1,2, \cdots, 6\}$, and $u_{i} \geq 0$ for all $i \in\left[\lambda_{2}+1, n\right]=\{7,8, \cdots, 16\}$.


Fig. 2. An example of Case (i).

## B. Proof of $\boldsymbol{x}=\boldsymbol{x}^{\prime}$ for Case (ii)

For this case, we have $e_{1}<d_{1} \leq e_{2}<d_{2}$ or $e_{2}<d_{1}<$ $e_{1} \leq d_{2}$. If $e_{1}<d_{1} \leq e_{2}<d_{2}$, let $\lambda_{1}=e_{1}$ and $\lambda_{2}=e_{2}+1$; If $e_{2}<d_{1}<e_{1} \leq d_{2}$, let $\lambda_{1}=e_{2}$ and $\lambda_{2}=e_{1}$. Then for both cases, we always have $\lambda_{1}<d_{1}<\lambda_{2} \leq d_{2}$. Since $E\left(\boldsymbol{x}, d_{1}, e_{1}\right)=E\left(\boldsymbol{x}^{\prime}, d_{2}, e_{2}\right)$, analogous to (6), we can obtain

$$
x_{i}= \begin{cases}x_{i}^{\prime}, & \text { for } i \in\left[1, d_{1}-1\right] \backslash\left\{\lambda_{1}\right\},  \tag{11}\\ x_{i-1}^{\prime}, & \text { for } i \in\left[d_{1}+1, d_{2}\right] \backslash\left\{\lambda_{2}\right\}, \\ x_{i}^{\prime}, & \text { for } i \in\left[d_{2}+1, n\right]\end{cases}
$$

Moreover, we have $x_{\lambda_{1}} \neq x_{\lambda_{1}}^{\prime}$ and $x_{\lambda_{2}} \neq x_{\lambda_{2}-1}^{\prime}$ because of the substitution error. According to (11), this case can be illustrated by Fig. 3.


Fig. 3. Illustration of Case (ii).

By Remark 3, we have $w t(\boldsymbol{x})=\operatorname{wt}\left(\boldsymbol{x}^{\prime}\right)$ and $x_{d_{1}}=x_{d_{2}}^{\prime}$. Then by (11) or Fig. 3, and through a cancelling process similar to Case (i), we can obtain $0=\mathrm{wt}(\boldsymbol{x})-\mathrm{wt}\left(\boldsymbol{x}^{\prime}\right)=$ $\sum_{\ell=1}^{n} x_{\ell}-\sum_{\ell=1}^{n} x_{\ell}^{\prime}=x_{\lambda_{1}}+x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}-1}^{\prime}-x_{d_{2}}^{\prime}=$ $x_{\lambda_{1}}+x_{\lambda_{2}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}-1}^{\prime}$ and

$$
u_{i}= \begin{cases}0, & \text { for } i \in\left[1, \lambda_{1}\right] ;  \tag{12}\\ x_{\lambda_{2}}-x_{\lambda_{2}-1}^{\prime}, & \text { for } i \in\left[\lambda_{1}+1, d_{1}\right] \\ x_{i}+x_{\lambda_{2}}-x_{\lambda_{2}-1}^{\prime}-x_{d_{2}}^{\prime}, \text { for } i \in\left[d_{1}+1, \lambda_{2}-1\right] \\ x_{i}-x_{d_{2}}^{\prime}, & \text { for } i \in\left[\lambda_{2}, d_{2}\right] \\ 0, & \text { for } i \in\left[d_{2}+1, n\right]\end{cases}
$$

Moreover, we have the following Claim.
Claim 2: Let $p_{1}=\lambda_{2}-1$. Then for each $j \in\{1,2\}$, either $u_{i} \geq 0$ for all $i \in\left[p_{j-1}+1, p_{j}\right]$ or $u_{i} \leq 0$ for all $i \in$ [ $p_{j-1}+1, p_{j}$ ], where $p_{0}=1$ and $p_{2}=n$. Moreover, we have $\left|u_{i}\right| \leq 2$ for all $i \in[n]$.

Proof of Claim 2: First consider $i \in\left[\lambda_{2}, n\right]$. By (12), $u_{i}=0$ or $u_{i}=x_{i}-x_{d_{2}}^{\prime}$. Clearly, if $x_{d_{2}}^{\prime}=0$, then $u_{i} \geq 0$ for all $i \in\left[\lambda_{2}, n\right]$; if $x_{d_{2}}^{\prime}=1$, then $u_{i} \leq 0$ for all $i \in\left[\lambda_{2}, n\right]$.

Now, consider $i \in\left[1, \lambda_{2}-1\right]$. Note that $x_{\lambda_{2}} \neq x_{\lambda_{2}-1}^{\prime}$. Then we have $x_{\lambda_{2}}-x_{\lambda_{2}-1}^{\prime} \in\{-1,1\}$. We need to consider the following two subcases.

Case (ii.1): $x_{\lambda_{2}}-x_{\lambda_{2}-1}^{\prime}=1$.
By (12), we have

$$
u_{i}=\left\{\begin{array}{lc}
0, & \text { for } i \in\left[1, \lambda_{1}\right] \\
1, & \text { for } i \in\left[\lambda_{1}+1, d_{1}\right] \\
x_{i}+1-x_{d_{2}}^{\prime}, & \text { for } i \in\left[d_{1}+1, \lambda_{2}-1\right]
\end{array}\right.
$$

Note that $x_{i} \geq 0$ and $1-x_{d_{2}}^{\prime} \geq 0$ (because $x_{d_{2}}^{\prime} \in\{0,1\}$ ). Then $u_{i} \geq 0$ for all $i \in\left[1, \lambda_{2}-1\right]$.

Case (ii.2): $x_{\lambda_{2}}-x_{\lambda_{2}-1}^{\prime}=-1$.

By (12), we have

$$
u_{i}=\left\{\begin{array}{l}
0, \quad \text { for } i \in\left[1, \lambda_{1}\right] \\
-1, \quad \text { for } i \in\left[\lambda_{1}+1, d_{1}\right] \\
x_{i}-1-x_{d_{2}}^{\prime}, \text { for } i \in\left[d_{1}+1, \lambda_{2}-1\right]
\end{array}\right.
$$

Note that $x_{i}-1 \leq 0$ (because $x_{d_{2}}^{\prime} \in\{0,1\}$ ) and $-x_{d_{2}}^{\prime} \leq 0$. Then $u_{i} \leq 0$ for all $i \in\left[1, \lambda_{2}-1\right]$.

Thus, $p_{1}=\lambda_{2}-1$ satisfies the desired property.
Finally, note that $\left|x_{\lambda_{2}}-x_{\lambda_{2}-1}^{\prime}\right| \leq 1$ and $\left|x_{i}-x_{d_{2}}^{\prime}\right| \leq 1$. Then it is easy to see from (12) that $\left|u_{i}\right| \leq 2$ for all $i \in[n]$, which proves Claim 2.

Similar to Case (i), by (2) and Claim 2, for $j=1,2$, we have

$$
\left|f_{j}(\boldsymbol{x})-f_{j}\left(\boldsymbol{x}^{\prime}\right)\right| \leq \sum_{i=1}^{n}\left|u_{i}\right| i^{j-1} \leq \sum_{i=1}^{n} 2 i^{j-1}<2 n^{j}
$$

On the other hand, by (C1) and (C2) of Definition 1, we have $f_{j}(\boldsymbol{x}) \equiv f_{j}\left(\boldsymbol{x}^{\prime}\right)\left(\bmod 2 n^{j}\right)$, so $f_{j}(\boldsymbol{x})=f_{j}\left(\boldsymbol{x}^{\prime}\right)$. Then by Claim 2 and Lemma 2, we have $\boldsymbol{x}=\boldsymbol{x}^{\prime}$.

Example 2: Consider an example with

$$
\begin{aligned}
x & =1001011101001110 \\
\boldsymbol{y} & =110111101001110 \\
\boldsymbol{x}^{\prime} & =1101111000011010
\end{aligned}
$$

where $n=16$. We can check that $\boldsymbol{y}$ can be obtained from $\boldsymbol{x}$ by deleting $x_{5}=0$ and substituting $x_{2}=0$ with $y_{2}=\bar{x}_{2}=1$, and $\boldsymbol{y}$ can also be obtained from $\boldsymbol{x}^{\prime}$ by deleting $x_{14}^{\prime}=0$ and substituting $x_{9}^{\prime}=0$ with $y_{9}=\bar{x}_{9}^{\prime}=1$. Hence, we have $\boldsymbol{y}=E(\boldsymbol{x}, 5,2)=E\left(\boldsymbol{x}^{\prime}, 14,9\right)$, that is, $d_{1}=5, e_{1}=2$, $d_{2}=14$ and $e_{2}=9$. Since $e_{1}<d_{1}<e_{2}<d_{2}$, we take $\lambda_{1}=e_{1}=2$ and $\lambda_{2}=e_{2}+1=10$. For this example, $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime}$ can be illustrated by Fig. 4, which is an instance of Fig. 3. It is easy to check that:

- For $i \in\left[\lambda_{1}\right]=\{1,2\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{\lambda_{1}}+$ $x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime}-x_{d_{2}}^{\prime}=0$;
- For $i \in\left[\lambda_{1}+1, d_{1}\right]=\{3,4,5\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-$ $\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{d_{1}}+x_{\lambda_{2}}-x_{\lambda_{2}-1}^{\prime}-x_{d_{2}}^{\prime}=x_{\lambda_{2}}-x_{\lambda_{2}-1}^{\prime}=1 ;$
- For $i \in\left[d_{1}+1, \lambda_{2}-1\right]=\{6,7,8,9\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-$ $\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{i}+x_{\lambda_{2}}-x_{\lambda_{2}-1}^{\prime}-x_{d_{2}}^{\prime}=x_{i}+1 \in\{1,2\} ;$
- For $i \in\left[\lambda_{2}, d_{2}\right]=\{10,11,12,13,14\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-$ $\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{i}-x_{d_{2}}^{\prime}=x_{i} \in\{0,1\} ;$
- For $i \in\left[d_{2}+1, n\right]=\{15,16\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-$ $\sum_{\ell=i}^{n} x_{\ell}^{\prime}=0 ;$
In summary, we have

$$
\begin{aligned}
& \left(u_{1}, u_{2}, \cdots, u_{n}\right) \\
& =(0,0,1,1,1,2,2,2,1,1,0,0,1,1,0,0)
\end{aligned}
$$

We can see that $u_{i} \geq 0$ for all $i \in\left[1, \lambda_{2}-1\right]=\{1,2, \cdots, 9\}$, and $u_{i} \geq 0$ for all $i \in\left[\lambda_{2}, n\right]=\{10,11, \cdots, 16\}$. Note that in this example, we have $u_{i} \geq 0$ for all $i \in[n]$, which is stronger than Claim 2. However, this is not the case in general.


Fig. 4. An example of Case (ii).

## C. Proof of $\boldsymbol{x}=\boldsymbol{x}^{\prime}$ for Case (iii)

For this case, we have $e_{1}<d_{1} \leq d_{2}<e_{2}$ or $e_{2}<d_{1} \leq$ $d_{2}<e_{1}$. Let $\lambda_{1}=\min \left\{e_{1}, e_{2}\right\}$ and $\lambda_{2}=\max \left\{e_{1}, e_{2}\right\}$. Then we have $\lambda_{1}<d_{1} \leq d_{2}<\lambda_{2}$. Since $E\left(\boldsymbol{x}, d_{1}, e_{1}\right)=$ $E\left(\boldsymbol{x}^{\prime}, d_{2}, e_{2}\right)$, analogous to (6), we can obtain

$$
x_{i}= \begin{cases}x_{i}^{\prime}, & \text { for } i \in\left[1, d_{1}-1\right] \backslash\left\{\lambda_{1}\right\}  \tag{13}\\ x_{i-1}^{\prime}, & \text { for } i \in\left[d_{1}+1, d_{2}\right] \\ x_{i}^{\prime}, & \text { for } i \in\left[d_{2}+1, n\right] \backslash\left\{\lambda_{2}\right\}\end{cases}
$$

Moreover, we have $x_{\lambda_{1}} \neq x_{\lambda_{1}}^{\prime}$ and $x_{\lambda_{2}} \neq x_{\lambda_{2}}^{\prime}$ because of the substitution error. According to (13), this case can be illustrated by Fig. 5.


Fig. 5. Illustration of Case (iii).
By Remark 3, we have $w t(\boldsymbol{x})=\mathrm{wt}\left(\boldsymbol{x}^{\prime}\right)$ and $x_{d_{1}}=x_{d_{2}}^{\prime}$. Then by (13) or Fig. 5, and through a cancelling process similar to Case (i), we can obtain $0=\mathrm{wt}(\boldsymbol{x})-\mathrm{wt}\left(\boldsymbol{x}^{\prime}\right)=$ $\sum_{\ell=1}^{n} x_{\ell}-\sum_{\ell=1}^{n} x_{\ell}^{\prime}=x_{\lambda_{1}}+x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime}-x_{d_{2}}^{\prime}=$ $x_{\lambda_{1}}+x_{\lambda_{2}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime}$ and

$$
u_{i}= \begin{cases}0, & \text { for } i \in\left[1, \lambda_{1}\right]  \tag{14}\\ x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}, & \text { for } i \in\left[\lambda_{1}+1, d_{1}\right] \\ x_{i}+x_{\lambda_{2}}-x_{d_{2}}^{\prime}-x_{\lambda_{2}}^{\prime}, \text { for } i \in\left[d_{1}+1, d_{2}\right] \\ x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}, & \text { for } i \in\left[d_{2}+1, \lambda_{2}\right] \\ 0, & \text { for } i \in\left[\lambda_{2}+1, n\right]\end{cases}
$$

Then we have the following Claim.
Claim 3: Either $u_{i} \geq 0$ for all $i \in[n]$ or $u_{i} \leq 0$ for all $i \in[n]$. Moreover, $\left|u_{i}\right| \leq 2$ for all $i \in[n]$.

Proof of Claim 3: Note that $x_{\lambda_{2}} \neq x_{\lambda_{2}}^{\prime}$. Then we have $x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime} \in\{-1,1\}$. To prove Claim 3, similar to Case (ii), we consider the following two subcases.

Case (iii.1): $x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}=1$.
By (14), we can obtain

$$
u_{i}= \begin{cases}0, & \text { for } i \in\left[1, \lambda_{1}\right] ; \\ 1, & \text { for } i \in\left[\lambda_{1}+1, d_{1}\right] \\ x_{i}-x_{d_{2}}^{\prime}+1, \text { for } i \in\left[d_{1}+1, \lambda_{2}-1\right] \\ 1, & \text { for } i \in\left[\lambda_{2}, d_{2}\right] \\ 0, & \text { for } i \in\left[d_{2}+1, n\right]\end{cases}
$$

Note that $1-x_{d_{2}}^{\prime} \geq 0$ and $x_{i} \geq 0$ for all $i \in[n]$. Then $u_{i} \geq 0$ for all $i \in[n]$.

Case (iii.2): $x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}=-1$.

By (14), we can obtain

$$
u_{i}=\left\{\begin{array}{lc}
0, & \text { for } i \in\left[1, \lambda_{1}\right] ; \\
-1, & \text { for } i \in\left[\lambda_{1}+1, d_{1}\right] ; \\
x_{i}-x_{d_{2}}^{\prime}-1, & \text { for } i \in\left[d_{1}+1, \lambda_{2}-1\right] ; \\
-1, & \text { for } i \in\left[\lambda_{2}, d_{2}\right] \\
0, & \text { for } i \in\left[d_{2}+1, n\right] .
\end{array}\right.
$$

Since $-x_{d_{2}}^{\prime} \leq 0$ and $x_{i}-1 \leq 0$ for all $i \in[n]$, then $u_{i} \leq 0$ for all $i \in[n]$.

Thus, either $u_{i} \geq 0$ for all $i \in[n]$ or $u_{i} \leq 0$ for all $i \in[n]$.
Note that $\left|x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}\right| \leq 1$ and $\left|x_{i}-x_{d_{2}}^{\prime}\right| \leq 1$. It is easy to see from (14) that $\left|u_{i}\right| \leq 2$ for all $i \in[n]$, which proves Claim 3.

Similar to Case (i), by (2) and by Claim 3, for $j=1,2$,

$$
\left|f_{j}(\boldsymbol{x})-f_{j}\left(\boldsymbol{x}^{\prime}\right)\right| \leq \sum_{i=1}^{n}\left|u_{i}\right| i^{j-1} \leq \sum_{i=1}^{n} 2 i^{j-1}<2 n^{j}
$$

On the other hand, by (C1) and (C2) of Definition 1, we have $f_{j}(\boldsymbol{x}) \equiv f_{j}\left(\boldsymbol{x}^{\prime}\right)\left(\bmod 2 n^{j}\right)$, so $f_{j}(\boldsymbol{x})=f_{j}\left(\boldsymbol{x}^{\prime}\right)$. Then by Claim 3 and Lemma 2, we have $\boldsymbol{x}=\boldsymbol{x}^{\prime}$.

Example 3: Consider an example with

$$
\begin{aligned}
x & =1001010101001111 \\
y & =100110101001101 \\
x^{\prime} & =1101101010001101
\end{aligned}
$$

where $n=16$. We can check that $\boldsymbol{y}$ can be obtained from $\boldsymbol{x}$ by deleting $x_{5}=0$ and substituting $x_{15}=1$ with $y_{14}=$ $\bar{x}_{15}=0$, and $\boldsymbol{y}$ can also be obtained from $\boldsymbol{x}^{\prime}$ by deleting $x_{10}^{\prime}=0$ and substituting $x_{2}^{\prime}=1$ with $y_{2}=\bar{x}_{2}^{\prime}=0$. Hence, $\boldsymbol{y}=E(\boldsymbol{x}, 5,15)=E\left(\boldsymbol{x}^{\prime}, 10,2\right)$, that is, $d_{1}=5, e_{1}=15$, $d_{2}=10$ and $e_{2}=2$. Since $e_{2}<e_{1}$, we take $\lambda_{1}=e_{2}=2$ and $\lambda_{2}=e_{1}=15$. For this example, $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime}$ can be illustrated by Fig. 6, which is an instance of Fig. 5. It is easy to check that:

- For $i \in\left[\lambda_{1}\right]=\{1,2\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{\lambda_{1}}+$ $x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime}-x_{d_{2}}^{\prime}=0$;
- For $i \in\left[\lambda_{1}+1, d_{1}\right]=\{3,4,5\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-$ $\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{d_{1}}+x_{\lambda_{2}}-x_{d_{2}}^{\prime}-x_{\lambda_{2}}^{\prime}=x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}=1$;
- For $i \in\left[d_{1}+1, d_{2}\right]=\{6,7,8,9,10\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-$ $\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{i}+x_{\lambda_{2}}-x_{d_{2}}^{\prime}-x_{\lambda_{2}}^{\prime}=x_{i}+1 \in\{1,2\}$;
- For $i \in\left[d_{2}+1, \lambda_{2}\right]=\{11,12,13,14,15\}, u_{i}=$ $\sum_{\ell=i}^{n} x_{\ell}-\sum_{\ell=i}^{n} x_{\ell}^{\prime}=x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}=1$;
- For $i \in\left[\lambda_{2}+1, n\right]=\{16\}, u_{i}=\sum_{\ell=i}^{n} x_{\ell}-\sum_{\ell=i}^{n} x_{\ell}^{\prime}=0$;

In summary, we have

$$
\begin{aligned}
& \left(u_{1}, u_{2}, \cdots, u_{n}\right) \\
& =(0,0,1,1,1,2,1,2,1,2,1,1,1,1,1,0)
\end{aligned}
$$

We can see that $u_{i} \geq 0$ for all $i \in[n]=\{1,2, \cdots, 16\}$.


Fig. 6. An example of Case (iii).

## D. Proof of $\boldsymbol{x}=\boldsymbol{x}^{\prime}$ for Case (v)

For this case, we have $d_{1}<e_{1} \leq d_{2}<e_{2}$ or $d_{1} \leq e_{2}<$ $d_{2}<e_{1}$. If $d_{1}<e_{1} \leq d_{2}<e_{2}$, let $\lambda_{1}=e_{1}$ and $\lambda_{2}=e_{2}$; If $d_{1} \leq e_{2}<d_{2}<e_{1}$, let $\lambda_{1}=e_{2}+1$ and $\lambda_{2}=e_{1}$. Then for both cases, we always have $d_{1}<\lambda_{1} \leq d_{2}<\lambda_{2}$. Since $E\left(\boldsymbol{x}, d_{1}, e_{1}\right)=E\left(\boldsymbol{x}^{\prime}, d_{2}, e_{2}\right)$, analogous to (6), we can obtain

$$
x_{i}= \begin{cases}x_{i}^{\prime}, & \text { for } i \in\left[1, d_{1}-1\right],  \tag{15}\\ x_{i-1}^{\prime}, & \text { for } i \in\left[d_{1}+1, d_{2}\right] \backslash\left\{\lambda_{1}\right\}, \\ x_{i}^{\prime}, & \text { for } i \in\left[d_{2}+1, n\right] \backslash\left\{\lambda_{2}\right\} .\end{cases}
$$

Moreover, we have $x_{\lambda_{1}} \neq x_{\lambda_{1}-1}^{\prime}$ and $x_{\lambda_{2}} \neq x_{\lambda_{2}}^{\prime}$ because of the substitution error. According to (15), this case can be illustrated by Fig. 7.


Fig. 7. Illustration of Case (v).
By Remark 3, we have $w t(\boldsymbol{x})=\mathrm{wt}\left(\boldsymbol{x}^{\prime}\right)$ and $x_{d_{1}}=x_{d_{2}}^{\prime}$. Then by (15) or Fig. 7, and through a cancelling process similar to Case (i), we can obtain $0=\mathrm{wt}(\boldsymbol{x})-\mathrm{wt}\left(\boldsymbol{x}^{\prime}\right)=$ $\sum_{\ell=1}^{n} x_{\ell}-\sum_{\ell=1}^{n} x_{\ell}^{\prime}=x_{\lambda_{1}}+x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{1}-1}^{\prime}-x_{\lambda_{2}}^{\prime}-x_{d_{2}}^{\prime}=$ $x_{\lambda_{1}}+x_{\lambda_{2}}-x_{\lambda_{1}-1}^{\prime}-x_{\lambda_{2}}^{\prime}$ and

$$
u_{i}=\left\{\begin{array}{l}
0, \quad \text { for } i \in\left[1, d_{1}\right] ;  \tag{16}\\
x_{i}-x_{d_{2}}^{\prime}, \quad \text { for } i \in\left[d_{1}+1, \lambda_{1}-1\right] ; \\
x_{i}+x_{\lambda_{2}}-x_{d_{2}}^{\prime}-x_{\lambda_{2}}^{\prime}, \text { for } i \in\left[\lambda_{1}, d_{2}\right] ; \\
x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}, \quad \text { for } i \in\left[d_{2}+1, \lambda_{2}\right] \\
0, \quad \text { for } i \in\left[\lambda_{2}+1, n\right]
\end{array}\right.
$$

Then we have the following Claim.
Claim 4: Let $p_{1}=\lambda_{1}-1$. Then for each $j \in\{1,2\}$, either $u_{i} \geq 0$ for all $i \in\left[p_{j-1}+1, p_{j}\right]$ or $u_{i} \leq 0$ for all $i \in$ $\left[p_{j-1}+1, p_{j}\right]$, where $p_{0}=1$ and $p_{2}=n$. Moreover, we have $\left|u_{i}\right| \leq 2$ for all $i \in[n]$.

Proof of Claim 4: For $i \in\left[1, \lambda_{1}-1\right]$, by (16), $u_{i}=0$ or $u_{i}=x_{i}-x_{d_{2}}^{\prime}$. Clearly, if $x_{d_{2}}^{\prime}=0$, then $u_{i} \geq 0$ for all $i \in\left[1, \lambda_{1}-1\right]$; if $x_{d_{2}}^{\prime}=1$, then $u_{i} \leq 0$ for all $i \in\left[1, \lambda_{1}-1\right]$.

For $i \in\left[\lambda_{1}, n\right]$, similar to Case (ii), we need to consider the following two subcases.

Case (v.1): $x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}=1$.
By (16), we have

$$
u_{i}=\left\{\begin{array}{l}
x_{i}-x_{d_{2}}^{\prime}+1, \text { for } i \in\left[\lambda_{1}, d_{2}\right] \\
1, \\
\text { for } i \in\left[d_{2}+1, \lambda_{2}\right] \\
0, \quad \text { for } i \in\left[\lambda_{2}+1, n\right]
\end{array}\right.
$$

Note that $1-x_{d_{2}}^{\prime} \geq 0$ and $x_{i} \geq 0$ for all $i \in[n]$. Then $u_{i} \geq 0$ for all $i \in\left[\lambda_{1}, n\right]$.

Case (v.2): $x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}=-1$.
By (16), we have

$$
u_{i}=\left\{\begin{array}{l}
x_{i}-x_{d_{2}}^{\prime}-1, \text { for } i \in\left[\lambda_{1}, d_{2}\right] \\
-1, \quad \text { for } i \in\left[d_{2}+1, \lambda_{2}\right] \\
0, \quad \text { for } i \in\left[\lambda_{2}+1, n\right]
\end{array}\right.
$$

Since $-x_{d_{2}}^{\prime} \leq 0$ and $x_{i}-1 \leq 0$ for all $i \in[n]$, then $u_{i} \leq 0$ for all $i \in\left[\lambda_{1}, n\right]$.

Thus, $p_{1}=\lambda_{1}-1$ satisfies the desired property.
Note that $\left|x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}\right| \leq 1$ and $\left|x_{i}-x_{d_{2}}^{\prime}\right| \leq 1$. It is easy to see from (16) that $\left|u_{i}\right| \leq 2$ for all $i \in[n]$, which proves Claim 4.

Similar to Case (i), by (2) and by Claim 4, for $j=1,2$,

$$
\left|f_{j}(\boldsymbol{x})-f_{j}\left(\boldsymbol{x}^{\prime}\right)\right| \leq \sum_{i=1}^{n}\left|u_{i}\right| i^{j-1} \leq \sum_{i=1}^{n} 2 i^{j-1}<2 n^{j}
$$

On the other hand, by (C1) and (C2) of Definition 1, we have $f_{j}(\boldsymbol{x}) \equiv f_{j}\left(\boldsymbol{x}^{\prime}\right)\left(\bmod 2 n^{j}\right)$, so $f_{j}(\boldsymbol{x})=f_{j}\left(\boldsymbol{x}^{\prime}\right)$. Then by Claim 4 and Lemma 2, we have $\boldsymbol{x}=\boldsymbol{x}^{\prime}$.
E. Proof of $\boldsymbol{x}=\boldsymbol{x}^{\prime}$ for Case (vi)

For this case, we have $d_{1} \leq d_{2}<e_{1}$ and $d_{1} \leq d_{2}<e_{2}$. Similar to Case (i), if $e_{1}=e_{2}$, then $\boldsymbol{x}=\boldsymbol{x}^{\prime}$. Therefore, we assume $e_{1} \neq e_{2}$. Let $\lambda_{1}=\min \left\{e_{1}, e_{2}\right\}$ and $\lambda_{2}=$ $\max \left\{e_{1}, e_{2}\right\}$. Then we have $d_{1} \leq d_{2}<\lambda_{1}<\lambda_{2}$. Since $E\left(\boldsymbol{x}, d_{1}, e_{1}\right)=E\left(\boldsymbol{x}^{\prime}, d_{2}, e_{2}\right)$, analogous to (6), we can obtain

$$
x_{i}= \begin{cases}x_{i}^{\prime}, & \text { for } i \in\left[1, d_{1}-1\right]  \tag{17}\\ x_{i-1}^{\prime}, & \text { for } i \in\left[d_{1}+1, d_{2}\right] \\ x_{i}^{\prime}, & \text { for } i \in\left[d_{2}+1, n\right] \backslash\left\{\lambda_{1}, \lambda_{2}\right\}\end{cases}
$$

Moreover, we have $x_{\lambda_{1}} \neq x_{\lambda_{1}}^{\prime}$ and $x_{\lambda_{2}} \neq x_{\lambda_{2}}^{\prime}$ because of the substitution error. According to (17), this case can be illustrated by Fig. 8.


Fig. 8. Illustration of Case (vi).
By Remark 3, we have $\mathrm{wt}(\boldsymbol{x})=\mathrm{wt}\left(\boldsymbol{x}^{\prime}\right)$ and $x_{d_{1}}=x_{d_{2}}^{\prime}$. Then by (17) or Fig. 8, and through a cancelling process similar to Case (i), we can obtain $0=\mathrm{wt}(\boldsymbol{x})-\mathrm{wt}\left(\boldsymbol{x}^{\prime}\right)=$ $\sum_{\ell=1}^{n} x_{\ell}-\sum_{\ell=1}^{n} x_{\ell}^{\prime}=x_{\lambda_{1}}+x_{\lambda_{2}}+x_{d_{1}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime}-x_{d_{2}}^{\prime}=$ $x_{\lambda_{1}}+x_{\lambda_{2}}-x_{\lambda_{1}}^{\prime}-x_{\lambda_{2}}^{\prime}$ and

$$
u_{i}= \begin{cases}0, & \text { for } i \in\left[1, d_{1}\right]  \tag{18}\\ x_{i}-x_{d_{2}}^{\prime}, & \text { for } i \in\left[d_{1}+1, d_{2}\right] \\ 0, & \text { for } i \in\left[d_{2}+1, \lambda_{1}\right] \\ x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}, & \text { for } i \in\left[\lambda_{1}+1, \lambda_{2}\right] \\ 0, & \text { for } i \in\left[\lambda_{2}+1, n\right]\end{cases}
$$

Then we have the following Claim.

Claim 5: Let $p_{1}=d_{2}$. Then for each $j \in\{1,2\}$, either $u_{i} \geq$ 0 for all $i \in\left[p_{j-1}+1, p_{j}\right]$ or $u_{i} \leq 0$ for all $i \in\left[p_{j-1}+1, p_{j}\right]$, where $p_{0}=1$ and $p_{2}=n$. Moreover, $\left|u_{i}\right| \leq 1$ for all $i \in[n]$.

Proof of Claim 5: For $i \in\left[1, d_{2}\right]$, by (18), we have $u_{i}=0$ or $u_{i}=x_{i}-x_{d_{2}}^{\prime}$. If $x_{d_{2}}^{\prime}=0$, then $u_{i} \geq 0$ for all $i \in\left[1, d_{2}\right]$; if $x_{d_{2}}^{\prime}=1$, then $u_{i} \leq 0$ for all $i \in\left[1, d_{2}\right]$.

For $i \in\left[d_{2}+1, n\right]$, by (18), we have $u_{i}=0$ or $u_{i}=$ $x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}$. If $x_{\lambda_{2}}^{\prime}=0$, then $u_{i} \geq 0$ for all $i \in\left[d_{2}+1, n\right]$; if $x_{\lambda_{2}}^{\prime}=1$, then $u_{i} \leq 0$ for all $i \in\left[d_{2}+1, n\right]$.

Thus, $p_{1}=d_{2}$ satisfies the desired property.
Note that $\left|x_{\lambda_{2}}-x_{\lambda_{2}}^{\prime}\right| \leq 1$ and $\left|x_{i}-x_{d_{2}}^{\prime}\right| \leq 1$. It is easy to see from (18) that $\left|u_{i}\right| \leq 1$ for all $i \in[n]$, which proves Claim 5.

Similar to Case (i), by (2) and by Claim 5, for $j=1,2$,

$$
\left|f_{j}(\boldsymbol{x})-f_{j}\left(\boldsymbol{x}^{\prime}\right)\right| \leq \sum_{i=1}^{n}\left|u_{i}\right| i^{j-1} \leq \sum_{i=1}^{n} i^{j-1}<n^{j}
$$

On the other hand, by (C1) and (C2) of Definition 1, we have $f_{j}(\boldsymbol{x}) \equiv f_{j}\left(\boldsymbol{x}^{\prime}\right)\left(\bmod 2 n^{j}\right)$, so $f_{j}(\boldsymbol{x})=f_{j}\left(\boldsymbol{x}^{\prime}\right)$. Then by Claim 5 and Lemma 2, we have $\boldsymbol{x}=\boldsymbol{x}^{\prime}$.

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[^0]:    ${ }^{1}$ For example, let $\boldsymbol{x}=0110001$ and let $\boldsymbol{y}=011001$ be obtained from $\boldsymbol{x}$ by deleting $x_{5}=0$. Then $\boldsymbol{y}$ can also be obtained from $\boldsymbol{x}$ by deleting $x_{3}=1$ and substituting $x_{4}=0$ with $\bar{x}_{4}=1$. In general, if $\boldsymbol{x}$ is not the all-zero sequence and $\boldsymbol{y}$ can be obtained from $\boldsymbol{x}$ by deleting a 0 , then we can always find a 0 (denoted by $\hat{0}$ ) in the same run with the deleted 0 that is adjacent to a 1 (denoted by $\hat{1}$ ). Then $\boldsymbol{y}$ can always be viewed as being obtained from $\boldsymbol{x}$ by deleting the 1 and substituting the 0 with 1 .

[^1]:    ${ }^{2}$ If $e_{1}=e_{2}<d_{1} \leq d_{2}$, then there is a $\boldsymbol{y}^{\prime} \in\{0,1\}^{n-1}$ such that $\boldsymbol{y}^{\prime}$ can be obtained from $\boldsymbol{x}$ (resp. $\boldsymbol{x}^{\prime}$ ) by a single deletion. By (C1) of Definition 1, $\mathscr{C}_{n}\left(c_{0}, c_{1}, c_{2}\right)$ is a single-deletion correcting code, so we can obtain $\boldsymbol{x}=\boldsymbol{x}^{\prime}$.
    ${ }^{3}$ In fact, let $\boldsymbol{y}=E\left(\boldsymbol{x}, d_{1}, e_{1}\right)=E\left(\boldsymbol{x}^{\prime}, d_{2}, e_{2}\right)$, which means that $\boldsymbol{y}$ can be obtained from $\boldsymbol{x}$ by deleting $x_{d_{1}}$ and substituting $x_{e_{1}}$ with $\bar{x}_{e_{1}}=1-x_{e_{1}}$, and $\boldsymbol{y}$ can also be obtained from $\boldsymbol{x}^{\prime}$ by deleting $x_{d_{2}}^{\prime}$ and substituting $x_{e_{2}}^{\prime}$ with $\bar{x}_{e_{2}}^{\prime}=1-x_{e_{2}}^{\prime}$. Then (6) can be obtained by comparing the elements of $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime}{ }^{2}$ with the elements of $\boldsymbol{y}: x_{i}=y_{i}=x_{i}^{\prime}$ for $i \in\left[1, d_{1}-1\right] \backslash\left\{\lambda_{1}, \lambda_{2}\right\} ; x_{i}=$ $y_{i-1}=x_{i-1}^{\prime}$ for $i \in\left[d_{1}+1, d_{2}\right]$; and $x_{i}=y_{i}=x_{i}^{\prime}$ for $i \in\left[d_{2}+1, n\right]$.

