Non-standard linear recurring sequence subgroups and automorphisms of irreducible cyclic codes

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Abstract—Let \mathcal{U} be the multiplicative group of order n in the splitting field \mathbb{F}_{q^m} of $x^n - 1$ over the finite field \mathbb{F}_q . Any map of the form $x \to cx^t$ with $c \in \mathcal{U}$ and $t = q^i$, $0 \leq i < m$, is \mathbb{F}_q -linear on \mathbb{F}_{q^m} and fixes \mathcal{U} set-wise; maps of this type will be called standard. Occasionally there are other, *non-standard* \mathbb{F}_q -linear maps on \mathbb{F}_{q^m} fixing \mathcal{U} set-wise, and in that case we say that the pair (n, q) is non-standard. We show that an irreducible cyclic code of length n over \mathbb{F}_q has "extra" permutation automorphisms (others than the standard permutations generated by the cyclic shift and the Frobenius mapping that every such code has) precisely when the pair (n,q)is non-standard; we refer to such irreducible cyclic codes as nonstandard or NSIC-codes. In addition, we relate these concepts to that of a non-standard linear recurring sequence subgroup as investigated in a sequence of papers by Brison and Nogueira. We present several families of NSIC-codes, and two constructions called "lifting" and "extension" to create new NSIC-codes from existing ones. We show that all NSIC-codes of dimension two can be obtained in this way, thus completing the classification for this case started by Brison and Nogueira.

Index Terms—linear recurrence relation, linear recurring sequence, *f*-sequence, *f*-subgroup, linear recurring sequence subgroup, non-standard sequence subgroup, cyclic code, irreducible cyclic code, permutation automorphism

I. INTRODUCTION

The general problem of determining the automorphism group of any cyclic code is a difficult problem, and [1, Section 3.5] suggests to investigate special cases such as irreducible cyclic codes. Usually, the only permutation automorphisms of an irreducible cyclic code are those generated by the cyclic shift and the Frobenius mapping; we refer to these codes as standard and to the exceptional ones that possess "extra" permutation automorphisms as non-standard. For briefness, we will refer to non-standard irreducible cyclic codes as NSICcodes. Examples include the q-ary simplex codes, certain evenweight codes, and the duals of the binary and ternary Golay codes. The ultimate goal is to obtain a full classification of the NSIC-codes. We present several families of NSICcodes, together with construction techniques called *lifting* and extension to construct new NSIC-codes from existing ones. As one of our main results, we show that every NSIC-code of dimension two can be obtained in this way.

Interestingly, the notion of an NSIC-code can be related to various other notions. In a series of papers, see, e.g., [2]–[5], Brison and Nogueira investigated the concept of a *non-standard linear recurring sequence subgroup* or *NSLRS-group*, a multiplicative subgroup in an extension of a finite field \mathbb{F}_q with the property that the elements can be represented by a non-cyclic linear recurring sequence with characteristic polynomial over \mathbb{F}_q (for precise definitions of these and other notions, we refer to the next two sections). We show that the notion of an NSIC-codes coincides with that of a NSLRS-group in the case where the characteristic polynomial of the linear recurrence relation is required to be *irreducible* over \mathbb{F}_q .

Let \mathcal{U} be the multilicative subgroup of order n in an extension \mathbb{F}_{q^m} of \mathbb{F}_q , with m minimal. Usually, the collection $\mathcal{L}(n,q)$ of \mathbb{F}_q -linear maps on \mathbb{F}_{q^m} that fix \mathcal{U} set-wise consists of the maps $x \to ux^{q^i}$ ($u \in \mathcal{U}, 0 \leq i < m$) only. But occasonally, $\mathcal{L}(n,q)$ contains other, *non-standard* maps; in that case, we refer to the pair (n,q) as *non-standard*. We show that an irreducible cyclic code C of length n over \mathbb{F}_q is NSIC-code precisely when (n,q) is non-standard; in fact, we show that $\mathcal{L}(n,q)$ is a group and $\mathcal{L}(n,q) \cong PAut(C)$, the permutation automorphism group of C.

The contents of this paper are the following. In Section II we introduce the notation used in this paper. In Section III, we sketch the required background on linear recurring sequences and provide definitions of some of the notions used above. In Section IV we prove the equivalence of NSIC-codes and non-standard pairs. Lifting and extension is discussed in Section V. In Section VI we introduce equally-spaced polynomials and a related class of NSIC-codes based on a new type of cyclic product codes. Section VII lists the NSIC-codes known to us, and in Section VIII we classify the NSIC-codes of dimension 2. Finally, in Section IX we discuss our results.

We have space to include just a few complete proofs, for all other proofs, further background, and any unexplained notation, we refer to [6] and [7]. Together, these two papers form an improved and extended version of the arxiv paper [8].

II. NOTATION AND PRELIMINARIES

Throughout this paper, q denotes a power of a prime, n is a positive integer such that gcd(n,q) = 1, and $m = ord_n(q)$, the smallest positive integer for which $n \mid q^m - 1$. The finite field of size q is denoted by \mathbb{F}_q , and we use \mathbb{F}_q^* to denote both the non-zero elements of the field and the multiplicative subgroup of \mathbb{F}_q . We let $\overline{\mathbb{F}}_q$ denote the algebraic closure of \mathbb{F}_q . For a

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subset \mathcal{H} of a group \mathcal{G} , we write $\mathcal{H} \leq \mathcal{G}$ to denote that \mathcal{H} is a subgroup of \mathcal{G} . There is a *unique* multiplicative subgroup in $\overline{\mathbb{F}}_q$ of order n, consisting of the n-th roots of unity in $\overline{\mathbb{F}}_q$, which we denote by $\mathcal{U}_{n,q}$. Note that $\mathcal{U}_{n,q} \leq \mathbb{F}_{q^m}^*$, and by the definition of m, \mathbb{F}_{q^m} is the smallest extension of \mathbb{F}_q that contains $\mathcal{U}_{n,q}$. We let ξ denote a fixed element in $\mathbb{F}_{q^m}^*$ of order n, and we mostly use g to denote the minimal polynomial of ξ over \mathbb{F}_q ; note that the degree deg(g) of g equals m. Then $\mathcal{U}_{n,q} = \langle \xi \rangle$, the multiplicative group generated by ξ .

An important notion in this paper is the *q*-order. The *q*-order of ξ , denoted by $\delta_q(\xi)$, is the smallest positive integer *d* for which $\xi^d \in \mathbb{F}_q$. It is not difficult to show that $\delta_q(\xi) = \delta_q(n) := n/\gcd(n, q-1)$, so depends only on *n*. Writing $d = \delta_q(n)$ and e = (n, q-1), we have n = de with $e \mid q-1$ and $\gcd(d, (q-1)/e) = 1$; as a consequence, the number $m = \operatorname{ord}_n(q)$ depends only on *d* and is the smallest positive integer such that $d \mid (q^m - 1)/(q - 1)$. For proofs and more information on the *q*-order, see [6].

In this paper, we investigate the group $\mathcal{L}(n,q)$ of all \mathbb{F}_q linear maps L on \mathbb{F}_{q^m} that fix the group $\mathcal{U}_{n,q}$ of order nin $\mathbb{F}_{q^m}^*$ set-wise, that is, such that $L(\mathcal{U}_{n,q}) = \mathcal{U}_{n,q}$. Note that $\mathcal{L}(n,q)$ is indeed a group: since $m = \operatorname{ord}_n(q)$, the \mathbb{F}_q -span of the elements $1, \xi, \ldots, \xi^{m-1}$ of $\mathcal{U}_{n,q} = \langle \xi \rangle$ is \mathbb{F}_{q^m} , so if $L \in \mathcal{L}(n,q)$, then $\mathrm{Im}(L) = \mathbb{F}_{q^m}$, hence L is invertible. We refer to an \mathbb{F}_q -linear map on \mathbb{F}_{q^m} of the form $L: x \to cx^{q^j}$, for some $j \in [m] := \{0, 1, \dots, m-1\}$ and $c \in \mathbb{F}_{q^m}$, as standard. Note that such a standard map is contained in $\mathcal{L}(n,q)$ if and only if $c \in \mathcal{U}_{n,q}$. We write $\mathcal{L}_{\mathrm{st}}(n,q)$ to denote the subgroup of $\mathcal{L}(n,q)$ consisting of the standard maps in $\mathcal{L}(n,q)$. In certain exceptional cases, $\mathcal{L}(n,q)$ is stricktly larger than $\mathcal{L}_{st}(n,q)$; in that case, we refer to the pair (n,q) as *non-standard* and to the group $\mathcal{U}_{n,q}$ as *non-standard* (over \mathbb{F}_q); in the usual case where $\mathcal{L}(n,q) = \mathcal{L}_{st}(n,q)$, we refer to both (n,q) and $\mathcal{U}_{n,q}$ as standard.

Let S_n denote the symmetric group on n symbols, the group of all permutations on [n]. Define the map $\Psi : \mathcal{L}(n,q) \to S_n$ by $\Psi(L) = \pi$ if $L(\xi^i) = \xi^{\pi(i)}$ for i = 0, ..., n-1. It is easily seen that Ψ is a one-to-one group-homomorphism. We will write $S(n,q) = \Psi(\mathcal{L}(n,q))$ and $S_{st}(n,q) = \Psi(\mathcal{L}_{st}(n,q))$ to denote the images in S_n under Ψ of $\mathcal{L}(n,q)$ and $\mathcal{L}_{st}(n,q)$. By the above, $\mathcal{L}(n,q) \cong S(n,q)$ and $\mathcal{L}_{st}(n,q) \cong S_{st}(n,q) =$ $\{x \to q^i x + a \pmod{n} \mid a \in \mathbb{Z}_n, i \in [m]\}$; in particular, we note that $\mathcal{L}_{st}(n,q)$ has size nm. For missing proofs and more details, we refer to [6], [7].

III. LINEAR RECURRING SEQENCE SUBGROUPS

In this section, we establish the relation between nonstandard pairs and non-standard linear recurring relation subgroups [3]. For more background on linear recurring sequence, see, e.g., [9] or [10].

A sequence $\mathbf{s} = s_0, s_1, \dots$ in $\overline{\mathbb{F}}_q$ is called an *m*th order *linear recurring sequence* if it satisfies a (homogeneous) linear recurrence relation of the form

$$s_k = \sigma_{m-1}s_{k-1} + \dots + \sigma_1s_{k-m+1} + \sigma_0s_{k-m}$$
(1)

for all integers $k \geq m$, where $m \geq 1$, $\sigma_0 \in \mathbb{F}_q^*$, and $\sigma_1, \ldots, \sigma_{m-1} \in \mathbb{F}_q$. The monic polynomial

$$f(x) = x^{m} - \sigma_{m-1}x^{m-1} - \dots - \sigma_{1}x - \sigma_{0}$$
(2)

in $\mathbb{F}[x]$ with $f(0) = -\sigma_0 \neq 0$ is called the *characteristic* polynomial of the recurrence relation (1) and a sequence s that satisfies (1) is called an f-sequence. Such a sequence is necessarily periodic, and we denote the (smallest) period of an f-sequence s by per(s). We say that an f-sequence s in $\overline{\mathbb{F}}_q$ is cyclic if there exists $\alpha \in \overline{\mathbb{F}}_q$ such that $s_{k+1}/s_k = \alpha$ for all $k \ge 0$; note that then, necessarily, $f(\alpha) = 0$. Recall that $\xi \in \mathbb{F}_{q^m}$ denotes an element of order n, and $\mathcal{U}_{n,q} = \langle \xi \rangle$ is the group of n-th roots of unity. If there exists an f-sequence s in \mathbb{F}_q with per(s) = n such that $\mathcal{U}_{n,q} = \{s_0, s_1, \dots, s_{n-1}\},\$ then we say that $\mathcal{U}_{n,q}$ is an *f*-subgroup, and that $\mathcal{U}_{n,q}$ is represented by s. Sometimes, such a group is referred to as a linear recurring sequence subgroup [2]. A rather uninteresting way for $\mathcal{U}_{n,q}$ to be an *f*-subgroup is when *f* is the minimal polynomial of ξ over \mathbb{F}_q , representing $\mathcal{U}_{n,q}$ by the cyclic fsequence s with $s_k = \xi^k$ ($k \ge 0$). But sometimes there exist *non-cyclic* f-sequences that represent $\mathcal{U}_{n,q}$; in that case, if f is a polynomial over \mathbb{F}_q , we refer to $\mathcal{U}_{n,q}$ as a non-standard f-subgroup or a non-standard linear recurring sequence subgroup (NSLRS-group) over \mathbb{F}_q .

In this paper, we will be mainly interested in the case where the characteristic polynomial of the linear recurrence relation is *irreducible* over \mathbb{F}_q . Our approach is based on the following results. For precise proofs, we refer to [6].

Theorem 3.1: Let g be irreducible over \mathbb{F}_q of degree m, and let ξ be a zero of g of order n in \mathbb{F}_{q^m} , so that $m = \operatorname{ord}_n(q)$. Then a sequence s is a g-sequence in \mathbb{F}_{q^m} if and only if there are $L_0, \ldots, L_{m-1} \in \mathbb{F}_{q^m}$ such that

$$s_k = L_0 \xi^k + L_1 \xi^{qk} + \dots + L_{m-1} \xi^{q^{m-1}k} \qquad (k \ge 0).$$
(3)

Proof: (Sketch) Since $\xi, \xi^q, \ldots, \xi^{q^{m-1}}$ are all zeros of g, sequences of the form as in the theorem are indeed g-sequences. A dimension argument shows that these are indeed all the g-sequences over \mathbb{F}_{q^m} .

Theorem 3.2: With the assumptions as in Theorem 3.1, a g-subgroup \mathcal{U} in an extension of \mathbb{F}_q is unique, and has the form $\mathcal{U} = \langle \xi \rangle = \mathcal{U}_{n,q}$.

Proof: (Sketch) All zeros ξ^{q^*} $(i \in [m])$ of g have the same period n, and so every non-zero g-sequence s has minimal period per(s) = n. \Box For a generalization of this result to f-subgroups for general f, and for additional background and references, see [9]. If we now combine the above results, we obtain the following.

Theorem 3.3: With the assumptions as in Theorem 3.1, a group \mathcal{U} is a g-subgroup if and only if $\mathcal{U} = \mathcal{U}_{n,q} = \langle \xi \rangle$. In addition, there is a one-to-one correspondence between gsequences representing $\mathcal{U}_{n,q}$ and \mathbb{F}_q -linear maps $L \in \mathcal{L}(n,q)$, where s corresponds to L if $s_k = L(\xi^k)$ $(k \ge 0)$; moreover, s is cyclic if and only if the corresponding map L is in $\mathcal{L}_{st}(n,q)$.

Proof: (Sketch) It is well-known that there is a one-toone correspondence between \mathbb{F}_q -linear maps on \mathbb{F}_{q^m} and qpolynomials on \mathbb{F}_{q^m} , maps of the form $L(x) = L_0 x + L_1 x^q +$ $\dots + L_{m-1}x^{q^{m-1}}$ with $L_0, L_1, \dots, L_{m-1} \in \mathbb{F}_{q^m}$. Now the result essentially is a direct consequence of the expression (3) for g-sequences in \mathbb{F}_{q^m} . \Box As a consequence, the notion of a non-standard pair (n,q) coincides with that of a non-standard g-subgroup over \mathbb{F}_q with g irreducible of order n.

IV. AUTOMORPHISMS OF CYCLIC CODES

A linear $[n, k]_q$ -code C is an \mathbb{F}_q -linear k-dimensional subspace of \mathbb{F}_q^n . We sometime refer to vectors in C as *code words*. The dual code C^{\perp} of C is the collection of all vectors $\mathbf{a} \in \mathbb{F}_q^n$ for which $(\mathbf{a}, \mathbf{c}) := a_0 c_0 + \cdots + a_{n-1} c_{n-1} = 0$ for all $\mathbf{c} \in C$. A permutation $\pi \in S_n$ induces a permutation on \mathbb{F}_q^n by mapping a vector $\mathbf{c} = (c_0, c_1, \dots, c_{n-1})$ to the vector

$$\mathbf{c}^{\pi} = (c_{\pi^{-1}(0)}, c_{\pi^{-1}(1)}, \dots, c_{\pi^{-1}(n-1)}).$$

The group of permutation automorphisms of C, denoted by $\operatorname{PAut}(C)$, is the collection of all permutations $\pi \in S_n$ with the property that if $\mathbf{c} \in C$, then $\mathbf{c}^{\pi} \in C$. For later use, we remark that $\operatorname{PAut}(C^{\perp}) = \operatorname{PAut}(C)$ [11, Lemma 1.3, (i)]. The cyclic shift is the permutation $\sigma = (0, 1, \ldots, n-1) \in S_n$, mapping *i* to $i + 1 \pmod{n}$ ($i \in [n]$). A linear code $C \subseteq \mathbb{F}_q^n$ is cyclic if $\sigma \in \operatorname{PAut}(C)$. By identifying a vector $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}_q^n$ with the polynomial c(x) = $c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$ in $\mathcal{R}_{n,q} := \mathbb{F}_q[x] \mod x^n - 1$, a code of length *n* over \mathbb{F}_q corresponds to a subset of $\mathcal{R}_{n,q}$. A linear code is cyclic if and only if the corresponding subset is an *ideal* in $\mathcal{R}_{n,q}$. Now any ideal of $\mathcal{R}_{n,q}$ is principal, that is, generated by a unique monic polynomial. If C = (g(x))is a cyclic code of length *n*, then g(x) is called the generator polynomial of *C* and $h(x) = (x^n - 1)/g(x)$ is called the parity-check polynomial of *C*.

Let $\xi \in \mathbb{F}_{q^m}$ have order n and degree $m = \operatorname{ord}_n(q)$ over \mathbb{F}_q . Let $\operatorname{Tr}(x) = x + x^q + \cdots + x^{q^{m-1}}$ denote the trace of \mathbb{F}_{q^m} over \mathbb{F}_q . The code $C_{n,q}$ consisting of all code words

$$\mu(a) = (\operatorname{Tr}(a\xi^0), \operatorname{Tr}(a\xi^1), \dots, \operatorname{Tr}(a\xi^{n-1}))$$

with $a \in \mathbb{F}_{q^m}$ is called the *irreducible* (or *minimal*) cyclic code of length n over \mathbb{F}_q . Note that $C_{n,q}$ is unique up to a permutation, and has dimension m. The dual $C_{n,q}^{\perp}$ of $C_{n,q}$ has as generator polynomial the minimal polynomial g(x) of ξ over \mathbb{F}_q , which has degree m; such a code is called *maximal* [12]. It is well-known and easily verified that beside the cyclic shift σ , also the Frobenius mapping $\phi : x \to qx \pmod{n}$ (considered as an element of S_n) is an automorphisms of $C_{n,q}$ (to see this, note that if $\mathbf{c} \in \mathbb{F}_q^n$, then $c(x)^q = c(x^q) = c^{\phi}(x)$ corresponds to \mathbf{c}^{ϕ}). We write PAut_{st} $(C_{n,q})$ to denote the group $\langle \sigma, \phi \rangle$ generated by σ and ϕ ; note that $|\text{PAut}_{st}(C_{n,q})| =$ nm. Recall that, by definition, $C_{n,q}$ is a NSIC-code if and only if PAut $(C_{n,q}) \supseteq \text{PAut}_{st}(C_{n,q})$. The following result shows some unexpected connections between the concept of a NSICcode and various other notions of non-standardness.

Theorem 4.1: An irreducible cyclic code $C_{n,q}$ is a NSICcode if and only if (n,q) is a non-standard pair. More precise, we have that $PAut(C_{n,q}) = PAut(C_{n,q}^{\perp}) = S(n,q)$ and $PAut_{st}(C_{n,q}) = PAut_{st}(C_{n,q}^{\perp}) = S_{st}(n,q)$. *Proof:* First, let $L \in \mathcal{L}(n,q)$, with $\Psi(L) = \pi \in S_n$. Recall that $C_{n,q}^{\perp}$ is the cyclic code with defining zero ξ . So if $\mathbf{c} \in C_{n,q}^{\perp}$, then $c(\xi) = 0$, hence

$$0 = L(0) = L(\sum_{i=0}^{n-1} c_i \xi^i) = \sum_{i=0}^{n-1} c_i \xi^{\pi(i)} = c^{\pi}(\xi),$$

and we conclude that $\mathbf{c}^{\pi} \in C_{n,q}^{\perp}$. Since $\mathbf{c} \in C_{n,q}^{\perp}$ was arbitrary, $\pi \in \operatorname{PAut}(C_{n,q}^{\perp})$.

On the other hand, let $\pi \in \text{PAut}(C_{n,q}^{\perp})$. Since $(1,\xi,\ldots,\xi^{m-1})$ is a basis for \mathbb{F}_{q^m} over \mathbb{F}_q , we can define an \mathbb{F}_q -linear map L on \mathbb{F}_{q^m} by setting $L(\xi^i) = \xi^{\pi(i)}$ for $i = 0, \ldots, m-1$, and then extending L by \mathbb{F}_q -linearity. We claim that $L \in \mathcal{L}(n,q)$ with $\Psi(L) = \pi$. To see this, let $j \in \{m,\ldots,n-1\}$. There are $a_0,\ldots,a_{m-1} \in \mathbb{F}_q$ such that $\xi^j = a_0 + a_1\xi + \cdots + a_{m-1}\xi^{m-1}$. This has two consequences. First, by definition of L,

$$L(\xi^{j}) = a_0 \xi^{\pi(0)} + \dots + a_{m-1} \xi^{\pi(m-1)}.$$
 (4)

Next, the vector $\mathbf{c} = (a_0, \ldots, a_{m-1}, 0, \ldots, 0, -1, 0, \ldots, 0)$, with the entry -1 in position j, has $c(\xi) = 0$, so $\mathbf{c} \in C_{n,q}^{\perp}$, and since $\pi \in \text{PAut}(C_{n,q}^{\perp})$, we have $\mathbf{c}^{\pi} \in \text{PAut}(C_{n,q}^{\perp})$, that is,

$$0 = \sum_{i=0}^{n-1} c_i \xi^{\pi(i)} = a_0 \xi^{\pi(0)} + \dots + a_{m-1} \xi^{m-1} - \xi^{\pi(j)}.$$
 (5)

Combining (4) and (5), we conclude that $L(\xi) = \xi^{\pi(j)}$. Since j was arbitrary, we conclude that $L(\xi^j) = \xi^{\pi(j)}$ for all $j \in [n]$, that is, $\Psi(L) = \pi$.

V. LIFTING AND EXTENSION

In this section, we discuss two methods to create new nonstandard pairs from existing ones.

Let ξ have order n and degree m over \mathbb{F}_q , and let t be a positive integer for which gcd(m,t) = 1. Since $gcd(n,q^{ti} - 1) = gcd(n,q^{ti} - 1,q^m - 1) = gcd(n,q^{gcd(m,ti)} - 1) = gcd(n,q^{gcd(m,i)} - 1)$, we have $m = ord_n(q) = ord_n(q^t)$, hence the minimal polynomial g(x) of ξ over \mathbb{F}_q is also the minimal polynomial of ξ over \mathbb{F}_{q^t} . So in view of the results in Section III, the following result is not too surprising.

Theorem 5.1: (Lifting) Let m and t be as above. Then $S(n,q) = S(n,q^t)$ and $S_{st}(n,q) = S_{st}(n,q^t)$. So (n,q) is non-standard if and only if (n,q^t) is non-standard.

Proof: (Sketch) Let $L \in \mathcal{L}(n,q)$, so let $L(x) = \sum_{i=0}^{m-1} L_i x^{q^i}$ on \mathbb{F}_{q^m} with $L_i \in \mathbb{F}_{q^m}$ $(i \in [m])$, where we consider the indices modulo m. Define $\tilde{L}(x) = \sum_{i=0}^{m-1} L_{it} x^{q^{ti}}$ $(x \in \mathbb{F}_{q^{mt}})$. Then $L = \tilde{L}$ on \mathbb{F}_{q^m} , and \tilde{L} is \mathbb{F}_{q^t} -linear on $\mathbb{F}_{q^{tm}}$. Moreover, since $\langle \xi \rangle \subseteq \mathbb{F}_{q^m}$, we have $\tilde{L}(\xi^i) = L(\xi^i) = \xi^{\pi(i)}$ for all $i \in [n]$, where $\pi = \Psi(L)$, so $\tilde{L} \in \mathcal{L}(n, q^t)$ and $\Psi(\tilde{L}) = \Psi(L)$. Conversely, if $\tilde{L} \in \mathcal{L}(n, q^t)$, then \tilde{L} is \mathbb{F}_{q^t} -linear, hence \mathbb{F}_q -linear, on $\mathbb{F}_{q^{mt}}$, and since \tilde{L} fixes $\langle \xi \rangle$ setwise, \tilde{L} also fixes $\mathbb{F}_q(\xi) = \mathbb{F}_{q^m}$ set-wise. So the restriction L of \tilde{L} to \mathbb{F}_{q^m} is in $\mathcal{L}(n, q)$. The claims are now obvious. \Box We will refer to the operation of passing from a non-standard pair (n, q) to a non-standard pair (n, q^t) with $\gcd(m, t) = 1$ as *lifting*. In coding terms, note that if \tilde{C} is the \mathbb{F}_{q^t} -span

of the code $C_{n,q}$, then $\operatorname{PAut}(\tilde{C}) = \operatorname{PAut}(C_{n,q})$. And we have $C_{n,q^t} \subseteq \tilde{C}$, with equality if and only if the parity-check polynomial g of $C_{n,q}$ (and of \tilde{C}) is also irreducible over \mathbb{F}_{q^t} , which holds if and only if $\operatorname{gcd}(m,t) = 1$.

The above operation on a non-standard pair (n, q) changed the value of q. Our next operation changes n. Recall that if $\xi \in \mathbb{F}_{q^m}^*$ has order n, then $d = \delta_q(n) = n/\gcd(n, q - 1)$ is the smallest positive integer such that $\xi^d \in \mathbb{F}_q$; in addition, n = de with $e = \gcd(n, q - 1)$ and $\gcd(d, (q - 1)/e) = 1$. First, we prove a lemma.

Lemma 5.2: Let $\xi \in \mathbb{F}_{q^m}$ have order n and q-order $\delta_q(\xi) = \delta_q(n) = d = n/\gcd(n, q-1)$, and let $\mathcal{G} \leq \mathbb{F}_q^*$ with $\xi^d \in \mathcal{G}$. Then $\mathcal{G}\langle \xi \rangle \leq \mathbb{F}_{q^m}^*$ and $|\mathcal{G}\langle \xi \rangle| = d|\mathcal{G}|$.

Proof: Obviously, $\mathcal{G}\langle\xi\rangle \leq \mathbb{F}_{q^m}^*$. Since $\xi^d \in \mathcal{G}$, every element of $\mathcal{G}\langle\xi\rangle$ has the form $g\xi^i$ with $g \in \mathcal{G}$ and $i \in [d]$. By definition of the q-order, all these elements are distinct, hence $\mathcal{G}\langle\xi\rangle$ has order $d|\mathcal{G}|$.

Now we have the following 'extension" result.

Theorem 5.3: (Extension) Let n = de with $e = \gcd(n, q-1)$ and $d = \delta_q(n)$, so that $\gcd(d, (q-1)/e) = 1$. For a positive integer f, we have $d = \delta_q(n) = \delta_q(nf)$ if and only if $f \mid (q-1)/e$. Let $f \mid (q-1)/e$ and let θ have order nf in an extension of \mathbb{F}_q . Then $\theta \in \mathbb{F}_{q^m}^*$, and $\langle \theta \rangle \mathcal{L}(n,q) \leq \mathcal{L}(nf,q)$; in particular, if (n,q) is non-standard and $f \mid d(q-1)/n$, then (nf,q) is also non-standard.

Proof: With the assumptions in the theorem, we have that $\delta_q(nf) = def/\gcd(def, q-1) = df/\gcd(df, (q-1)/e) = df/\gcd(df, (q-1)/e) = d$ if and only if $f \mid (q-1)/e$. Next, assume that $f \mid (q-1)/e$. Let $\xi \in \mathbb{F}_{q^m}^*$ have order n, and let θ have order nf. Now $\mathcal{G} = \langle \theta^d \rangle \leq \mathbb{F}_q^*$ and $|\mathcal{G}| = ef$, hence $\langle \theta \rangle = \mathcal{G} \langle \xi \rangle$ by Lemma 5.2. Finally, if $g \in \mathcal{G}$ and $L \in \mathcal{L}(n,q)$, then $gL(\langle \theta \rangle) = gL(\mathcal{G} \langle \xi \rangle) \subseteq g\mathcal{G} \langle \xi \rangle \subseteq \langle \theta \rangle$, hence $gL \in \mathcal{L}(n,q)$. Obviously, $\mathcal{L}_{st}(nf,q) = \langle \theta \rangle \mathcal{L}_{st}(n,q)$, so the last conclusion is immediate.

Extension is related to the following code construction technique. Let C be a cyclic code of length n over \mathbb{F}_q , with paritycheck polynomial $g(x) \in \mathbb{F}_q[x]$, so that $g(x) \mid x^n - 1$, and let $\nu \in \mathbb{F}_q^*$ with $\nu^{nf} = 1$. Let $\tilde{C} \subseteq \mathbb{F}_q^{nf}$ be the collection of words

$$\tilde{\mathbf{c}} = (c_0, \nu c_1, \dots, \nu^{n-1} c_{n-1}, \nu^n c_0, \dots, \nu^{nf-1} c_{n-1})$$

with $\mathbf{c} = (c_0, \ldots, c_{n-1}) \in C$. If $c(x) = a(x)(x^n - 1)g(x)$ for some $a(x) \in \mathbb{F}_q[x]$ with $\deg(a) < \deg(g)$, then $\tilde{c}(x) = c(\nu x)((\nu x)^{nf} - 1)/((\nu x)^n - 1) = a(\nu x)(x^{nf} - 1)/g(\nu x)$, hence \tilde{C} is a cyclic code of length nf over \mathbf{F}_q , with paritycheck polynomial $(x^{nf} - 1)/g(\nu x)$. Note that if g is irreducible with zero ξ , then $C = C_{n,q}$, and if the zero ξ/ν of $g(\nu x)$ has order nf, then $\tilde{C} = C_{nf,q}$. Remark also that if $\tilde{\nu}$ has order f, if C_0 is the cyclic code consisting of all scalar multiples of the word $(1, \tilde{\nu}, \tilde{\nu}^2, \ldots, \tilde{\nu}^{f-1})$, and if C_1 is the code with code words $(c_0, \nu c_1, \ldots, \nu^{n-1}c_{n-1})$ with $(c_0, c_1, \ldots, c_{n-1}) \in C$, then C_1 is equivalent to C and \tilde{C} is equivalent to the product code $C_0 \times C_1$. For more information on this product code construction and its relation with extension, we refer to [6].

In [6] we show that multiple lifts and extensions can always be obtained by a *single* lift, followed by a *single* extension.

VI. EQUALLY-SPACED POLYNOMIALS

A polynomial $f(x) \in \mathbb{F}_q[x]$ is called *equally-spaced* if it is of the form $f(x) = g(x^k)$ for some integer $k \ge 2$. Then $\deg(f) = k \cdot \deg(g)$ and if $g(0) \ne 0$, then $\operatorname{ord}(f) = k \cdot$ $\operatorname{ord}(g)$ and $\operatorname{ord}_q(f) = k \cdot \operatorname{ord}_q(g)$ (see [6]). Equally-spaced polynomials form a rich source of non-standard examples [5].

Theorem 6.1: Let $k \ge 2$ be a positive integer with gcd(k,q) = 1, let $g(x) \in \mathbb{F}_q[x]$ be monic with $g(0) \ne 0$, of order $n \ge 1$ and degree $m \ge 1$, and suppose that $f(x) = g(x^k)$ is irreducible over \mathbb{F}_q . Then the pair (nk,q) is non-standard except when $f(x) = x^2 + 1$.

For a proof, we refer to [5] or [6]. In [6] we also show that if C is cyclic of length n over \mathbb{F}_q , with generator polynomial g(x), then the product code $\mathbb{F}_q^k \times C$ is again cyclic, with generator polynomial $g(x^k)$.

Note that under the conditions of Theorem 6.1, the polynomial $f(x) = g(x^k)$ is irreducible over \mathbb{F}_q if and only if $\operatorname{ord}_{nk}(q) = mk$. The next theorem states when this occurs.

Theorem 6.2: Let k, g(x), n, and m be as in Theorem 6.1, so with gcd(nk, q) = 1 and with g irreducible, so with $m = ord_n(q)$. Let P(n,q) be the set of numbers that have only prime factors r for which r|n and $gcd(r, (q^m - 1)/n) = 1$. Then the polynomial $f(x) = g(x^k)$ is irreducible if and only if $k \in P(n,q)$, with $4 \not\mid k$ if $2 \in P(n,q)$ and $n \equiv 2 \mod 4$. The attraction of Theorem 6.2 is that it is *constructive*. For a proof, we refer to [6].

VII. KNOWN NON-STANDARD PAIRS AND THEIR CODES

The following non-standard pairs (n, q) and corresponding irreducible (NSIC) and maximal cyclic codes are known to us. 1) Pairs (n, q) with $n \ge 5$ prime and $m = \operatorname{ord}_n(q) = n - 1$. The corresponding codes are the *repetition codes* and their duals, the *even-weight codes*, with corresponding polynomials $f(x) = (x^n - 1)/(x - 1)$. Here, $\mathcal{L}(n, q) = \mathcal{S}_n$.

2) Pairs (n,q) with $n = q^m - 1$, where m > 2 or m = 2, q > 2. The corresponding codes are the *q*-ary simplex codes and their duals, the primitive BCH codes with designed distance 2, with as polynomials the *primitive* polynomials over \mathbb{F}_q . Here, $\mathcal{L}(n,q) = \mathrm{GL}(m,q)$.

3) The pair (n,q) = (23,2), where $m = \operatorname{ord}_n(q) = 11$, with corresponding code the (dual of the) *binary Golay code*, with group $\mathcal{L}(23,2) \cong M_{23}$.

4) The pair (n,q) = (11,3), where m = 5, with corresponding codes the (dual of the) *ternary Golay code*, with group $\mathcal{L}(11,3) \cong PSL(2,11)$.

5) Pairs $(n,q) = (kn_0,q)$ with $m = \operatorname{ord}_n(q) = km_0$, where $n_0 > 1$ and $k \ge 2$ are integers with k > 2 if $n_0 = 2$ and $m_0 = \operatorname{ord}_{n_0}(q)$. (See Theorem 6.2 for the conditions under which this is the case.) The corresponding codes are $\mathbb{F}_q^k \times C_{n_0,q}$ and their duals, with group $\mathcal{L}(n,q) \cong S_k \wr \operatorname{PAut}(C_{n_0,q})$ (wreath product), and with corresponding polynomials the irreducible *equally-spaced* polynomials $f(x) = g(x^k)$ with $g(x) \in \mathbb{F}_q[x]$ and $f(x) \neq x^2 + 1$.

6) In addition, for every non-standard pair (n,q) with m =ord_n(q), we have non-standard pairs (nf, q^t) for every positive integer t such that gcd(t,m) = 1 and every positive integer f

such that $f \mid (q^t - 1) / \gcd(n, q^t - 1)$ that can be obtained from (n, q) by *lifting and extension* as described in Section V.

Possibly, the above examples exhaust all possibilities. In the next section, we show this to hold for the case where m = 2.

VIII. CLASSIFICATION FOR m = 2

For degree m = 2, we have the following result.

Theorem 8.1: The non-standard pairs (n,q) with $m = \operatorname{ord}_n(q) = 2$ and q-order $d = n/\operatorname{gcd}(n, q-1)$ are the ones listed in Section VII for which m = 2 and are the following. 1) Pairs (n,q) where n = 2e > 4 with $e \ge 1$ integer for which $e \mid q-1$, with both q and (q-1)/e odd (the equally-spaced case); here d = 2 and $\mathcal{L}(2e,q) \cong \mathbb{Z}_e \wr S_2$ (wreath product). 2) Pairs (n,q) where $n = q^2 - 1$ and $q \ge 3$ (the primitive case); here $d = q + 1 \ge 4$ and $\mathcal{L}(n,q) = \operatorname{GL}(2,q)$. And in addition, all pairs (nf,q^t) obtained from a non-standard pair $(n = q^2 - 1, q)$ as above by lifting and extension, so with t odd and $f \mid (q^t - 1)/(q - 1)$; here d = q + 1and $\mathcal{L}(f(q^2 - 1), q^t) = \langle \theta \rangle \operatorname{GL}(2,q)$, where $\theta \in \mathbb{F}_{q^t}^*$ has order $f(q^2 - 1)$.

Proof: (Sketch) If d = 2, then the corresponding polynomial is of the form $x^2 - \sigma$ with $\sigma \in \mathbb{F}_q^*$ and is irreducible over \mathbb{F}_q . A simple analysis, using Theorem 6.2, quickly leads to the non-standard pairs as listed in case 1). Next, let $\xi \in \mathbb{F}_{q^2}$ be an element of degree m = 2, order n, and q-order d > 2, so with minimal polynomial $g(x) = x^2 - \sigma_1 x - \sigma_0$, where $\sigma_0, \sigma_1 \in \mathbb{F}_q^*$. Let $T \in \mathcal{L}(n, q)$ be the "cyclic shift" map defined by $T(x) = \xi x$ on \mathbb{F}_{q^2} , and let $L \in \mathcal{L}(n, q)$ with L(1) = 1 and $L(\xi) = \omega + \nu \xi$, say. We proceed in several steps.

1. (Normalization) The element $\xi = \xi/\sigma_1$ has minimal polynomial $\tilde{g}(x) = x^2 - x - \lambda$, where $\lambda = \sigma_0/\sigma_1^2$. Obviously, the *q*-order of $\tilde{\xi}$ again equals *d*. Writing $\tilde{\omega} = \omega/\sigma_1$, we have $T(1) = \sigma_1 \tilde{\xi}, T(\tilde{\xi}) = \sigma_1 \tilde{\xi}^2 = \sigma_1(\tilde{\xi} + \lambda)$ and $L(1) = 1, L(\tilde{\xi}) = \nu \tilde{\xi} + \tilde{\omega}$. Put $D = \text{diag}(1, \sigma_1^{-1})$ and define

$$\tilde{L} = L^D = \begin{bmatrix} 1 & \tilde{\omega} \\ 0 & \nu \end{bmatrix}, \qquad \tilde{T} = \sigma_1^{-1} T^D = \begin{bmatrix} 0 & \lambda \\ 1 & 1 \end{bmatrix}.$$

Note that \tilde{L} and \tilde{T} are the matrices of the maps L and $\sigma_1^{-1}T$ on \mathbb{F}_{q^2} , with respect to the basis $(1, \tilde{\xi})$.

2. (Subgroup of $\operatorname{PGL}(2,q)$) Next, we identify the points of the 1-dimensional projective geometry $\operatorname{PG}(1,q)$ over \mathbb{F}_q with the sets $\mathbb{F}_q^* \alpha$ ($\alpha \in \mathbb{F}_{q^2}^*$), and we consider the group $\mathcal{G} = \langle \tilde{T}, \tilde{L} \rangle$ generated by the matrices \tilde{T} and \tilde{L} as a subgroup of $\operatorname{PGL}(2,q)$. Since $\tilde{\xi}^d \in \mathbb{F}_q^*$, every power $\tilde{\xi}^j$ of $\tilde{\xi}$ is of the form $a\tilde{\xi}^i$ for some $i \in \{0, 1, \ldots, d-1\}$ and some $a \in \mathbb{F}_q^*$, hence $\mathcal{G} \leq \operatorname{PGL}(2,q)$ fixes the set $O = \{1, \xi, \ldots, \xi^{d-1}\}$, considered as a subset of $\operatorname{PG}(1,q)$, set-wise.

3. (Subgroup structure of PGL(2, q)) Now we use the known subgroup structure of PGL(2, q), see, e.g., [14], [15], and a separate analysis for the cases $3 \le d \le 5$, to conclude that either L is standard, or \mathcal{G} is conjugate in PGL(2, q) to one of its subgroups $PGL(2, q_0)$ or $PSL(2, q_0)$. Assuming the last case, we can then show that $d = q_0 + 1$, where $q = q_0^t$ with todd, and $\lambda, \tilde{\omega}, \nu \in \mathbb{F}_{q_0}$ with $\mathbb{F}_{q_0} = \mathbb{F}_p(\lambda)$, hence $\tilde{\xi} \in \mathbb{F}_{q_0}^2$.

4. As a consequence, L fixes $\mathbb{F}_{q_0^2}$, and a simple analysis reveals that the restriction of L to $\mathbb{F}_{q_0^2}$ is non-standard if L itself is

non-standard. Moreover, since L fixes both $\mathbb{F}_{q_0}^*$ and $\langle \xi \rangle$ setwise, it also fixes the intersection $\mathbb{F}_{q_0}^* \cap \langle \xi \rangle = \langle \xi^{\delta} \rangle$, where δ is the q_0 -order of ξ . So, writing $\theta = \xi^{\delta}$, we conclude that the group $\langle \theta \rangle$ is non-standard over \mathbb{F}_{q_0} , as witnessed by the non-standard map L on $\mathbb{F}_{q_0}^2$.

5. We have shown that $\overset{\theta}{\theta}$ is non-standard over \mathbb{F}_{q_0} , with (maximal) q_0 -order $q_0 + 1$. Now by [4, Theorem 2.4], θ is primitive in $\mathbb{F}_{q_0^2}$.

6. Finally, it can be shown that, as expected, the non-standard group $\langle \xi \rangle$ over \mathbb{F}_q can be obtained for the non-standard group $\langle \theta \rangle$ over \mathbb{F}_{q_0} by lifting and extension. In [7] we shown that a nonstandard element ξ of order n and degree $m = \operatorname{ord}_n(q)$ over \mathbb{F}_q with maximal q-order $d = (q^m - 1)/(q-1)$ is necessarily primitive, so has order $n = q^m - 1$, thus generalizing [4, Theorem 2.4] for m = 2 to all m. The proof of the generalization is completely different and uses the recently completed classification of 1/2-transitive linear groups [13]. We take this opportunity to remark that this result *does not* follow from [16, Theorem 2.3] since the given proof is only valid when the subgroup G in that theorem is *proper*.

IX. DISCUSSION AND CONCLUSIONS

Let n and q be positive integers with gcd(n,q) = 1, and let m be the multiplicative order of q molulo n, the smallest positive integer such that $n \mid q^m - 1$. We say that the pair (n, q)is non-standard if the collection $\mathcal{L}(n,q)$ of \mathbb{F}_q -linear maps on \mathbb{F}_{q^m} that fixes the group $\mathcal{U}_{n,q}$ of *n*-th roots of unity set-wise contains maps not of the form $x \to cx^{q^j}$ with $c \in \mathbb{F}_{q^m}$ and $j \in [m]$. In this paper, we have first linked this notion to that of non-standard linear recurring sequence subgroups [2]–[5]. Then we showed that a pair (n,q) is non-standard precisely when an irreducible cyclic code of length n over \mathbb{F}_q has "extra" permutation automorphisms (others than those generated by the cyclic shift and some Frobenius mapping); in this paper, we refer to such codes as NSIC-codes. A result by Brison and Nogueira from [4] for the case m = 2 states that a nonstandard pair (n, q) with (maximal) q-order $d = \delta_q(n) = q + 1$ is necessarily primitive, so has $n = q^2 - 1$. Using the known subgroup structure of PG(2, q) in combination with this result, we have finished the classification of the non-standard pairs and NSIC-codes for the case of dimension m = 2 that was initiated by Brison and Nogueira in [2]-[4].

Substantial information is available on the subgroup structure of PGL(m,q) for m = 3, and to a lesser extend also for m = 4, see, e.g., [17]. By using similar methods, in combination with our generalization of [4, Theorem 2.4] to *all* m in [7], we expect that classification is also possible for the case m = 3, and perhaps even for the case m = 4.

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