A Generalization of the Stratonovich's Value of Information and Application to Privacy-Utility Trade-off

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Abstract—The Stratonovich's value of information (VoI) is quantity that measure how much inferential gain is obtained from a perturbed sample under information leakage constraint. In this paper, we introduce a generalized VoI for a general loss function and general information leakage. Then we derive an upper bound of the generalized VoI. Moreover, for a classical loss function, we provide a achievable condition of the upper bound which is weaker than that of in previous studies. Since VoI can be viewed as a formulation of a privacy-utility trade-off (PUT) problem, we provide an interpretation of the achievable condition in the PUT context.

I. INTRODUCTION

Research on decision-making under a constraint of information leakage has been studied in 1960s in the academy of sciences of the Soviet Union (USSR Academy). In particular, Stratonovich's work [1] is pioneering, however, it does not appear to be widely known¹. In [1] and [2], he introduced *Value of Information* (VoI) to quantify how much inferential gain is obtained from a perturbed sample Y which contains some information about original sample X. His formulation of the VoI was based on the Shannon's mutual information (MI) I(X;Y) in the information theory [4] and a loss (cost) function $\ell(x, a)$, where a is some action (e.g. point estimation on X, hypothesis testing on p_X , prediction), in the statistical decision theory (see, e.g., [5]).

Since Shannon's proposal of MI, various information leakage measures have been proposed. Some examples are Arimoto's MI [6], Sibson's MI [7], and Csiszár's MI [8]. Recently, new information leakage measures have been proposed in the privacy-utility trade-off (PUT) problem, such as *finformation* [9] and *f*-leakage [10], as privacy measures. In addition to these measures, by assuming a "guessing" adversary, information leakage measures that have operational meanings have been proposed. For example, Asoodeh *et al.* introduced *probability of correctly guessing* in [11], [12]. In [13]–[15], Issa *et al.* introduced *maximal leakage* which quantifies the maximal logarithmic gain of correctly guessing any arbitrary function of the original sample. Extending the maximal leakage, Liao *et al.* introduced α -leakage and α maximal leakage in [10], [16]–[18]. Liao *et al.* also showed the relationships between the (maximal) α -leakage and both Arimoto's MI and Sibson's MI. It is worth noting that Liao *et al.* introduced an α -loss to define the α -leakage.

In this study, we first introduce an information leakage measure in a general manner by extracting common properties from these specific information leakage measures. Then we define a generalized VoI for the information leakage measure and a general loss function containing the α -loss. For the generalized VoI, we derive an upper bound next. Moreover, for a classical loss function $\ell(x, a)$, we also provide an achievable condition of the upper bound which is weaker than that of in previous studies [1], [2] and [19]. We also show basic properties of the achievable upper bound and some extended results. Finally, since VoI can be viewed as a formulation of a PUT problem in a certain situation, based on our prior work [20], we provide an interpretation of the achievable condition in the PUT context.

II. PRELIMINARY

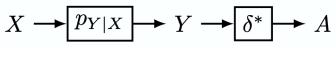


Fig. 1. System Model

In this section, we first review the statistical decision theory and the concept of *information leakage* in information theory on the system model in Figure 1. For simplicity, unless otherwise stated, we will assume that all alphabets are finite.

A. Notations

Let X, Y and A be random variables on alphabets \mathcal{X}, \mathcal{Y} and \mathcal{A} . Let $p_{X,Y} = p_X \times p_{Y|X}$ and p_Y be a given joint distribution of (X, Y) and a marginal distribution of Y, respectively. Let $\delta^* : \mathcal{Y} \times \mathcal{A} \to [0, 1]$ and $\delta : \mathcal{Y} \to \mathcal{A}$ be a randomized decision rule and a deterministic decision rule, respectively. Since $\delta^*(y, a)$ is equivalent to a conditional probability $p_{A|Y}(a \mid y)$, we will use these notations interchangeably. The classical notation for a loss function in the statistical decision theory is $\ell(x, a)$, which represents a loss for making an action A = a

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¹Recently, his book containing this research has been translated into English [2]. It is worth noting that similar approach have been studied by Kanaya and Nakagawa [3].

when the true state is X = x. In this study, however, we extend the concept of the loss function to a loss for making an action A = a from a sample Y = y using the (randomized) decision rule δ^* when the true state is X = x, denoted as $\ell(x, y, a, \delta^*)$. Finally, we use log to represent the natural logarithm.

B. Statistical decision theory

We review the basic concepts and results in the statistical decision theory next.

Definition 1. The loss function for a randomized decision rule $\delta^* : \mathcal{Y} \times \mathcal{A} \rightarrow [0, 1]$ is defined as

$$L(x,\delta^*(y,\cdot)) := \mathbb{E}_A\left[\ell(x,y,A,\delta^*) \mid Y=y\right]$$
(1)

$$=\sum_{a} p_{A|Y}(a \mid y)\ell(x, y, a, \delta^*).$$
(2)

Definition 2. The risk function and the Bayes risk function for a randomized decision rule δ^* is defined as

$$R(x,\delta^*,p_{Y|X}) := \mathbb{E}_Y[L(x,\delta^*(Y,\cdot)) \mid X = x]$$
(3)

$$=\sum_{y} p_{Y|X}(y \mid x) L(x, \delta^*(y, \cdot)), \quad (4)$$

$$r(\delta^*, p_{Y|X}) := \mathbb{E}_X \left[R(X, \delta^*, p_{Y|X}) \right]$$
(5)

$$=\sum_{x} p_X(x) R(x, \delta^*, p_{Y|X}).$$
(6)

Proposition 1 ([20, Prop 1]). The minimal Bayes risk is given by

$$\inf_{\delta^*} r(\delta^*, p_{Y|X}) = r(\delta^{*, \text{Bayes}}, p_{Y|X})$$
(7)

$$= \mathbb{E}_Y \left[\inf_{\delta(y,\cdot)} \mathbb{E}_X \left[L(X, \delta^*(Y, \cdot)) \mid Y \right] \right], \quad (8)$$

with the optimal randomized decision rule $\delta^{*,Bayes}$ given by

$$\delta^{*,\operatorname{Bayes}}(y,\cdot) := \operatorname{arginf}_{\delta^{*}(y,\cdot)} \mathbb{E}_{X} \left[L(X,\delta^{*}(y,\cdot)) \mid Y = y \right], \quad (9)$$

where infimum is over all randomized decision rule $\delta^*(y, \cdot) = p_{A|Y}(\cdot | y)$ for fixed y. In particular, when a channel is $p_{Y|X} = p_Y$ (i.e., X and Y are independent, denoted by $X \perp Y$),

$$\inf_{\delta^*} r(\delta^*, p_Y) = r(\delta^{*, \text{Bayes}}, p_Y)$$
(10)

$$= \mathbb{E}_{Y} \left[\inf_{\delta^{*}(y,\cdot)} \mathbb{E}_{X} \left[L(X, \delta^{*}(Y, \cdot)) \right] \right].$$
(11)

Remark 1. The corresponding result of the Proposition 1 for a deterministic decision rule δ and a classical loss function $\ell(x, a)$ is given by

$$\delta^{\text{Bayes}}(y) := \operatorname*{arginf}_{a} \mathbb{E}_{X} \left[\ell(X, a) \mid Y = y \right], \tag{12}$$

$$\inf_{\delta} r(\delta, p_{Y|X}) = r(\delta^{\text{Bayes}}, p_{Y|X}) = \mathbb{E}_Y \left[\inf_{a} \mathbb{E}_X \left[\ell(X, a) \right] \middle| Y \right]$$
(13)

$$\inf_{\delta} r(\delta, p_Y) = r(\delta^{\text{Bayes}}, p_Y) = \inf_{a} \mathbb{E}_X \left[\ell(X, a) \right].$$
(14)

C. Information leakage

In this study, we introduce information leakage measure, denoted as $\mathcal{L}(X \to Y)$, to quantify how much information Y leak about X. To this end, we extract some properties in common to well-known information leakage measures in information theory².

Definition 3. The information leakage $\mathcal{L}(X \to Y) = \mathcal{L}(p_X, p_{Y|X})$ is defined as a functional of p_X and $p_{Y|X}$ that satisfies following properties:

1) Non-negativity:

$$\mathcal{L}(X \to Y) \ge 0. \tag{15}$$

2) Data Processing Inequality (DPI): If X - Y - Z forms a Markov chain, then

$$\mathcal{L}(X \to Z) \le \mathcal{L}(X \to Y). \tag{16}$$

3) Independence:

$$\mathcal{L}(X \to Y) = 0 \Longleftrightarrow X \perp Y. \tag{17}$$

1) Examples of the information leakage: Table I shows the typical information leakage measures in information theory that have these properties and their references, where

- α ∈ (0,1)∪(1,∞). Note that the value of the information leakage measures in the table are extended by continuity to α = 1 and α = ∞.
- to $\alpha = 1$ and $\alpha = \infty$. • $H_{\alpha}(X) := \frac{\alpha}{1-\alpha} \log \left(\sum_{x} p_{X}(x)^{\alpha} \right)^{\frac{1}{\alpha}}$ is the Rényi entropy of order α .
- $H^{\alpha}_{\alpha}(X|Y) := \frac{\alpha}{1-\alpha} \log \sum_{y} (\sum_{x} p_{X,Y}(x,y)^{\alpha})^{\frac{1}{\alpha}}$ is Arimoto's conditional entropy of X given Y of order α .
- $D_{\alpha}(p||q) := \frac{1}{\alpha-1} \log \left(\sum_{z} p^{\alpha}(z) q^{1-\alpha}(z) \right)$ is the Rényi divergence of order α .
- U represents an arbitrary (potentially random) function of X and \hat{U} represents its estimator.
- $D_f(p||q) := \sum_{z \in \mathbb{Z}} q(z) f\left(\frac{p(z)}{q(z)}\right)$ is the *f*-divergence, where $f : \mathbb{R}_+ \to \mathbb{R}$ is a convex function such that f(1) = 0 and strictly convex at t = 1, where $\mathbb{R}_+ := [0, \infty)$.

Note that relationships between these information leakage measures are given as follows:

- $I(X;Y) = I_1^A(X;Y) = I_1^S(X;Y) = I_1^C(X;Y) = I_f(X;Y)$, where $f(t) = t \log t$.
- $I^{A}_{\alpha}(X;Y) = \mathcal{L}_{\alpha}(X \to Y)$ (see [10, Thm 1]).
- $I_{\alpha}(X, Y) = \mathcal{L}_{\alpha}(X \to Y)$ • $\mathcal{L}_{\alpha}^{\max}(X \to Y)$ = $\begin{cases} \sup_{p_{\tilde{X}}} I_{\alpha}^{A}(\tilde{X}; Y) = \sup_{p_{\tilde{X}}} I_{\alpha}^{S}(\tilde{X}; Y), & \alpha > 1, \\ \mathcal{L}_{\max}(X \to Y), & \alpha = \infty, \\ I(X; Y), & \alpha = 1, \end{cases}$ where $p_{\tilde{X}}$ is a probability distribution over support

where $p_{\tilde{X}}$ is a probability distribution over support of p_X . See [10, Thm 2] for detail.

Most of the non-negativity properties 1) in the Table I follow from the non-negativity of $D_{\alpha}(p||q)$ and $D_{f}(p||q)$. Note that properties of α -leakage $\mathcal{L}_{\alpha}(X \to Y)$ follows

 $^{^{2}}$ Note that these properties are part of requirements for reasonable information leakage measures proposed by Issa *et al.* [15].

| Name | Definition | 1) | 2) | 3) |
|-------------------------------------|---|------------------|---|-------------------------------|
| mutual information (MI) [4] | $I(X;Y) := H(X) - H(X \mid Y)$ | 1 | ✓ [21, Thm 2.8.1] | ✓ [21, Eq (2.90)] |
| Arimoto's MI of order α [6] | $I^{\mathbf{A}}_{\alpha}(X;Y) := H_{\alpha}(X) - H^{\mathbf{A}}_{\alpha}(X \mid Y)$ | √ [6, Thm 2] | [10, Footnote 4],[22, Cor 1] | √ [6, Thm 2] |
| Arimoto's MI of order ∞ [10] | $I^{\mathcal{A}}_{\infty}(X;Y) := \log \frac{\sum_{y} \max_{x} p_{X,Y}(x,y)}{\max_{x} p_{X}(x)}$ | 1 | ✓ [22, Cor 1] | × [23, Sec 6.6] |
| Sibson's MI of order α [7] | $I_{\alpha}^{S}(X;Y) := \min_{q_{Y}} D_{\alpha}(p_{X,Y} p_{X}q_{Y})$ | 1 | ✓ [24, Eq (55)] | 1 |
| Sibson's MI of order ∞ [10] | $I_{\infty}^{\mathbf{S}}(X;Y) := \log \sum_{y} \max_{x} p_{Y X}(y \mid x)$ | 1 | 1 | 1 |
| Csiszár's MI of order α [8] | $I_{\alpha}^{\mathbb{C}}(X;Y) := \min_{q_Y} \mathbb{E}_X \left[D_{\alpha}(p_Y _X(\cdot \mid X) q_Y) \right]$ | 1 | ✓ [8, Eq (22)] | 1 |
| f-information [9] | $I_f(X;Y) := D_f(p_{X,Y} p_X p_Y)$ | 1 | ✓ [25, Thm 7.9] | [26, Lem 4], [25, Thm 7.3] |
| f-leakage [10] | $\mathcal{L}_f(X \to Y) := \min_{q_Y} D_f(p_{X,Y} p_X q_Y)$ | 1 | 1 | 1 |
| maximal leakage [15] | $\mathcal{L}_{\max L}(X \to Y) := \sup_{U = X - Y} \log \frac{\max_{p_{\hat{U} Y}} \mathbb{E}_{U,Y} \left[p_{\hat{U} Y}(U Y) \right]}{\max_{u} p_{U}(u)}$ | ✓ [15, Lem 1] | ✓ [15, Lem 1] | ✓ [15, Cor 2] |
| α -leakage [10] | $\mathcal{L}_{\alpha}(X \to Y) := \frac{\alpha}{\alpha - 1} \log \frac{\max_{p_{\hat{X} Y}} \mathbb{E}_{X,Y} \left[p_{\hat{X} Y}(X Y)^{\frac{\alpha - 1}{\alpha}} \right]}{\max_{p_{\hat{X}}} \mathbb{E}_{X} \left[p_{\hat{X}}(X)^{\frac{\alpha - 1}{\alpha}} \right]}$ | 1 | V | 1 |
| maximal α -leakage [10] | $\mathcal{L}^{\max}_{\alpha}(X \to Y) := \sup_{U - X - Y} \mathcal{L}_{\alpha}(U \to Y)$ | 1 | ✓ [10, Thm 3] | 1 |
| mmse-leakage [This study] | $\mathcal{L}_{mmse}(X \to Y) := \mathbb{V}(X) - \mathbb{E}_Y \left[\mathbb{V}(X \mid Y) \right]$ | ✓ [Prop 2] | ✓ [Prop 2] | × [Prop 2] |
| TABLE I | | | | |

TYPICAL INFORMATION LEAKAGE MEASURES IN INFORMATION THEORY

from that of Arimoto's MI $I^{A}_{\alpha}(X;Y)$ because of their identity mentioned above. Independence property 3) of maximal α -leakage $\mathcal{L}^{\max}_{\alpha}(X \to Y)$ follows from the property in the α -leakage, while the property of Sibson's MI follows can be derived in a similar manner of [6, Thm 2]. Csiszár's MI $I^{C}_{\alpha}(X;Y)$ and f-leakage $\mathcal{L}_{f}(X \to Y)$ also have the independence property 3). In fact, for Csiszár's MI, it follows from the non-negativity of the α -divergence that $I^{C}_{\alpha}(X;Y) = \mathbb{E}_{X}\left[D_{\alpha}(p_{Y|X}(\cdot \mid X)||q^{C,*}_{Y})\right] = 0 \iff$ $D_{\alpha}(p_{Y|X}(\cdot|X)||q^{C,*}_{Y}) = 0$ a.s. $\iff p_{Y|X}(y|x) = q^{C,*}_{Y}(y) =$ $p_{Y}(y), \forall x \in \text{supp}(p_{X}), y \in \mathcal{Y} \iff X \perp Y$, where $q^{C,*}_{Y} := \arg\min_{q_{Y}} \mathbb{E}_{X}\left[D_{\alpha}(p_{Y|X}(\cdot \mid X)||q_{Y})\right]$, $\operatorname{supp}(p_{X}) :=$ $\{x \in \mathcal{X} \mid p_{X}(x) > 0\}$ (support of p_{X}) and a.s. means almost surely. For f-leakage, it can be shown in a similar way. Finally, DPI property of f-leakage follows from [26, Lem 4], [25, Thm 7.2], and a discussion in [24, Sec V].

2) *mmse-leakage:* In addition to the typcal information leakage measures, we can define a new information leakage measure, *minimum mean squared error-leakage* $\mathcal{L}_{mmse}(X \rightarrow Y)$, which has the properties 1), 2) but does *not* satisfy 3) in general. Note that we assume that alphabets are continuous here, i.e., $\mathcal{X} = \mathcal{Y} = \mathbb{R}$.

Definition 4 (Minimum mean squared error-leakage). The minimum mean squared error-leakage $\mathcal{L}_{mmse}(X \to Y)$ is defined as

$$\mathcal{L}_{\mathsf{mmse}}(X \to Y) := \mathbb{V}(X) - \inf_{f \colon \mathcal{Y} \to \mathcal{X}} \mathbb{E}_{X,Y} \left[(X - f(Y))^2 \right]$$
(18)

$$= \mathbb{V}(X) - \mathbb{E}_{Y}\left[\mathbb{V}(X \mid Y)\right], \tag{19}$$

where infimums is over all (measurable) function $f: \mathcal{Y} \to \mathcal{X}$ and $\mathbb{V}(X) := \mathbb{E}_X \left[(X - \mathbb{E}_X[X])^2 \right], \mathbb{V}(X \mid Y = y) :=$ $\mathbb{E}_X \left[(X - \mathbb{E}_X [X | Y = y])^2 \right]$ are variance of X and conditional variance of X given Y = y, respectively.

Proposition 2. $\mathcal{L}_{mmse}(X \to Y)$ has the properties 1), 2) but does not satisfy 3) in general.

Proof. Property 1) is trivial from the definition of the quantity. Property 2) can be proved as follows: If X - Y - Z forms a Markov chain, then $\mathcal{L}_{mmse}(X \to Y) - \mathcal{L}_{mmse}(X \to Z) = \mathbb{E}_{Y,Z} \left[(\mathbb{E}_X[X | Y] - \mathbb{E}_X[X | Z])^2 \right] \ge 0$, where we used the *the orthogonal principle*³ in the first equality⁴. Finally, it follows from *the law of total variance* $\mathbb{V}(X) = \mathbb{E}_Y [\mathbb{V}(X | Y)] + \mathbb{V}(\mathbb{E}_X [X | Y])$ that $\mathcal{L}_{mmse}(X \to Y) = 0 \iff \mathbb{V}(\mathbb{E}_Y [X | Y]) = \mathbb{E}_Y \left[(\mathbb{E}_X [X | Y] - \mathbb{E}_X [X])^2 \right] = 0 \iff \mathbb{E}_X [X | Y] = \mathbb{E}_X [X] \ a.s.$ The equality condition $\mathbb{E}_X [X | Y] = \mathbb{E}_X [X] \ a.s.$ is often called a *mean indepen dence*, which is known as a weaker condition than independence 3), i.e., $X \perp Y \implies \mathbb{E}_X [X | Y] = \mathbb{E}_X [X] \ a.s.$ ⁵

Remark 2. As with the mmse-leakage $\mathcal{L}_{mmse}(X \to Y)$, Arimoto's MI of order $\alpha = \infty$, i.e., $I^{A}_{\infty}(X;Y)$ does not have the independence property 3) (see [23, Sec 6.6]).

III. A GENERALIZATION OF THE VALUE OF INFORMATION

In this section, we introduce *the Stratonovich's Value of Information* (VoI) in a general manner to formulate the leakage-

³For any function f(Y), $\mathbb{E}_{X,Y}[(X - \mathbb{E}_X[X \mid Y])f(Y)] = 0$.

⁴This proof is borrowed from [27, Thm 11]. Interestingly, unlike the DPI for mutual information I(X; Y), $\mathcal{L}_{mmse}(X \to Y) = \mathcal{L}_{mmse}(X \to Z)$ does *not* imply that Z is a sufficient statistic of Y for X. The equality holds iff $\mathbb{E}_{X}[X \mid Y] = \mathbb{E}_{X}[X \mid Z]$ *a.s.*

⁵On the other hand, the mean independence is a stronger condition than *uncorrelatedness*, i.e., $\mathbb{E}_X[X \mid Y]$ *a.s.* $\Longrightarrow \rho(X,Y) = 0$, where $\rho(X,Y) := (\mathbb{E}_{X,Y}[XY] - \mathbb{E}_X[X]\mathbb{E}_Y[Y])/\sqrt{\mathbb{V}(X)}\sqrt{\mathbb{V}(Y)}]$ is the coefficient of correlation between X and Y.

utility trade-off problem. We also show that the generalized VoI can be viewed as an analogue of *the distortion-rate function* and *the information bottleneck*.

A. Average gain

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We first introduce *average gain* to quantify the utility of using Y for a decision-making as largest reduction of the minimal Bayes risk compared to independent case.

Definition 5 (Average gain). The average gain of using Y on X for making an action A when a loss function is $\ell(x, y, a, \delta^*)$ is defined as

$$gain^{\ell}(X;Y) := \inf_{\delta^{*}} r(\delta^{*}, p_{Y}) - \inf_{\delta^{*}} r(\delta^{*}, p_{Y|X})$$
(20)
$$= \mathbb{E}_{Y} \left[\inf_{\delta^{*}(y,\cdot)} \mathbb{E}_{X} \left[L(X, \delta^{*}(Y, \cdot)) \right] \right]$$
$$- \mathbb{E}_{Y} \left[\inf_{\delta^{*}(y,\cdot)} \mathbb{E}_{X} \left[L(X, \delta^{*}(Y, \cdot)) \mid Y \right] \right],$$
(21)

where $p_Y(y) := \sum_x p_X(x)p_{Y|X}(y \mid x)$ is a marginal distribution on Y. Note that the last equality follows from Proposition 1. In particular, the average gain with a deterministic decision rule a classic loss function $\ell(x, a)$ is given as

$$\operatorname{gain}^{c}(X;Y) = \inf_{a} \mathbb{E}_{X} \left[\ell(X,a) \right] - \mathbb{E}_{Y} \left[\inf_{a} \mathbb{E}_{X} \left[\ell(X,a) \mid Y \right] \right].$$
(22)

Remark 3. Note that the average gain is a statistical decisiontheoretic counterpart of *the average cost gain* ΔC defined in [28].

Using the similar argument as in [29, Sec V.F], it follows that the average gain satisfies the DPI.

Proposition 3 ([29, Sec V.F]). For any loss function $\ell(x, y, a, \delta^*)$, the average gain gain $\ell(X; Y)$ satisfies DPI.

Example 1. When a decision maker's action is to estimate X deterministically under a squared-loss, i.e., $A = \hat{X} = \delta(Y), \ell_{sq}(x, \hat{x}) := (x - \hat{x})^2$, $gain^{\ell_{sq}}(X; Y) = \mathcal{L}_{mmse}(X \to Y)$.

Example 2. When a decision maker's action is to estimate X randomly under an α -loss proposed by Liao *et al.* in [10, Def 3]⁶, i.e., $A = \hat{X}, \ell_{\alpha}(x, y, \hat{x}, \delta^*) := \frac{\alpha}{\alpha-1} \left(1 - \delta^*(y, \hat{x})^{\frac{-1}{\alpha}} \mathbb{1}_{\{\hat{x}=x\}}\right),$

$$\operatorname{gain}^{\ell_{\alpha}}(X;Y) = \begin{cases} \frac{\alpha}{\alpha-1} \left(e^{\frac{1-\alpha}{\alpha} \cdot H_{\alpha}^{A}(X|Y)} - e^{\frac{1-\alpha}{\alpha} \cdot H_{\alpha}(X)} \right), & \alpha > 1 \\ H(X) - H(X \mid Y) = I(X;Y), & \alpha = 1. \end{cases}$$
(23)

where (23) follows from [10, Lem 1].

Intuitively, the optimal decision rule (9) seems not to depend on y when the independent channel p_Y is used, however,

it is not the case in general loss function $\ell(x, y, a, \delta^*)$. Thus we restrict the loss function to the following *standard loss* class.

Definition 6 (Standard loss). The loss function $\ell(x, y, a, \delta^*)$ is said to be a *standard loss* if there exists a function $\tilde{\ell} \colon \mathcal{X} \times \mathcal{A} \times [0, 1] \to \mathbb{R}_+; (x, a, p) \mapsto \tilde{\ell}(x, a, p)$ such that for all x, y, a and δ^* ,

$$\ell(x, y, a, \delta^*) = \tilde{\ell}(x, a, \delta^*(y, a)).$$
(24)

Example 3. The classical loss function $\ell(x, a)$ and the α -loss $\ell_{\alpha}(x, y, \hat{x}, \delta^*)$ in the Example 2 are typical examples of the standard loss.

Proposition 4. For a standard loss $\ell(x, y, a, \delta^*)$, the optimal decision rule (9) does not depend on y when a channel is independent.

$$\inf_{\delta^*(y,\cdot)} \mathbb{E}_X[L(X,\delta^*(y,\cdot))] = \inf_{\delta^*(y,\cdot)} \sum_x p_X(x) \sum_a \delta^*(y,a) \tilde{\ell}(x,a,\delta^*(y,a))$$
(25)

is constant regardless of the value of y, the optimal decision rule (9) does not depend on y.

B. A Generalization of the Value of Information

We define VoI for information leakage to formulate the leakage-utility trade-off problem. In the following, we assume that the information leakage $\mathcal{L}(X \to Y)$ is bounded above, i.e., there exists an upper bound K(X) that can depend on p_X such that for all $p_{Y|X}$, $\mathcal{L}(X \to Y) \leq K(X)$.

Definition 7. Let the loss function $\ell(x, y, a, \delta^*)$ be a standard loss. For $0 \le R \le K(X)$, the *generalized value of information for information leakage* $\mathcal{L}(X \to Y)$ is defined as

$$\mathbf{V}_{\mathcal{L}}^{\ell}(R;\mathcal{Y}) := \sup_{\substack{p_{Y|X}:\\\mathcal{L}(X \to Y) \leq R}} \operatorname{gain}^{\ell}(X;Y) \quad (26)$$

$$= \inf_{\delta^{*}(y,\cdot)} \mathbb{E}_{X} \left[L(X,\delta^{*}(y,\cdot)) \right]$$

$$- \inf_{\substack{p_{Y|X}:\\\mathcal{L}(X \to Y) \leq R}} \mathbb{E}_{Y} \left[\inf_{\delta^{*}(y,\cdot)} \mathbb{E}_{X} \left[L(X,\delta^{*}(Y,\cdot)) \mid Y \right] \right]. \quad (27)$$

In particular, VoI for a deterministic decision rule and a classical loss function $\ell(x, a)$ is given as

$$V_{\mathcal{L}}^{\ell}(R;\mathcal{Y}) = \inf_{a} \mathbb{E}_{X} \left[\ell(X,a)\right] - \inf_{\substack{p_{Y\mid X:\\\mathcal{L}_{\alpha}(X \to Y) \leq R}} \mathbb{E}_{Y} \left[\inf_{a} \mathbb{E}_{X} \left[\ell(X,a) \mid Y\right]\right].$$
(28)

Remark 4. Stratonovich's original formulation of VoI is when $\mathcal{L}(X \to Y) = I(X;Y)$ and classical loss $\ell(x,a)$. Note that the second term of the generalized VoI $U(R;\mathcal{Y}) := \inf_{\substack{p_{Y|X}:\\ \mathcal{L}(X\to Y) \leq R}} \mathbb{E}_Y \left[\inf_{\delta^*(y,\cdot)} \mathbb{E}_X \left[L(X,\delta^*(Y,\cdot)) \mid Y \right] \right]$ will be the distortion-rate function $D(R;\mathcal{Y})$ under a non-standard loss function $\ell(x, y, a, \delta^*) = d(x, y)$, where d(x, y) is a distortion

⁶Technically, Liao *et al.* call $L_{\alpha}(x, \delta^{*}(y, \cdot)) := \mathbb{E}_{\hat{X}}[\ell_{\alpha}(x, y, \hat{X}, \delta^{*}) | Y = y]$ itself as α -loss. Note that the value of $L_{\alpha}(x, \delta^{*}(y, \cdot))$ is extended by continuity to $\alpha = 1$ and $\alpha = \infty$.

function, which is not appropriate loss for a decision-making context since it only measures the distortion between x and y.

Example 4. From the Example 1 and Example 2, it follows immediately that

$$\mathsf{V}_{\mathcal{L}_{\mathsf{mmse}}}^{\ell_{\mathsf{sq}}}(R;\mathcal{Y}) = R, \qquad 0 \le R \le \mathbb{V}(X), \tag{29}$$

$$\mathsf{V}_{I}^{\ell_{\alpha=1}}(R;\mathcal{Y}) = R, \qquad 0 \le R \le H(X), \tag{30}$$

for all alphabet \mathcal{Y} .

Example 5. When an action is to estimate U correlated only with X, i.e., $A = \hat{U}$ under $\alpha = 1$ -loss $\ell^U_{\alpha=1}(u, y, \hat{u}, \delta^*) :=$ $\frac{\alpha}{\alpha-1} \left(1 - \delta^*(y, \hat{u})^{\frac{-1}{\alpha}} 1\!\!1_{\{\hat{u}=u\}} \right) \text{ and the information leakage constraint } \mathcal{L}(X \to Y) = I(X; Y) \leq R, \text{ the generalized VoI}$ is given as

$$\mathsf{V}_{I}^{\ell_{\alpha=1}^{U}}(R;\mathcal{Y}) := \sup_{\substack{p_{Y|X}:\\I(X;Y) \leq R}} \mathsf{gain}^{\ell_{\alpha=1}^{U}}(U;Y) \tag{31}$$

$$= \sup_{\substack{p_{Y|X}:\\I(X;Y) \le R}} I(U;Y). \tag{32}$$

Note that this quantity is the well-known information bottleneck [30].

IV. MAIN RESULTS

The main results of this paper are an upper bound of the VoI for a standard loss and a fundamental limit of the VoI for a classical loss.

A. Upper bound and Fundamental Limit

For a standard loss $\ell(x, y, a, \delta^*)$, following upper bound holds.

Proposition 5. For a standard loss $\ell(x, y, a, \delta^*)$, define a function as follows:

$$\mathbf{V}_{\mathcal{L}}^{\ell}(R;\mathcal{Y}) := \inf_{\substack{\delta^{*}(y,\cdot)\\ \mathcal{L}(X\to A) \leq R}} \mathbb{E}_{X} \left[L(X,\delta^{*}(y,\cdot)) \right] - \inf_{\substack{p_{Y|X},\delta^{*}:\\ \mathcal{L}(X\to A) \leq R}} \mathbb{E}_{X,Y} \left[L(X,\delta^{*}(Y,\cdot)) \right].$$
(33)

Then $\bar{\mathsf{V}}^{\ell}_{\mathcal{L}}(0) = 0$ and for $0 \leq R \leq K(X)$ and arbitrary alphabet \mathcal{Y} ,

$$\mathsf{V}_{\mathcal{L}}^{\ell}(R;\mathcal{Y}) \le \bar{\mathsf{V}}_{\mathcal{L}}^{\ell}(R;\mathcal{Y}). \tag{34}$$

Proof. See Appendix A.

Note that the upper bound (34) still depends on the alphabet \mathcal{Y} . Interestingly, when it comes to the classical loss function $\ell(x, a)$, corresponding upper bound is independent on the alphabet \mathcal{Y} and it is even achievable.

Theorem 1. For a classical loss $\ell(x, a)$, define a function as follows:

$$\mathsf{V}_{\mathcal{L}}^{\ell}(R) := \inf_{a} \mathbb{E}_{X} \left[\ell(X, a) \right] - \inf_{\substack{p_{A|X}:\\\mathcal{L}(X \to A) \le R}} \mathbb{E}_{X, A} \left[\ell(X, A) \right].$$
(35)

Then $V_{\mathcal{L}}^{\ell}(0) = 0$ and for $0 \leq R \leq K(X)$ and arbitrary alphabet \mathcal{Y} ,

$$\mathsf{V}^{\ell}_{\mathcal{L}}(R;\mathcal{Y}) \le \mathsf{V}^{\ell}_{\mathcal{L}}(R). \tag{36}$$

Moreover, let t(A) be a sufficient statistic of A for X and $t(\mathcal{A})$ be a set of all values of the statistic. Then the equality in the inequality (36) holds when $\mathcal{Y} = t(\mathcal{A})$ and the optimal mechanism is given by

$$p_{Y|X}^{*}(y \mid x) := \sum_{a} p_{A|X}^{*}(a \mid x) 1_{\{y=t(a)\}}, \qquad (37)$$

where $p_{A|X}^* = \operatorname{arginf}_{p_{A|X}: \mathcal{L}(X \to A) \leq R} \mathbb{E}_{X,A} [\ell(X, A)].$ The statement above can be summarized as follows:

$$\sup_{\mathcal{Y}} \mathsf{V}^{\ell}_{\mathcal{L}}(R;\mathcal{Y}) = \mathsf{V}^{\ell}_{\mathcal{L}}(R).$$
(38)

Proof. See Appendix B.

Remark 5. Stratonovich call $V_I^{\ell}(R)$ as Value of Shannon's Information in [2, Chapter. 9.3]. Thus we call $\mathsf{V}_{I_{\alpha}^{\ell}}^{\ell}(R)$ (resp. $\mathsf{V}_{I_{\alpha}^{\ell}}^{\ell}(R), \mathsf{V}_{I_{\alpha}^{\ell}}^{\ell}(R), \mathsf{V}_{I_{f}}^{\ell}(R)$) and $\mathsf{V}_{\mathcal{L}_{\alpha}}^{\ell}(R)$ (resp. $\mathsf{V}^{\ell}_{\mathcal{L}^{\max}}(R), \mathsf{V}^{\ell}_{\mathcal{L}_{\mathfrak{s}}}(R)$) as Value of Arimoto's (resp. Sibson's, Csiszár's, f-) Information and Value of α - (resp. maximal α -, f-) leakage.

Let the alphabet \mathcal{X} be $\mathcal{X} := \{1, 2, \dots, m\}$ and $\mathcal{P}(X)$ be a probability simplex in \mathbb{R}^m . In Storatonovich's original proof of the achievability, he showed the equality condition as $\mathcal{Y} =$ $\mathcal{P}(X)$ and $Y = (p_{X|A}(1 \mid A), p_{X|A}(2 \mid A), \dots, p_{X|A}(m \mid A))$ $(A)) \in \mathcal{P}(X)$. In [19], Raginsky gave much shorter proof with $\mathcal{Y} = \mathcal{A}$ and $Y = \mathcal{A}$. Note that both equality conditions are special cases of the Theorem 1, i.e., following holds.

Proposition 6. t(A) = A is a sufficient statistic of A for X. Moreover, if a family of distributions ${p_{A|X}(\cdot \mid x)}_{x \in \mathcal{X}}$ have the same support, then t(A) = $(p_{X|A}(1|A), p_{X|A}(2|A), \dots, p_{X|A}(m \mid A))$ is also sufficient for X.

Proof. See Appendix C.

Remark 6. Even though mmse-leakage $\mathcal{L}_{mmse}(X \to Y)$ and Arimoto's MI of order $\alpha = \infty$, i.e., $I^{A}_{\infty}(X;Y)$ does not have the independence property 3), almost the same result holds for $\mathsf{V}^{\ell}_{\mathcal{L}_{\mathsf{mmse}}}(R)$ and $\mathsf{V}^{\ell}_{I^{\mathsf{A}}_{\mathsf{A}}}(R)$ since the only part that we use the independence property is to prove $V_{\mathcal{L}}^{\ell}(0) = 0$. Note that $\mathsf{V}^{\ell}_{\mathcal{L}_{\mathsf{mmse}}}(0) \geq 0 \text{ and } \mathsf{V}^{\ell}_{I^{\mathsf{A}}_{\infty}}(0) \geq 0 \text{ in general.}$

B. Basic properties of the Fundamental Limit

The following basic properties hold for the fundamental limit $\mathsf{V}^{\ell}_{\mathcal{L}}(R)$.

Proposition 7.

- $\begin{array}{ll} 1) \ \ \mathsf{V}^\ell_{\mathcal{L}}(R) \ \, \text{is non-decreasing in } R. \\ 2) \ \ \mathsf{V}^\ell_{\mathcal{L}}(R) \ \, \text{is concave (resp. quasi-concave) if } \mathcal{L}(X \to A) \end{array}$ is convex (resp. quasi-convex) in $p_{A|X}$.

3) Let $\mathcal{L}_1(X \to Y), \mathcal{L}_2(X \to Y)$ be information leakage measures. If there exists a constant c > 0 such that $\mathcal{L}_1(X \to Y) \leq c\mathcal{L}_2(X \to Y)$, then

$$\mathsf{V}_{\mathcal{L}^{(2)}}^{\ell}(R) \le \mathsf{V}_{\mathcal{L}^{(1)}}^{\ell}(cR), \tag{39}$$

$$\mathsf{V}^{\ell}_{\mathcal{L}^{(2)}}(R/c) \le \mathsf{V}^{\ell}_{\mathcal{L}^{(1)}}(R). \tag{40}$$

Proof. See Appendix D.

Corollary 1. From the property 2) above, following holds.

- $V_I^{\ell}(R)$ is concave since I(X; A) is convex in $p_{A|X}$ for
- fixed p_X (see, e.g., [21, Thm 2.7.4]) $\mathsf{V}_{\mathcal{L}_{\alpha}}^{\ell}(R) = \mathsf{V}_{I_{\alpha}}^{\ell}(R)$ is quasi-concave since $\mathcal{L}_{\alpha}(X \to \mathcal{L}_{\alpha})$ $A = I^{A}_{\alpha}(X; A)$ is quasi-convex in $p_{A|X}$ for fixed p_X (see [10, Footnote 3])
- For $\alpha > 0$, $V_{I^{s}}^{\ell}(R)$ is quasi-concave since $I_{\alpha}^{s}(X;A)$ is quasi-convex in $p_{A|X}$ for fixed p_X . For $0 < \alpha \leq 1$, $\mathsf{V}^{\ell}_{I^{\mathsf{S}}_{\alpha}}(R)$ is concave since $I^{\mathsf{S}}_{\alpha}(X;A)$ is convex in $p_{A|X}$ for fixed p_X (see [31, Thm 10])
- For $0 < \alpha \leq 1$, $V_{I_{\alpha}}^{\ell}(R)$ is concave since $I_{\alpha}^{C}(X; A)$ is convex in $p_{A|X}$ for fixed p_X (see [32, Thm 9 (c)]) • $\mathsf{V}_{I_f}^\ell(R)$ and $\mathsf{V}_{\mathcal{L}_f}^\ell(R)$ are concave since $I_f(X; A)$ and
- $\mathcal{L}_{f}^{T}(X \to A)$ are both convex in $p_{A|X}^{T}$ for fixed p_{X}
- For $\alpha > 0$, $\mathsf{V}^{\ell}_{\mathcal{L}^{\max}_{\alpha}}(R)$ is quasi-concave since $\mathcal{L}^{\max}_{\alpha}(X \to \mathbb{C}^{\ell})$ A) is quasi-convex in $p_{A|X}$ for fixed support of p_X (see [10, Thm 3]). For $0 < \alpha \le 1$, $V_{\mathcal{L}_{\alpha}^{\max}}^{\ell}(R)$ is concave since $\mathcal{L}_{\alpha}^{\max}(X \to A)$ is convex in $p_{A|X}$ for fixed support of p_X^{8}

Figure 2 shows a graph of the value of Shannon's information.

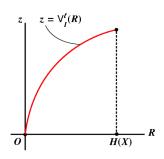


Fig. 2. Value of Shannon's information

C. Extension: logarithmic value of information

Instead of the average gain in Definition 5, we can consider logarithmic gain to capture utility.

Definition 8. The *logarithmic gain* of using Y on X for making an action A when a loss function is $\ell(x, a)$ and the logarithmic value of information are defined as follows:

$$\mathsf{Lgain}^{\ell}(X;Y) := \log \frac{\inf_{\delta^*} r(\delta^*, p_Y)}{\inf_{\delta^*} r(\delta^*, p_{Y|X})}$$
(41)

$$= \log \frac{\inf_{a} \mathbb{E}_{X} \left[\ell(X, a) \right]}{\mathbb{E}_{Y} \left[\inf_{a} \mathbb{E}_{X} \left[\ell(X, a) \mid Y \right] \right]}, \quad (42)$$

$$\mathsf{LV}^{\ell}_{\mathcal{L}}(R;\mathcal{Y}) := \sup_{\substack{p_{Y|X}:\\\mathcal{L}(X \to Y) \le R}} \mathsf{Lgain}^{\ell}(X;Y).$$
(43)

Example 6. Let $A = \hat{X}$ and $\ell_{sq}(x, \hat{x}) = (x - \hat{x})^2$. Then

$$\mathsf{Lgain}^{\ell_{\mathsf{sq}}}(X;Y) = \log \frac{\mathbb{V}(X)}{\mathbb{E}_{Y}\left[\mathbb{V}(X \mid Y)\right]} =: \mathcal{L}_{\mathsf{MS}}(X \to Y).$$
(44)

From Proposition 2, it follows that Lgain $\ell_{sq}(X;Y) =$ $\mathcal{L}_{MS}(X \to Y)$ has properties 1),2) and does not have the independence property 3).

Remark 7. It is worth noting that Issa et al. introduce maximal versions of the logarithmic gain in [15]. For example, they inrotoduced the variance leakage $\mathcal{L}^{v}(X \to Y)$ as follows:

$$\mathcal{L}^{v}(X \to Y) := \sup_{U - X - Y} \mathcal{L}_{\mathsf{MS}}(U \to Y)$$
(45)

$$= -\log(1 - \rho_m(X;Y)),$$
 (46)

(see [15, Def 10 and Lem 16]) where

$$\rho_m(X;Y) := \sup_{\substack{f,g:\\ \mathbb{E}[f(X)] = \mathbb{E}[g(X)] = 0,\\ \mathbb{E}[f(X)^2] = \mathbb{E}[g(X)^2] = 1}} \mathbb{E}[f(X)g(Y)]$$
(47)

is the maximal correlation. Note that the variance leakage $\mathcal{L}^{v}(X \to Y)$ have all properties 1), 2) and 3) in Definition 3 (see [23, Prop 5.2]). They also introduced a maximal version of all the logarithmic gain, called maximal cost leakage $\mathcal{L}^{c}(X \to Y)$, as follows:

$$\mathcal{L}^{c}(X \to Y) := \sup_{\substack{U-X-Y\\\hat{U},\ell: \ X \times \hat{\mathcal{X}} \to \mathbb{R}_{+}}} \mathsf{Lgain}^{\ell}(U;Y)$$
(48)

$$= -\log \sum_{y} \min_{x \in \operatorname{supp}(p_X)} p_{Y|X}(y \mid x)$$
(49)

(see [15, Def 11 and Thm 15]). Note also that the maximal cost gain $\mathcal{L}^{c}(X \to Y)$ have have all properties 1),2) and 3) in Definition 3 (see [15, Cor 5]). In addition to these loss (cost) based information leakage measures, they also introduced several utility9 based information leakage measures and showed relationships to the maximal information leakage $\mathcal{L}_{MaxL}(X \to Y)$. See [15] for detail.

For the logarithmic gain, a similar result as in Theorem 1 holds as follows.

⁷From the convexity of f-divergence [26, Lem 4.1], one can derive the convexity of $I_f(X; A)$ and $\mathcal{L}_f(X \to A)$ in $p_{A|X}$.

⁸Convexity of $\mathcal{L}^{\max}_{\alpha}(X \to A)$ in $p_{A|X}$ follows from [10, Thm 2] and [31, Thm 10].

⁹Here we used the term 'utility' in a statistical decision-theoretic sense. Note that Issa et al. call 'utility based information leakage' as 'gain based information leakage'.

Corollary 2. For a classical loss $\ell(x, a)$, define a function as follows:

$$\mathsf{LV}_{\mathcal{L}}^{\ell}(R) := \log \inf_{a} \mathbb{E}_{X} \left[\ell(X, a) \right] - \inf_{\substack{p_{A|X}:\\\mathcal{L}(X \to A) \leq R}} \log \mathbb{E}_{X, A} \left[\ell(X, A) \right].$$
(50)

Then, following holds.

$$\sup_{\mathcal{V}} \mathsf{LV}^{\ell}_{\mathcal{L}}(R;\mathcal{Y}) = \mathsf{LV}^{\ell}_{\mathcal{L}}(R).$$
(51)

V. APPLICATION TO PRIVACY-UTILITY TRADE-OFF

In this section, we provide an interpretation of the achievability condition in Theorem 1 in the PUT context. We assume three parties: data curator (Alice), a legitimate user (Bob), and an adversary (Eve). Alice has the original data X and disclose perturbed data Y through a privacy mechanism $p_{Y|X}$ to prevent information leakage to Eve. A privacy constraint is represented as $\mathcal{L}(X \to Y) \leq R$, where the information leakage measure $\mathcal{L}(X \to Y)$ is chosen arbitrarily by Alice. While Bob's purpose of using the published data Y is represented as an action, a deterministic decision rule and a loss function, i.e., $A = \delta(Y)$ and $\ell(x, a)$, respectively. Suppose that Alice knows the Bob's purpose of using the published data Y before disclosure. We also assume that Bob make his action with the optimal decision rule δ^{Bayes} under the loss functions $\ell(x, a)$.

In the situation above, Theorem 1 states that in order to maximize utility measured by $gain^{\ell}(X;Y)$ under the privacy constraint $\mathcal{L}(X \to Y) \leq R$, Alice should take the following steps:

1) Find the channel $p_{A|X}^*$ such that

$$p_{A|X}^* = \operatorname*{arginf}_{p_{A|X}: \ \mathcal{L}(X \to A) \le R} \mathbb{E}_{X,A} \left[\ell(X, A) \right].$$
(52)

- 2) Generate a random variable \tilde{A} drawn to $p_{A|X}^*$.
- 3) Finally, disclose $Y = t(\tilde{A})$, a sufficient statistic of \tilde{A} for X, to public.

Remark 8. When Alice assumes Eve's purpose of using Y, say δ_{eve}^* and $\ell_{\text{eve}}(x, y, a, \delta_{\text{eve}}^*)$, she can chose a privacy constraint as an average gain for Eve, i.e., $\mathcal{L}(X \to Y) := \text{gain}^{\ell_{\text{eve}}}(X;Y)$. Note that she can even adopt the privacy constraint as the *maximal gain* Mgain $^{\ell_{\text{eve}}}(X;Y)$ defined as follows, which is the inferential gain for using Y in the most favorable situation for Eve.

Definition 9 (Maximal gain). For a standard loss $\ell(x, y, a, \delta^*)$, the maximal gain of using Y on X for making an action A is defined as

$$\operatorname{\mathsf{Mgain}}^{\ell}(X;Y) := \mathbb{E}_{Y} \left[\inf_{\delta^{*}(y,\cdot)} \mathbb{E}_{X} \left[L(X,\delta^{*}(Y,\cdot)) \right] \right] - \min_{y} \inf_{\delta^{*}(y,\cdot)} \mathbb{E}_{X} \left[L(X,\delta^{*}(Y,\cdot)) \mid Y = y \right].$$
(53)

Note that it follows immediately from [33, Prop 23] that the maximal gain satisfies DPI.

VI. CONCLUSION

In this study, we generalized the Stratonovich's VoI to formulate a problem of decision-making under a general information leakage constraint and a general loss function. We derived upper bound for the VoI and showed weaker achievability condition than ever for a classical loss function. We presented an interpretation of these results in the PUT context and some extended results. Future work includes deriving calculation algorithms for the upper bound.

APPENDIX A PROOF OF PROPOSITION 5

Proof. Define $\tilde{U}_{\mathcal{L}}^{\ell}(R; \mathcal{Y})$ and $\bar{U}_{\mathcal{L}}^{\ell}(R)$ as the second terms of the RHS in (28) and (35), respectively, i.e.,

$$\tilde{U}_{\mathcal{L}}^{\ell}(R;\mathcal{Y}) := \inf_{\substack{p_{Y|X}:\\\mathcal{L}(X \to Y) \le R}} \mathbb{E}_{Y} \left[\inf_{\delta^{*}(y,\cdot)} \mathbb{E}_{X} \left[L(X,\delta^{*}(Y,\cdot)) \mid Y \right] \right]$$

$$\bar{U}_{\mathcal{L}}^{\ell}(R;\mathcal{Y}) := \inf_{\substack{p_{Y|X},\delta^*:\\\mathcal{L}(X \to A) \le R}} \mathbb{E}_{X,Y}\left[L(X,\delta^*(Y,\cdot))\right].$$
(55)

It suffices to show that $\tilde{U}_{\mathcal{L}}^{\ell}(R; \mathcal{Y}) \geq \bar{U}_{\mathcal{L}}^{\ell}(R; \mathcal{Y})$ for arbitrary alphabet \mathcal{Y} . Define the privacy mechanism $\tilde{p}_{Y|X}$ and the optimal randomized decision rule $\tilde{\delta}^{*,\text{Bayes}} = \tilde{p}_{A|Y}$ as

$$\tilde{p}_{Y|X} := \underset{\substack{p_{Y|X}:\\\mathcal{L}(X \to Y) \le R}}{\operatorname{arginf}} \quad \tilde{U}_{\mathcal{L}}^{\ell}(R; \mathcal{Y}), \tag{56}$$

$$\tilde{\delta}^{*,\text{Bayes}}(y,a) = \tilde{p}_{A|Y}(a \mid y)$$

$$:= \operatorname{arginf}_{\delta^{*}(y,\cdot)} \sum_{x,a} \ell(x,y,a,\delta^{*}) \delta^{*}(y,a) \tilde{p}_{X|Y}(x \mid y),$$
(58)

where $\tilde{p}_{X|Y}(x \mid y) := \frac{p_X(x)\tilde{p}_{Y|X}(y|x)}{p_Y(y)}$. Since X - Y - A forms a Markov chain for the distributions $\tilde{p}_{Y|X}$ and $\tilde{\delta}^{*,\text{Bayes}} = \tilde{p}_{A|Y}$,

$$\mathcal{L}(X \to A) \le \mathcal{L}(X \to Y) \le R \tag{59}$$

holds from DPI (16) and (56). Then from (59),

$$\bar{U}_{\mathcal{L}}^{\ell}(R;\mathcal{Y}) = \inf_{\substack{p_{Y\mid X}, \delta^*:\\\mathcal{L}(X \to A) \le R}} \mathbb{E}_{X,Y} \left[L(X, \delta^*(Y, \cdot)) \right]$$
(60)
$$\leq \sum_{x,y,a} p_X(x) \tilde{p}_{Y\mid X}(y \mid x) \tilde{\delta}^{*, \text{Bayes}}(y, a) \ell(x, y, a, \tilde{\delta}^{*, \text{Bayes}})$$
(61)

$$= \tilde{U}_{\mathcal{L}}^{\ell}(R; \mathcal{Y}).$$
(62)

APPENDIX B Proof of Theorem 1

Based on [2, Chapter. 9.7] and a refined proof in [19], we prove Theorem 1 as follows.

Proof. Define $U_{\mathcal{L}}^{\ell}(R; \mathcal{Y})$ and $U_{\mathcal{L}}^{\ell}(R)$ as the second terms of RHS in (28) and (35), respectively, i.e.,

$$U_{\mathcal{L}}^{\ell}(R;\mathcal{Y}) := \inf_{\substack{p_{Y\mid X}:\\\mathcal{L}(X \to Y) \le R}} \mathbb{E}_{Y} \left[\inf_{a} \mathbb{E}_{X} \left[\ell(X,a) \mid Y \right] \right], \quad (63)$$

$$U_{\mathcal{L}}^{\ell}(R) := \inf_{\substack{p_{A|X}:\\\mathcal{L}(X \to A) \le R}} \mathbb{E}_{X,A}\left[\ell(X,A)\right].$$
(64)

(Converse part): It suffices to show that $U_{\mathcal{L}}^{\ell}(R; \mathcal{Y}) \geq U_{\mathcal{L}}^{\ell}(R)$ to prove $V_{\mathcal{L}}^{\ell}(R; \mathcal{Y}) \leq V_{\mathcal{L}}^{\ell}(R)$ for arbitrary \mathcal{Y} . This can be proved in a similar way to that in the proof of Proposition 5 (see [20, Appendix D]).

(Achievable part): Let $\mathcal{Y} := t(\mathcal{A})$. It suffices to show that $U^{\ell}_{\mathcal{L}}(R; t(\mathcal{A})) \leq U^{\ell}_{\mathcal{L}}(R)$. Define $p^*_{A|X}, p^*_A$ and $p^*_{X|A}$ as follows:

$$p_{A|X}^* := \underset{\substack{p_{A|X}:\\\mathcal{L}(X \to A) \le R}}{\operatorname{argmin}} \mathbb{E}_{X,A}\left[\ell(X,A)\right], \quad (65)$$

$$p_A^*(a) := \sum_x p_X(x) p_{A|X}^*(a \mid x), \tag{66}$$

$$p_{X|A}^{*}(x \mid a) := \frac{p_{X}(x)p_{A|X}^{*}(a \mid x)}{p_{A}^{*}(a)}.$$
(67)

Let \tilde{A} be a random variable drawn to p_A^* . Since $X - \tilde{A} - Y := t(\tilde{A})$ forms a Markov chain,

$$\mathcal{L}(X \to Y) \le \mathcal{L}(X \to \tilde{A}) \le R \tag{68}$$

holds from DPI (16) and (65). Now, define a privacy mechanism $p^*_{Y|X}$ as

$$p_{Y|X}^*(y \mid x) := \sum_{a} p_{A|X}^*(a \mid x) \mathbf{1}_{\{y=t(a)\}}.$$
 (69)

Then

$$U_{\mathcal{L}}^{\ell}(R; t(\mathcal{A})) := \inf_{\substack{p_{Y|X}:\\\mathcal{L}(X \to Y) \le R}} \mathbb{E}_{Y} \left[\inf_{a} \mathbb{E}_{X} \left[\ell(X, a) \mid Y \right] \right]$$
(70)

$$\leq \mathbb{E}_{Y}\left[\inf_{a} \mathbb{E}_{X}^{p_{X\mid Y}^{*}}\left[\ell(X, a) \mid Y\right]\right],\tag{71}$$

where the expectation $\mathbb{E}_X^{p_{Y|X}^*}[\cdot]$ is taken over the distribution $p_{X|Y}^*(x|y) = p_X(x)p_{Y|X}^*(y|x)/p_Y^*(y)$. Now, we will evaluate $\inf_a \mathbb{E}_X^{p_{X|Y}^*}[\ell(X,a) \mid Y = t(a')]$ from above.

$$\inf_{a} \mathbb{E}_{X}^{p_{X|Y}^{*}} \left[\ell(X,a) \mid Y = t(a') \right]$$

$$\stackrel{(*)}{=} \inf_{a} \mathbb{E}_{X}^{p_{X|A}^{*}} \left[\ell(X,a) \mid \tilde{A} = a' \right]$$
(72)

$$\leq \mathbb{E}_X^{p_{X|A}^*} \left[\ell(X, a') \mid \tilde{A} = a' \right], \tag{73}$$

where the equality (*) follows from the sufficiency of $t(\hat{A})$ 10 . Thus we have

$$\mathbb{E}_{Y}\left[\inf_{a} \mathbb{E}_{X}^{p_{X|Y}^{*}}\left[\ell(X,a)\right] \mid Y\right] = \mathbb{E}_{\tilde{A}}\left[\inf_{a} \mathbb{E}_{X}^{p_{X|A}^{*}}\left[\ell(X,a)\right] \mid \tilde{A}\right]$$
(74)

$$\leq \mathbb{E}_{X,\tilde{A}}^{p_{X|A}^{*}}\left[\ell(X,\tilde{A})\right] = \inf_{\substack{p_{A|X}:\\\mathcal{L}(X \to A) \leq R}} \mathbb{E}_{X,A}\left[\ell(X,A)\right]$$
(75)

$$= U_{\mathcal{L}}^{\ell}(R). \tag{76}$$

By combining with (71), $U_{\mathcal{L}}^{\ell}(R; t(\mathcal{A})) \leq U_{\mathcal{L}}^{\ell}(R)$.

APPENDIX C PROOF OF PROPOSITION 6

The sufficiency of t(A) = A is trivial. To prove the sufficiency of $t(A) = (p_{X|A}(1|A), p_{X|A}(2|A), \dots, p_{X|A}(m|A))$, we first introduce the following lemmas.

Lemma 1 ([34, Thm 6.12]). Assume that a family of distributions $\{p_{A|X}(\cdot | x)\}_{x \in \mathcal{X}}$ have the same support. Then

$$s(A) = \left(\frac{p_{A|X}(A \mid 2)}{p_{A|X}(A \mid 1)}, \dots, \frac{p_{A|X}(A \mid m)}{p_{A|X}(A \mid 1)}\right)$$
(77)

is a minimal sufficient statistic of A for X.

Lemma 2. Let $T_1 = t_1(A)$ be a sufficient statistic of A for X. If there exists a (measurable) function f such that $T_1 = f(t_2(A))$, then $T_2 = t_2(A)$ is also sufficient for X.

Proof. The statement follows immediately from the Fisher's factorization theorem (see, e.g., [34, Thm 6.5]) or DPI for Shannon's mutual information (see e.g., [21, Eq (2.124)]).

Lemma 3.

$$e(A) = \left(\frac{p_{X|A}(2 \mid A)}{p_{X|A}(1 \mid A)}, \dots, \frac{p_{X|A}(m \mid A)}{p_{X|A}(1 \mid A)}\right)$$
(78)

is a (minimal) sufficient statistic of A for X.

Proof. ¹¹ Since $s(A) := \left(\frac{p_{A|X}(A|2)}{p_{A|X}(A|1)}, \dots, \frac{p_{A|X}(A|m)}{p_{A|X}(A|1)}\right) = \left(\frac{p_X(1)}{p_X(2)} \cdot \frac{p_{X|A}(2|A)}{p_{X|A}(1|A)}, \dots, \frac{p_X(1)}{p_X(m)} \cdot \frac{p_{X|A}(m|A)}{p_{X|A}(1|A)}\right)$ is a function of e(A), it follows from Lemma 2 that e(A) is also sufficient. The minimality follows immediately as follows: For arbitrary $a, b \in \mathcal{A}$, it holds that $s(a) = s(b) \iff e(a) = e(b)$.

Making use of these results, we prove Proposition 6 as follows.

Proof. Since $e(A) = \left(\frac{p_{X|A}(2|A)}{p_{X|A}(1|A)}, \dots, \frac{p_{X|A}(m|A)}{p_{X|A}(1|A)}\right)$ is a function of $t(A) = (p_{X|A}(1|A), p_{X|A}(2|A), \dots, p_{X|A}(m|A))$, from Lemma 2, t(A) is also sufficient for X.

¹⁰It follows immediately from $p_{X|\tilde{A}}(x|a') = \sum_{y} p_{X,Y|\tilde{A}}(x,y|a') = \sum_{y} p_{Y|\tilde{A}}(y|a')p_{X|Y}(x|y) = p_{X|Y}(x|y)\mathbb{1}_{\{y=t(a')\}} = p_{X|Y}(x \mid t(a')),$ where we used the sufficiency of $Y = t(\tilde{A})$ in the second equality. ¹¹This proof is based on [35, Prop 3.3].

APPENDIX D

PROOF OF PROPOSITION 7

Proof. The property 1) is trivial. To prove the property 2), it suffices to show that $U(R) := \inf_{\substack{p_{A|X}: \\ \mathcal{L}(X \to A) \leq R \\ \mathcal{L}(X \to A) \leq R \\ \mathcal{L}(X \to A) = R \\$ $\mathcal{L}(p_X, p_{A|X})$ is convex (resp. quasi-convex). We will only prove the convexity. For arbitrary $0 \leq \lambda \leq 1$ and $0 \leq$ $R_1, R_2 \leq K(X)$, define

$$p_{A|X}^{*,1} := \underset{\substack{p_{A|X}:\\\mathcal{L}(X \to A) \le R_1}}{\operatorname{arginf}} \mathbb{E}_{X,A} \left[\ell(X,A) \right], \tag{79}$$

$$p_{A|X}^{*,2} := \underset{\substack{p_{A|X}:\\\mathcal{L}(X \to A) \le R_2}}{\operatorname{arginf}} \mathbb{E}_{X,A}\left[\ell(X,A)\right], \tag{80}$$

$$p_{A|X}^{*,\lambda} := \lambda p_{A|X}^{*,1} + (1-\lambda) p_{A|X}^{*,2}.$$
(81)

Then let denote $\mathcal{L}^{*,1}(X \to A), \mathcal{L}^{*,2}(X \to A)$ and $\mathcal{L}^{*,\lambda}(X \to A)$ as the α -leakages defined by $p_{A|X}^{*,1}, p_{A|X}^{*,2}$ and $p_{A|X}^{*,\lambda}$, respectively. Then

$$\mathcal{L}^{*,\lambda}(X \to A) \le \lambda \mathcal{L}^{*,1}(X \to A) + (1-\lambda)\mathcal{L}^{*,2}(X \to A) \qquad (82)$$
$$\le \lambda R_1 + (1-\lambda)R_2. \qquad (83)$$

$$\leq \lambda R_1 + (1 - \lambda)R_2. \tag{83}$$

Therefore,

$$U(\lambda R_1 + (1 - \lambda)R_2) \le \mathbb{E}_{X,A}^{p_{A|X}^{*,\Lambda}} \left[\ell(X, A)\right]$$
(84)

$$=\sum_{x,a} p_X(x) p_{A|X}^{*,\lambda}(a \mid x) \ell(x,a) \quad (85)$$

$$= \lambda U(R_1) + (1 - \lambda)U(R_2).$$
 (86)

The quasi-convexity can be proved in a similar way.

To prove the property 3), it suffices to show that

$$\mathsf{V}_{\mathcal{L}_{2}}^{\ell}(R;\mathcal{Y}) \le \mathsf{V}_{\mathcal{L}_{1}}^{\ell}(cR;\mathcal{Y}) \tag{87}$$

for arbitrary alphabet \mathcal{Y} . To this end, define

$$p_{Y|X}^{*,2} := \underset{\substack{p_{Y|X}:\\\mathcal{L}_2(X \to Y) \le R}}{\operatorname{argsup}} \operatorname{gain}^{\ell}(X;Y)$$
(88)

for arbitrary alphabet \mathcal{Y} . Since

$$\mathcal{L}_1(p_X, p_{Y|X}^{*,2}) \le c \mathcal{L}_2(p_X, p_{Y|X}^{*,2}) \le cR,$$
(89)

it holds that

$$\mathsf{V}_{\mathcal{L}_{1}}^{\ell}(cR;\mathcal{Y}) := \sup_{\substack{p_{Y|X}:\\\mathcal{L}_{1}(X \to Y) \le cR}} \mathsf{gain}^{\ell}(X;Y) \tag{90}$$

$$\leq \operatorname{gain}^{\ell}(p_X, p_{Y|X}^{*,2}) = \mathsf{V}_{\mathcal{L}_2}^{\ell}(R; \mathcal{Y}), \qquad (91)$$

where $\operatorname{gain}^{\ell}(p_X, p_{Y|X}^{*,2}) := r(\delta^{*,\operatorname{Bayes}}, p_Y^{*,2}) - r(\delta^{*,\operatorname{Bayes}}, p_{Y|X}^{*,2})$ and $p_Y^{*,2}(y) := \sum_x p_X(x) p_{Y|X}^{*,2}(y \mid x).$

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