# Multichannel Optimal Tree-Decodable Codes are Not Always Optimal Prefix Codes 

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#### Abstract

The theory of multichannel prefix codes aims to generalize the classical theory of prefix codes. Although singleand two-channel prefix codes always have decoding trees, the same cannot be said when there are more than two channels. One question is of theoretical interest: Do there exist optimal tree-decodable codes that are not optimal prefix codes? Existing literature, which focused on generalizing single-channel results, covered little about non-tree-decodable prefix codes since they have no single-channel counterparts. In this work, we study the fundamental reason behind the non-tree-decodability of prefix codes. By investigating the simplest non-tree-decodable structure, we obtain a general sufficient condition on the channel alphabets for the existence of optimal tree-decodable codes that are not optimal prefix codes.


## I. Introduction

Prefix-free codes, or conventionally called prefix codes, are a class of zero-error uniquely decodable source code being applied in a wide range of scenarios including the country codes [1], UTF-8 [2], [3], and most importantly, data compression [4]-[6]. In a traditional sense, an optimal prefix code is a symbol-by-symbol prefix code having the lowest redundancy when the probability of the information source is known Huffman code [6] is an iconic optimal prefix code which can be encoded in linear time (in the support size) for sorted probability [11] and be decoded in linear time (in the codeword length) by using the constructed decoding tree. To better understand the nature of prefix codes, literature also include theoretical research such as those for infinite sources [12]-[15].

Another theoretical generalization of prefix codes is to use more channels [16] ${ }^{2}$ Not all single-channel results hold when generalized to the multichannel setting. One example is that the satisfiability of the multichannel Kraft inequality [16] is insufficient for the existence of prefix codes. Instead, a rectangle packing formulation is needed to capture the geometry of prefix codes [18], [19]. Worse, [16] showed an example of a prefix code having no decoding tree. While single- and two-channel prefix codes are tree-decodable [20],

[^0]for three or more channels, we are only assured that treedecodable codes are prefix codes [16], but not the converse. Non-tree-decodable prefix codes are not well-studied, to say the least, as they have no single-channel counterparts.

When the channel alphabet sizes are the same, an optimal tree-decodable code is an optimal prefix code [16], which can be constructed by manipulating the decoding tree of the singlechannel Huffman code. Much less is known for differing alphabet sizes. To begin, although a modified Huffman procedure can produce an optimal multichannel tree-decodable code [20], the procedure is not known to be efficiently computable at this moment. Furthermore, it is unclear whether the resulting tree-decodable code is also optimal as a prefix code. One question is of interest: is an optimal tree-decodable code also an optimal prefix code in general?

In this work, we first study the fundamental reason the interweave structures - behind the non-tree-decodability of prefix codes. After that, we make use of the simplest interweave structure to construct a class of non-tree-decodable prefix codes called the selvage codes. By investigating the selvage codes, we prove that if the channel alphabets satisfy a "separable" property, then there exists an optimal tree-decodable code which is not an optimal prefix code.

## II. Preliminaries

Denote by $\mathbb{N}$ and $\mathbb{Z}^{+}$the sets of non-negative integers and positive integers respectively. For any $q \in \mathbb{Z}^{+}$, define $\mathbb{Z}_{q}=$ $\{0,1, \ldots, q-1\}$. We always count objects from the 0 -th. The notation $\uplus$ denotes the multiset sum. Let $\epsilon$ be the empty string.

## A. Multichannel Prefix Codes

Suppose there are $n$ channels. For each $i \in \mathbb{Z}_{n}$, each symbol sent on the $i$-th channel is from the alphabet $\mathcal{Z}_{i}$ where $\left|\mathcal{Z}_{i}\right|=q_{i} \geq 2$. Since we can map $\mathcal{Z}_{i}$ to $\mathbb{Z}_{q_{i}}$ bijectively, we assume $\mathcal{Z}_{i}=\mathbb{Z}_{q_{i}}$ in the rest of this work. For any $k \in \mathbb{Z}^{+}$, we write $\mathbb{Z}_{q_{i}}^{k}$ as the set of strings of $k$ symbols from $\mathbb{Z}_{q_{i}}$. Define $\mathbb{Z}_{q_{i}}^{0}=\{\epsilon\}$.

For any $n \in \mathbb{Z}^{+}$, define $Q_{n}:=\left(q_{0}, q_{1}, \ldots, q_{n-1}\right)=$ $\left(q_{i}\right)_{i \in \mathbb{Z}_{n}}$. We drop the subscript of $Q_{n}$ when it is clear from context. A $Q_{n}$-ary word is an $n$-tuple where the $i$-th component of the word is in $\mathcal{Z}_{i}^{*}:=\bigcup_{k=0}^{\infty} \mathbb{Z}_{q_{i}}^{k}$, the set of all possible strings built using the alphabet $\mathbb{Z}_{q_{i}}$. For any word $\mathbf{c}$, the $i$-th component is denoted by $\mathbf{c}(i)$.

Let $Z$ be an information source and $\mathcal{Z}$ be the alphabet of $Z$. A $Q_{n}$-ary source code for $Z$ is a map $\mathcal{Q}: \mathcal{Z} \rightarrow \prod_{i \in \mathbb{Z}_{n}} \mathcal{Z}_{i}^{*}$. For each $z \in \mathcal{Z}, \mathcal{Q}(z)$ is the codeword for $z$. The $i$-th


Fig. 1: The relations between certain classes of source codes.
component of any codeword is sent through the $i$-th channel. When we send more than one codeword, the codewords are concatenated component-wise so that the boundaries of the codewords are not explicit. The image of $\mathcal{Q}$ is called the codebook of the source code. For convenience, we also refer a source code to its codebook. The codeword matrix $M$ of a $Q_{n^{-}}$ ary source code $C=\left\{\mathbf{c}_{j}\right\}_{j \in \mathbb{Z}_{m}}$ is an $m \times n$ matrix where the $j$-th row of $M$ is $\mathbf{c}_{j}$. If the multiset $P$ is the probability of $Z$, then the source code for $Z$ is also called a source code on $P$.

Two words $\mathbf{c}, \mathbf{c}^{\prime}$ are prefix-free, denoted by $\mathbf{c} \not \subset \mathbf{c}^{\prime}$, if there exists a channel $i$ such that $\mathbf{c}(i)$ and $\mathbf{c}^{\prime}(i)$ are prefix-free.

Definition 1 (Prefix Codes [16]). A $Q$-ary prefix code is a $Q$-ary source code such that every pair of codewords are prefix-free.

A (multichannel) decoding tree is a tree that every non-leaf node belongs to a class. A class $i$ node means that this node is associated with the $i$-th channel. Each branch of a class $i$ node corresponds to a distinct symbol in $\mathbb{Z}_{q_{i}}$, i.e., a class $i$ node has at most $q_{i}$ children. Every leaf corresponds to a codeword and associates with a source symbol. To decode a codeword, we traverse the tree from the root node. When we reach a class $i$ node, we read a symbol of the codeword from the $i$-th channel and traverse through the corresponding branch. We can decode the codeword once we reach a leaf. An example of decoding tree can be found in Fig. 4] where the codewords are $(0, \epsilon),(1,0)$ and $(11,1)$.

Definition 2 (Tree-Decodable Code [16]). A source code is a tree-decodable code if it has a decoding tree such that every codeword can be decoded by this tree. A codeword matrix $M$ of a source code $C$ is called tree decodable if and only if $C$ is tree decodable.

Every tree-decodable code is a prefix code but not the converse [16]. All single- and two-channel prefix codes are tree decodable [20].

As the alphabets of the channels can be different, we need to use a unified unit to measure the amount of information. In this work, we use the unit "nat" (natural unit of information), i.e., we use $\ln$, the natural logarithm, in the evaluation of entropy. All results in this work are valid if we use another base for the logarithm.

Definition 3 (Lengths). The length tuple of a $Q_{n}$-ary word $\mathbf{c}$ is denoted by len $(\mathbf{c}):=\left\langle\ell_{0}, \ell_{1}, \ldots, \ell_{n-1}\right\rangle=\left\langle\ell_{i}\right\rangle_{i \in \mathbb{Z}_{n}}$, where $\ell_{i}$ is the number of symbols in $\mathbf{c}(i)$. The $Q_{n^{-}}$ descriptive length of a length tuple $\left\langle\ell_{i}\right\rangle_{i \in \mathbb{Z}_{n}}$ is defined as
$\left|\left\langle\ell_{i}\right\rangle_{i \in \mathbb{Z}_{n}}\right|_{Q_{n}}:=\sum_{i \in \mathbb{Z}_{n}} \ell_{i} \ln q_{i}$. The descriptive length of a $Q_{n}$-ary word $\mathbf{c}$ is defined as $|\mathbf{c}|:=|\operatorname{len}(\mathbf{c})|_{Q_{n}}$.

The descriptive length of a word is the number of nats we need to represent the word. The multichannel entropy bound [16], [18] below states that the expected descriptive codeword length is no less than the entropy of the source, which is consistent with the single-channel one.
If the finite sequences of codewords of any two distinct finite sequences of source symbols are different, then the source code is a uniquely decodable code. Any codeword of a uniquely decodable code can be decoded without referring to the symbols of any future codewords if the code is tree-decodable [16], [18].
Entropy Bound: For any uniquely decodable code $\left\{\mathbf{c}_{j}\right\}_{j \in \mathbb{Z}_{m}}$ for a source random variable with probability $\left\{p_{j}\right\}_{j \in \mathbb{Z}_{m}}$, $\sum_{j \in \mathbb{Z}_{m}} p_{j}\left|\mathbf{c}_{j}\right| \geq-\sum_{j \in \mathbb{Z}_{m}} p_{j} \ln p_{j}$. The equality holds if and only if $\left|\mathbf{c}_{j}\right|=-\ln p_{j}, \forall j \in \mathbb{Z}_{m}$.
The value $\left|\mathbf{c}_{j}\right|+\ln p_{j}$ is the local redundancy of the codeword $\mathbf{c}_{j}$ and the $\operatorname{sum} \sum_{j \in \mathbb{Z}_{m}} p_{j}\left(\left|\mathbf{c}_{j}\right|+\ln p_{j}\right)$ is the redundancy of the source code. An optimal code of a certain class of codes (e.g., tree-decodable codes, prefix codes, etc.) on a multiset $P$ of probabilities is a (symbol-by-symbol) code of that class having the lowest redundancy.

Definition 4 (Tree Line). A tuple $Q$ of alphabet sizes is said to be above tree line on a multiset of probabilities $P$ if there exists an optimal $Q$-ary tree-decodable code $C$ on $P$ such that $C$ is not an optimal $Q$-ary prefix code on $P$. If $Q$ is above tree line on some $P$, we say that $Q$ is above tree line, otherwise it is below tree line.

As a brief summary, the relations between the classes of source codes we have mentioned are illustrated in Fig. 1 .

## B. Rectangle Packing Graph

The rectangle packing graph $(R P G)$ [18], [19] is a graphical tool to visualize the geometric nature of prefix codes. Consider a $Q_{n}$-ary source code $C$. For each $\mathbf{c} \in C$, let len $(\mathbf{c})=\left\langle\ell_{i}^{\mathbf{c}}\right\rangle_{i \in \mathbb{Z}_{n}}$. Let $\ell_{i}^{\max }:=\max _{\mathbf{c} \in C} \ell_{i}^{\mathbf{c}}$. The size of a $w_{0} \times w_{1} \times \ldots \times w_{n-1}$ hyper-rectangle is denoted by $\left\langle w_{i}\right\rangle_{i \in \mathbb{Z}_{n}}$.

To draw an RPG for $C$, we first draw a container $R$, which is a hyper-rectangle of size $\left\langle q_{i}^{\ell_{i}^{\text {max }}}\right\rangle_{i \in \mathbb{Z}_{n}}$. For the $i$-th dimension, the edge is an interval $\left[0, q_{i}^{\ell_{i}^{\text {max }}}\right)$. We write each integer $s$ in the above interval in its unique $q_{i}$-ary numeral representation as a string of length $\left\lceil\log _{q_{i}}(s+1)\right\rceil$, then left-pad 0 's to the string until the length becomes $\ell_{i}^{\max }$. That is, each unit hyper-cube in the RPG, which is called a cell, corresponds to a word in $\prod_{i \in \mathbb{Z}_{n}} \mathbb{Z}_{q_{i}}^{\ell_{i}^{\mathrm{max}}}$.
Next, each codeword $\mathbf{c}$ corresponds to a hyper-rectangle of size $\left\langle q_{i}^{\ell_{i}^{\text {max }}-\ell_{i}^{c}}\right\rangle_{i \in \mathbb{Z}_{n}}$, which is called a block. The symbols of $\mathbf{c}$ specify the location where the block is put into the container. Concretely, the block occupies exactly those cells whose common prefix of length len $(\mathbf{c})$ is given by the codeword $\mathbf{c}$.
Example 1. The RPG of a binary codebook $\{0,10,110\}$ is illustrated in Fig. 2 The container is of length 8. The blocks

Fig. 2: The RPG in Example 1


Fig. 3: The RPG in Example 2
for 0 (red), 10 (blue) and 110 (green) have length 4, 2 and 1 respectively.

Example 2. The RPG of a $(2,2)$-ary codebook $\{(0, \epsilon)$, $(1,0),(11,1)\}$ is illustrated in Fig. 3, where the horizontal and vertical axes corresponds to the 0 -th and 1 -st channels respectively. The container is a $4 \times 2$ rectangle. The blocks for $(0, \epsilon)$ (red), $(1,0)$ (blue) and $(11,1)$ (green) are $2 \times 2$, $2 \times 1$ and $1 \times 1$ rectangles respectively.

All the blocks $b_{\mathbf{c}}$ for $\mathbf{c} \in C$ do not overlap with each other if and only if $C$ is a prefix code [18]. We can write $b_{\mathbf{c}} \cap b_{\mathbf{c}^{\prime}}=\emptyset$ to indicate that these two blocks do not overlap, but we cannot write $\mathbf{c} \cap \mathbf{c}^{\prime}=\emptyset$ as codewords are not sets. So, we use the notation $\mathbf{c} \not \subset \mathbf{c}^{\prime}$, which symbolizes that the corresponding blocks do not overlap.

Suppose $C$ is a prefix code. Let $\operatorname{Vol}(R)=\prod_{i \in \mathbb{Z}_{n}} q_{i}^{\ell_{i}^{\text {max }}}$ be the hyper-volume of the container. The hyper-volume of the block $b_{\mathbf{c}}$ is $\operatorname{Vol}(\mathbf{c})=\operatorname{Vol}(R) \exp (-|\mathbf{c}|)$. As the blocks do not overlap, the sum of volumes of the blocks must be no larger than $\operatorname{Vol}(R)$. This coincides with the multichannel Kraft inequality [16], [18].
Kraft Inequality: For any uniquely decodable code $\left\{\mathbf{c}_{j}\right\}_{j \in \mathbb{Z}_{m}}$, the descriptive lengths of the codewords satisfy $\sum_{j \in \mathbb{Z}_{m}} \exp \left(-\left|\mathbf{c}_{j}\right|\right) \leq 1$.

## III. Non-Tree-Decodable Prefix Codes

## A. Guillotine-Cuts and Tree Decodability

In the view of RPG, a decoding tree is a variant of a k-d tree constructed as follows. The root node corresponds to the container and is assigned a class $i \in \mathbb{Z}_{n}$. Each internal or leaf node, corresponding to a subspace of the container, is constructed recursively using the following procedure: For each node of class $i$, we guillotine-cut the space corresponding to the node by $\left(q_{i}-1\right)$ hyper-planes perpendicular to the $i$-th dimension into $q_{i}$ subspaces of equal size, and assign a node to each subspace. Upon completion, each leaf corresponds to a block, i.e., a codeword. Summarizing, the decoding tree is a k-d tree formed by performing equal-space partitioning where the number of children and the orientation depend on the class of the node.

Example 3. We take the RPG in Example 2 (Fig. 3) as an example. The decoding tree is illustrated in Fig. 4 The number in each non-leaf node is the class of the node. Each color of the branches corresponds to the guillotine-cuts with the same color in the RPG below the tree. The orientation of each guillotine-cut depends on the class of the node.

Theorem 1. A prefix code is tree decodable if and only if all the blocks representing the codewords in the rectangle packing graph can be obtained by guillotine-cuts.

## B. Interweave Structures

We need to use a more-than-3-D RPG in the following discussion. To illustrate a 4-D hyper-rectangle of size $\langle 2\rangle_{i \in \mathbb{Z}_{4}}$, we concatenate the 0 -th and the 1 -st channels into one dimension and use a Gray code to let consecutive symbols of the same channel adjacent to each other. Then, we can transform the RPG into a cuboid as shown in Fig. 5$]^{3}$ For the 5-D hyperrectangle of size $\langle 2\rangle_{i \in \mathbb{Z}_{5}}$, besides merging the first two channels, we merge the 2 -nd and the 3 -rd channels by a Gray code. Similarly, we can transform the RPG into a cuboid as shown in Fig. $6{ }^{4}$ Note that a block can be sheared into subblocks.

We consider a space containing more than one block such that no guillotine-cut is possible without cutting through a block. For each dimension, there must be a block of length matching the one of the space at this dimension, or otherwise we can further guillotine-cut the space due to the constraints on the location and the size of the blocks. By removing the common prefix of the blocks, we can treat the space as the container itself and thus the blocks form a prefix code. If there is a dimension such that the length of every block in this dimension matches the one of the space, we can safely remove this whole dimension. This dimension corresponds to a dummy channel, which is unused by any codeword.

In the following discussion, we regard the resulting space after performing the actions described above as the container, i.e., the common prefix is removed from all the codewords and then the dummy channels are removed. Let $T$ be the set of all codewords in this space and $t$ be the number of dimensions of this space. Without loss of generality, assume these $t$ channels are the first $t$ channels.

Definition 5. An $\epsilon$-locating function for an $m \times n$ coding matrix $M$ is a function $\mathcal{E}_{M}: \mathbb{Z}_{n} \rightarrow 2^{\mathbb{Z}_{m}}$ defined as $\mathcal{E}_{M}(i)=\left\{j: M_{j, i}=\epsilon\right\}$.

Theorem 2. Let $M$ be an n-column codeword matrix. If $\mathcal{E}_{M}(i) \neq \emptyset$ for all $i \in \mathbb{Z}_{n}$, then $M$ is not tree decodable.

Example 4. Suppose the space which cannot be guillotine-cut contains the codewords $(1,01,10,0),(10,0,11,0)$ and $(11,00,1,0)$. By removing the common prefix, the codewords become $(\epsilon, 1,0, \epsilon),(0, \epsilon, 1, \epsilon)$ and $(1,0, \epsilon, \epsilon)$. By removing the dummy channel, the codewords become $(\epsilon, 1,0),(0, \epsilon, 1)$ and $(1,0, \epsilon)$. By Theorem 2, these codewords cannot form a decoding tree. This fact can be visualized in the RPG of this code illustrated in Fig. 7] The blocks interweave with each

[^1]

Fig. 4: An example of the relation between guillotine-cuts and RPG.

(a) The cylinder structure after merging the 0 -th and the 1 -st channels.

(b) The top view of the cylinder (c) The rectangle representation after structure.

cutting the red plane between 00 and 10 .

Fig. 5: The representation of a 4-D RPG by a 3-D rectangle.

(a) The bicycle tube formed after merging the 0 -th and the 1 -st channels, and merging the 2 -nd and the 3 -rd

(b) One of the cross sections of the bicycle tube. channels.

(c) The cylinder after cutting at one side of the cross section shown in Fig. 6b

(d) The rectangle formed after further cutting the cylinder at the red plane.

Fig. 6: The representation of a 5-D RPG by a 3-D rectangle.
other so that it is impossible to separate these blocks using guillotine-cuts. This is one of the interweave structures which makes prefix codes not tree decodable.

Note that it is not necessary to have $t$ interweaving blocks to form a $t$-channel non-tree-decodable prefix code. Also, we can have more than one $\epsilon$ per column or per row in the codeword matrix. To demonstrate these, we need to consider a higher dimensional hyper-rectangle. Below is an example using a 5 -D RPG.
Example 5. Consider a (2, 2, 2, 2, 2)-ary codeword matrix

$$
\underset{\text { gray }}{\text { bred }} \underset{\text { green }}{\text { yellow }}\left(\begin{array}{ccccc}
\epsilon & \epsilon & 1 & 0 & 0 \\
0 & \epsilon & \epsilon & 1 & 0 \\
0 & 0 & \epsilon & \epsilon & 1 \\
1 & 0 & 0 & \epsilon & \epsilon \\
\epsilon & 1 & 0 & 0 & \epsilon
\end{array}\right) .
$$


(a) $Q=(2,2,2)$.

(b) $Q=(5,3,2)$.

Fig. 7: The RPG of the $Q$-ary codebook $\{(\epsilon, 1,0),(0, \epsilon, 1),(1,0, \epsilon)\}$ in Example 4

We have two $\epsilon$ 's per row and per column. The RPG is illustrated in Fig. 8 We can see that if we remove the blue and the yellow blocks, the remaining three blocks can still make the 5-D container not guillotine-cutable, i.e., the code is still not tree decodable. In this new codeword matrix, we still have two $\epsilon$ 's in the first column, one $\epsilon$ in each other column, and each row has two $\epsilon$ 's.

## IV. Main Result

In this section, we will show a sufficient condition for the channel alphabets $Q$ being above tree line. At the core of our result is the construction of a $Q$-ary prefix code, called the $Q$-ary selvage code, which achieves the entropy bound on a special multiset of probabilities called the $Q$-ary selvage probability assembly (SPA).

## A. Selvage Code and Selvage Probability Assembly (SPA)

We now construct the $Q$-ary selvage code and then assign probabilities, i.e., the $Q$-ary SPA, to the codewords so that the code achieves the entropy (bound) on these probabilities.

To ensure that the $Q$-ary selvage code is not tree-decodable, our strategy is to construct its codeword matrix such that it consists of an $n \times n$ cyclic codeword submatrix as the interweave core, called the selvage core, where the diagonal is filled with $\epsilon$, the off-diagonal above the main one is filled with 1 cyclically, and the remaining entries are 0 . Then, we assign probability $p_{j}$ to the $j$-th codeword in the core such that the local redundancy $\left|\mathbf{c}_{j}\right|+\ln p_{j}$ is zero. Concretely, we set $p_{j}:=q_{j} \prod_{i \in \mathbb{Z}_{n}} q_{i}^{-1}$.
Example 6. Below is the codeword matrix of a (2, 2, 2, 2)-ary selvage core with its RPG shown in Fig. 9

$$
\underset{\text { green }}{\underset{\text { glue }}{\text { blue }}}\left(\begin{array}{cccc}
\epsilon & 1 & 0 & 0 \\
0 & \epsilon & 1 & 0 \\
0 & 0 & \epsilon & 1 \\
1 & 0 & 0 & \epsilon
\end{array}\right) \leftarrow \text { probability }\left(q_{1} q_{2} q_{3}\right)^{-1}, \begin{aligned}
& \leftarrow \text { probability }\left(q_{0} q_{2} q_{3}\right)^{-1} \\
& \leftarrow \text { probability }\left(q_{0} q_{1} q_{3}\right)^{-1} \\
& \leftarrow \text { probability }\left(q_{0} q_{1} q_{2}\right)^{-1}
\end{aligned}
$$

We can see from the above example that the RPG of a selvage code has a simple interweave structure: Each block in the interweave core has exactly one edge fitting an edge of the container in the same dimension. The RPGs in Fig. 7 fall in this category. For each plane, this structure looks like a selvage at the edges of denim, so we name our construction "selvage".

After that, we pad sufficiently many codewords of length $\langle 1, \ldots, 1\rangle$, each assigned with a probability $\prod_{i \in \mathbb{Z}_{n}} q_{i}^{-1}$, so that the sum of probabilities becomes 1 while the local


Fig. 8: The RPG of the $(2,2,2,2,2)$ ary codebook in Example 5


Fig. 9: A (2, 2, 2, 2)-ary selvage core.

Table I
The Descriptive Codeword Lengths of Optimal Codes on SPA

|  | Optimal Prefix (Entropy) | Optimal Tree-Decodable |
| :---: | :---: | :---: |
| $(2,2,2)$-ary | 1.559581 | 1.559581 |
| $(5,3,2)$-ary | 2.976887 | 2.980124 |
| $(6,3,2)$-ary | 3.154833 | 3.154833 |

redundancy is zero everywhere. This corresponds to filling the RPG with as many unit hyper-cubes as possible until the container is full. The fully-filled RPG corresponds to a prefix code as the blocks do not overlap. Since the local redundancy of each codeword is zero, the overall redundancy of the code is also zero. In other words, the $Q$-ary selvage code achieves the entropy on the $Q$-ary SPA, which is the multiset of the probabilities assigned above. On the other hand, since for each channel $i$ there exists a codeword in the interweave core such that its $i$-th component is $\epsilon$, we know from Theorem 2 that the selvage code is not tree-decodable.

We now formally state the above construction.
Definition 6 (Selvage Code). Let $n \geq 3$. The $Q$-ary selvage core, denoted by $C_{Q}^{\odot}$, is a codebook $\left\{\mathbf{c}_{j}^{\odot}\right\}_{j \in \mathbb{Z}_{n}}$ where for every $j \in \mathbb{Z}_{n}$,

$$
\mathbf{c}_{j}^{\odot}(i)= \begin{cases}\epsilon & \text { if } i=j \\ 1 & \text { if } i=(j+1) \bmod n \\ 0 & \text { otherwise }\end{cases}
$$

The $Q$-ary selvage code, denoted by $C_{Q}^{\S}$, is the codebook $C_{Q}^{\odot} \cup\left\{\mathbf{s} \in \prod_{i \in \mathbb{Z}_{n}} \mathbb{Z}_{q_{i}}: \mathbf{s} \not \subset \mathbf{c}, \forall \mathbf{c} \in C_{Q}^{\odot}\right\}$.
Theorem 3. Let $n \geq$ 3. Then, $\left|C_{Q}^{\S} \backslash C_{Q}^{\odot}\right|=$ $\prod_{i \in \mathbb{Z}_{n}} q_{i}-\sum_{i \in \mathbb{Z}_{n}} q_{i}$. Also, both $C_{Q}^{\odot}$ and $C_{Q}^{\S}$ are $Q$ ary non-tree-decodable prefix codes.
Given the number of unit hyper-cubes introduced, we can define the $Q$-ary SPA on which the $Q$-ary selvage code achieves the entropy.
Definition 7 (SPA). Let $n \geq 3$ and $k:=\left|C_{Q}^{\S} \backslash C_{Q}^{\odot}\right|$. The $Q$-ary selvage probability assembly (SPA) is the multiset of probabilities

$$
\left(\biguplus_{j \in \mathbb{Z}_{n}}\left\{q_{j} \prod_{i \in \mathbb{Z}_{n}} q_{i}^{-1}\right\}\right) \uplus\left(\biguplus_{j \in \mathbb{Z}_{k}}\left\{\prod_{i \in \mathbb{Z}_{n}} q_{i}^{-1}\right\}\right)
$$

Theorem 4. Let $n \geq 3$. The $Q$-ary selvage code $C_{Q}^{\S}$ achieves the entropy on the $Q$-ary SPA.

## B. Sufficient Condition

Before we start, we first examine the expected descriptive codeword lengths of the (entropy-achieving) $Q$-ary selvage
code and an optimal $Q$-ary tree-decodable code on the $Q$-ary SPA for some choices of $Q$. Table $\square$ shows the expected lengths for $Q$ being $(2,2,2),(5,3,2)$, or $(6,3,2)$, where the optimal tree-decodable codes are found using the multichannel Huffman procedure described in [20].

For the $(2,2,2)$-ary case, we observe that optimal treedecodable codes are entropy-achieving. This is as expected since, when all channel alphabets in $Q$ are of the same size $q$, a $Q$-ary code is equivalent to a single-channel $q$-ary code [16], and single-channel Huffman codes are optimal prefix codes.

For the $(5,3,2)$-ary case, we observe that all 170,625 possible trees produced by the multichannel Huffman procedure [20] have non-zero redundancy. On the other hand, for the $(6,3,2)$-ary case, some of the $1,467,357$ trees achieve the entropy. This suggests that besides the case where all channels having the same alphabet size, there are other $Q$ 's which are below tree line. In fact, we know that $Q=(6,3,2)$ is below tree line because each 6 -ary symbol can be split into a tuple consisting of a 2 -ary and a 3 -ary symbol, and all two-channel prefix codes are tree-decodable [20].

We now define the notion of $t$-separation which will serve as a sufficient condition for $Q$ being above tree line. For convenience, we regard the tuple $Q$ as a multiset of the channel alphabets.
Definition 8 ( $t$-Separation). Let $Q^{\prime} \subseteq Q$ and write $\bar{Q}^{\prime}:=Q \backslash Q^{\prime}$. We say that $Q^{\prime}$ is separated from $Q$ if any non-negative integral solution $\left(x_{q}\right)_{q \in Q} \in \mathbb{N}^{|Q|}$ to the equation

$$
\begin{equation*}
\prod_{q \in Q} q^{x_{q}}=\prod_{q \in \bar{Q}^{\prime}} q \tag{1}
\end{equation*}
$$

satisfies $x_{q}=0$ for all $q \in Q^{\prime}$. Let $\left\{Q_{j}\right\}_{j \in \mathbb{Z}_{t}}$ be a partition of $Q$, i.e., $Q=\biguplus_{j \in \mathbb{Z}_{t}} Q_{j}$. We say that $\left\{Q_{j}\right\}_{j \in \mathbb{Z}_{t}}$ is a $t$-separation of $Q$ if, for each $j \in \mathbb{Z}_{t}, Q_{j}$ is separated from $Q$. We say that $Q$ is $t$-separable if there exists a $t$-separation of $Q$.

A linear-algebraic interpretation of Definition 8 is to view Eq. (1) as a system of linear Diophantine equations $\mathbf{A}_{Q} \mathbf{x}=\mathbf{A}_{Q} \mathbf{e}_{\bar{Q}^{\prime}}$ defined as follows: $\mathbf{A}_{Q}$ is a matrix determined by $Q$ with rows indexed by prime factors of elements in $Q$ and columns indexed by elements in $Q$. The $(p, q)$-th entry of $\mathbf{A}_{Q}$, where $p$ is a prime and $q \in Q$, is the exponent of $p$ in the unique prime factorization of $q$. The vectors $\mathbf{x}$ and $\mathbf{e}_{\bar{Q}^{\prime}}$ have their entries indexed by elements in $Q$, and $\mathbf{e}_{\bar{Q}^{\prime}}$ is the binary vector where those entries indexed by $\bar{Q}^{\prime}$ are 1 and other entries are 0 . The separation condition is equivalent to that if $\mathbf{A}_{Q} \mathbf{x}=\mathbf{A}_{Q} \mathbf{e}_{\bar{Q}^{\prime}}$ for $\mathbf{x} \in \mathbb{N}^{|Q|}$, then the entries of $\mathbf{x}$ indexed by $\bar{Q}^{\prime}$ must be 0 .
Example 7. Consider $Q=(4,6,10,15)$ and hence

$$
\mathbf{A}_{Q}=\underset{\mathrm{p}=5}{\mathrm{p}=2}\left(\begin{array}{llll}
2 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{array}\right) \stackrel{\text { row op. }}{\sim} \mathbf{A}_{Q}^{\prime}=\left(\begin{array}{llll}
2 & 2 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{array}\right) .
$$

We can easily see that $\{4\}$ is not separated from $Q$ because

$$
\mathbf{A}_{Q}(1,0,0,2)^{\top}=\mathbf{A}_{Q}(0,1,1,1)^{\top}
$$

where $(0,1,1,1)^{\top}=\mathbf{e}_{Q \backslash\{4\}}$. Similarly, we observe that each of $\{6\},\{10\}$ and $\{15\}$ is also not separated from $Q$. We therefore conclude that $Q$ has no 3 - nor 4 -separation as each of them must contain a singleton chunk. On the other hand, by transforming $\mathbf{A}_{Q}$ to $\mathbf{A}_{Q}^{\prime}$ via elementary row operations, we see that any solution $\mathbf{x} \in \mathbb{N}^{4}$ satisfying

$$
(2,2,0,0) \mathbf{x}=(2,2,0,0)(0,0,1,1)^{\top}
$$

must have its first two entries, i.e., those corresponding to $\{4,6\}$, set to 0 . Similar holds for

$$
(0,0,1,1) \mathbf{x}=(0,0,1,1)(1,1,0,0)^{\top}
$$

which concerns the entries of x corresponding to $\{10,15\}$. We therefore conclude that $\{\{4,6\},\{10,15\}\}$ is a 2 -separation of $Q$.

The following lemma gives a natural class of separations of $Q$, which can be used to identify certain special cases with ease.

Lemma 1 (Natural Separation). A partition $\left\{Q_{j}\right\}_{j \in \mathbb{Z}_{t}}$ of $Q$ is a $t$-separation of $Q$ if, for each $j \in \mathbb{Z}_{t}$ and for every $q^{\prime} \in Q_{j}$, there exists a prime $p_{q^{\prime}}$ such that $p_{q^{\prime}} \mid q^{\prime}$ but $p_{q^{\prime}} \nmid q$ for all $q \in \biguplus_{k \in \mathbb{Z}_{t} \backslash\{j\}} Q_{k}$.

If for each $j \in \mathbb{Z}_{t}$, there exists a prime $p_{j}$ such that $p_{j} \mid q$ for all $q \in Q_{j}$ but $p_{j} \nmid q$ for all $q \in \biguplus_{k \in \mathbb{Z}_{t} \backslash\{j\}} Q_{k}$, we can also apply Lemma 1 to conclude the $t$-separation of $Q$. This lemma also covers the $n$-separation where $Q=\left(q_{i}\right)_{i \in \mathbb{Z}_{n}}$ is a tuple of distinct primes. For such $Q$, a natural $n$-separation consists of $Q_{i}=\left\{q_{i}\right\}$ for all $i \in \mathbb{Z}_{n}$. There are, however, separations of $Q$ which do not satisfy the condition of Lemma 11 e.g., the separation $\{\{6\},\{4\},\{3\}\}$ of $Q=(6,4,3)$. A reason is that the power of $q$ on the RHS of Eq. (1) is fixed to 1 , which means that we do not require the set of columns of $\mathbf{A}_{Q}$ to be a basis.
Theorem 5. Let $t \geq 3$. If $Q$ is $t$-separable, then $Q$ is above tree line.

We remark that $Q$ being $t$-separable does not imply that any superset $Q^{\prime} \supset Q$ is $t$-separable. Indeed, we can check that $(4,6,10)$ is 3 -separable but $(4,6,10,15)$ is not. This aligns with our geometric intuition that, when there are more dimensions, it is easier to pack blocks into a container without them interweaving with each other.

The proof of Theorem [5 can be done by showing that if $Q$ is $t$-separable, then $Q$ is above tree line on the $Q^{\times}$-ary SPA where $Q^{\times}:=\left(\prod_{q \in Q_{j}} q\right)_{j \in \mathbb{Z}_{t}}$ is the product channel alphabets. In the following, we state a partial converse - if $Q$ is above tree line on the $Q^{\times}$-ary SPA then $Q$ is $t$-separable. We note, however, that this is not the converse of Theorem 5 since, for $Q$ to be above tree line, it could be above tree line for some $P$ different from the $Q^{\times}$-ary SPA.

Theorem 6. Let $\left\{Q_{j}\right\}_{j \in \mathbb{Z}_{t}}$ be a t-partition of $Q$ for some $t \geq 3$. If $Q$ is above tree line on the $Q^{\times}$-ary SPA, where $Q^{\times}=\left(\prod_{q \in Q_{j}} q\right)_{j \in \mathbb{Z}_{t}}$, then $\left\{Q_{j}\right\}_{j \in \mathbb{Z}_{t}}$ is a $t$-separation.

We outline the high level idea of the proof of Theorem 6. Suppose $Q_{j^{*}}$ is not separated from $Q$, then
$\prod_{q \in Q} q^{x_{q}}=\prod_{\bar{Q}_{j}} q$ admits a solution $\left(x_{q}^{*}\right)_{q \in Q}$ where $x_{q}^{*} \neq 0$ for some $q \in Q_{j^{*}}$. We show how to construct a $Q$-ary tree-decodable code which is entropy-achieving on the $Q^{\times}$-ary SPA, contradicting the assumption. The basic idea is to first interpret the $Q^{\times}$-ary selvage code as a $Q$-ary code, then use the above equation to move codeword symbols in the selvage core across different channels without affecting the descriptive length of each codeword. The second step "disentangles" the selvage core, making it tree-decodable.

More concretely, suppose $Q^{+}$denotes the set of $q$ for which $x_{q}^{*}>0$. We consider two cases: 1) $q_{i^{*}}>2$ for some $q_{i^{*}} \in Q_{j^{*}} \cap Q^{+}$;2) $q=2$ for all $q \in Q_{j^{*}} \cap Q^{+}$.

For Case 1, we change the $j^{*}$-th codeword in the selvage code so that its $i^{*}$-th component is an all- 2 string of length $x_{q_{i^{*}}}^{*}$, its $i$-th component is an all-0 string of length $x_{q_{i}}^{*}$ for all $i \in Q^{+} \backslash\left\{q_{i^{*}}\right\}$, and all other components are empty strings. Consequently, the $i^{*}$-th components of all codewords are all non-empty, which allows us to build a decoding tree with a root assigned to channel $i^{*}$.

For Case 2, we can derive that all $q \in \bar{Q}_{j^{*}} \backslash Q^{+}$are powers of 2 . Therefore we can write $q_{i^{\dagger}}=q_{i^{*}}^{r}$ for some $q_{i^{\dagger}} \in \bar{Q}_{j^{*}} \backslash Q^{+}$and some $q_{i^{*}} \in Q_{j^{*}} \cap Q^{+}$. Using this relation, we can replace each $q_{i^{\dagger}}$-ary symbol in the selvage code with $r q_{i^{*}}$-ary symbol (while choosing symbols carefully so that prefix-freeness is preserved). Consequently, channel $i^{\dagger}$ becomes a dummy channel and the $i^{*}$-th component of all codewords are non-empty. The latter again allows us to build a decoding tree with a root assigned to channel $i^{*}$.

## V. CONCLUDING REMARKS

We investigated the interweave structure of non-treedecodable prefix codes and then filled a theoretical gap by giving a sufficient condition for the existence of optimal multichannel tree-decodable codes that are not optimal prefix codes. We leave proving that (a relaxation of) the sufficient condition is necessary as one of the future research directions on the theory of multichannel source coding.

## Appendix I <br> Proof of Theorem 1

Proof: If a prefix code is not tree decodable, then it means that there is a subspace $S$ guillotine-cut from the container, or $S$ is the container itself, such that $S$ is occupied by more than one block but no guillotine-cut is possible without cutting through a block. That is, there is no way to obtain any block in $S$ by guillotine-cuts.

Conversely, suppose some blocks cannot be obtained by guillotine-cuts. We consider a subset $T$ of these blocks having the same prefix where $|T|>1$ such that the smallest space $S$ containing the blocks in $T$ is not possible to be separated by guillotine-cuts without cutting through a block. Suppose the prefix code is tree decodable, then there is a non-leaf node corresponds to the space $S$. However, there is no way to further guillotine-cut $S$, so we cannot find a class for this node, which contradicts that the code is tree decodable.

## Appendix II Proof of Theorem 2

Proof: Suppose $M$ is tree decodable. The root node of the decoding tree must belong to one of the classes in $\mathbb{Z}_{n}$. However, for each possible class $k$, there exists at least one codeword that is not a descendant of the root node. This contradicts that $M$ is tree decodable.

## Appendix III <br> Proof of Theorem 3

Proof: Recall that, for every channel $i \in \mathbb{Z}_{n}$ and every codeword $j \in \mathbb{Z}_{n}$ in the core, we have

$$
\mathbf{c}_{j}^{\odot}(i)= \begin{cases}\epsilon & \text { if } i=j \\ 1 & \text { if } i=(j+1) \bmod n \\ 0 & \text { otherwise }\end{cases}
$$

For any $j, k \in \mathbb{Z}_{n}$ with $k \notin\{j,(j+1) \bmod n\}$, we have $\mathbf{c}_{j}^{\odot}((j+1) \bmod n)=1$ and $\mathbf{c}_{k}^{\odot}((j+1) \bmod n)=0$, which means $\mathbf{c}_{j}^{\odot} \not \subset \mathbf{c}_{k}^{\odot}$ in the $((j+1) \bmod n)$-th channel. On the other hand, for $k=(j+1) \bmod n$, we have $\mathbf{c}_{j}^{\odot}((k+1) \bmod n)=0$ and $\mathbf{c}_{k}^{\odot}((k+1) \bmod n)=1$, which means $\mathbf{c}_{j}^{\odot} \not \subset \mathbf{c}_{k}^{\odot}$ in the $((k+1) \bmod n)$-th channel. Summarizing, we have $\mathbf{c}_{j}^{\odot} \not \subset \mathbf{c}_{k}^{\odot}$ for all $j, k \in \mathbb{Z}_{n}$ with $j \neq k$, i.e., $C_{Q}^{\odot}$ is a prefix code. Let $M$ be the codeword matrix of $C_{Q}^{\odot}$ where the $j$-th row is $\mathbf{c}_{j}^{\odot}$. It is clear that $\mathcal{E}_{M}(i)=\{i\}$ for all $i \in \mathbb{Z}_{n}$, so $C_{Q}^{\odot}$ is not tree decodable by Theorem 2

Now, we consider $C_{Q}^{\S}$. Note that the words in $\prod_{i \in \mathbb{Z}_{n}} \mathbb{Z}_{q_{i}}$ are distinct but have the same length tuple $\langle 1\rangle_{i \in \mathbb{Z}_{n}}$, so every pair of words in this set are prefix-free. As $C_{Q}^{\odot}$ is a prefix code, $C_{Q}^{\S}$ is also a prefix code by its definition. Let $M^{\prime}$ be the codeword matrix of $C_{Q}^{\S}$ where the first $t$ rows are the matrix $M$. Again, we have $\mathcal{E}_{M^{\prime}}(i)=\{i\}$, so by Theorem 2, $C_{Q}^{\S}$ is also not tree decodable.

We now calculate the size of $\left|C_{Q}^{\S} \backslash C_{Q}^{\odot}\right|$. Recall that $C_{Q}^{\S}$ is a prefix code. In this code, $\ell_{i}^{\max }=1$ for all $i \in \mathbb{Z}_{n}$, so the container in its RPG is a hyper-rectangle of size $\left\langle q_{i}\right\rangle_{i \in \mathbb{Z}_{n}}$. The codeword $\mathbf{c}_{j}^{\odot} \in C_{Q}^{\odot}$ corresponds to a block of size $\left\langle q_{i}^{1-\delta_{i, j}}\right\rangle_{i \in \mathbb{Z}_{n}}$, where $\delta_{i, j}$ is the Kronecker delta, so $\operatorname{Vol}\left(\mathbf{c}_{j}^{\odot}\right)=$ $q_{j}$. The smallest possible block has size $\langle 1\rangle_{i \in \mathbb{Z}_{n}}$, which corresponds to a codeword having the same length as any of those in $C_{Q}^{\S} \backslash C_{Q}^{\odot}$. That is, $\operatorname{Vol}(\mathbf{c})=1$ for any $\mathbf{c} \in C_{Q}^{\S} \backslash C_{Q}^{\odot}$. As a prefix code, the blocks for $C_{Q}^{\S}$ must packable in the container. The construction of $C_{Q}^{\S}$ includes all codewords of length $\langle 1\rangle_{i \in \mathbb{Z}_{n}}$ which are prefix-free to those in $C_{Q}^{\odot}$, which means that besides the blocks for $C_{Q}^{\odot}$, we put as many blocks of size $\langle 1\rangle_{i \in \mathbb{Z}_{n}}$ as possible to fully fill the container. In other words, the sum of volumes of the blocks equals the volume of the container, i.e.,

$$
\sum_{i \in \mathbb{Z}_{n}} \operatorname{Vol}\left(\mathbf{c}_{i}^{\odot}\right)+\sum_{\mathbf{c} \in C_{Q}^{\S} \backslash C_{Q}^{\odot}} \operatorname{Vol}(\mathbf{c})=\prod_{i \in \mathbb{Z}_{n}} q_{i} .
$$

The proof is done by reordering the terms.

## Appendix IV <br> Proof of Theorem4

Proof: We define the source code as follows. For each $j \in \mathbb{Z}_{n}$, we map the probability $q_{j} \prod_{i \in \mathbb{Z}_{n}} q_{i}^{-1}$ to $\mathbf{c}_{j}^{\odot} \in C_{Q}^{\odot}$. We have

$$
\begin{equation*}
\left|\mathbf{c}_{j}^{\odot}\right|=\sum_{i \in \mathbb{Z}_{n}} \ln q_{i}-\ln q_{j}=-\ln \left(q_{j} \prod_{i \in \mathbb{Z}_{n}} q_{i}^{-1}\right) \tag{2}
\end{equation*}
$$

Next, we map each of the remaining probabilities $\prod_{i \in \mathbb{Z}_{n}} q_{i}^{-1}$ to an arbitrary codeword $\mathbf{c} \in C_{Q}^{\S} \backslash C_{Q}^{\odot}$ bijectively, which is possible according to Theorem 3. Recall that $\operatorname{len}(\mathbf{c})=\langle 1\rangle_{i \in \mathbb{Z}_{n}}$, so we have

$$
\begin{equation*}
|\mathbf{c}|=\sum_{i \in \mathbb{Z}_{n}} \ln q_{i}=-\ln \prod_{i \in \mathbb{Z}_{n}} q_{i}^{-1} \tag{3}
\end{equation*}
$$

Eqs. (2) and (3) imply that the condition for the equality of the entropy bound holds.

## Appendix V

PROOF OF LEMMA 1
Proof: Fix any $j \in \mathbb{Z}_{t}$. Let $\left(x_{q}\right)_{q \in Q} \in \mathbb{N}^{|Q|}$ solves Eq. (1) where $Q^{\prime}=Q_{j}$. If $x_{q^{\prime}} \neq 0$ for some $q^{\prime} \in Q_{j}$, then $p_{q^{\prime}} \mid \prod_{q \in Q} q^{x_{q}}$ but $p_{q^{\prime}} \nmid \prod_{q \in \bar{Q}^{\prime}} q$, which contradicts that $\left(x_{q}\right)_{q \in Q}$ solves Eq. (1).

## Appendix VI <br> Proof of Theorem 5

Proof: Let $\left\{Q_{j}\right\}_{j \in \mathbb{Z}_{t}}$ be a $t$-separation of $Q$, $Q^{\times}=\left(q_{i}^{\times}\right)_{i \in \mathbb{Z}_{t}}$ be the product channel alphabets where $q_{i}^{\times}=\prod_{q \in Q_{i}} q$ for $i \in \mathbb{Z}_{t}$, and $P$ be the $Q^{\times}$-ary SPA. By Theorem 4 the $Q^{\times}$-ary selvage code $C_{Q^{\times}}^{\S}$ on $P$ achieves the entropy bound. Interpreting $C_{Q^{\times}}^{\S}$ as a $Q$-ary code, we obtain an entropy-achieving $Q$-ary prefix code on $P$.
Suppose there is an entropy-achieving $Q$-ary treedecodable code $C$ on $P$. Recall that $q_{j}^{\times} / \prod_{i \in \mathbb{Z}_{t}} q_{i}^{\times} \in P$ for all $j \in \mathbb{Z}_{t}$. Let $\left\langle x_{i, j}\right\rangle_{i \in \mathbb{Z}_{n}}$ be the length of the codeword for the probability $q_{j}^{\times} / \prod_{i \in \mathbb{Z}_{t}} q_{i}^{\times}$. Since $C$ achieves the entropy bound, we have $\prod_{i \in \mathbb{Z}_{n}} q_{i}^{x_{i, j}}=\left(\prod_{i \in \mathbb{Z}_{t}} q_{i}^{\times}\right) / q_{j}^{\times}=\prod_{q \in Q \backslash Q_{j}} q$. Since $\left\{Q_{j}\right\}_{j \in \mathbb{Z}_{t}}$ is a $t$-separation of $Q$, for each $j \in \mathbb{Z}_{t}$, we must have $x_{i, j}=0$ for all $q_{i} \in Q_{j}$. In other words, for every channel $i \in \mathbb{Z}_{n}$, there exists a codeword in $C$ such that its $i$-th component is $\epsilon$. By Theorem 2, $C$ is not tree-decodable, which is a contradiction.

## Appendix VII <br> PROOF OF THEOREM 6

Proof: In the proof below, we adopt the following notion. For any strings $a, b$, denote by $a \| b$ the concatenation of $a$ and $b$. To represent a string formed by duplicating the same symbol, we bold the symbol and write the number of repetitions as its exponent. We will always use $i$ and $j$ as running variables over $\mathbb{Z}_{n}$ and $\mathbb{Z}_{t}$ respectively.

We prove Theorem 6 by contrapositive. Suppose for some $j^{*} \in \mathbb{Z}_{t}, Q_{j}$ is not separated from $Q$, i.e., there is some $\left(x_{q}^{*}\right)_{q \in Q} \in \mathbb{N}^{|Q|}$ solving Eq. (1) such that $x_{q}^{*} \neq 0$ for
some $q \in Q_{j^{*}}$. We construct an entropy-achieving $Q$-ary tree-decodable code on the $Q^{\times}$-ary SPA.

Since $\left(x_{q}^{*}\right)_{q \in Q}$ solves Eq. (1), we have

$$
\prod_{q \in Q} q^{x_{q}^{*}}=\prod_{q \in \bar{Q}_{j^{*}}} q
$$

Define $Q^{+}:=\left\{q \in Q: x_{q}^{*}>0\right\} \subseteq Q$. We have

$$
\begin{equation*}
\prod_{q \in Q^{+}} q^{x_{q}^{*}}=\prod_{q \in \bar{Q}_{j^{*}}} q \tag{4}
\end{equation*}
$$

Note also that $Q_{j^{*}} \cap Q^{+} \neq \emptyset$ and $\bar{Q}_{j^{*}} \backslash Q^{+} \neq \emptyset$.
For $j \in \mathbb{Z}_{t}$, let $q_{j}^{\times}=\prod_{q \in Q_{j}} q$ be the product channel alphabet sizes. By Theorem 4 the $Q^{\times}$-ary selvage code $C^{\S}$ is entropy-achieving on the $Q^{\times}$-ary SPA. The $Q^{\times}$-ary selvage core $C^{\odot}$ can be interpreted as a $Q$-ary code using the following transform: For each $Q^{\times}$-ary codeword $\mathbf{c}^{\times}$of $C^{\S}$, define the $Q$-ary codeword $\mathbf{c}$ by $\mathbf{c}(i)=\mathbf{c}^{\times}(j)$ where $q_{i} \in Q_{j}$ for all $i \in \mathbb{Z}_{n}$ and $j \in \mathbb{Z}_{t}$. This is possible as $\mathbf{c}^{\times}(j) \in\{\epsilon, 0,1\}$. We denote this $Q$-ary form of the selvage core by $\overline{C^{\odot}}$. It is easy to check that $\bar{C}$ is a prefix code whose codewords have one-toone correspondence to those of $C^{\odot}$, and each codeword in $\overline{C \odot}$ has the same descriptive length as that of its counterpart in $C^{\odot}$.

Case 1: $q_{i^{*}}>2$ for some $q_{i^{*}} \in Q_{j^{*}} \cap Q^{+}$. We modify the core $\overline{C^{\odot}}=\left\{\overline{\mathbf{c}_{j}^{\top}}\right\}_{j \in \mathbb{Z}_{t}}$ to construct a new codebook $\widehat{C \odot}=\left\{\widehat{\mathbf{c}_{j}^{\odot}}\right\}_{j \in \mathbb{Z}_{t}}$ by $\widehat{\mathbf{c}_{j}^{\odot}}=\overline{\mathbf{c}_{j}^{\odot}}$ for $j \neq j^{*}$, and

$$
\widehat{\mathbf{c}_{j^{*}}}(i)= \begin{cases}\mathbf{2}^{x_{q_{i}}^{*}} & \text { if } i=i^{*} \\ \mathbf{0}^{x_{q_{i}}^{*}} & \text { if } q_{i} \in Q^{+} \backslash\left\{q_{i^{*}}\right\}, \\ \epsilon & \text { otherwise }\end{cases}
$$

By Eq. (4), we know that $\left|\widehat{\mathbf{c}_{j^{*}}}\right|=\left|\overline{\mathbf{c}_{j^{*}}}\right|$.
As $\overline{C^{\odot}}$ is a prefix code, $\overline{C^{\odot}} \backslash\left\{\overline{\mathbf{c}_{j^{*}}^{\odot}}\right\}$ also is. From the $i^{*}$-th component, we know that $\widehat{\mathbf{c}_{j^{*}}} \not \subset \widehat{\mathbf{c}_{j}^{\odot}}$ for all $j \in \mathbb{Z}_{t} \backslash\left\{j^{*}\right\}$. Therefore, $\widehat{C \odot}$ is a prefix code.

Next, we extend the core $\overline{C^{\odot}}$ into a prefix code $\widehat{C}$ by building its decoding tree:

1) The root node belongs to class $i^{*}$.
2) The 0 -child of the root node is the root of a subtree constructed by first building a tree whose leaves are labelled by the set $\{0\} \times \prod_{q \in Q \backslash\left\{q_{i^{*}}\right\}} \mathbb{Z}_{q}$, and then removing the subtree under the node labelled by $\widehat{\mathbf{c}_{j}^{\odot}}$ for all $j \in$ $\mathbb{Z}_{t} \backslash\left\{j^{*},\left(j^{*}-1\right) \bmod t\right\}$. Let $S_{0}$ denote the set of leaves of this subtree except $\widehat{\mathbf{c}_{j}^{\odot}}$ for all $j \in \mathbb{Z}_{t} \backslash\left\{j^{*},\left(j^{*}-1\right) \bmod t\right\}$.
3) The 1 -child of the root node is the root of a subtree constructed by first building a tree whose leaves are labelled by the set $\{1\} \times \prod_{q \in Q \backslash\left\{q_{i^{*}}\right\}} \mathbb{Z}_{q}$, and then removing the subtree under the node labelled by $\mathbf{c}_{\left(j^{*}-1\right) \bmod t}^{\odot}$. Let $S_{1}$ denote the set of leaves of this subtree except $\mathbf{c}_{\left(j^{*}-1\right) \bmod t}^{\odot}$.
4) The 2 -child of the root node is the root of a subtree constructed by first building a tree whose leaves are labelled by the set $\{2\} \times \mathbb{Z}_{q_{i^{*}}}^{x_{q_{i}}^{*}-1} \times \prod_{q \in Q^{+} \backslash\left\{q_{i^{*}}\right\}} \mathbb{Z}_{q}^{x_{q}^{*}} \times \prod_{q \in Q_{j}} \mathbb{Z}_{q}$, and
then removing the subtree under the node labelled by $\widehat{\mathbf{c}_{j^{*}}}$. Let $S_{2}$ denote the set of leaves of this subtree except $\mathbf{c}_{j^{*}}^{\odot}$.
5) For $i=3, \ldots, q_{i^{*}}-1$, the $i$-child of the root node is the root of a subtree whose leaves are labelled by the set $\{i\} \times \prod_{q \in Q \backslash\left\{q_{i^{*}}\right\}} \mathbb{Z}_{q}$. Let $S_{3}$ denote the set of leaves of these subtrees.
We now count $|\widehat{C} \backslash \widehat{C \odot}|=\sum_{i \in \mathbb{Z}_{4}}\left|S_{i}\right|$. We observe that

$$
\begin{aligned}
\left|S_{0}\right| & =\prod_{i \in \mathbb{Z}_{n} \backslash\left\{i^{*}\right\}} q_{i}-\sum_{j \in \mathbb{Z}_{t} \backslash\left\{j^{*},\left(j^{*}-1\right) \bmod t\right\}} q_{j}^{\times} \\
\left|S_{1}\right| & =\prod_{i \in \mathbb{Z}_{n} \backslash\left\{i^{*}\right\}} q_{i}-q_{\left(j^{*}-1\right) \bmod t}^{\times}, \\
\left|S_{2}\right| & =q_{i_{i^{*}}^{*}}^{x_{i^{*}}-1} \prod_{q \in Q^{+} \backslash\left\{q_{\left.i^{*}\right\}}\right.} q^{x_{q}^{*}} \prod_{q \in Q_{j}} q-q_{j^{*}}^{\times} \\
& =q_{i^{*}}^{-1} \prod_{q \in \bar{Q}_{j^{*}}} q \prod_{q \in Q_{j}} q-q_{j^{*}}^{\times} \\
& =\prod_{i \in \mathbb{Z}_{n} \backslash\left\{i^{*}\right\}} q_{i}-q_{j^{*}}^{\times}, \\
\left|S_{3}\right| & =\left(q_{i^{*}}-3\right) \prod_{i \in \mathbb{Z}_{n} \backslash\left\{i^{*}\right\}} q_{i} .
\end{aligned}
$$

Summing them up, we have $|\widehat{C} \backslash \widehat{C \odot}|=\prod_{i \in \mathbb{Z}_{n}} q_{i}-\sum_{j \in \mathbb{Z}_{t}} q_{j}^{\times}$, which is the same size as $\left|C_{Q}^{\S} \backslash C_{Q}^{\odot}\right|=\prod_{j \in \mathbb{Z}_{t}} q_{j}^{\times}-\sum_{j \in \mathbb{Z}_{t}} q_{j}^{\times}$ according to Theorem 3 Therefore, we have a bijective mapping from $C_{Q}^{\S}$ to $\widehat{C}$ where the descriptive length of every codeword is preserved. That is, $\widehat{C}$ is an entropy-achieving $Q$-ary tree-decodable code on the $Q^{\times}$-ary SPA.

Case 2: $q=2$ for all $q \in Q_{j^{*}} \cap Q^{+}$. We can see from Eq. (4) that every $q \in \bar{Q}_{j^{*}} \backslash Q^{+}$is a power of 2. Pick any $i^{*}, i^{\dagger} \in \mathbb{Z}_{n}$ and $j^{\dagger} \in \mathbb{Z}_{t}$ such that $q_{i^{*}} \in Q_{j^{*}} \cap Q^{+}$and $q_{i^{\dagger}} \in \bar{Q}_{j^{*}} \backslash Q^{+}$. We have

$$
\begin{equation*}
q_{i^{\dagger}}=q_{i^{*}}^{r} \tag{5}
\end{equation*}
$$

for some $r \in \mathbb{Z}^{+}$. We modify the core $\overline{C \odot}=\left\{\overline{\mathbf{c}_{j}^{\odot}}\right\}_{j \in \mathbb{Z}_{t}}$ to construct a new codebook $\widehat{C \odot}=\left\{\widehat{\mathbf{c}_{j}^{\odot}}\right\}_{j \in \mathbb{Z}_{t}}$ by setting $\widehat{\mathbf{c}_{j^{\dagger}}}=\overline{\mathbf{c}_{j^{\dagger}}^{\odot}}$,

$$
\mathbf{c}_{\left(j^{\dagger}-2\right) \bmod t}^{\odot}(i)= \begin{cases}\overline{\mathbf{c}_{j}^{\odot}}(i) \| \mathbf{1}^{r} & \text { if } i=i^{*}, \\ \epsilon & \text { if } i=i^{\dagger}, \\ \overline{\mathbf{c}_{j}^{\odot}}(i) & \text { if } i \in \mathbb{Z}_{n} \backslash\left\{i^{*}, i^{\dagger}\right\}\end{cases}
$$

and

$$
\widehat{\mathbf{c}_{j}^{\odot}}(i)= \begin{cases}\overline{\mathbf{c}_{j}^{\odot}}(i) \| \mathbf{0}^{r} & \text { if } i=i^{*} \\ \epsilon & \text { if } i=i^{\dagger}, \\ \overline{\mathbf{c}_{j}^{\odot}}(i) & \text { if } i \in \mathbb{Z}_{n} \backslash\left\{i^{*}, i^{\dagger}\right\}\end{cases}
$$

for $j \in \mathbb{Z}_{t} \backslash\left\{j^{*}, j^{\dagger}\right\}$. Note that $\left|\widehat{\mathbf{c}_{j}^{\odot}}\right|=\left|\overline{\mathbf{c}_{j}^{\odot}}\right|$ for all $j \in \mathbb{Z}_{t}$ due to Eq. (5). Moreover, we have $\widehat{\mathbf{c}_{j}^{\odot}}\left(i^{\dagger}\right)=\epsilon$ for all $j \in \mathbb{Z}_{t}$ by construction, i.e., channel $i^{\dagger}$ becomes dummy.

Except for $\left(j, j^{\prime}\right)=\left(\left(j^{\dagger}-2\right) \bmod t,\left(j^{\dagger}-1\right) \bmod t\right)$, for all other distinct $j, j^{\prime} \in \mathbb{Z}_{t}$, the original codewords $\overline{\mathbf{c}_{j}^{\odot}}$ and $\overline{\mathbf{c}_{j^{\prime}}}$ are prefix-free in some channel $i \neq i^{\dagger}$, so do the modified codewords $\widehat{\mathbf{c}_{j}^{\odot}}$ and $\widehat{\mathbf{c}_{j^{\prime}}^{\odot}}$. For $\left(j, j^{\prime}\right)=\left(\left(j^{\dagger}-2\right) \bmod t,\left(j^{\dagger}-\right.\right.$

1） $\bmod t$ ），we note that $\widehat{\mathbf{c}_{j}^{\odot}}\left(i^{*}\right)=\overline{\mathbf{c}_{j}^{\odot}}\left(i^{*}\right) \| \mathbf{1}^{r}$ and $\widehat{\mathbf{c}_{j^{\prime}}}\left(i^{*}\right)=$ $\overline{\mathbf{c}_{j^{\prime}}}\left(i^{*}\right) \| \mathbf{0}^{r}$ ．There are four possibilities：$\left(\overline{\mathbf{c}_{j}^{\odot}}\left(i^{*}\right), \overline{\mathbf{c}_{j^{\prime}}^{\odot}}\left(i^{*}\right)\right) \in$ $\{(\epsilon, 0),(1, \epsilon),(0,1),(0,0)\}$ ．Clearly，in each case，$\widehat{\mathbf{c}_{j}^{\odot}}\left(i^{*}\right)$ and $\widehat{\mathbf{c}_{j^{\prime}}}\left(i^{*}\right)$ are prefix－free．We conclude that $\widehat{C \odot}$ is a prefix code．

Next，we extend the core $\widehat{C \odot}$ into a prefix code $\widehat{C}$ by building its decoding tree．The tree is constructed by first building a tree whose leaves are labelled by the set $\mathbb{Z}_{q_{i^{*}}}^{1+r} \times \mathbb{Z}_{q_{i} \dagger}^{0} \times \prod_{i \in \mathbb{Z}_{n} \backslash\left\{i^{*}, i^{\dagger}\right\}} \mathbb{Z}_{q_{i}}$ ，and then removing the subtrees under the nodes labelled by $\widehat{\mathbf{c}_{j}^{\ominus}}$ ，for $j \in \mathbb{Z}_{t}$ ．

Observe that

$$
|\widehat{C} \backslash \widehat{C \odot}|=q_{i^{*}}^{1+r} \prod_{i \in \mathbb{Z}_{n} \backslash\left\{i^{*}, i^{\dagger}\right\}} q_{i}-\sum_{i \in \mathbb{Z}_{t}} q_{i}^{\times}=\prod_{i \in \mathbb{Z}_{n}} q_{i}-\sum_{i \in \mathbb{Z}_{t}} q_{i}^{\times}
$$

By the same argument as in Case $1, \widehat{C}$ is an entropy－achieving $Q$－ary tree－decodable code on the $Q^{\times}$－ary SPA．

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    ${ }^{1}$ It is possible for a non-symbol-by-symbol prefix code to achieve a lower redundancy, e.g., the arithmetic codes [7], [8]. It is also possible for a symbol-by-symbol non-instantaneous code to do so, e.g., the AIFV codes [9], [10].
    ${ }^{2}$ Multichannel prefix codes can be interpreted as multidirectional context sets [17]. Although some structural results in context sets are isomorphic to those in prefix codes, e.g., context trees versus decoding trees, the optimization problems on the relevant structures are quite different.

[^1]:    ${ }^{3}$ The RPG is a cylinder with a hole in the middle as shown in Fig. 5a When we perform a guillotine-cut on the 0 -th channel, we need to cut through both sides of the cylinder, i.e., cut both the red planes in the figure. Similarly, we need to cut both blue planes to guillotine-cut the 1 -st channel. For a simpler illustration, we can cut one of the red plane and bend the structure into a rectangle as shown in Fig. 5c This way, each guillotine-cut on the 0 -th or the 1 -st channel in the 4 -D space becomes two guillotine-cuts in this rectangle.
    ${ }^{4}$ The RPG is a bicycle tube: a torus with thickness and with a hollow tube inside. Figs. 6a and 6billustrate the bicycle tube and one of its cross sections respectively. By cutting Fig. 6 at one side of the cross section shown in Fig. 6b we bend the tube into a cylinder structure as in the 4-D RPG shown in Fig. 6c By further cutting one side (the red plane) of the cylinder, we can bend the structure into a rectangle shown in Fig. 6d

