# Centralised multi link measurement compression with side information

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#### **Abstract**

We prove new one shot achievability results for measurement compression of quantum instruments with side information at the receiver. Unlike previous one shot results for this problem, our one shot bounds are nearly optimal and do not need catalytic randomness. In fact, we state a more general problem called centralised multi link measurement compression with quantum side information and provide one shot achievability results for it. As a simple corollary, we obtain one shot measurement compression results for quantum instruments with side information that we mentioned earlier. All our one shot results lead to the standard results for this problem in the asymptotic iid setting. We prove our achievability bounds by first proving a novel sequential classical quantum multipartite covering lemma, which should be of independent interest.

## 1 Introduction

In order to obtain important statistical information about quantum systems, an experimenter must perform *measure-ments* on the system. This information can be used in subsequent physical operations on the system. However, the measurement process is inherently noisy. This noise could arise due to *extrinsic* noise in the measurement procedure itself (which is uncorrelated to the state) and/or *intrinsic* noise arising from the uncertainty introduced by the quantum state being measured. An important information theoretic problem is to filter out the extrinsic noise so that we can compress the number of bits required to describe the measurement outcome.

We model the measurement procedure using the positive operator valued measure (POVM) formalism. Consider the following setting: The experimenter Alice possesses a quantum state  $\rho$  and a POVM  $\Lambda$ . The action of the POVM on the quantum state is to produce a classical register (in Alice's possession) which contains an index corresponding to the measurement outcome, as well as a post measurement quantum state (to which Alice does not have access). The task is for Alice to send the contents of her classical register to Bob, such that the correlations between this register and the post measurement state are conserved, using as few bits as possible. This problem is formally known as measurement compression.

In his seminal paper, Winter [21] observed that the above task can be achieved if one can construct an approximate decomposition of the POVM of the following form :

$$\Lambda \stackrel{\varepsilon}{\approx} \sum_{c} P_{\text{COIN}}(c) \Lambda(c)$$

where each  $\Lambda(c)$  is a POVM with fewer possible classical outcomes as compared to  $\Lambda$ . The variable c corresponds to the outcome of a coin toss which is distributed according to  $P_{\text{COIN}}$ . As long as Alice and Bob both have access to the coin c, Alice only needs to send enough bits to be able to describe the set of possible outcomes of  $\Lambda(c)$ . Note that this distribution is independent of the state  $\rho$ , and thus is a source of extrinsic noise. The measurement procedure can then be summarised as follows:

1. Toss a coin  $\sim P_{\rm COIN}$ . Let the outcome be c.

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#### 2. Measure $\rho$ with $\Lambda(c)$

Let the number of bits needed to describe the state of the coin be C and the number of bits of communication from Alice to Bob be R. Winter characterised the region of all (R,C) pairs for which the above decomposition exists. A cleaner exposition of Winter's results can be found in a later paper by Wilde et al. [20], who also extend Winter's original contribution by considering the case when Bob has access to some part of the quantum state being measured. This protocol is known as Measurement Compression with Quantum Side Information or MC-QSI in short. In this scenario Alice and Bob share the quantum state  $\rho^{AB}$ , where Alice possesses the system A and bob possesses the system A. Alice then measures her share of the state and sends her classical register to Bob. Wilde et al. showed that in this setting, Bob can use his share of the quantum state to reduce the number of bits Alice needs to send to him even further.

All of the above works consider the measurement compression problem in the asymptotic iid regime i.e. in the limit of a large number of repeated identical measurements on several copies of the original quantum state. A more general variant of the problem would be if only one copy of the state was available for measurement. This is the so called *one-shot* regime. In a recent paper, Anshu, Jain and Warsi [3] provide an achievable region for MC-QSI in this regime. In contrast to all previous works, that paper shows a reduction of the MC-QSI problem to the well studied problem of *quantum message compression with side information* [2], which allows them to leverage known tools for the latter problem. However, even though Anshu et al.'s results imply the best known asymptotic iid bounds for measurement compression, their one-shot bounds are unsatisfactory, due to the following issues:

- 1. The protocol uses shared randomness in a catalytic manner i.e. it needs some a large amount of shared randomness to begin with but regenerates part of this randomness at the end of the protocol. In the asymptotic iid limit with n copies, this protocol is applied iteratively n/m times to blocks of states containing m copies each, where m < n. By choosing m appropriately, the authors could show that that the amount of extra initial randomness required is o(n). Thus, the amount of extra initial shared randomness per iteration goes to zero in the asymptotic iid limit. Nevertheless in the one shot scenario the extra initial shared randomness can potentially be very large, almost as large as the alphabet size;
- 2. The second and perhaps more serious issue is that the bounds achieved by Anshu et al. are optimal only in the case when the probability distribution induced by the original POVM is uniform on the alphabet X. In all other cases, there is a trade-off between the amount of shared randomness required to run the protocol and the extent to which Alice can compress her message to Bob.

In this paper we define an even more general problem called *centralised multi-link measurement compression with side information* and prove a one shot achievability result for it. The problem that we discussed above will be called the *point to point* message compression problem. As corollaries of our result for centralised multi link measurement compression, we obtain

- 1. New one shot achievability result for point to point measurement compression with side information in both feedback as well as non-feedback cases;
- 2. Our protocol does not require any extra initial shared randomness to begin with;
- 3. It recovers the optimal bounds of Wilde et al. in the asymptotic iid limit;
- 4. It achieves the natural one-shot bounds that one expects for this problem, for any probability distribution on the set of outcomes, without making any compromises between the amount of shared randomness required and the rate of compression.

To motivate this problem we consider a natural generalisation of the measurement compression, to the case of quantum instruments. Instruments are the most general model for quantum measurements, which include both a classical output and a post measurement quantum state [6, 7]. A quantum instrument is a CPTP map  $\mathcal{N}_{INSTR}$  of the following form:

$$\mathcal{N}_{\text{INSTR}}(\rho) \coloneqq \sum_{x} \left| x \right\rangle \left\langle x \right|^{X} \otimes \mathcal{N}_{x}(\rho)$$

where each  $\mathcal{N}_x$  is a completely positive trace non-increasing map with Kraus decompositions as follows

$$\mathcal{N}_x(
ho) = \sum_y \mathcal{N}_{x,y} \ 
ho \, \mathcal{N}_{x,y}^\dagger$$

and

$$\sum_{y} \mathcal{N}_{x,y}^{\dagger} \ \rho \ \mathcal{N}_{x,y} \leq \mathbb{I}$$

One can then ask the question whether we can prove a compression theorem for quantum instruments as well. This theorem was claimed without proof in [9] and proven rigorously in [20]. Wilde et al. solve this problem by suitably reducing it to an instance of the original measurement compression problem. Their approach at simulating the action of  $\mathcal{N}_{\text{INSTR}}$  is to identify its operation as a tracing out of the Y-register of the state

$$\sum_{x,y} |x\rangle \langle x|^X \otimes |y\rangle \langle y|^Y \otimes \mathcal{N}_{x,y} \rho \mathcal{N}_{x,y}^{\dagger}.$$

Naturally, they design a POVM simulation protocol wherein Alice simulates the POVM  $\{\mathcal{N}_{x,y}^{\dagger}\mathcal{N}_{x,y}\}_{x,y}$ . Recognizing that Bob only needs to recover the X-register they propose Alice discards her simulation of the Y-register and thereby achieve rates that one would naturally expect.

We now define our new problem called the centralised multilink measurement compression problem with side information. Taking a cue from the discussion above we consider a POVM  $\Lambda_{XY} = \{\Lambda_{x,y}\}$  which outputs two classical symbols x and y according to some joint distribution which depends on the state  $\rho$ . There exist two separate noiseless channels, called X and Y channels, from Alice to Bob, and two independent public coin registers, called X and Y public coins, between them. We use the word link to refer to a noiseless channel. During the protocol at most one of the links may be turned OFF by an adversary without Alice or Bob's knowledge. We ask whether we can design a single simulation protocol that enables Bob to recover either X or Y or both X, Y depending on which link is ON. See Fig. 1.

To be precise, we seek *one* simulation protocol that enables Bob to recover (i) the X register when only the X-link of rate  $R_1$  is active and they share  $C_1$  bits of randomness, (ii) the Y-register when only the Y-link of rate  $R_2$  is active and they share  $C_2$  bits of randomness, and (iii) recover both X and Y registers when both links are active and they share a total of  $C_1 + C_2$  random bits. We require that Alice's and Bob's strategies should be agnostic to which links are operational i.e. their encoding and decoding strategies should continue to work even if one link fails.

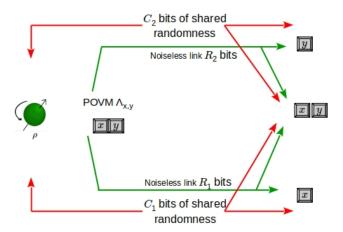


Figure 1: Fig.1

A careful consideration of this problem reveals the following challenge. The requirement that each link enable recovery of the corresponding classical register precludes the use of designing conditional codebooks for the POVM simulation protocol. Otherwise, if the codebook for Y were to be conditionally dependant on the codebook for X,

Bob would be unable to recover Y when the link of rate  $R_1$  is inactive, and vice-versa. Therefore, this strategy, which works well for the purposes of Wilde et al., fails in our case.

On the other hand, if the codebooks have to be designed independently, then it is unclear how Bob would be able to recover both X, Y when both links are active. We overcome this issue by proving a novel quantum covering lemma, referred to herein as measure transformed sequential covering lemma. This lemma is one of the main technical contributions of our work. The issue in proving a covering lemma of this sort is that most arguments invariably run into the issue of simultaneous smoothing, which is an outstanding open problem in quantum information theory [10]. Recently however, Chakraborty, Nema and Sen [5] showed how one can get around the smoothing problem in certain situations. We adapt their construction and generalise it for our purposes. In fact, the proof of the covering lemma presented in this work is much easier and far more general than that in [5], and it may be useful in other applications as well.

Before stating our one shot achievability results for centralised measurement compression, we need to recall the concept of rate splitting in the quantum setting [4]. The joint probability distribution on the measurement outcomes (x,y) of the given POVM  $\Lambda$  is defined by  $p(x,y) := \mathrm{Tr}[((\Lambda_{x,y} \otimes \mathbb{I}^{BR})(\rho))^{ABR}]$ , which in turn defines p(x) and p(y|x) in the natural fashion. Now, the splitting function creates two new random variables U and V supported on alphabet  $\mathcal{X}$ , defines  $x := \max\{u,v\}$ , and splits the probability distribution p(x) into a joint distribution  $p^{\theta}(u,v)$  parametrised by a real number  $\theta \in [0,1]$ . This splitting function satisfies the following properties for any  $\theta \in [0,1]$ :

- 1.  $U^{\theta}$ ,  $V^{\theta}$  are independent random variables;
- 2. The probability distribution of the random variable  $\max\{U^{\theta}, V^{\theta}\}$  is the same as that of the random variable X;
- 3. When  $\theta = 0$ ,  $U^0 = X$  and  $V^0$  is constant. When  $\theta = 1$ ,  $V^1 = X$  and  $U^1$  is constant.

We prove the following theorem for one shot achievability results for centralised measurement compression. The theorem is stated in terms of smooth one shot entropic quantities like  $H_{\max}^{\varepsilon}(\cdot)$ ,  $I_{\max}^{\varepsilon}(\cdot:\cdot)$ ,  $I_{H}^{\varepsilon}(\cdot:\cdot)$ , the definitions of which can be found in the full version of this paper. They are the natural one shot analogues of Shannon entropic quantities like entropy and mutual information.

**Theorem 1.1.** (Centralised multi-link measurement compression) Consider the  $\varepsilon$ -error centralised multi-link measurement compression problem with side information at Bob in the feedback case. One achievable rate region is obtained as the union over a parameter  $\theta \in [0, 1]$  of the regions  $S_{\theta}$  defined by:

$$R_{X} = R_{U} + R_{V}$$

$$R_{U} > I_{\max}^{\varepsilon}(U:RB) - I_{H}^{\varepsilon}(U:B)$$

$$+ O(\log \varepsilon^{-1})$$

$$R_{Y} > I_{\max}^{\varepsilon}(Y:RBU) - I_{H}^{\varepsilon}(Y:B)$$

$$+ O(\log \varepsilon^{-1})$$

$$S_{\theta} : R_{V} > I_{\max}^{\varepsilon}(V:RBUY) - I_{H}^{\varepsilon}(V:B)$$

$$+ O(\log \varepsilon^{-1})$$

$$C_{X} = C_{U} + C_{V}$$

$$C_{U} + R_{U} > H_{\max}^{\varepsilon}(U) - I_{H}^{\varepsilon}(U:B)$$

$$C_{Y} + R_{Y} > H_{\max}^{\varepsilon}(Y) - I_{H}^{\varepsilon}(Y:B)$$

$$C_{V} + R_{V} > H_{\max}^{\varepsilon}(V) - I_{H}^{\varepsilon}(V:B),$$

where the entropic quantities are calculated for the control state

$$\sum_{(u,v,y)\in\mathcal{X}\times\mathcal{X}\times\mathcal{Y}} p^{\theta}(u)p^{\theta}(v)p(y|u,v) |u,v,y\rangle \langle u,v,y|^{UVY}$$

$$\otimes \frac{((\Lambda_{\max\{u,v\},y}\otimes \mathbb{I}^{BR})(\rho))^{ABR}}{\operatorname{Tr}[((\Lambda_{\max\{u,v\},y}\otimes \mathbb{I}^{BR})(\rho))^{ABR}]}.$$

The above state is obtained by splitting random variable X into independent random variables U, V in the state  $\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}}|x,y\rangle\langle x,y|^{XY}\otimes((\Lambda_{x,y}\otimes\mathbb{I}^{BR})(\rho))^{ABR}\text{ according to the parameter }\theta.$  Another achievability region is obtained by rate splitting Y instead of X. The total achievable region is the union of the two regions. The encoding

obtained by rate splitting Y instead of X. The total achievable region is the union of the two regions. The encoding and decoding strategies are agnostic to which links are actually functioning.

Our general theorem above lends itself to two important corollaries which are stated below.

**Corollary 1.2.** In the asymptotic iid setting of the centralised measurement compression problem, the following rate region per channel use is achievable:

$$R_X > I(X:BR) - I(X:B)$$
 $R_Y > I(Y:BR) - I(Y:B)$ 
 $R_X + R_Y > I(XY:BR) + I(X:Y)$ 
 $-I(X:B) - I(Y:B)$ 
 $R_X + C_X > H(X) - I(X:B)$ 
 $R_Y + C_Y > H(Y) - I(Y:B)$ 

where all entropic quantities are computed with respect to  $\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} |x,y\rangle \langle x,y|^{XY} \otimes ((\Lambda_{x,y}\otimes\mathbb{I}^{BR})(\rho))^{ABR}$ .

**Corollary 1.3.** For the point to point measurement compression problem with side information for quantum instruments, an achievable region can be obtained by assuming that the Y links is not functional. In the one shot setting we get

$$R_X > I_{\max}^{\varepsilon}(X:RB) - I_H^{\varepsilon}(X:B) + O(\log \varepsilon^{-1})$$

and

$$R_X + C_X > H_{\max}^{\varepsilon}(X) - I_H^{\varepsilon}(X:B).$$

In the asymptotic iid setting this reduces to

$$R_X > I(X:RB) - I(X:B) + O(\log \varepsilon^{-1})$$

and

$$R_X + C_X > H(X) - I(X:B).$$

Above, all entropic quantities are computed with respect to  $\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} |x,y\rangle \langle x,y|^{XY} \otimes ((\Lambda_{x,y}\otimes\mathbb{I}^{BR})(\rho))^{ABR}$ .

#### 1.1 Measurement Compression with Quantum Side Information

In this section we provide a brief exposition of the ideas which allow us to prove Corollary 1.3. The techniques developed in this section allow us to design a measurement compression protocol in which the receiver Bob can use quantum side information to reduce the umber of bits that Alice needs to send him. The setup is as follows:

Alice and Bob share a joint quantum state  $a^{AB}$  where Alice has access to system A and Bob has access to system

Alice and Bob share a joint quantum state  $\rho^{AB}$ , where Alice has access to system A and Bob has access to system B. Alice is given a POVM  $\Lambda^A = \{\Lambda_x^A\}$ , where each POVM element  $\Lambda_x$  acts on the system A and corresponds to a classical outcome  $x \in \mathcal{X}$ . We define the *true* post measurement state

$$\sigma^{RBX\hat{X}} := \operatorname{Tr}_{A} \left[ \left( \mathbb{I}^{RB} \otimes \Lambda^{A} \right) \varphi^{RAB} \right]$$
$$= \sum_{x \in \mathcal{X}} |x, x\rangle \langle x, x|^{X\hat{X}} \otimes$$
$$\operatorname{Tr}_{A} \left[ \left( \mathbb{I}^{RB} \otimes \Lambda_{x}^{A} \right) \varphi^{RAB} \right]$$

We allow Alice and Bob access to a noiseless forward classical channel and also to public coins. The system  $\hat{X}$  is used by Bob to store his guess for the outcome of the measurement. In the case of the true post measurement state, one can imagine that Alice sent Bob the full description of the outcome through the classical channel, which Bob then stored in  $\hat{X}$ . Now suppose that Alice and Bob use a protocol  $\mathcal{P}$  which allows them to reduce the number of bits which need to be communicated to Bob. Let the post measurement state created by  $\mathcal{P}$  be  $\tilde{\sigma}^{RBX\hat{X}}$ . We require that for any such protocol,

$$\|\sigma - \tilde{\sigma}\|_1 \le \varepsilon$$

The first idea is that Alice designs a class of POVMs  $\{\theta_k\}$ , where each POVM  $\theta_k$  is indexed by the setting of the public coin  $k \in [2^C]$ . Each  $\theta_k$  contains fewer outcomes than the original  $\Lambda$  which allows Alice to save on the number of random bits that she needs to describe the outcome. In particular, each  $\theta_k$  acts on the system A and produces as output an index in the set  $[2^{R+B}]$  i.e.

$$\theta_k \coloneqq \{\theta_k(\ell) \mid \ell \in [2^{R+B}]\}$$

. Winter's original construction of these POVMs relied on the Operator Chernoff [1] bound to show the existence of these POVMs. Unfortunately, the Operator Chernoff bound is not easily adapted to the one shot setting. Instead we use a new one shot covering lemma (Section 3.1) along with a one shot operator inequality (Section 3.2) to show the existence of the compressed POVM. These two techniques are new and are powerful enough to allow us to emulate the Operator Chernoff bound even in the one shot setting. We believe that these two new tools maybe useful elsewhere. We note that recently Padakandla [13] provided another alternate way of designing the compressed POVM in the one shot setting.

The second idea is that after Alice applies a POVM  $\theta_k$  and observes some outcome  $\ell$ , she hashes the outcome into a bit string by using a hash function  $f:[2^{R+B}] \to [2^R]$ , which is chosen uniformly at random from a 2-universal hash family. This hash value is what Alice sends to Bob. Suppose that the hash value sent by Alice is m.

Upon receiving Alice's bit string, Bob chooses a POVM  $\gamma(m,k)$ , based on Alice's message and the setting of the public coin. Bob will use this POVM to measure his system B to find the index which is consistent with the outcome  $\ell$ . The idea is that as long as the set  $f^{-1}(m)$  is not too large, Bob should be able to find the correct measurement outcome with a high probability of success.

The above protocol is essentially a classical data compression with quantum side information protocol (CDC-QSI), which has been studied in the asymptotic iid setting by Devetak and Winter [8] and Renes [14] and in the one shot setting by Renes and Renner [15]. However, it is not immediately obvious how one can adapt Renes and Rener's protocol to our setting. Instead, we design a new one shot protocol for CDC-QSI, which is more suitable for our setting. The one shot rates our new protocol achieves are different from those of Renes and Renner, but recover to the same asymptotic iid bounds achieved by them. The upshot of our protocol is that it is *composable* with the measurement compression protocol, which is in general not true. The formal statement is as follows:

#### Lemma 1.4. Classical Message Compression with Quantum Side Information

Given a classical quantum state of the form

$$\sigma^{XB} := \sum_{x} P_X(x) |x\rangle \langle x|^X \otimes \sigma^B$$

where Alice possesses the system X and Bob possess the system B, there exists a CDC-QSI protocol with probability of error at most  $\varepsilon$  and the rate of communication  $R_A$  from Alice to Bob is at least

$$R_A \ge H_{\max}^{\varepsilon}(X) - I_H^{\varepsilon}(X:B) + O\left(\log \frac{1}{\varepsilon}\right)$$

where all entropic quantities are computed in terms of the state  $\sigma^{XB}$ .

## 1.2 Centralised Measurement Compression

In this section we briefly explain our strategy to design a protocol for centralised multi link measurement compression. Due to the composability of our CDC-QSI lemma with any measurement compression scheme, we combine these two

protocols to achieve the bounds claimed in Theorem 1.1. The technique that we use to design the compressed POVMs  $\theta_k$  first requires the creation of a random codebook  $\mathcal{C} := \{X(k,\ell)\}$  sampled iid from the distribution  $P_X(x) := \operatorname{Tr}[\rho \Lambda_x]$ . We further require that, on expectation over choices of codebook, the following *sample average state* is close to

$$\frac{1}{L} \sum_{\ell \in [L]} \frac{1}{P_X(x(k,l))} \sqrt{\rho} \Lambda_{x(k,\ell)} \sqrt{\rho} \stackrel{\varepsilon}{\approx} \rho$$

In the centralised case however, since we are forced to construct the two codebooks for the X and Y links independently from the distributions  $P_X$  and  $P_Y$ , the *sample average matrices* that we must consider are of the form

$$\frac{1}{L_1 \cdot L_2} \sum_{\ell_1, \ell_2} \frac{1}{P_X(x(k_1, \ell_1)) P_Y(y(k_2, \ell_2))} \sqrt{\rho} \lambda_{\substack{x(k_1, \ell_1) \\ y(k_2, \ell_2)}} \sqrt{\rho}$$

Even though the above matrix may not even be a quantum state, we still require that in expectation, it should be close to  $\rho$ . We show this using our *measure transformed sequential quantum covering lemma*:

#### Lemma 1.5. Measure Transformed Sequential Covering Lemma

Suppose we are given a joint distribution  $P_{XY}$  on classical alphabets  $\mathcal{X} \otimes \mathcal{Y}$ , with marginals  $P_X$  and  $P_Y$ . Suppose we are also given the following quantum state:

$$\rho^{XYE} := \sum P_{XY}(x, y) |x\rangle \langle x|^X \otimes |y\rangle \langle y|^Y \otimes \rho_{x, y}$$

Let  $\{x(1), x(2), \dots, x(K)\}$  and  $\{y(1), y(2), \dots, y(L)\}$  be iid samples from the distribution  $P_X \otimes P_Y$ . Then

$$\mathbb{E}_{\substack{x(1),x(2),...,x(K)\\y(1),y(2),...,y(L)}} \left\| \frac{1}{K \cdot L} \sum_{k,l} \frac{P_{XY}(x(k),y(l))}{P_{X}(x(k)) \cdot P_{Y}(y(l))} \rho_{x(k),y(l)} - \sigma^{E} \right\|_{1} \leq \varepsilon$$

whenever

$$\log K > I_{\max}^{\varepsilon}(X:E)$$
 and  $\log L > I_{\max}^{\varepsilon}(Y:XE)$ 

where

$$\sigma^E := \sum_{x,y} P_{XY}(x,y) \rho_{x,y}^E$$

## 1.3 Organisation

The paper is organised as follows: In Section 2 we define the one shot quantities that we will need for the proof, and state some useful facts. In Section 3 we present the main technical tools that we will use in the proof of our measurement compression theorem, including the measure transformed sequential covering lemma Section 3.1, a one shot operator inequality that arises from this covering lemma Section 3.2 and the one shot classical message compression protocol with quantum side information Section 3.3. Then in Section 4 we prove a version of our centralised multi link measurement compression theorem in the case when there is no side information available to the receiver Bob. In Section 4.2, we show how one can compose a measurement compression theorem with the protocol for classical message compression with quantum side information. Finally, we prove our full centralised multi link measurement compression theorem with side information in Section 4.3. In Section 5 we show how our one-shot bounds can be extended to the desired rate region in the asymptotic iid setting.

# **2** One Shot Entropic Quantities

In this section we present the important one shot entropic quantities that we will use throughout the paper, and some useful facts regarding these quantities.

## 2.1 Smooth Max Entropy

#### **Definition 2.1. Smooth Max Entropy**

Given a distribution  $P_X$  on the alphabet  $\mathcal{X}$  and some error parameter  $\varepsilon > 0$ , the  $\varepsilon$ -smooth max entropy or just smooth max entropy of  $P_X$ , denoted by  $H_{\max}^{\varepsilon}(X)$  is defined as

$$H^{\varepsilon}_{\max}(X) \coloneqq \log \sum_{x} \lambda^{*}(x)$$

where vector  $\lambda^*$  (indexed by x) is the optimiser for the following linear program :

$$\min \sum_{x} \lambda(x)$$

$$\sum_{x} P_X(x) \cdot \lambda(x) \ge 1 - \varepsilon$$

$$\lambda(x) \le 1$$

$$\lambda(x) \ge 0$$
(1)

**Fact 2.2.** Given the distribution  $P_X$ , consider an ordering  $\{x_0, x_1, \dots x_{|\mathcal{X}|}\}$  on the alphabet  $\mathcal{X}$  such that

$$x_i < x_j \implies P_X(x_i) \le P_X(x_j)$$

Define the set  $S \subset \mathcal{X}$  such that

$$\mathcal{S} \coloneqq \left\{ x_i \mid \sum_{x_i} P_X(x_i) \le \varepsilon \land x_i \le x_{i+1} \right\}$$

Then, the linear program in Eq. (1) is optimised by a vector  $\lambda^*$  such that

$$\lambda^*(x) = 0 \qquad x \in S$$

$$\in (0,1) \qquad x = x_{|S|+1}$$

$$= 1 \qquad \text{otherwise}$$

#### **Definition 2.3. Optimising Sub-distribution**

Given the setup of Fact 2.2, we define the sub-distribution  $P'_X$  as

$$P'_X(x) = P_X(x) x \in S^c$$
= 0 otherwise

**Fact 2.4.** Given the setup of Fact 2.2 and the sub-distribution P',

$$||P' - P||_1 \le \varepsilon$$

and for all x in the support of P'(x),

$$P_X(x) \ge \frac{1}{2^{H_{\max}^{\varepsilon}(X)}} \cdot (1 - \varepsilon)$$

#### 2.2 Hypothesis Testing Relative Entropy

**Definition 2.5. Smooth Hypothesis Testing Relative Entropy** *Given quantum state*  $\rho^A$  *and*  $\sigma^A$ , *the smooth hypothesis testing relative entropy between*  $\rho$  *and*  $\sigma$  *is defines as* 

$$D_H^{\varepsilon}(\rho^A \mid\mid \sigma^A) \coloneqq -\log \mathsf{OPT}$$

where OPT is the optimum attained for the following semi-definite program

$$\begin{aligned} \max & \operatorname{Tr}[\Pi \sigma] \\ & \operatorname{Tr}[\Pi \rho] \geq 1 - \varepsilon \\ & \Pi \geq 0 \\ & \Pi \leq \mathbb{I} \end{aligned}$$

**Definition 2.6. Hypothesis Testing Mutual Information** Given the quantum state  $\rho^{AB}$ , the hypothesis testing mutual information  $I_H^{\varepsilon}(A:B)$  is defined as

$$I_H^{\varepsilon}(A:B)_{\rho} := D_H^{\varepsilon}(\rho^{AB} \mid\mid \rho^A \otimes \rho^B)$$

Fact 2.7. Given a classical quantum state

$$\sum_{x} P_X(x) |x\rangle \langle x|^X \otimes \rho_x^B$$

the operator  $\Pi_{OPT}$  which is the optimiser for  $I^{\varepsilon}_H(X:B)_{\rho}$  is of the form

$$\Pi_{\text{OPT}} = \sum_{x} |x\rangle \langle x|^X \otimes \Pi_x^B$$

where for each x,

$$0 \le \Pi_x \le \mathbb{I}$$

## 2.3 Smooth Max Relative Entropy

**Definition 2.8.** Max Relative Entropy Given a quantum states  $\rho^A$  and  $\sigma^A$ , the max relative entropy  $D_{\max}(\rho^A \mid\mid \sigma^A)$  is defined as

$$D_{\max}(\rho^A \mid\mid \sigma^A) \coloneqq \inf\left\{\lambda \mid \rho^A \le 2^{\lambda} \sigma^A\right\}$$

**Definition 2.9. Purified Distance** Given two quantum states  $\rho^A$  and  $\sigma^A$ , the purified distance  $P(\rho, \sigma)$  between the two states is defined as

$$P(\rho, \sigma) := \sqrt{1 - F^2(\rho, \sigma)}$$

where  $F(\cdot | \cdot)$  is the fidelity.

**Definition 2.10.**  $\varepsilon$ **-Ball** Given a quantum state  $\rho^A$ , the  $\varepsilon$ -ball around this state is defined as

$$B^{\varepsilon}(\rho^{A}) \coloneqq \left\{ \rho'^{A} \mid P(\rho, \rho') \le \varepsilon \right\}$$

**Definition 2.11. Smooth Max Relative Entropy** Given two quantum states  $\rho^A$  and  $\sigma^A$ , the smooth max relative entropy  $D_{\max}^{\varepsilon}(\rho^A \mid\mid \sigma^A)$  is defined as

$$D_{\max}^{\varepsilon}(\rho^A \mid\mid \sigma^A) \coloneqq \inf_{\rho'^A \in B^{\varepsilon}(\rho^A)} D_{\max}(\rho'^A \mid\mid \sigma^A)$$

**Definition 2.12. Smooth Max Information** Given a state  $\rho^{AB}$  the smooth max information  $I_{\max}^{\varepsilon}(A:B)_{\rho}$  is given by

$$I_{\max}^{\varepsilon}(A:B)_{\rho} := D_{\max}^{\varepsilon}(\rho^{AB} \mid\mid \rho^{A} \otimes \rho^{B})$$

We will need a slightly perturbed version of this quantity to state our theorems. We call this the tilde smooth max information.

**Definition 2.13. Tilde Smooth Max Information** Given a state  $\rho^{AB}$  the tilde smooth max information between A and B is defined as

$$\tilde{I}_{\max}^{\varepsilon}(A:B)_{\rho} := \inf_{\rho'^{AB}: P(\rho^{AB}, \rho'^{AB}) < \varepsilon} D_{\max}(\rho'^{AB} \mid\mid \rho^{A} \otimes \rho'^{B})$$

## 2.4 Quantum Asymptotic Equipartition Property

We will find the following facts useful while extending our results to the asymptotic iid setting. We collectively refer to them as the Quantum Asymptotic Equipartition Property (QAEP). The proofs of these facts can be found in [17, 18, 12, 16].

**Fact 2.14.** Given a classical probability distribution  $P_X$  and some integer  $n \in \mathbb{N}$ , the following holds

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} H_{\max}^{\varepsilon}(X) = H(X)$$

**Fact 2.15. Asymptotic Equipartition Property** Given the quantum states  $\rho^A$  and  $\sigma^A$ , and some integer  $n \in \mathbb{N}$ , the following hold:

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n} \mid\mid \sigma^{\otimes n}) = D(\rho \mid\mid \sigma)$$
$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} D_{\max}^{\varepsilon}(\rho^{\otimes n} \mid\mid \sigma^{\otimes n}) = D(\rho \mid\mid \sigma)$$

where  $D(\rho || \sigma)$  is the quantum von Neumann relative entropy.

#### 2.5 Other Useful Facts

**Fact 2.16.** [19] For any state  $\rho^{AB}$ ,  $\varepsilon > 0$  and  $0 < \gamma < \varepsilon$ , the following is true

$$\tilde{I}_{\max}^{\varepsilon}(A:B)_{\rho} \le I_{\max}^{\varepsilon-\gamma}(A:B)_{\rho} + \log \frac{3}{\gamma^2}$$

## Fact 2.17. [2, 19]Smoothed Convex Split Lemma

Let  $\rho^{AB}$  be any quantum state and let  $\tau^{A_1...A_KB}$  be the so called convex split state, defined as

$$\tau^{A_1\dots A_K B} \coloneqq \frac{1}{K} \sum_{k \in [K]} \rho^{A_k B} \bigotimes \rho^{A_{[K] \setminus k}}$$

Then for all

$$\log K \ge \tilde{I}_{\max}^{\sqrt{\varepsilon}-\eta} (A:B)_{\rho} + 2\log \frac{1}{\eta}$$

the following holds true

$$\left\| \tau^{A_1 \dots A_K B} - \rho^B \bigotimes \rho^{A_{[K]}} \right\|_{1} \le 2 \cdot \left( 2\sqrt{\varepsilon} - \eta \right)$$

where  $\varepsilon \in (0,1)$  and  $\eta \in (0,\sqrt{\varepsilon})$ .

## 3 Technical Tools

In this section we will prove the two main technical tools we will need to prove our centralised multi link measurement compression theorem:

- 1. the measure transformed sequential covering lemma,
- 2. a new operator inequality implied by the covering lemma, and
- 3. the protocol for classical message compression with quantum side information.

## 3.1 A Measure Transformed Sequential Covering Lemma

In this section we will prove Lemma 1.5. To do this we will first need to prove the following sequential convex split lemma:

**Theorem 3.1. Successive Cancellation Convex Split** Suppose we are given the tripartite state  $\rho^{ABR}$ . Define the convex split state as follows

$$\tau^{A_1...A_KB_1...B_LR} \coloneqq \frac{1}{K \cdot L} \sum_{k,\ell \in [K],[L]} \rho^{A_kB_\ell R} \bigotimes \rho^{A_{[K] \setminus k}} \bigotimes \rho^{B_{[L] \setminus \ell}}$$

Then whenever

$$\log K \ge I_{\max}^{\frac{\sqrt{\varepsilon}}{4}} (A:BR)_{\rho} + O(\log \frac{1}{\varepsilon})$$
$$\log L \ge I_{\max}^{\frac{\sqrt{\varepsilon}}{4}} (B:R)_{\rho} + O(\log \frac{1}{\varepsilon})$$

the following holds

$$\left\| \tau^{A_1 \dots A_K B_1 \dots B_L R} - \rho^R \bigotimes_{i \in [K]} \rho^{A_i} \bigotimes_{j \in [L]} \rho^{B_j} \right\|_1 \le 6\sqrt{\varepsilon}$$

Proof. Consider the expression

$$\left\| \tau^{A_1 \dots A_K B_1 \dots B_L R} - \rho^R \bigotimes_{i \in [K]} \rho^{A_i} \bigotimes_{j \in [L]} \rho^{B_j} \right\|_{1}$$

$$\leq \left\| \tau - \frac{1}{L} \sum_{\ell \in [L]} \rho^{B_{\ell} R} \bigotimes \rho^{B_{[L] \setminus \ell}} \bigotimes_{i \in [K]} \rho^{A_i} \right\|_{1} + \left\| \frac{1}{L} \sum_{\ell \in [L]} \rho^{B_{\ell} R} \bigotimes \rho^{B_{[L] \setminus \ell}} \bigotimes_{i \in [K]} \rho^{A_i} - \rho^R \bigotimes_{i \in [K]} \rho^{A_i} \bigotimes_{j \in [L]} \omega^{B_j} \right\|_{1}$$

$$(2)$$

Let us first consider the first term in the RHS. Note that we can write this term as follows

$$\left\| \tau - \frac{1}{L} \sum_{\ell \in [L]} \rho^{B_{\ell}R} \bigotimes_{j \neq \ell} \rho^{B_{[L] \setminus \ell}} \bigotimes_{i \in [K]} \rho^{A_i} \right\|_{1}$$

$$\leq \frac{1}{L} \sum_{\ell \in [L]} \left\| \frac{1}{K} \sum_{k \in [K]} \rho^{A_k B_{\ell}R} \bigotimes \rho^{A_{[K] \setminus k}} \bigotimes \rho^{B_{[L] \setminus \ell}} - \rho^{B_{\ell}R} \bigotimes \rho^{B_{[L] \setminus \ell}} \bigotimes_{i \in [K]} \rho^{A_i} \right\|_{1}$$

Taking the systems  $B_1 \dots B_{\ell-1} B_{\ell+1} \dots B_L$  systems from each of the corresponding norm expressions outside the norm, we see that RHS

$$= \frac{1}{L} \sum_{\ell \in [L]} \left\| \frac{1}{K} \sum_{k \in [K]} \rho^{A_k B_\ell R} \bigotimes \rho^{A_{[K] \setminus k}} - \rho^{B_\ell R} \bigotimes_{i \in [K]} \rho^{A_i} \right\|_1$$

By the convex split lemma, for

$$\log K \ge \tilde{I}_{\max}^{\sqrt{\varepsilon} - \eta} (A : BR)_{\rho} + 2 \log \frac{1}{\eta}$$

we see that, for all  $\ell \in [L]$ , the terms

$$\left\| \frac{1}{K} \sum_{k \in [K]} \rho^{A_k B_\ell R} \bigotimes \rho^{A_{[K] \setminus k}} - \rho^{B_\ell R} \bigotimes_{i \in [K]} \rho^{A_i} \right\|_1 \le 2 \cdot (2\sqrt{\varepsilon} - \eta)$$

where  $0 < \eta < \sqrt{\varepsilon}$ . We will now consider the second term in the RHS of Eq. (1). Notice that,

$$\begin{split} & \left\| \frac{1}{L} \sum_{\ell \in [L]} \rho^{B_{\ell}R} \bigotimes_{j \neq \ell} \rho^{B_{j}} \bigotimes_{i \in [K]} \rho^{A_{k}} - \rho^{R} \bigotimes_{i \in [K]} \rho^{A_{i}} \bigotimes_{j \in [L]} \rho^{B_{j}} \right\|_{1} \\ & = \left\| \frac{1}{L} \sum_{\ell \in [L]} \rho^{B_{\ell}R} \bigotimes_{j \neq \ell} \rho^{B_{j}} - \rho^{R} \bigotimes_{j \in [L]} \rho^{B_{j}} \right\|_{1} \end{split}$$

Then by another application of the convex split lemma, for all

$$\log L \ge \tilde{I}_{\max}^{\sqrt{\varepsilon} - \eta}(B:R)_{\rho} + 2\log \frac{1}{\eta}$$

the following holds

$$\left\| \frac{1}{L} \sum_{\ell \in [L]} \rho^{B_{\ell} R} \bigotimes_{j \neq \ell} \rho^{B_j} - \rho^R \bigotimes_{j \in [L]} \rho^{B_j} \right\|_{1} \leq 2 \cdot (2\sqrt{\varepsilon} - \eta)$$

Collating all these arguments we see that

$$\left\| \tau^{A_1 \dots A_K B_1 \dots B_L R} - \rho^R \bigotimes_{i \in [K]} \rho^{A_i} \bigotimes_{j \in [L]} \rho^{B_j} \right\|_1 \le 4 \cdot \left( 2\sqrt{\varepsilon} - \eta \right)$$

Finally, invoking Fact 2.16 we see that the above bound holds true if we set

$$\begin{split} \log K &\geq I_{\max}^{(\sqrt{\varepsilon} - (\eta + \gamma))} (A:BR)_{\rho} + 2\log\frac{1}{\eta} + \log\frac{3}{\gamma^2} \\ \log L &\geq I_{\max}^{(\sqrt{\varepsilon} - (\eta + \gamma))} (B:R)_{\rho} + 2\log\frac{1}{\eta} + \log\frac{3}{\gamma^2} \end{split}$$

whenever  $\gamma \in (0, \sqrt{\varepsilon} - \eta)$ . To simplify the bounds we set

$$\eta = \frac{\sqrt{\varepsilon}}{2}$$
$$\gamma = \frac{\sqrt{\varepsilon}}{4}$$

which gives us the bounds

$$\log K \ge I_{\max}^{\frac{\sqrt{\varepsilon}}{4}} (A : BR)_{\rho} + O(\log \frac{1}{\varepsilon})$$
$$\log L \ge I_{\max}^{\frac{\sqrt{\varepsilon}}{4}} (B : R)_{\rho} + O(\log \frac{1}{\varepsilon})$$

with a corresponding error of  $6\sqrt{\varepsilon}$  in the 1-norm. This concludes the proof.

We will now use Theorem 3.1 to prove Lemma 1.5 as a corollary. We restate the lemma below for convenience.

#### Corollary 3.2. Measure Transformed Sequential Covering Lemma

Suppose we are given a joint distribution  $P_{XY}$  on classical alphabets  $\mathcal{X} \otimes \mathcal{Y}$ , with marginals  $P_X$  and  $P_Y$ . Suppose we are also given the following quantum state:

$$\rho^{XYE} := \sum P_{XY}(x, y) |x\rangle \langle x|^X \otimes |y\rangle \langle y|^Y \otimes \rho_{x,y}$$

Let  $\{x(1), x(2), \dots, x(K)\}$  and  $\{y(1), y(2), \dots, y(L)\}$  be iid samples from the distribution  $P_X \otimes P_Y$ . Then

$$\mathbb{E}_{\substack{x(1), x(2), \dots, x(K) \\ y(1), y(2), \dots, y(L)}} \left\| \frac{1}{K \cdot L} \sum_{k,l} \frac{P_{XY}(x(k), y(l))}{P_X(x(k) \cdot P_Y(y(l)))} \rho_{x(k), y(l)} - \sigma^E \right\|_{1} \le \varepsilon$$

whenever

$$\log K > I_{\max}^{\varepsilon}(X:E)$$
$$\log L > I_{\max}^{\varepsilon}(Y:XE)$$

where

$$\sigma^E \coloneqq \sum_{x,y} P_{XY}(x,y) \rho_{x,y}^E$$

*Proof.* We apply Theorem 3.1 after instantiating terms appropriately. For our case, the classical systems X and Y play the roles of the systems A and B in Theorem 3.1. Then, recall that

$$\tau^{X_1...X_KY_1...Y_LR} \coloneqq \frac{1}{K \cdot L} \sum_{k,\ell \in [K],[L]} \rho^{X_kY_\ell R} \bigotimes \rho^{X_{[K] \setminus k}} \bigotimes \rho^{Y_{[L] \setminus \ell}}$$

where

$$\rho^{X} := \sum_{x} P_{X}(x) |x\rangle \langle x|^{X}$$
$$\rho^{Y} := \sum_{y} P_{Y}(y) |y\rangle \langle y|^{Y}$$

Note that the states  $\rho^X$  and  $\rho^Y$  are marginals of the control state  $\rho^{XYE}$ . Let us consider one term in the expansion of  $\tau$ , for some fixed  $k \in [K]$  and  $\ell \in [L]$ 

$$\rho^{X_k Y_\ell R} \bigotimes \rho^{X_{[K] \setminus k}} \bigotimes \rho^{Y_{[L] \setminus \ell}} \tag{1}$$

To describe this fixed term we use the following convention: we write  $x_i$  to denote a sample from the system  $X_i$ , and similarly for  $y_i$ . Let us fix a certain setting of these samples

$$\prod_{i \in [K]} |x_i\rangle \langle x_i|^{X_i} \otimes \prod_{j \in [L]} |y_j\rangle \langle y_j|^{Y_j}$$

Since the  $X_k$  and the  $Y_\ell$  systems are entangled with the R system for the state in Eq. (1), the term that appears in the expansion of the state in Eq. (1) is

$$P_{XY}(x_k, y_\ell) |x_k\rangle \langle x_k|^{X_k} \otimes |y_\ell\rangle \langle y_\ell|^{Y_\ell} \otimes \prod_{\substack{i \in [K], \ j \in [L] \\ i \neq k, \ j \neq \ell}} P_X(x_i) P_Y(y_j) \otimes |x_i\rangle \langle x_i|^{X_i} |y_j\rangle \langle y_j|^{Y_j} \otimes \rho_{x_k, y_\ell}^R$$
(2)

The full expansion of the state in Eq. (1) is a sum over all strings  $x_1, x_2, \ldots, x_K$  and  $y_1, y_2, \ldots, y_L$ , of terms like the one in Eq. (2). Now, when we sum over all fixed states as in Eq. (1) to get  $\tau$ , we can group together those matrices of the kind in Eq. (2) which have the same string  $x_1, x_2, \ldots, x_K, y_1, y_2, \ldots, y_L$ . Note that this choice of fixed string fixes the pure states on the systems  $X_1X_2 \ldots X_KY_1Y_2 \ldots Y_L$ . However, the corresponding state on the system R will be of the following form

$$\frac{1}{K \cdot L} \sum_{k,\ell} P_{XY}(x_k, y_\ell) \prod_{\substack{i \neq k \\ j \neq \ell}} P_X(x_i) \cdot P_Y(y_j) \rho_{x_k, y_\ell}^R$$

the above argument, along with Theorem 3.1, essentially implies the following:

$$\left\| \tau^{X_{1}X_{2}...X_{K}Y_{1}Y_{2}...Y_{L}R} - \sigma^{R} \bigotimes_{i \in [K]} \rho^{X_{i}} \bigotimes_{j \in [L]} \rho^{Y_{j}} \right\|_{1}$$

$$= \left\| \sum_{\substack{x_{1}, x_{2}, ...x_{K} \\ y_{1}, y_{2}, ...y_{L}}} x_{1}, x_{2}, ...x_{k} \otimes y_{1}, y_{2}, ...y_{L} \otimes \frac{1}{K \cdot L} \sum_{k, \ell} P_{XY}(x_{k}, y_{\ell}) \prod_{\substack{i \neq k \\ j \neq \ell}} P_{X}(x_{i}) \cdot P_{Y}(y_{j}) \rho^{R}_{x_{k}, y_{\ell}}$$

$$- \sum_{\substack{x_{1}, x_{2}, ...x_{K} \\ y_{1}, y_{2}, ...y_{L}}} x_{1}, x_{2}, ...x_{k} \otimes y_{1}, y_{2}, ...y_{L} \otimes \prod_{\substack{i \in [K], j \in [L]}} P_{X}(x_{i}) \cdot P_{Y}(y_{j}) \sigma^{R} \right\|_{1}$$

where we have omitted the ket bra notation on the classical string for brevity. Then, using the block diagonal nature of these matrices, we see that the above expression is

$$= \sum_{\substack{x_1, x_2, \dots x_K \\ y_1, y_2, \dots y_L}} \prod_{i \in [K], j \in [L]} P_X(x_i) \cdot P_Y(y_j) \left\| \frac{1}{K \cdot L} \sum_{k, \ell} \frac{P_{XY}(x_k, y_\ell)}{P_X(x_k) \cdot P_Y(y_\ell)} \rho_{x_k, y_\ell}^R - \sigma^R \right\|_{1}$$

We now change notation from  $x_i$  and  $y_j$  to x(i) and y(j) to emphasise the fact that they are the *i*-th and *j*-th samples drawn from the distributions  $P_X$  and  $P_Y$  respectively. Then, invoking Theorem 3.1, we see that the above expression is

$$= \underset{\substack{x(1), x(2), \dots x(K) \\ y(1), y(2), \dots y(L)}}{\mathbb{E}} \left\| \frac{1}{K \cdot L} \sum_{k, \ell} \frac{P_{XY}(x(k), y(\ell))}{P_{X}(x(k)) \cdot P_{Y}(y(\ell))} \rho_{x(k), y(\ell)}^{R} - \sigma^{R} \right\|_{1}$$

$$\leq \varepsilon$$

for appropriately chosen values of K and L.

This concludes the proof of the claim.

#### 3.2 An Operator Inequality

In this section we prove an operator inequality that we will find useful in proving our measurement compression theorem.

**Lemma 3.3. Operator Inequality for the Covering Lemma** Suppose we are given some states  $\rho_1^A, \rho_2^A, \dots, \rho_L^A$  and a distribution P(i) on the set of indices [L] such that

$$\left\| \sum_{i} P(i)\rho_{i}^{A} - \rho^{A} \right\|_{1} \leq \varepsilon$$

for a given average state  $\rho$ . Then there exist a good subset GOOD  $\subset [L]$  such that

$$\Pr_{P}[\text{GOOD}] \ge 1 - O(\varepsilon^{1/4})$$

and quantum states  $\{\rho'_i \mid i \in GOOD\}$  such that

$$\left\| \rho_i^{'A} - \rho_i \right\|_1 \le 2\varepsilon^{1/4}$$

and

$$\sum_{i \in GOOD} P(i)\rho_i^{'A} \le (1 + \varepsilon^{1/4})\rho^A$$

*Proof.* Let the state  $|\rho\rangle^{AB}$  be a purification of  $\rho^A$ . Similarly, let  $|\rho_i\rangle^{AB}$  denote purifications of the states  $\rho_i^A$ . Consider the purification :

$$|\psi\rangle^{ABC} \coloneqq \sum_{i} \sqrt{P(i)} |\rho_{i}\rangle^{AB} |i\rangle^{C}$$

of the sample average state

$$\sum_{i} P(i) \rho_i^A$$

Then by Uhlmann's theorem and the closeness in 1-norm of the sample average state to  $\rho^A$ , we see that there exists a pure state  $|\varphi\rangle^{ABC}$  such that

$$\|\varphi - \psi\|_1 \le 2\sqrt{\varepsilon}$$

We will expand  $|\varphi\rangle^{ABC}$  by fixing the computational basis on C. This expansion can be written in the following form :

$$|\varphi\rangle^{ABC} = \sum_{i} |v_{i}\rangle^{AB} |i\rangle^{C}$$

where  $|v_i\rangle$ 's are some vectors on which we make no assumptions. We claim that each vector  $|v_i\rangle^{AB}$  has length at most 1. This is not hard to see, since taking the inner product of  $|\varphi\rangle^{ABC}$  with itself, we see that

$$\sum_{i} \langle v_i | v_i \rangle = 1$$

The above observation can be used to define a distribution Q(i) on [L] in the following way:

$$Q(i) := \langle v_i | v_i \rangle$$

We also define vectors  $\left|\rho_i'\right\rangle^{AB}$  by normalising the  $\left|v_i\right\rangle$  's :

$$|\rho_i'\rangle^{AB} \coloneqq \frac{1}{\sqrt{Q(i)}} |v_i\rangle^{AB}$$

This allows us to express  $|\varphi\rangle^{ABC}$  as follows :

$$|\varphi\rangle^{ABC} = \sum_{i} \sqrt{Q(i)} |\rho'_{i}\rangle^{AB} |i\rangle^{C}$$

Note that by definition each  $|\rho'_i\rangle^{AB}$  is a vector of length 1, and hence by the Schmidt decomposition and tracing out the system B, can be seen as a purification of some quantum state  $\rho'_i$ .

**Claim 3.4.** There exists a subset of [L] of probability (under P) of at least  $1 - O(\varepsilon^{1/4})$  on which

$$P(i) \le (1 + \varepsilon^{1/4})Q(i)$$

*Proof.* Note that by the  $2\sqrt{\varepsilon}$  closeness of  $\varphi$  and  $\psi$  and the monotonicity of 1-norm :

$$\sum_{i} Q(i) \left| 1 - \frac{P(i)}{Q(i)} \right| \le 2\sqrt{\varepsilon}$$

Define the set

$$\mathrm{INDEX} \coloneqq \left\{ i \mid \left| 1 - \frac{P(i)}{Q(i)} \right| \leq \varepsilon^{1/4} \right\}$$

From Markov's inequality, we see that

$$\Pr_{O}\left[\text{INDEX}^{c}\right] \leq 2\varepsilon^{1/4}$$

Since  $||Q - P||_1 \le 2\sqrt{\varepsilon}$ , this implies that

$$\Pr_{P}\left[\text{INDEX}^{c}\right] \leq O(\varepsilon^{1/4})$$

Therefore for all  $i \in INDEX$ , we have that

$$P(i) \le (1 + \varepsilon^{1/4})Q(i)$$

This proves the claim.

**Claim 3.5.** There exists a subset of [L] of probability under P at least  $1 - O(\varepsilon^{1/4})$  such that

$$\|\rho_i' - \rho_i\|_1 \le 2\varepsilon^{1/4}$$

Proof. Consider the expression

$$\|\varphi - \psi\|_1 \le \sqrt{\varepsilon}$$

We will measure the matrices inside the 1-norm along the computational basis on the system C. This produces states that are block diagonal. Appealing to the monotonicity of the 1-norm we see that this implies

$$\sum_{i} Q(i) \left\| \rho_{i}' - \frac{P(i)}{Q(i)} \rho_{i} \right\|_{1} \le 2\sqrt{\varepsilon}$$

Define the set

CLOSE := 
$$\left\{ i \mid \left\| \rho_i' - \frac{P(i)}{Q(i)} \rho_i \right\|_1 \le \varepsilon^{1/4} \right\}$$

By arguments similar to those used in Claim 3.4, we see that

$$\Pr_{P}\left[\mathtt{CLOSE}^{c}\right] \leq O(\varepsilon^{1/4})$$

Then, recalling the definition of the set INDEX from Claim 3.4, we see that

$$\Pr_{P}\left[\mathrm{INDEX}\bigcap\mathrm{CLOSE}\right] \geq 1 - O(\varepsilon^{1/4})$$

Define

$$GOOD := INDEX \bigcap CLOSE$$

Then, for all  $i \in GOOD$  we see that

$$\|\rho_i' - \rho_i\|_1 \le \|\rho_i' - \frac{P(i)}{Q(i)}\rho_i\| + \|\frac{P(i)}{Q(i)}\rho_i - \rho_i\|_1$$
$$\le \varepsilon^{1/4} + \varepsilon^{1/4}$$
$$= 2\varepsilon^{1/4}$$

We now define  $|\tilde{\varphi}\rangle$  by throwing away those indices from the expansion of  $|\varphi\rangle$  which are not in GOOD.

$$\left| \tilde{\varphi} \right\rangle^{ABC} \coloneqq \sum_{i \in \mathrm{GOOD}} \sqrt{Q(i)} \left| \rho_i' \right\rangle^{AB} \left| i \right\rangle$$

Then note that

$$\operatorname{Tr}_{BC}\left[\tilde{\varphi}\right] = \sum_{i \in \operatorname{GOOD}} Q(i) \rho_{i}^{'A} \leq \sum_{i \in [L]} Q(i) \rho_{i}^{'A} = \rho$$

Finally, by using the properties of GOOD, we observe that

$$\sum_{i \in \text{good}} P(i) \rho_i^{'A} \le (1 + \varepsilon^{1/4}) \sum_{i \in \text{good}} Q(i) \rho_i^{'A}$$
$$\le (1 + \varepsilon^{1/4}) \rho$$

This concludes the proof.

**Corollary 3.6.** Suppose we are given some positive semi-definite matrices  $\sigma_1^A, \sigma_2^A, \dots, \sigma_L^A$  and a distribution P(i) on the set of indices [L] such that

$$\left\| \sum_{i} P(i)\sigma_{i}^{A} - \rho^{A} \right\|_{1} \le \varepsilon$$

for a given average state  $\rho$ . Define

$$P'(i) := \frac{P(i) \cdot \operatorname{Tr}[\sigma_i]}{\sum_{i} P(i) \cdot \operatorname{Tr}[\sigma_i]}$$

Then there exist a good subset  $GOOD \subset [L]$  such that

$$\Pr_{P'}[\text{GOOD}] \ge 1 - O(\varepsilon^{1/4})$$

and quantum states  $\{\rho'_i \mid i \in GOOD\}$  such that

$$\left\| \rho_i^{'A} - \frac{\sigma_i}{\text{Tr}[\sigma_i]} \right\|_1 \le 2\varepsilon^{1/4}$$

and

$$\sum_{i \in GOOD} P(i) \cdot \text{Tr}[\sigma_i] \cdot \rho_i^{'A} \le (1 + O(\varepsilon^{1/4})) \rho^A$$

*Proof.* Firstly note that from the hypothesis given in the statement, using monotonicity of the 1-norm

$$\left\| \sum_{i} P(i) \cdot \text{Tr}[\sigma_i] - 1 \right\|_1 \le \varepsilon$$

which implies that

$$1 - \varepsilon \le \sum_{i} P(i) \cdot \text{Tr}[\sigma_i] \le 1 + \varepsilon$$

Let us define the distribution

$$P'(i) := \frac{P(i) \cdot \text{Tr}[\sigma_i]}{\sum_i P(i) \cdot \text{Tr}[\sigma_i]}$$

We also define

$$\rho_i := \frac{\sigma_i}{\text{Tr}[\sigma_i]}$$

We will need the following fact:

**Fact 3.7.** Given a state  $\rho$  and a positive semi-definite matrix  $\sigma$ , the following holds:

$$\left\| \rho - \frac{\sigma}{\text{Tr}[\sigma]} \right\|_1 \le 2 \left\| \rho - \sigma \right\|_1$$

Using this fact we can now rewrite the condition in the theorem statement as:

$$\left\| \sum_{i} P'(i)\rho_i - \rho \right\|_1 \le 2\varepsilon$$

Then, invoking Lemma 3.3, we can infer the existence of a set of indices GOODsuch that

$$\Pr_{P'}[\mathsf{GOOD}] \ge 1 - O(\varepsilon^{1/4})$$

and states  $\rho'_i$  such that

$$\left\| \rho_i' - \rho_i \right\|_1 \le O(\varepsilon^{1/4})$$

and

$$\sum_{i \in GOOD} P'(i)\rho'_i \le (1 + O(\varepsilon^{1/4}))\rho$$

Recall that

$$P'(i) = \frac{P(i) \cdot \text{Tr}[\sigma_i]}{\sum_{i} P(i) \cdot \text{Tr}[\sigma_i]} \ge \frac{1}{1 + \varepsilon} P(i) \cdot \text{Tr}[\sigma_i]$$

Therefore, we conclude that

$$\sum_{i \in \text{GOOD}} P(i) \cdot \text{Tr}[\sigma_i] \cdot \rho_i^{'A} \le (1 + O(\varepsilon^{1/4})) \rho^A$$

## 3.3 Classical Message Compression with Quantum Side Information

In this section we will prove Lemma 1.4. To define the above task, abbreviated as CQSI, we consider two parties Alice and Bob and a classical quantum control state of the form

$$\rho^{XB} := \sum_{x} P_X(x) |x\rangle \langle x|^X \otimes \rho_x^B$$

where the classical system X belongs to Alice and the quantum system B belongs to Bob. Alice and Bob also share a forward noiseless classical channel from Alice to Bob. Alice wishes to send the contents of the classical register to Bob using as little classical communication as possible.

To be precise, Alice and Bob wish to create the state  $\sigma^{X\hat{X}B}$  via classical communication, where the system X belongs to Alice and  $\hat{X}B$  belongs to Bob such that

$$\left\| \rho^{X\hat{X}B} - \sigma^{X\hat{X}B} \right\|_1 \le \varepsilon$$

where we define

$$\rho^{X\hat{X}B} := \sum_{x} P_{X}(x) |x\rangle \langle x|^{X} \otimes |x\rangle \langle x|^{\hat{X}} \otimes \rho_{x}^{B}$$

We also allow Alice and Bob to share public random coins and they are allowed to use private coin algorithms for encoding and decoding. We wish to minimise the amount of classical communication from Alice to Bob. For convenience we restate Lemma 1.4 below.

#### **Theorem 3.8.** Given the control state

$$\rho^{XB} = \sum_{x} P_X(x) |x\rangle \langle x|^X \otimes \rho_x^B$$

there exists a classical message compression protocol with quantum side information whenever the rate of communication R satisfies

$$R \ge H_{\max}^{\varepsilon}(X) - I_H^{\varepsilon}(X:B)_{\rho} + O(\log \frac{1}{\varepsilon})$$

*Proof.* The rough idea is as follows:

- 1. Alice hashes the set  $\mathcal{X}$  into a smaller set  $\mathcal{L}$  using a 2-universal hash family  $\mathcal{F}$ , where each  $f \in \mathcal{F}$  maps  $\mathcal{X} \to \mathcal{L}$ .
- 2. Each index  $\ell \in \mathcal{L}$  then corresponds to a subset of symbols, usually called a 'bucket', which is essentially the pre-image of  $\ell$  with respect to some randomly chosen hash function  $f \in \mathcal{F}$ .
- 3. Alice and Bob are both provided the description of this randomly chosen has function before the protocol starts.

- 4. One can imagine that Alice never receives a symbol which has low probability. Thus, we can essentially discard those x's of low probability, which have at most  $\varepsilon$ -mass under the distribution  $P_X$ . This implies that we can imagine that Alice receives her symbols from the sub-distribution P'(x) instead of P(x).
- 5. Upon receiving the symbol x, Alice sends the hash f(x) to Bob via the classical noiseless channel.
- 6. Bob then knows the subset  $f^{-1}(x) \subset \mathcal{X}$ . To decode the correct x sent by Alice, Bob does a measurement on his quantum system B.
- 7. To be able to distinguish among the members  $x' \in f^{-1}(x)$ , the quantum states  $\rho_{x'}$  need to be sufficiently far apart.
- 8. We show that this condition holds as long as

$$|f^{-1}(x)| \le 2^{I_H^{\varepsilon}(X:B)_{\rho}}$$

9. Recall that the support of P' is of size at most  $2^{H_{\max}^{\varepsilon}(X)}$ . Thus, the hash function essentially divides this set into  $2^{H_{\max}^{\varepsilon}(X)-I_H^{\varepsilon}(X:B)_{\rho}}$  buckets. This concludes the protocol.

The reason why the above protocol works is the 'random codebooks' created by the hash function. Without this randomness, we could partition the support of P' arbitrarily. But then we would not be able to guarantee that the symbols in the pre-image of any hash is distinguishable on average over the choice of the hash function.

#### **Bob's Decoding**

We wish to analyse the probability of a decoding error. However, we will not work with the distribution  $P_X$  but with the sub-distribution  $P_X'$ . To see that this only incurs an extra  $\varepsilon$  error, note that

$$\begin{split} \Pr[\operatorname{decoding\ error}] &= \sum_{x} P_X(x) \cdot \Pr[\operatorname{decoding\ error} \mid x] \\ &= \sum_{x \in \operatorname{supp}(P')} P_X(x) \cdot \Pr[\operatorname{decoding\ error} \mid x] + \sum_{x \notin \operatorname{supp}(P')} P_X(X)(x) \cdot \Pr[\operatorname{decoding\ error} \mid x] \\ &\leq \sum_{x} P'_X(x) \cdot \Pr[\operatorname{decoding\ error} \mid x] + \varepsilon \end{split}$$

We first define the bucket corresponding to the the hash function f and the hash  $\ell$  as

$$\mathcal{A}(f,\ell) \coloneqq \big\{ x \mid f(x) = \ell, x \in \operatorname{supp}(P') \big\}$$

Let us denote the elements in  $\mathcal{A}(f,\ell)$  as  $\left\{a_1^\ell,a_2^\ell,\ldots,a_{|\mathcal{A}|}^\ell\right\}$ . Now consider the operator  $\Pi_{\mathsf{OPT}}$  from the definition of  $I_H^\varepsilon(X:B)_\rho$  and the associated operators  $\Pi_x$ . Upon receiving the has  $\ell$ , Bob sequentially measures his system B with the operators  $\Pi_{a^\ell}$ .

We wish to analyse the probability of a decoding error. However, we will not work with the distribution  $P_X$  but with the sub-distribution  $P_X'$ . To see that this only incurs an extra  $\varepsilon$  error, note that

$$\begin{split} \Pr[\operatorname{decoding\ error}] &= \sum_{x} P_X(x) \cdot \Pr[\operatorname{decoding\ error} \mid x] \\ &= \sum_{x \in \operatorname{supp}(P')} P_X(x) \cdot \Pr[\operatorname{decoding\ error} \mid x] + \sum_{x \notin \operatorname{supp}(P')} P_X(X)(x) \cdot \Pr[\operatorname{decoding\ error} \mid x] \\ &\leq \sum_{x} P'_X(x) \cdot \Pr[\operatorname{decoding\ error} \mid x] + \varepsilon \end{split}$$

Suppose that when the symbol x is sent, the corresponding index of this symbol in the set  $\mathcal{A}(f,\ell)$  is  $a_m^\ell$ . Then, conditioned on Alice having received x, the probability of incorrect decoding is given by

$$\begin{split} &1 - \operatorname{Tr}\left[\Pi_{a_m^{\ell}}(\mathbb{I} - \Pi_{a_{m-1}^{\ell}}) \dots (\mathbb{I} - \Pi_{a_1^{\ell}}) \cdot \rho_{a_m^{\ell}}\right] \\ &= \operatorname{Tr}\left[\rho_{a_m^{\ell}}\right] - \operatorname{Tr}\left[\Pi_{a_m^{\ell}}(\mathbb{I} - \Pi_{a_{m-1}^{\ell}}) \dots (\mathbb{I} - \Pi_{a_1^{\ell}}) \cdot \rho_{a_m^{\ell}}\right] \end{split}$$

Using Sen's non-commutative union bound, the above expression can by bounded by

$$\leq \sqrt{\operatorname{Tr}\left[(\mathbb{I} - \Pi_{a_m^\ell})\rho_{a_m^\ell}\right] + \sum_{i=1}^{m-1}\operatorname{Tr}\left[\Pi_{a_i^\ell}\rho_{a_m^\ell}\right]}$$

Now notice that, the sets  $\mathcal{A}(f,\ell)$  form a disjoint cover of the set SUPP  $(P_X')$  over the indices  $\ell$ . Thus, taking an average over the elements of the set  $\bigcup_{\ell} \mathcal{A}(f,\ell)$  is the same as taking an average over the set SUPP  $(P_X')$ . Using this observation along with the concavity of the square root, we see that the average error probability over choices of x is at most

$$\sqrt{\sum_{\ell \in \mathcal{L}} \sum_{a_m^{\ell} \in \mathcal{A}(f,l)} P_X'(a_m^{\ell}) \left( \operatorname{Tr} \left[ (\mathbb{I} - \Pi_{a_m^{\ell}}) \rho_{a_m^{\ell}} \right] + \sum_{i=1}^{m-1} \operatorname{Tr} \left[ \Pi_{a_i^{\ell}} \rho_{a_m^{\ell}} \right] \right)} \\
= \sqrt{\sum_{x} P_X'(x) \operatorname{Tr} \left[ (\mathbb{I} - \Pi_x) \rho_x \right] + \sum_{\ell \in \mathcal{L}} \sum_{a_m^{\ell} \in \mathcal{A}(f,l)} P_X'(a_m^{\ell}) \sum_{i=1}^{m-1} \operatorname{Tr} \left[ \Pi_{a_i^{\ell}} \rho_{a_m^{\ell}} \right]}$$

The first term inside the square root is at most  $\varepsilon$ , by the property of  $\Pi_{OPT}$  that

$$\operatorname{Tr}\left[\Pi_{\operatorname{OPT}}\sum_{x}P_{X}(x)\left|x\right\rangle \left\langle x\right|^{X}\otimes\rho_{x}\right]\geq1-\varepsilon$$

which implies that

Tr 
$$\left[\sum_{x} P_X(x) |x\rangle \langle x|^X \otimes \Pi_x \rho_x\right] \ge 1 - \varepsilon$$

It is easy to see from the above that

$$\sum_{x} P_X'(x) \operatorname{Tr} \left[ (\mathbb{I} - \Pi_x) \rho_x \right]$$

$$\leq \sum_{x} P_X(x) \operatorname{Tr} \left[ (\mathbb{I} - \Pi_x) \rho_x \right]$$

$$< \varepsilon$$

To analyse the second term inside the square root, consider the following:

$$\begin{split} &\sum_{\ell \in \mathcal{L}} \sum_{a_m^\ell \in \mathcal{A}(f,l)} P_X'(a_m^\ell) \sum_{i=1}^{m-1} \operatorname{Tr} \left[ \Pi_{a_i^\ell} \rho_{a_m^\ell} \right] \\ &\leq \sum_{\ell \in \mathcal{L}} \sum_{a_m^\ell \in \mathcal{A}(f,l)} P_X'(a_m^\ell) \sum_{i \neq m} \operatorname{Tr} \left[ \Pi_{a_i^\ell} \rho_{a_m^\ell} \right] \\ &= \sum_{x} P_X'(x) \sum_{\substack{x' \neq x \\ x' \in \operatorname{SUPP}(P')}} I_{\{f(x') = f(x)\}} \operatorname{Tr} \left[ \Pi_{x'} \rho_x \right] \end{split}$$

where  $I_{x'\neq x}$  is the indicator for when f(x') = f(x). We will now take an expectation over the choice of the hash function f. Note that the above term is inside a square root, so to do this we use the concavity if square root:

$$\mathbb{E}\left[\sum_{x} P_X'(x) \sum_{x' \neq x} I_{\{x' \neq x\}} \operatorname{Tr}\left[\Pi_{x'}\rho_{x}\right]\right]$$

$$= \sum_{x} P_X'(x) \sum_{x' \neq x} \mathbb{E}\left[I_{\{f(x') = f(x)\}}\right] \operatorname{Tr}\left[\Pi_{x'}\rho_{x}\right]$$

$$= \sum_{x} P_X'(x) \sum_{x' \neq x} \operatorname{Pr}\left[f(x') = f(x)\right] \operatorname{Tr}\left[\Pi_{x'}\rho_{x}\right]$$

$$\leq 2^{-R} \sum_{x} P_X'(x) \sum_{x' \neq x} \operatorname{Tr}\left[\Pi_{x'}\rho_{x}\right]$$

$$\leq 2^{-R} \sum_{x} P_X'(x) \sum_{x' \in \operatorname{SUPP}(P')} \operatorname{Tr}\left[\Pi_{x'}\rho_{x}\right]$$

$$\leq 2^{-R} \sum_{x} P_X'(x) \sum_{x' \in \operatorname{SUPP}(P')} \operatorname{Tr}\left[\Pi_{x'}\rho_{x}\right]$$

To bound this term, we multiply and divide by  $P_X'(x')$  inside the second summation. This shows us that the above expression is equal to

$$=2^{-R} \sum_{x} P_X'(x) \sum_{x' \in \text{SUPP}(P')} \frac{P_X'(x')}{P_X'(x')} \operatorname{Tr} \left[ \Pi_{x'} \rho_x \right]$$

$$\leq 2^{-R} \cdot 2^{H_{\max}^{\varepsilon}(X)} \sum_{x} P_X'(x) \operatorname{Tr} \left[ \sum_{x' \in \text{SUPP}(P')} \left( P_X'(x') \Pi_{x'} \right) \rho_x \right]$$

where we have used Fact 2.4 to upper bound each  $\frac{1}{P_X'(x')}$  term by  $2^{H_{\max}^{\varepsilon}(X)}$ . We will now switch back to the distribution  $P_X$  by adding the terms corresponding to the x's which not in the support of  $P_X'$ . This implies that the above expression can be upper bounded by

$$\begin{split} &\leq 2^{-R} \cdot 2^{H_{\max}^{\varepsilon}(X)} \sum_{x} P_{X}(x) \operatorname{Tr} \left[ \sum_{x'} \left( P_{X}(x') \Pi_{x'} \right) \rho_{x} \right] \\ &= 2^{-R + H_{\max}^{\varepsilon}(X)} \operatorname{Tr} \left[ \left( \sum_{x'} P_{X}(x') \Pi_{x'} \right) \left( \sum_{x} P_{X}(x) \rho_{x} \right) \right] \\ &= 2^{-R + H_{\max}^{\varepsilon}(X)} \operatorname{Tr} \left[ \left( \sum_{x'} P_{X}(x') \left| x' \right\rangle \left\langle x' \right|^{X} \otimes \Pi_{x'}^{B} \right) \mathbb{I}^{X} \otimes \left( \sum_{x} P_{X}(x) \rho_{x}^{B} \right) \right] \\ &= 2^{-R + H_{\max}^{\varepsilon}(X)} \operatorname{Tr} \left[ \left( \sum_{x'} \left| x' \right\rangle \left\langle x' \right|^{X} \otimes \Pi_{x'}^{B} \right) \left( \sum_{x''} P_{X}(x'') \left| x'' \right\rangle \left\langle x'' \right|^{X} \right) \otimes \left( \sum_{x} P_{X}(x) \rho_{x}^{B} \right) \right] \\ &= 2^{-R + H_{\max}^{\varepsilon}(X)} \operatorname{Tr} \left[ \Pi_{\text{OPT}} \rho^{X} \otimes \rho^{B} \right] \\ &< 2^{-R + H_{\max}^{\varepsilon}(X) - I_{H}^{\varepsilon}(X; B) \rho} \end{split}$$

Thus, this shows that as long as

$$R \ge H_{\max}^{\varepsilon}(X) - I_H^{\varepsilon}(X:B)_{\varrho} - \log \varepsilon$$

the average decoding error over choices of x and the hash function f is at most  $\sqrt{2\varepsilon} + \varepsilon$ .

To finish the proof, consider the left polar decomposition of the operator

$$\Pi_{a_m^\ell}(\mathbb{I} - \Pi_{a_{m-1}^\ell}) \dots (\mathbb{I} - \Pi_{a_1^\ell}) = U_{a_m^\ell} \sqrt{\Theta_{a_m^\ell}}$$

where  $\Theta_{a_m^\ell}$  is some positive operator. Then, when Bob recovers the correct symbol  $x=a_m^\ell$ , the post measurement state is given by

$$\frac{1}{\operatorname{Tr}\left[\Theta_{a_{m}^{\ell}}\rho_{a_{m}^{\ell}}\right]}U_{a_{m}^{\ell}}\sqrt{\Theta_{a_{m}^{\ell}}}\rho_{a_{m}^{\ell}}\sqrt{\Theta_{a_{m}^{\ell}}}\left(U_{a_{m}^{\ell}}\right)^{\dagger}$$

Bob then applies the unitary  ${\cal U}_{a_m^\ell}$  to this state to get the state

$$\frac{1}{\operatorname{Tr}\left[\Theta_{a_{m}^{\ell}}\rho_{a_{m}^{\ell}}\right]}\sqrt{\Theta_{a_{m}^{\ell}}}\rho_{a_{m}^{\ell}}\sqrt{\Theta_{a_{m}^{\ell}}}$$

Suppose that  $a_m^\ell=x\in \operatorname{SUPP}(P_X')$ . Then, by the Gentle Measurement Lemma, we have that

$$\left\| \rho_x - \frac{1}{\operatorname{Tr}\left[\Theta_x \rho_x\right]} \sqrt{\Theta_x} \rho_x \sqrt{\Theta_x} \right\|_1 \le 2\sqrt{\operatorname{Tr}\left[\left(\mathbb{I} - \Theta_x\right) \rho_x\right]}$$

It is now easy to see that the following bounds hold

$$\sum_{x} P_{X}(x) \left\| \rho_{x} - \frac{1}{\operatorname{Tr}\left[\Theta_{x}\rho_{x}\right]} \sqrt{\Theta_{x}} \rho_{x} \sqrt{\Theta_{x}} \right\|_{1}$$

$$\leq \sum_{x} P'_{X}(x) \left\| \rho_{x} - \frac{1}{\operatorname{Tr}\left[\Theta_{x}\rho_{x}\right]} \sqrt{\Theta_{x}} \rho_{x} \sqrt{\Theta_{x}} \right\|_{1} + 2\varepsilon$$

$$\leq 2 \sum_{x} P'_{X}(x) \sqrt{\operatorname{Tr}\left[(\mathbb{I} - \Theta_{x})\rho_{x}\right]} + 2\varepsilon$$

$$\leq 2 \sqrt{\sum_{x} P'_{X}(x) \operatorname{Tr}\left[(\mathbb{I} - \Theta_{x})\rho_{x}\right]} + 2\varepsilon$$

which implies, by our previous computations that, the above expression can be upper bounded by

$$\leq 2\sqrt{\sqrt{2\varepsilon} + \varepsilon} + 2\varepsilon =: \varepsilon'$$

The protocol then is that Bob, after decoding with the above POVM elements, places the classical symbol in the system  $\hat{X}$ . Let  $\sigma^{X\hat{X}B}$  be the state after the protocol ends. We set the marginal  $\sigma^{\hat{X}B}$  to be some junk if the decoding failed. Note that  $\sigma$  is always classical on the system X. Then,

$$\begin{split} & \left\| \rho^{X\hat{X}B} - \sigma^{X\hat{X}B} \right\|_{1} \\ &= \sum_{x} P_{X}(x) \left\| |x\rangle \left\langle x|^{\hat{X}} \otimes \rho_{x}^{B} - \sigma^{\hat{X}B} \right\|_{1} \\ &= \underset{P_{X}}{\mathbb{E}} \left[ \left\| |x\rangle \left\langle x|^{\hat{X}} \otimes \rho_{x}^{B} - \sigma^{\hat{X}B} \right\|_{1} \right] \\ &= \underset{P_{X}}{\mathbb{E}} \left[ \left\| |x\rangle \left\langle x|^{\hat{X}} \otimes \rho_{x}^{B} - \sigma^{\hat{X}B} \right\|_{1} \right| \text{ correct decoding} \right] \cdot \text{Pr[correct decoding]} \\ &+ \underset{P_{X}}{\mathbb{E}} \left[ \left\| |x\rangle \left\langle x|^{\hat{X}} \otimes \rho_{x}^{B} - \sigma^{\hat{X}B} \right\|_{1} \right| \text{ incorrect decoding} \right] \cdot \text{Pr[incorrect decoding]} \\ &\leq \underset{x}{\sum} P_{X}(x) \left\| \rho_{x} - \frac{1}{\text{Tr} \left[\Theta_{x}\rho_{x}\right]} \sqrt{\Theta_{x}} \rho_{x} \sqrt{\Theta_{x}} \right\|_{1} + \varepsilon' \\ &\leq 2\varepsilon' \end{split}$$

This concludes the proof of the lemma.

## 4 Centralised Multi Link Measurement Compression

In this section we prove our centralised multi-link measurement compression theorem, in the presence of quantum side information. We precisely define the problem below: Suppose that Alice possesses register A of a pure state  $|\rho\rangle^{ABR}$ . Let  $\Lambda^{A\to AXY}:=\{\Lambda_{x,y}^{A\to A}\}_{x,y}$  be a POVM where for each classical outcome (x,y),  $\Lambda_{x,y}$  is a genuine POVM element i.e. a Hermitian operator on A with eigenvalues between zero and one. There exist two separate noiseless channels, called X and Y channels, from Alice to Bob, and two independent public coin registers, called X and Y public coins, between them. A noiseless channel together with its corresponding public coin is called a link. During the protocol at most one of the links may be turned OFF by an adversary without Alice or Bob's knowledge. Suppose Alice were to measure register A of state  $|\rho\rangle^{ABR}$  with the POVM  $\Lambda^{A\to AXY}$  obtaining classical outcome (x,y). In the centralised measurement compression protocol, Alice compresses the outcome pair and conveys the messages through the corresponding noiseless channels with the help of the corresponding public coins. Let  $R_X$ ,  $R_Y$  denote the rates of the noiseless channels for X and Y links, and  $C_X$ ,  $C_Y$  the rates for the public coins for X and Y links. We require Bob to be able to decode with the help of the public coins and produce a state on ABRXY with the following properties:

- 1. If the X link is ON and Y link is OFF, the state at the end of the protocol should be  $\varepsilon$ -close in Schatten  $\ell_1$ -norm to  $\sum\limits_{x\in\mathcal{X}}\left|x\right\rangle \left\langle x\right|^X\otimes ((\Lambda_x\otimes\mathbb{I}^{BR})(\rho))^{ABR},$  where  $\Lambda_x^{A\to A}:=\sum\limits_y\Lambda_{x,y};$
- 2. If the Y link is ON and X link is OFF, the state at the end of the protocol should be  $\varepsilon$ -close in Schatten  $\ell_1$ -norm to  $\sum\limits_{y\in\mathcal{Y}}\left|y\right\rangle\left\langle y\right|^Y\otimes((\Lambda_y\otimes\mathbb{I}^{BR})(\rho))^{ABR},$  where  $\Lambda_y^{A\to A}:=\sum\limits_x\Lambda_{x,y};$
- 3. If both links are ON, the state at the end of the protocol should be  $\varepsilon$ -close in Schatten  $\ell_1$ -norm to the state  $\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}}|x,y\rangle\left\langle x,y\right|^{XY}\otimes((\Lambda_{x,y}\otimes\mathbb{I}^{BR})(\rho))^{ABR}.$

Moreover, Alice's and Bob's strategies should be agnostic to which links are operational i.e. their encoding and decoding strategies should continue to work even if one link fails.

We will prove Theorem 1.1 and Corollary 1.2. We do this in several steps:

- 1. We will first prove the centralised multi link measurement compression theorem in the *one shot setting* when Bob does *does* not posses any qunatum side information. To that end we will first show that a corner point in the rate region claimed in Theorem 1.1 (in the case when there is no side information) is achievable. See Proposition 4.1.
- 2. Using the techniques of Proposition 4.1 we show that a rate region of the kind claimed in Theorem 1.1 (in the case when there is no side information) is achievable by using the rate splitting technique shown in [5]. See Proposition 4.6.
- 3. We then derive the rate region when quantum side information is present at the decoder by composing the measurement compression theorem with our protocol to do classical message compression with quantum side information. This is a highly non-trivial task and we do this in Section 4.2.
- 4. Finally, we observe that the rate region claimed in Corollary 1.2 is easily obtained by using the Quantum Asymptotic Equipartition Property (Section 2.4) for all the one shot quantities used in the proof of the one shot theorem.

#### 4.1 One Shot Centralised Multi Link Measurement Without Side Information

**Proposition 4.1.** Given the state  $\rho^A$  and the POVM  $\Lambda_{XY} := \{\Lambda_{xy}^A\}$ , the following rate point is achievable for centralised multi-link measurement compression:

$$R_X + C_X > H_{\max}^{\varepsilon}(X)$$

$$R_Y + C_Y > H_{\max}^{\varepsilon}(Y)$$

$$R_X > I_{\max}^{\varepsilon}(X:R)$$

$$R_Y > I_{\max}^{\varepsilon}(Y:RX)$$

where all entropic quantities are computed with respect to the state

$$\sum_{x,y} |x\rangle \langle x|^X \otimes |y\rangle \langle y|^Y \otimes \sqrt{\rho} \Lambda_{xy} \sqrt{\rho}^R$$

#### **Proof. POVM Construction**

We are given the POVM

$$\Lambda_{XY}\coloneqq\left\{\Lambda_{xy}^A\right\}$$

and the state  $\rho^A$ . Consider the true post measurement state

$$\sum_{x,y} |x\rangle \langle x|^X \otimes |y\rangle \langle y|^Y \otimes \sqrt{\rho} \Lambda_{xy} \sqrt{\rho}^R$$

We define

$$P_{XY}(x,y) := \text{Tr}[\Lambda_{xy}\rho]$$

Let  $\log K_1$  and  $\log K_2$  be the number of public coins available to Alice and Bob with respect to the X- and Y- channels respectively. Similarly, let  $\log L_1$  and  $\log L_2$  be the number of bits that Alice needs to send Bob along the two channels respectively. We will show that the following region is achievable for centralised multi link measurement compression.

$$\log K_1 + \log L_1 > H_{\max}^{\varepsilon}(X)$$

$$\log K_2 + \log L_2 > H_{\max}^{\varepsilon}(Y)$$

$$\log L_1 > I_{\max}^{\varepsilon}(X:R)$$

$$\log L_2 > I_{\max}^{\varepsilon}(Y:RX)$$

Construct  $C_X := \{x(k_1, \ell_1)\}$  and  $C_Y := \{y(k_2, \ell_2)\}$  randomly and independently from the distributions  $P_X$  and  $P_Y$ , where  $k_1 \in [K_1], \ell_1 \in [L_1]$  and  $k_2 \in [K_2], \ell_2 \in [L_2]$ . Define

$$\rho(k_1, k_2, \ell_1, \ell_2) := \frac{\sqrt{\rho} \Lambda_{x(k_1, \ell_1), y(k_2, \ell_2)} \sqrt{\rho}}{P_{XY}(x(k_1, \ell_1), y(k_2, \ell_2))}$$

From the choices of  $L_1$  and  $L_2$  and the measure transformed successive cancellation covering lemma we have that

$$\mathbb{E}_{\mathcal{C}_{X} \times \mathcal{C}_{Y}} \sum_{k_{1}, k_{2}} \frac{1}{K_{1} \cdot K_{2}} \left\| \frac{1}{L_{1} \cdot L_{2}} \sum_{\ell_{1}, \ell_{2}} \frac{P_{XY}(x(k_{1}, \ell_{1}), y(k_{2}, \ell_{2}))}{P_{X}(x(k_{1}, \ell_{1})) \cdot P_{Y}(y(k_{2}, \ell_{2}))} \rho(k_{1}, k_{2}, \ell_{1}, \ell_{2}) - \rho \right\|_{1} \leq \varepsilon$$
 (close)

**Definition 4.2.** We call a block  $(k_1, k_2)$  'nice' if the following condition holds for that block, with respect to some fixed codebook  $C_X \times C_Y$ :

$$\left\| \frac{1}{L_1 \cdot L_2} \sum_{\ell_1, \ell_2} \frac{P_{XY}(x(k_1, \ell_1), y(k_2, \ell_2))}{P_X(x(k_1, \ell_1)) \cdot P_Y(y(k_2, \ell_2))} \rho(k_1, k_2, \ell_1, \ell_2) - \rho \right\|_1 \le \sqrt{\varepsilon}$$

Let us fix a nice block  $(k_1, k_2)$  with respect to some fixed codebook. To ease the notation define

$$t(k_1, k_2, \ell_1, \ell_2) := \frac{P_{XY}(x(k_1, \ell_1), y(k_2, \ell_2))}{P_X(x(k_1, \ell_1)) P_Y(y(k_2, \ell_2))}$$

We are now in a position to apply Corollary 3.6. To do this we define Q to be the uniform distribution on the set  $[L_1] \times [L_2]$ . Then let

$$Q'(k_1, k_2, \ell_1, \ell_2) := \frac{Q(\ell_1, \ell_2) \cdot t(k_1, k_2, \ell_1, \ell_2)}{\sum_{\ell_1, \ell_2} Q(\ell_1, \ell_2) \cdot t(k_1, k_2, \ell_1, \ell_2)}$$

Then, Corollary 3.6 implies that there exists a subset GOOD  $\subset [L_1] \times [L_2]$  such that

$$\Pr_{O'}[\mathsf{GOOD}] > 1 - O(\varepsilon^{1/4})$$

and for all  $(\ell_1, \ell_2) \in GOOD$  there exist states  $\rho'(k_1, k_2, \ell_1, \ell_2)$  such that

$$\|\rho'(k_1, k_2, \ell_1, \ell_2) - \rho(k_1, k_2, \ell_1, \ell_2)\|_1 \le O(\varepsilon^{1/4})$$

and

$$\sum_{(\ell_1,\ell_2) \in \text{GOOD}} \frac{1}{L_1 \cdot L_2} \cdot t(k_1,k_2,\ell_1,\ell_2) \rho'(k_1,k_2,\ell_1,\ell_2) \le (1 + O(\varepsilon^{1/4})) \rho$$

For the fixed block indices  $k_1$  and  $k_2$  we define the POVM elements for  $(\ell_1, \ell_2) \in GOOD$ :

$$\Gamma_{\ell_1,\ell_2}(k_1,k_2) := \frac{1}{1 + O(\varepsilon^{1/4})} \frac{t(k_1,k_2,\ell_1,\ell_2)}{L_1 \cdot L_2} \rho^{-1/2} \rho'(k_1,k_2,\ell_1,\ell_2) \rho^{-1/2}$$

#### **Some Important Observations:**

$$\begin{split} \sum_{(\ell_1,\ell_2) \notin \text{GOOD}} \frac{t(k_1,k_2,\ell_1,\ell_2)}{L_1 \cdot L_2} &= \left( \sum_{\ell_1,\ell_2} Q(\ell_1,\ell_2) \cdot t(k_1,k_2,\ell_1,\ell_2) \right) \cdot \sum_{(\ell_1,\ell_2) \notin \text{GOOD}} Q'(k_1,k_2,\ell_1,\ell_2) \\ &\leq (1 + O(\varepsilon^{1/4})) \cdot O(\varepsilon^{1/4}) \\ &= O(\varepsilon^{1/4}) \end{split}$$

Then, we define the POVM

$$\Gamma(k_1, k_2) := \{ \Gamma_{\ell_1, \ell_2}(k_1, k_2) \mid (\ell_1, \ell_2) \in \text{GOOD} \} \bigcup \{ \Gamma_0(k_1, k_2) \}$$

where

$$\Gamma_0(k_1, k_2) \coloneqq \mathbb{I}_{\text{supp}(\rho)} - \sum_{(\ell_1, \ell_2) \in \text{GOOD}} \Gamma_{\ell_1, \ell_2}(k_1, k_2)$$

#### **Claim 4.3.**

$$\operatorname{Tr}[\Gamma_0(k_1, k_2)\rho] \le O(\varepsilon^{1/4})$$

Proof.

$$\operatorname{Tr}[\Gamma_{0}(k_{1}, k_{2})\rho] = \operatorname{Tr}[\rho - \sum_{(\ell_{1}, \ell_{2}) \in \operatorname{GOOD}} \frac{1}{1 + O(\varepsilon^{1/4})} \frac{t(k_{1}, k_{2}, \ell_{1}, \ell_{2})}{L_{1} \cdot L_{2}} \rho'(k_{1}, k_{2}, \ell_{1}, \ell_{2})]$$

$$= 1 - \frac{1}{1 + O(\varepsilon^{1/4})} \sum_{(\ell_{1}, \ell_{2}) \in \operatorname{GOOD}} \frac{t(k_{1}, k_{2}, \ell_{1}, \ell_{2})}{L_{1} \cdot L_{2}}$$

Recall that

$$\begin{split} \sum_{(\ell_1,\ell_2)} \frac{t(k_1,k_2,\ell_1,\ell_2)}{L_1 \cdot L_2} \geq & 1 - \varepsilon \\ \Longrightarrow \sum_{(\ell_1,\ell_2) \in \text{GOOD}} \frac{t(k_1,k_2,\ell_1,\ell_2)}{L_1 \cdot L_2} \geq & 1 - \varepsilon - \sum_{(\ell_1,\ell_2) \notin \text{GOOD}} \frac{t(k_1,k_2,\ell_1,\ell_2)}{L_1 \cdot L_2} \\ \Longrightarrow \sum_{(\ell_1,\ell_2) \in \text{GOOD}} \frac{t(k_1,k_2,\ell_1,\ell_2)}{L_1 \cdot L_2} \geq & 1 - \varepsilon - O(\varepsilon^{1/4}) \end{split}$$

Therefore,

$$\text{Tr}[\Gamma_0(k_1, k_2)\rho] \le 1 - \frac{1 - O(\varepsilon^{1/4})}{1 + O(\varepsilon^{1/4})} \le O(\varepsilon^{1/4})$$

#### **Closeness of the Post Measurement States**

Alice will use the compressed POVMs only for those blocks which are nice. For all other blocks she aborts the protocol. Notice that the compressed POVMs that we designed output a classical index  $(\ell_1, \ell_2)$ . There exist functions f and g such that

$$f(k_1, \ell_1) := x(k_1, \ell_1)$$
  
 $g(k_2, \ell_2) := y(k_2, \ell_2)$ 

The error expression is then given by

$$\mathbb{E}_{\mathcal{C}_{X} \times \mathcal{C}_{Y}} \left\| \sum_{x,y} |x\rangle \left\langle x\right|^{X} \otimes |y\rangle \left\langle y\right|^{Y} \otimes \sqrt{\rho} \Lambda_{x,y} \sqrt{\rho}^{R} - \sum_{\substack{(k_{1},k_{2}) \text{ nice} \\ (\ell_{1},\ell_{2}) \in \mathsf{GOOD} \cup \{0\}}} \frac{1}{K_{1} \cdot K_{2}} |x(k_{1},\ell_{1})\rangle \left\langle x(k_{1},\ell_{1})| \otimes |y(k_{2},\ell_{2})\rangle \left\langle y(k_{2},\ell_{2})| \otimes \sqrt{\rho} \Gamma_{\ell_{1},\ell_{2}}(k_{1},k_{2}) \sqrt{\rho} \right\|_{1} \tag{post-measurement}$$

#### **Definition 4.4.** We define random variable:

$$\begin{split} T(\mathcal{C}_{X},\mathcal{C}_{Y}) \coloneqq & \left\| \sum_{x,y} \left| x \right\rangle \left\langle x \right|^{X} \otimes \left| y \right\rangle \left\langle y \right|^{Y} \otimes \sqrt{\rho} \Lambda_{x,y} \sqrt{\rho}^{R} \right. \\ & \left. - \sum_{\substack{(k_{1},k_{2}) \text{ nice} \\ (\ell_{1},\ell_{2}) \in \mathsf{GOOD} \ \cup \{0\}}} \frac{1}{K_{1} \cdot K_{2}} \left| x(k_{1},\ell_{1}) \right\rangle \left\langle x(k_{1},\ell_{1}) \right| \otimes \left| y(k_{2},\ell_{2}) \right\rangle \left\langle y(k_{2},\ell_{2}) \right| \otimes \sqrt{\rho} \Gamma_{\ell_{1},\ell_{2}}(k_{1},k_{2}) \sqrt{\rho} \right\|_{1} \end{split}$$

and the event:

$$E := \left\{ \# \text{ of nice blocks } \ge (1 - \varepsilon^{1/4}) \cdot K_1 \cdot K_2 \right\}$$

Consider the following claim which is easy to prove:

#### Claim 4.5.

$$\Pr[E] \ge 1 - \varepsilon^{1/4}$$

The error expression can then be analysed as:

$$\begin{split} \mathbb{E}[T] &= \mathbb{E}[T \cdot 1_E] + \mathbb{E}[T \cdot 1_{E^c}] \\ &\leq \mathbb{E}[T \cdot 1_E] + 2 \cdot \Pr[E^c] \\ &\leq \mathbb{E}[T \cdot 1_E] + 2 \cdot \varepsilon^{1/4} \end{split}$$

The first inequality used the fact that T is at most 2, since it is the 1-norm between two states. We will thus bound  $T(\mathcal{C}_X, \mathcal{C}_Y)$  for only those codebooks for which the event E holds.

We will massage the second term inside the 1-norm expression above. First, we discard the 0-th outcome which adds at most  $O(\varepsilon^{1/4})$ . Next, notice that

$$\sqrt{\rho}\Gamma_{\ell_1,\ell_2}(k_1,k_2)\sqrt{\rho} = \frac{1}{1 + O(\varepsilon^{1/4})} \frac{t(k_1,k_2,\ell_1,\ell_2)}{L_1 \cdot L_2} \rho'(k_1,k_2,\ell_1,\ell_2)$$

We will apply the triangle inequality twice. In the first we replace all the  $\rho'(k_1, k_2, \ell_1, \ell_2)$  states with  $\rho(k_1, k_2, \ell_1, \ell_2)$ . This gives rise to the term

$$\frac{1}{1 + O(\varepsilon^{1/4})} \sum_{\substack{(k_1, k_2) \text{ is nice} \\ (\ell_1, \ell_2) \in \mathsf{GOOD}}} \frac{1}{K_1 \cdot K_2} \frac{t(k_1, k_2, \ell_1, \ell_2)}{L_1 \cdot L_2} \left\| \rho'(k_1, k_2, \ell_1, \ell_2) - \rho(k_1, k_2, \ell_1, \ell_2) \right\|_1$$

Due to the conditions in Corollary 3.6, this can be bounded above by

$$\frac{1}{1 + O(\varepsilon^{1/4})} \sum_{\substack{(k_1, k_2) \text{ is nice} \\ (\ell_1, \ell_2) \in \text{GOOD}}} \frac{1}{K_1 \cdot K_2} \frac{t(k_1, k_2, \ell_1, \ell_2)}{L_1 \cdot L_2} O(\varepsilon^{1/4})$$

We already know from Eq. (close) that for a nice block index  $(k_1, k_2)$ 

$$\sum_{\ell_1,\ell_2} \frac{t(k_1,k_2,\ell_1,\ell_2)}{L_1 \cdot L_2} \le 1 + \varepsilon$$

Therefore,

$$\frac{O(\varepsilon^{1/4})}{1 + O(\varepsilon^{1/4})} \sum_{\substack{(k_1, k_2) \text{ is nice} \\ (\ell_1, \ell_2) \in \mathsf{GOOD}}} \frac{1}{K_1 \cdot K_2} \frac{t(k_1, k_2, \ell_1, \ell_2)}{L_1 \cdot L_2} \leq \frac{O(\varepsilon^{1/4})}{1 + O(\varepsilon^{1/4})} (1 + \varepsilon)$$

$$\leq O(\varepsilon^{1/4})$$

These steps have allowed us to massage the second term inside the norm in Eq. (post-measurement) into

$$\frac{1}{1 + O(\varepsilon^{1/4})} \sum_{\substack{(k_1, k_2) \text{ is nice} \\ (\ell_1, \ell_2) \in \mathsf{GOOD}}} \frac{1}{K_1 \cdot K_2} \frac{t(k_1, k_2, \ell_1, \ell_2)}{L_1 \cdot L_2} \left| x(k_1, \ell_1) \right\rangle \left\langle x(k_1, \ell_1) \right|$$

$$\otimes |y(k_2,\ell_2)\rangle \langle y(k_2,\ell_2)| \otimes \rho(k_1,k_2,\ell_1,\ell_2)$$

We further massage this term by adding those terms in the sum above which correspond to those  $(\ell_1, \ell_2)$  which are not in the set GOOD. This adds an extra

$$\frac{1}{1 + O(\varepsilon^{1/4})} \sum_{(k_1, k_2) \text{ is nice}} \frac{1}{K_1 \cdot K_2} \sum_{(\ell_1, \ell_2) \notin \text{GOOD}} \frac{t(k_1, k_2, \ell_1, \ell_2)}{L_1 \cdot L_2} \le O(\varepsilon^{1/4})$$

Collating the arguments above, we see that the true post measurement state in Eq. (post-measurement) can be replaced with the state

$$T_{\text{GOOD}} \coloneqq \frac{1}{1 + O(\varepsilon^{1/4})} \sum_{\substack{(k_1, k_2) \text{ is nice} \\ (\ell_1, \ell_2)}} \frac{1}{K_1 \cdot K_2} \frac{t(k_1, k_2, \ell_1, \ell_2)}{L_1 \cdot L_2} \left| x(k_1, \ell_1) \right\rangle \left\langle x(k_1, \ell_1) \right|$$

Finally, we will add the terms corresponding to the blocks which are not nice, i.e,

$$T_{\text{BAD}} \coloneqq \frac{1}{1 + O(\varepsilon^{1/4})} \sum_{\substack{(k_1, k_2) \text{ not nice} \\ (\ell_1, \ell_2)}} \frac{1}{K_1 \cdot K_2} \frac{t(k_1, k_2, \ell_1, \ell_2)}{L_1 \cdot L_2} \left| x(k_1, \ell_1) \right\rangle \left\langle x(k_1, \ell_1) \right| \\ \otimes \left| y(k_2, \ell_2) \right\rangle \left\langle y(k_2, \ell_2) \right| \otimes \rho(k_1, k_2, \ell_1, \ell_2)$$

Recall that since we only consider those codebooks for which the event E holds, there are at most  $\varepsilon^{1/4} \cdot K_1 \cdot K_2$  blocks which are not nice. Consider the random variable

$$\|T_{\text{BAD}}\|_{1} = \frac{1}{1 + O(\varepsilon^{1/4})} \sum_{\substack{(k_{1}, k_{2}) \text{ not nice} \\ (\ell_{1}, \ell_{2})}} \frac{1}{K_{1} \cdot K_{2}} \frac{t(k_{1}, k_{2}, \ell_{1}, \ell_{2})}{L_{1} \cdot L_{2}}$$
 (bad\_blocks)

#### Bounding $||T_{BAD}||_1$

From the classical measure transformed covering lemma we know that

$$\mathbb{E}_{C_{X} \times C_{Y}} \left\| \sum_{x,y} |x\rangle \langle x|^{X} \otimes |y\rangle \langle y|^{Y} \otimes \sqrt{\rho} \Lambda_{x,y} \sqrt{\rho}^{R} - \sum_{\substack{(k_{1},k_{2}) \\ (\ell_{1},\ell_{2})}} \frac{t(k_{1},k_{2},\ell_{1},\ell_{2})}{K_{1}L_{1} \cdot K_{2}L_{2}} |x(k_{1},\ell_{1})\rangle \langle x(k_{1},\ell_{1})| \otimes |y(k_{2},\ell_{2})\rangle \langle y(k_{2},\ell_{2})| \otimes \rho(k_{1},k_{2},\ell_{1},\ell_{2}) \right\|_{1} \leq \varepsilon$$

This implies that

$$\mathbb{E}_{\mathcal{C}_{X} \times \mathcal{C}_{Y}} \left\| \sum_{x,y} |x\rangle \langle x|^{X} \otimes |y\rangle \langle y|^{Y} \otimes \sqrt{\rho} \Lambda_{x,y} \sqrt{\rho}^{R} \right.$$

$$- \sum_{\substack{(k_{1},k_{2}) \\ (\ell_{1},\ell_{2})}} \frac{t(k_{1},k_{2},\ell_{1},\ell_{2})}{K_{1}L_{1} \cdot K_{2}L_{2}} |x(k_{1},\ell_{1})\rangle \langle x(k_{1},\ell_{1})| \otimes |y(k_{2},\ell_{2})\rangle \langle y(k_{2},\ell_{2})| \otimes \rho(k_{1},k_{2},\ell_{1},\ell_{2}) \right\|_{1} \cdot 1_{E}$$

$$\leq \varepsilon$$

Thus a random codebook satisfies the event E and the condition

$$\left\| \sum_{x,y} |x\rangle \langle x|^X \otimes |y\rangle \langle y|^Y \otimes \sqrt{\rho} \Lambda_{x,y} \sqrt{\rho}^R \right\|_{L^2(\ell_1,\ell_2)}$$

$$- \sum_{\substack{(k_1,k_2) \\ (\ell_1,\ell_2)}} \frac{t(k_1,k_2,\ell_1,\ell_2)}{K_1 L_1 \cdot K_2 L_2} |x(k_1,\ell_1)\rangle \langle x(k_1,\ell_1)| \otimes |y(k_2,\ell_2)\rangle \langle y(k_2,\ell_2)| \otimes \rho(k_1,k_2,\ell_1,\ell_2) \right\|_{L^2(\ell_1,\ell_2)}$$

$$\leq \sqrt{\varepsilon}$$

with probability at least  $1-\sqrt{\varepsilon}$ . Fix such a codebook. Then for this good codebook it is easy to see that

$$\left\| \sum_{x,y} |x\rangle \langle x|^X \otimes |y\rangle \langle y|^Y \otimes \sqrt{\rho} \Lambda_{x,y} \sqrt{\rho}^R - \frac{1}{1 + O(\varepsilon^{1/4})} \sum_{\substack{(k_1, k_2) \\ (\ell_1, \ell_2)}} \frac{t(k_1, k_2, \ell_1, \ell_2)}{K_1 L_1 \cdot K_2 L_2} |x(k_1, \ell_1)\rangle \langle x(k_1, \ell_1)| \otimes |y(k_2, \ell_2)\rangle \langle y(k_2, \ell_2)| \otimes \rho(k_1, k_2, \ell_1, \ell_2) \right\|_{1}$$

$$\leq O(\varepsilon^{1/4})$$

We can write the second term in the 1-norm above as a sum  $T_{\text{GOOD}}$  and  $T_{\text{BAD}}$ . Then using the monotonicity of the 1-norm under trace, we see that

$$|1 - \text{Tr}[T_{\text{GOOD}}] - \text{Tr}[T_{\text{BAD}}]|_1 \le O(\varepsilon^{1/4})$$

From our previous analysis, we know that

$$\left| \operatorname{Tr}[T_{\text{GOOD}}] - \operatorname{Tr} \left( \sum_{\substack{(k_1, k_2) \text{ nice} \\ (\ell_1, \ell_2) \in \operatorname{GOOD} \cup \{0\}}} \frac{1}{K_1 \cdot K_2} |x(k_1, \ell_1)\rangle \langle x(k_1, \ell_1)| \otimes |y(k_2, \ell_2)\rangle \langle y(k_2, \ell_2)| \otimes \sqrt{\rho} \Gamma_{\ell_1, \ell_2}(k_1, k_2) \sqrt{\rho} \right) \right|_{1} \leq O(\varepsilon^{1/4})$$

Recall that for our choice of codebook, there are at most  $\varepsilon^{1/4}$  fraction of bad blocks. This implies that the second term in the expression above is at least  $1 - \varepsilon^{1/4}$ . Therefore, we can conclude that

$$\operatorname{Tr}[T_{\text{GOOD}}] \ge 1 - O(\varepsilon^{1/4})$$

Using this lower bound on the trace of  $T_{GOOD}$ , it is easy to see that

$$\operatorname{Tr}[T_{\text{BAD}}] \leq O(\varepsilon^{1/4})$$

Collating all the arguments above, we see that we can replace the true post measurement state in Eq. (post-measurement) to  $T_{\rm GOOD}+T_{\rm BAD}$  with an additive error of at most  $O(\varepsilon^{1/4})$ . Recall that we have already fixed a codebook which bounds the 1-norm between the ideal post measurement state and  $T_{\rm GOOD}+T_{\rm BAD}$  by  $\sqrt{\varepsilon}$ . Thus, this implies that there exists with probability at least  $1-\sqrt{\varepsilon}$  a codebook which ensures that

$$\mathbb{E}_{\mathcal{C}_X \times \mathcal{C}_Y}[T(\mathcal{C}_X, \mathcal{C}_Y)] \le O(\varepsilon^{1/4})$$

This concludes the proof.

We will now show how one can achieve a larger rate region in the one shot setting for centralised multi link measurement compression by using the technique of quantum rate splitting, as shown in [5].

**Proposition 4.6.** Suppose Alice is given the state  $\rho^A$  and the POVM  $\Lambda_X Y$  as in Proposition 4.1. Suppose that  $P_X Y$  is the distribution induced by the POVM. Let  $(P_U^{\theta}, P_V^{\theta}, \max)$  be a split of the marginal  $P_X$ , as defined in [5, 11], for some parameter  $\theta \in [0, 1]$ . Then one achievable rate region is obtained as the union over a parameter  $\theta \in [0, 1]$  of

the regions  $S_{\theta}$  defined by:

$$R_{X} = R_{U} + R_{V}$$

$$R_{U} > I_{\max}^{\varepsilon}(U:R) + O(\log \varepsilon^{-1})$$

$$R_{Y} > I_{\max}^{\varepsilon}(Y:RU) + O(\log \varepsilon^{-1})$$

$$S_{\theta} : R_{V} > I_{\max}^{\varepsilon}(V:RUY) + O(\log \varepsilon^{-1})$$

$$C_{X} = C_{U} + C_{V}$$

$$C_{U} + R_{U} > H_{\max}^{\varepsilon}(U)$$

$$C_{Y} + R_{Y} > H_{\max}^{\varepsilon}(Y),$$

where the entropic quantities are calculated for the control state

$$\sum_{\substack{(u,v,y)\in\mathcal{X}\times\mathcal{X}\times\mathcal{Y}\\ \\ \otimes \frac{((\Lambda_{\max\{u,v\},y}\otimes\mathbb{I}^R)(\rho))^{AR}}{\mathrm{Tr}[((\Lambda_{\max\{u,v\},y}\otimes\mathbb{I}^R)(\rho))^{AR}]}}}.$$

The above state is obtained by splitting random variable X into independent random variables U, V in the state  $\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}}|x,y\rangle\langle x,y|^{XY}\otimes((\Lambda_{x,y}\otimes\mathbb{I}^R)(\rho))^{AR} \text{ according to the parameter }\theta. \text{ Another achievability region is obtained}$ 

by rate splitting Y instead of X. The total achievable region is the union of the two regions. The encoding and decoding strategies are agnostic to which links are actually functioning.

*Proof.* First consider the post measurement state

$$\sum_{(x,y)\in\mathcal{X}\times\mathcal{V}} |x,y\rangle \langle x,y|^{XY} \otimes ((\Lambda_{x,y}\otimes \mathbb{I}^R)(\rho))^{AR}$$

This can be rewritten as:

$$\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} P_X(x) P_{Y|X}(y|x) |x,y\rangle \langle x,y|^{XY} \otimes \frac{((\Lambda_{x,y}\otimes \mathbb{I}^R)(\rho))^{AR}}{\operatorname{Tr}\left[((\Lambda_{x,y}\otimes \mathbb{I}^R)(\rho))^{AR}\right]}$$

where

$$P_{XY}(x,y) := \operatorname{Tr}\left[ ((\Lambda_{x,y} \otimes \mathbb{I}^R)(\rho))^{AR} \right]$$

One can then split the distribution  $P_X$  into the distributions  $p_U^{\theta}(u)$  and  $p_V^{\theta}(v)$  such that

$$\max{(U,V)} \sim P_X$$

where the  $p_U^{\theta}(u)$  and  $p_V^{\theta}(v)$  are both defined on the classical alphabet  $\mathcal{X}$ . Refer to [5] for details about this splitting operation. This leads to the control state:

$$\sum_{(u,v,y)\in\mathcal{X}\times\mathcal{X}\times\mathcal{Y}} p_U^{\theta}(u) p_V^{\theta}(v) p(y|u,v) |u,v,y\rangle \langle u,v,y|^{UVY}$$

$$\otimes \frac{((\Lambda_{\max\{u,v\},y}\otimes \mathbb{I}^R)(\rho))^{AR}}{\text{Tr}[((\Lambda_{\max\{u,v\},y}\otimes \mathbb{I}^R)(\rho))^{AR}]}.$$

where

$$p(y|u,v) \coloneqq P_{Y|X}(y|\max(u,v))$$

Next, we construct the POVM elements as follows:

- 1. Alice picks samples  $\{U(k_1,\ell_1) \mid k_1 \in [2^{C_U}], \ell_1 \in [2^{R_U}]\}$  iid from  $p_U^{\theta}$ ,  $\{Y(k_2,\ell_2) \mid k_1 \in [2^{C_Y}], \ell_1 \in [2^{R_Y}]\}$  iid from  $P_Y$  and  $\{V(k_3,\ell_3) \mid k_3 \in [2^{C_V}], \ell_1 \in [2^{R_V}]\}$  iid from  $p_V^{\theta}$ .
- 2. Define

$$\Lambda_{u,v,y} \coloneqq \Lambda_{\max(u,v),y}$$

The rest of the proof is similar to the proof of Proposition 4.1, where in this case we have to use the sequential covering lemma for 3 parties. Using the techniques in that proof, it is not hard to see that the rate region  $S_{\theta}$  is achievable. This concludes the proof. This concludes the proof.

## 4.2 Measurement Compression with Side Information

In this section we will show how to compose the centralised measurement compression theorem with our protocol for classical data compression with quantum side information. We will first demonstrate our technique for the simpler case when the POVM only outputs one classical symbol x i.e. the point to point case.

**Proposition 4.7.** Given the shares quantum state  $\rho^{AB}$ , where the receiver Bob possesses the B system, the following rates are achievable for one shot measurement compression with quantum side information:

$$R_X > I_{\max}^{\varepsilon}(X:RB) - I_H^{\varepsilon_0/2}(X:B) + O(\log \varepsilon^{-1}) + 1$$

and

$$R_X + C_X > H_{\max}^{\varepsilon}(X) - I_H^{\varepsilon_0/2}(X:B) + O(\log 1/\varepsilon) + 1.$$

where  $\varepsilon_0 := \varepsilon^{1/10}$ . Above, all entropic quantities are computed with respect to  $\sum_{x \in \mathcal{X}} |x\rangle \langle x|^X \otimes ((\Lambda_x \otimes \mathbb{I}^{BR})(\rho))^{ABR}$ .

*Proof.* We will first construct a point to point measurement compression protocol which ignores that Bob has any side information. This follows as a corollary of Proposition 4.1. Recall that this construction consists of a random codebook  $\mathcal{C}_X$  which is divided into K blocks, where each block contains L elements. Note that

$$\log K + \log L > H_{\max}^{\varepsilon}(X) - O(\log \varepsilon)$$
$$\log L > I_{\max}^{\varepsilon}(X : RB) - O(\log \varepsilon)$$

where the entropic quantities are computed with respect to the control state

$$\sum_{x \in \mathcal{X}} |x\rangle \langle x|^X \otimes ((\Lambda_x \otimes \mathbb{I}^{BR})(\rho))^{ABR}.$$

Suppose that  $C_X$  obeys the condition that the fraction of nice blocks is at least  $1 - O(\varepsilon^{1/4})$ , where a 'nice' block refers to a block which satisfies Definition 4.2. Suppose that for every nice block index k, let GOOD(k) be the set of indices  $\ell$  which obey the conditions of Lemma 3.3. Then, our protocol ensures that with respect to this codebook, the global post measurement state is

$$\tilde{\rho}^{KLRB} \coloneqq \frac{1}{\mathrm{Tr}[\tilde{\rho}]} \sum_{\substack{k \text{ is a nice block} \\ \ell \in \mathrm{GOOD}(k)}} \frac{1}{L \cdot K} \left| k \right\rangle \left\langle k \right|^K \otimes \left| \ell \right\rangle \left\langle \ell \right|^L \otimes \rho_{k,\ell}^{'RB}$$

with the promise that for all nice indices k,

$$\sum_{\ell \in \text{GOOD}(k)} \frac{1}{L} \rho_{k,\ell}^{'RB} \le (1 + \varepsilon^{1/4}) \operatorname{Tr}_A((\Lambda \otimes \mathbb{I}^{BR})(\rho))^{ABR}$$

Recall that conditioned on the codebook, there exists a deterministic map  $f:[K]\times[L]\to\mathcal{X}$  such that

$$\|\rho^{XBR} - \tilde{\rho}^{XBR}\| \le O(\varepsilon^{1/4})$$

Let  $\tilde{P}_X$  be the marginal of  $\tilde{\rho}^{XBR}$  on the system X. Then it is not hard to see that there exists a subset INDEX of x's with probability at least  $1 - O(\varepsilon^{1/8})$  under the distribution  $P_X$  such that, for all  $x \in \text{INDEX}$ 

$$\tilde{P}_X(x) \le (1 + O(\varepsilon^{1/8}))P_X(x)$$

We now require that Alice and Bob work with a post-measurement state which has  $\tilde{P}_X$  as the marginal distribution on the system X. To do this, Alice simply discards those  $(k,\ell)$  pairs which map to those x's under f which are not in INDEX. We will call this post measurement state  $\sigma^{KLB}$ :

$$\sigma^{KLRB} \coloneqq \frac{1}{\mathrm{Tr}[\sigma]} \sum_{\substack{k \text{ is nice} \\ \ell \in \mathrm{GOOD}(k) \\ f(k,\ell) \in \mathrm{INDEX}}} \frac{1}{K \cdot L} \left| k \right\rangle \left\langle k \right|^K \otimes \left| \ell \right\rangle \left\langle \ell \right|^L \otimes \sigma_{k,\ell}^{'RB}$$

where

$$\sigma_{k,\ell}^{'RB} \coloneqq \rho_{k,\ell}^{'RB}$$

and for all nice k,

$$\sum_{\substack{\ell \in \text{GOOD}(k) \\ f(k,\ell) \in \text{INDEX}}} \frac{1}{L} \sigma_{k,\ell}^{\prime RB} \le (1 + \varepsilon^{1/4}) \operatorname{Tr}_A((\Lambda \otimes \mathbb{I}^{BR})(\rho))^{ABR} \tag{op-ineq}$$

Since  $\Pr[INDEX] \ge 1 - O(\varepsilon^{1/8})$ , it is not hard to see that

$$\|\tilde{\rho}^{KLRB} - \sigma^{KLRB}\|_{1} \le O(\varepsilon^{1/8})$$

Note that the definition of  $\sigma$  implies that, under the map f:

$$\sigma^X \le \frac{1}{1 - O(\varepsilon^{1/8})} \rho^X$$

since the trace of  $\sigma$  is at least  $1 - O(\varepsilon^{1/8})$ .

For the tools we develop in this section, we will not require the system R. Thus we will only with the control state  $\sigma^{KLB}$ . Note that the operator inequality still holds since trace out preserves operator inequalities. Eq. (op-ineq) can now be written as:

$$\sum_{\substack{\ell \in \text{GOOD}(k) \\ f(k,\ell) \in \text{INDEX}}} \frac{1}{L} \sigma_{k,\ell}^{'B} \le (1 + \varepsilon^{1/4}) \rho^B$$

Now, consider the following claim:

#### Claim 4.8. Control state:

$$\sigma^{KK'LB} \coloneqq \frac{1}{\operatorname{Tr}[\sigma]} \sum_{\substack{k \text{ is a nice} \\ \ell \in \operatorname{GOOD}(k) \\ f(k,\ell) \in \operatorname{INDEX}}} \frac{1}{L \cdot K} |k\rangle \langle k|^K \otimes |k\rangle \langle k|^{K'} \otimes |\ell\rangle \langle \ell|^L \otimes \sigma_{k,\ell}^{'B}$$

with the property that for every nice index k,

$$\sum_{\substack{\ell \in \text{GOOD}(k) \\ f(k,\ell) \in \text{INDEX}}} \frac{1}{L} \sigma_{k,\ell}^{'B} \le (1 + \varepsilon^{1/4}) \rho^{B}$$

and under the map f

$$\sigma^X \le \frac{1}{1 - O(\varepsilon^{1/8})} \rho^X$$

Then:

$$I_H^{\varepsilon_0}(KL:BK')_{\sigma} \ge \log K + I_H^{\varepsilon_0}(X:B)_{\rho^{XB}} - 1$$

where

$$\varepsilon_0 := \varepsilon^{1/10}$$

*Proof.* Let  $\Pi_{OPT}$  be the optimising operator in the definition of

$$D_H^{\varepsilon_0}(\sigma^{KLB} \mid\mid \sigma^{KL} \otimes \rho^B)$$

Without loss of generality we can assume that

$$\Pi_{\mathrm{OPT}} = \sum_{\substack{k \text{ is a nice} \\ \ell \in \mathrm{GOOD}(k) \\ f(k,\ell) \in \mathrm{INDEX}}} \left| k \right\rangle \left\langle k \right|^K \otimes \left| k \right\rangle \left\langle k \right|^{K'} \otimes \left| \ell \right\rangle \left\langle \ell \right|^L \otimes \Pi^B_{k,\ell}$$

For the purposed of brevity we will refer to the  $(k, \ell)$  which obey the conditions in the definition of  $\sigma$  as 'fine'. Then

$$\Pi_{ ext{OPT}}\sigma^{KL}\otimes\sigma^{K'B}=c_{0}\cdot\Pi_{ ext{OPT}}\left(\sum_{\left(k,\ell
ight) ext{ fine }}rac{1}{L\cdot K}\left|k
ight
angle\left\langle k
ight|^{K}\otimes\left|\ell
ight
angle\left\langle \ell
ight|^{L}
ight)\otimes\left(\sum_{k ext{ fine }}rac{1}{K}\left|k'
ight
angle\left\langle k'
ight|^{K'}\otimes\sigma_{k'}^{'B}
ight)$$

where

$$\sigma_k^{'B} \coloneqq \sum_{\ell \text{ fine}} \frac{1}{L} \sigma_{k,\ell}^{'B}$$

and

$$c_0 := \frac{1}{(\text{Tr}[\sigma])^2} \le \frac{1}{(1 - O(\varepsilon^{1/8}))^2} \le \frac{1}{1 - O(\varepsilon^{1/8})}$$

Then RHS is equal to

$$= c_{0} \cdot \sum_{(k,\ell) \text{ fine}} \frac{1}{K^{2} \cdot L} |k\rangle \langle k|^{K} \otimes |k\rangle \langle k|^{K'} \otimes |\ell\rangle \langle \ell|^{L} \otimes \Pi_{k,\ell} \sigma_{k}^{B}$$

$$\leq c_{0} \cdot (1 + \varepsilon^{1/4}) \sum_{(k,\ell) \text{ fine}} \frac{1}{K^{2} \cdot L} |k\rangle \langle k|^{K} \otimes |k\rangle \langle k|^{K'} \otimes |\ell\rangle \langle \ell|^{L} \otimes \Pi_{k,\ell} \rho^{B}$$

$$(1)$$

Then taking trace on both sides and using the definition of  $\Pi_{OPT}$  we get

$$\begin{split} 2^{-I_H^{\varepsilon_0}(KL:K'B)} \leq & c_0 \cdot (1+\varepsilon^{1/4}) \operatorname{Tr} \left[ \sum_{(k,\ell) \text{ fine}} \frac{1}{K^2 \cdot L} \left| k \right\rangle \left\langle k \right|^K \otimes \left| k \right\rangle \left\langle k \right|^{K'} \otimes \left| \ell \right\rangle \left\langle \ell \right|^L \otimes \Pi_{k,\ell} \rho^B \right] \\ = & c_0 \cdot (1+\varepsilon^{1/4}) \cdot 2^{-\log K} \cdot 2^{-D_H^{\varepsilon_0}(\sigma^{KLB} \parallel \sigma^{KL} \otimes \rho^B)} \\ \leq & c_0 \cdot (1+\varepsilon^{1/4}) \cdot 2^{-\log K} \cdot 2^{-D_H^{\varepsilon_0}(\sigma^{XB} \parallel \sigma^X \otimes \rho^B)} \end{split}$$

where the last inequality is via the data processing inequality.

Next, suppose that  $\Pi^{XB}$  is the optimising operator for

$$D_H^{\varepsilon_0}(\sigma^{XB} \mid\mid \sigma^X \otimes \rho^B)$$

Also observe that

$$\begin{split} \operatorname{Tr}[\Pi^{XB}\sigma^{XB}] &\geq \operatorname{Tr}[\Pi^{XB}\rho^{XB}] - \left\|\sigma^{XB} - \rho^{XB}\right\|_1 \\ &\geq 1 - \varepsilon_0 - O(\varepsilon^{1/8}) \\ &\geq 1 - \frac{\varepsilon_0}{2} \end{split}$$

where the last line is due to the fact that  $\varepsilon_0 = \varepsilon^{1/10}$ . This implies that

$$2^{-I_H^{\varepsilon_0}(KL:K'B)} \le c_0 \cdot (1+\varepsilon^{1/4}) \cdot 2^{-\log K} \cdot 2^{-D_H^{\varepsilon_0/2}(\rho^{XB} \parallel \sigma^X \otimes \rho^B)}$$

Finally, since we know that

$$\sigma^X \leq \frac{1}{1 - O(\varepsilon^{1/8})} \rho^X$$

one can see that

$$2^{-I_H^{\varepsilon_0}(KL:K'B)} \le c_0 \cdot (1 + \varepsilon^{1/4}) \cdot \frac{1}{1 - O(\varepsilon^{1/8})} \cdot 2^{-\log K} \cdot 2^{-D_H^{\varepsilon_0/2}(\rho^{XB} \parallel \rho^X \otimes \rho^B)}$$

$$= c_0 \cdot (1 + \varepsilon^{1/4}) \cdot \frac{1}{1 - O(\varepsilon^{1/8})} \cdot 2^{-\log K} \cdot 2^{-I_H^{\varepsilon_0/2}(X:B)_{\rho^{XB}}}$$

For small enough  $\varepsilon$ , we can sue the following bound

$$c_0 \cdot (1 + \varepsilon^{1/4}) \cdot \frac{1}{1 - O(\varepsilon^{1/8})} \le 2$$

Therefore, it is now clear that

$$I_H^{\varepsilon_0}(KL:K'B)_{\sigma} \ge \log K + I_H^{\varepsilon_0/2}(X:B)_{\rho^{XB}} - 1$$

This concludes the proof.

#### **Composing the Two Protocols:**

We will now use the claims proved above to show that the measurement compression theorem and the one-shot protocol for classical message compression with side information can be composed in the case when Bob has some side information. In that case, the actual post measurement state that we will work with is

$$\sigma^{KLK'RB} := \frac{1}{\text{Tr}[\sigma]} \sum \frac{1}{K \cdot L} \left| k \right\rangle \left\langle k \right|^K \otimes \left| k \right\rangle \left\langle k \right|^{K'} \otimes \left| \ell \right\rangle \left\langle \ell \right|^L \otimes \sigma_{k,\ell}^{'RB}$$

where the summation is over the appropriate set of  $(k, \ell)$  as defined before. We will treat the systems K' and B as side information available to Bob. One can now see that the arguments of Claim 4.8 and Theorem 3.8 together imply that Alice has to send at most the following number of bits:

$$H_{\max}^{\varepsilon_0}(KL)_{\sigma} - I_H^{\varepsilon_0}(KL:K'R)_{\sigma}$$

$$\leq H_{\max}^{\varepsilon_0}(KL) - \log K - I_H^{\varepsilon_0/2}(X:B)_{\rho^{XB}} + 1$$

$$\leq \log K + \log L - \log K - I_H^{\varepsilon_0/2}(X:B)_{\rho^{XB}} + 1$$

$$= \log L - I_H^{\varepsilon_0/2}(X:B)_{\rho^{XB}} + 1$$

From the measurement compression theorem, we know that

$$\log L > I_{\max}^{\varepsilon}(X:RB) - O(\log \varepsilon)$$

Thus, this implies that Alice needs to send at most

$$I_{\max}^{\varepsilon}(X:RB) - I_{H}^{\varepsilon_0/2}(X:B)_{\rho^{XB}} - O(\log \varepsilon) + 1$$

bits. This concludes the proof.

**Remark 4.9.** It is important to note that Claim 4.8 holds even when the control state is a weighted sum of the following form:

$$\sigma^{KK'LB} = \frac{1}{\text{Tr}[\sigma]} \sum \frac{w(k,\ell)}{L \cdot K} \left| k \right\rangle \left\langle k \right|^K \otimes \left| k \right\rangle \left\langle k \right|^{K'} \otimes \left| \ell \right\rangle \left\langle \ell \right|^L \otimes \sigma_{k,\ell}^{'B}$$

This is precisely the case that we will face when we use Claim 4.8 to prove Theorem 1.1, due to the presence of the terms  $t(k_1, k_2, \ell_1, \ell_2)$  (refer to the proof of Proposition 4.1). However, it is easy to verify that even in this case the proof of Claim 4.8 remains unchanged.

We are finally ready to prove Theorem 1.1.

#### 4.3 Proof of Theorem 1.1

*Proof.* The proof is not hard given the discussion in the previous sections. We will show that the region  $S_{\theta}$ , when  $\theta=0$  is achievable. This argument can then be extended to other values of  $\theta$  using the similar reasoning to what we used in proving Proposition 4.6. To show that  $S_0$  is achievable, we first invoke Proposition 4.1 to show that there exists a measurement compression scheme which achieves the rates given by  $S_0$  in the case when there is no side information. To be precise,  $S_0$  (in the case when the system  $S_0$  is trivial) is given by:

$$R_X > I_{\max}^{\varepsilon}(X:RB)$$

$$R_Y > I_{\max}^{\varepsilon}(Y:XRB)$$

$$C_X + R_X > H_{\max}^{\varepsilon}(X)$$

$$C_Y + R_Y > H_{\max}^{\varepsilon}(Y)$$

where we have ignored the additive  $O(\log \frac{1}{\varepsilon})$  factors. We will call the above region

$$S_0^{\rm unassisted}$$

Now when Bob possesses some quantum side information in the B system, we can repeat the proof of Proposition 4.7 independently for the X- and Y- channels. This is possible because of two reasons:

- 1. The encoding maps for the X- and Y- channels are independent by design.
- 2. Even when both channels are ON, Bob can first decode the X- channel and then decode the Y- channel sequentially, since the first decoding does not perturb the state in the B system too much. To be precise, this will lead to at most an additive error of some  $g(\varepsilon)$ , where  $g(\varepsilon)$  is  $\varepsilon$  to some rational power. This is due to the Gentle Measurement Lemma and the proof of Theorem 3.8.

An important point to note that is that we cannot do any better in the sum rate, even though we could potentially get even more savings by repeating the proof of Proposition 4.7 in the case when both channels are ON. This is because we require the protocol to be agnostic to the actions of the adversary. For example, suppose Alice were to send a bit string along the X- channel which was shorter than the one prescribed by our protocol. She would then have to compensate by sending more bits along the Y- channel. However, the adversary could then simply turn OFF the Y- channel, leading to a situation where it is not certain whether Bob can decode.

This is why, the rate region will look

$$S_0^{\rm unassisted} - (I_H^{\varepsilon_0}(X:B), I_H^{\varepsilon_0}(Y:B))$$

The operation  $S - (x_0, y_0)$  is defined as

$$S - (x_0, y_0) := \{(a - x_0, b - y_0) \mid (a, b) \in S\}$$

For other values of  $\theta$ , we can now split the distribution  $P_X$ , as in the proof of Proposition 4.6. Bob can then repeat his sequential decoding algorithm pretending as if there are three channels namely the U- channel, the Y- channel and the V- channel. This shows that the rate region claimed in the statement of Theorem 1.1 is indeed achievable. This concludes the proof.

## 5 Asymptotic IID Analysis

In this section we will prove Corollary 1.2:

*Proof.* As in the proof of Theorem 1.1, we first show that all points in the region

$$S_0^{\text{assisted}} = S_0^{\text{unassisted}} - (I_H^{\varepsilon_0}(X:B), I_H^{\varepsilon_0}(Y:B))$$

is achievable when only 1 copy of the state  $\rho^{AB}$  is available. Now, we use the quantum asymptotic equipartition property for each of the one shot quantities in this expression to show that, when the state available for measurement is  $\rho^{AB\otimes n}$ , with  $n\to\infty$ . Thus, after normalising by n and taking the limit, the region  $S_0^{\text{assisted}}$  is equivalent to:

$$R_X > I(X : RB) - I(X : B)$$
  
 $R_Y > I(Y : XRB) - I(Y : B)$   
 $C_X + R_X > H(X) - I(X : B)$   
 $C_Y + R_Y > H(Y) - I(Y : B)$ 

One can similarly prove that the following region, which we call  $S_1^{\text{assisted}}$ , corresponding to  $\theta = 1$  is also achievable:

$$R_X > I(X : YRB) - I(X : B)$$
  
 $R_Y > I(Y : RB) - I(Y : B)$   
 $C_X + R_X > H(X) - I(X : B)$   
 $C_Y + R_Y > H(Y) - I(Y : B)$ 

One can then use a time sharing argument to show that the full achievable region is as follows:

$$R_X > I(X : RB) - I(X : B)$$

$$R_Y > I(Y : RB) - I(Y : B)$$

$$R_X + R_Y > I(XY : RB) + I(X : Y)$$

$$-I(X : B) - I(Y : B)$$

$$C_X + R_X > H(X) - I(X : B)$$

$$C_Y + R_Y > H(Y) - I(Y : B)$$

This concludes the proof.

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