# On the Weight Spectrum Improvement of Pre-transformed Reed-Muller Codes and Polar Codes 

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#### Abstract

Pre-transformation with an upper-triangular matrix (including cyclic redundancy check (CRC), parity-check (PC) and polarization-adjusted convolutional (PAC) codes) improves the weight spectrum of Reed-Muller (RM) codes and polar codes significantly. However, a theoretical analysis to quantify the improvement is missing. In this paper, we provide asymptotic analysis on the number of low-weight codewords of the original and pre-transformed RM codes respectively, and prove that pretransformation significantly reduces low-weight codewords, even in the order sense. For polar codes, we prove that the average number of minimum-weight codewords does not increase after pre-transformation. Both results confirm the advantages of pretransformation.


## I. Introduction

Polar codes [1], invented by Arikan, are a great breakthrough in coding theory. As code length approaches infinity, the synthesized channels become either noiseless or purenoise. Channel polarization occurs under successive cancellation (SC) decoding, which has a low complexity. However, the performance of polar codes under SC decoding is poor at short to moderate block lengths.

To boost finited-length performance, a successive cancellation list (SCL) decoding algorithm was proposed [2]. As list size $L$ increases, the performance of SCL decoding approaches that of maximum-likehood (ML) decoding. But the ML performance of polar codes is still inferior due to low minimum distance. Consequently, concatenation of polar codes with CRC [3] and PC [4] were proposed to improve weight spectrum. In Arıkan's PAC codes [5], convolutional precoding and RM rate-profiling were applied to approach binary input additive white Gaussian noise (BIAWGN) dispersion bound [6] under large list decoding [7].

CRC-Aided (CA) polar, PC-polar, and PAC codes can be viewed as pre-transformed polar codes with upper-triangular transformation matrices [8]. In polar codes, frozen bits are all zeros, while in pre-transformed polar codes, traditional frozen bits are replaced by dynamically frozen bits [9], whose value depends on previous bits. It is proved that any upper-triangular pre-transformation does not reduce minimum distance [8]. In [10], efficient recursive formulas were proposed to calculate
the average weight spectrum of pre-transformed polar codes with polynomial complexity rather than exponential complexity.

In this paper, we simplify the recursive formulas in [10] through the monomial representation of row vectors. From [8] [10], low-weight codewords are induced by low-weight row vectors. We further prove that, low-weight codewords are mainly induced by a small fraction of low-weight row vectors. Based on this discovery, we provide asymptotic analysis on the number of low-weight codewords of pre-transformed codes, and quantitatively analyze the improvement of weight spectrum.

This paper is organized as follows. In section II, we review polar codes and pre-transformed polar codes. In section III, we analyze the number of low-weight codewords of the original and pre-transformed RM codes respectively. Asymptotic analysis shows that low-weight codewords reduce significantly after pre-transformation. For polar codes, we prove that the average number of minimum-weight codewords does not increase after pre-transformation, as long as the code is decreasing [11]. Finally we draw some conclusions in section IV.

## II. BaCkground

## A. Polar Codes as Monomial Codes

Let $F=\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right), N=2^{m}$, and $F_{N}=F^{\otimes m}$. Starting from $N$ binary-input discrete memoryless channels (B-DMC) $W$, we obtain $N$ synthetic channels $W_{N}^{(i)}$ after polarization. Polar codes can be constructed by selecting the indices of $K$ most reliable information sub-channels, i.e., $K$ row vectors of $F_{N}$, as information set $\mathcal{I}$. Density evolution (DE) algorithm [12], Gaussian approximation (GA) algorithm [13] and the channel-independent polarization weight (PW) construction method [14] are efficient methods to find reliable sub-channels.

After determining the information set $\mathcal{I}$, its complement set is called the frozen set $\mathcal{F}$. Let $u_{1}^{N}=\left(u_{1}, u_{2}, \ldots, u_{N}\right)$ be the bit sequence to be encoded. $K$ bits are inserted into $u_{\mathcal{I}}$, and
all zeros are filled into $u_{\mathcal{F}}$. Then the codeword $c_{1}^{N}$ is obtained by $c_{1}^{N}=u_{1}^{N} F_{N}$.

Polar codes can also be expressed as monomial codes [11] and the monomial set is denoted by

$$
\mathcal{M}_{m} \stackrel{\text { def }}{=}\left\{x_{m-1}^{a_{m-1}} \cdots x_{0}^{a_{0}} \mid\left(a_{m-1}, \ldots, a_{0}\right) \in \mathbf{F}_{2}^{m}\right\}
$$

From this point of view, each row vector of $F_{N}$ corresponds to a monomial represented by $m$ binary variables $\left\{x_{i}\right\}, 0 \leq$ $i \leq m-1$. For instance, a monomial $f=x_{i_{r-1}} \cdots x_{i_{0}}$, where $0 \leq i_{0}<\ldots<i_{r-1} \leq m-1$, the degree of $f$ is $r$, denoted by $\operatorname{deg}(f)$. Denote $I_{f}=i$, if the monomial $f$ represents the $i$-th row vector of $F_{N}$.

To be specific, let $\left(a_{m-1}, \ldots, a_{0}\right)$ be the binary representation of $N-i$, i.e., $N-i=\sum_{j=0}^{m-1} 2^{j} a_{j}$, then the $i$-th row vector of $F_{N}$ can be represented by monomial $x_{m-1}^{a_{m-1}} \cdots x_{0}^{a_{0}}$. For example, the monomial representation of $F_{8}$ is shown in Fig. 1 .

$$
F_{8}=\left[\begin{array}{llllllll}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right] \begin{array}{llll}
a_{2} & a_{1} & a_{0} \\
1 & 1 & 1 & x_{2} x_{1} x_{0} \\
1 & 1 & 0 & x_{2} x_{1} \\
1 & 0 & 1 & x_{2} x_{0} \\
1 & 0 & 0 & x_{2} \\
0 & 1 & 1 & x_{1} x_{0} \\
0 & 1 & 0 & x_{1} \\
0 & 0 & 1 & x_{0} \\
0 & 0 & 0 & 1
\end{array}
$$

Fig. 1. The monomial representation of $F_{8}$.
From now on, the $i$-th row vector of $F_{N}$ and the corresponding monomial $x_{m-1}^{a_{m-1}} \cdots x_{0}^{a_{0}}$ are used interchangeably because they refer to the same thing.

## B. Decreasing Monomial Codes

It was revealed in [11] and [15] that the reliability of synthetic channels follows a partial order " $\preceq$ ". If $f, g \in \mathcal{M}_{m}$, $g \preceq f$ means $g$ is universally more reliable than $f$. For monomials of the same degree, partial order is defined as

$$
x_{i_{r-1}} \cdots x_{i_{0}} \preceq x_{j_{r-1}} \cdots x_{j_{0}} \Longleftrightarrow i_{s} \leq j_{s}, 0 \leq s \leq r-1
$$

and for monomials of different degrees

$$
g \preceq f \Longleftrightarrow \exists f^{*} \mid f, \operatorname{deg}\left(f^{*}\right)=\operatorname{deg}(g), \text { and } g \preceq f^{*} .
$$

Denote the monomial code with information set $\mathcal{I}$ by $\mathcal{C}(\mathcal{I})$, $\mathcal{C}(\mathcal{I})$ is a decreasing monomial code if $\mathcal{I}$ satisfies partial order, i.e.,

$$
\forall f \in \mathcal{I} \text { and } g \in \mathcal{M}_{m} \text {, if } g \preceq f \text { then } g \in \mathcal{I} \text {. }
$$

For example, the information set of $\mathrm{RM}(m, r)$ consists of monomials with degree no larger than $r$. By definition, RM codes are decreasing monomial codes.

## C. Pre-Transformed Polar Codes

$$
T=\left[\begin{array}{cccc}
1 & T_{12} & \cdots & T_{1 N} \\
0 & 1 & \cdots & T_{2 N} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{array}\right] .
$$

Let $G_{N}=T F_{N}$ be the generator matrix of pre-transformed polar codes, where $T$ is an upper triangular pre-transformation matrix defined above. The codeword of the pre-transformed polar codes is given by $c_{1}^{N}=u_{1}^{N} G_{N}=u_{1}^{N} T F_{N}$. In PAC codes, the pre-transformation matrix $T$ is a Toeplitz matrix.
In [16], a deterministic algorithm for computing the weight spectrum of any given pre-transformed polar code was proposed. However, the computational complexity is still exponential. In [10], an efficient algorithm was proposed to compute the average weight spectrum of pre-transformed polar codes with polynomial complexity. The code ensemble assumes that $T_{i j}, 1 \leq i<j \leq N$ are i.i.d. Bernoulli( $\frac{1}{2}$ ) r.v..

## III. The number of Low-weight codewords

In this section, we provide asymptotic analysis on the number of low-weight codewords of the original and pretransformed RM codes respectively. We prove that, for decreasing polar codes, the average number of minimum-weight codewords after pre-transformation is no larger than that of the original codes.

## A. Notations and definitions

In this paper, $\log x$ is the base- 2 logarithm of $x,\lceil x\rceil$ is the ceiling function of $x$, and $w(x)$ is the Hamming weight of $x$. The entropy function $h(x)=-x \log x-(1-x) \log (1-x)$, $0<x<1$, and $|\mathcal{S}|$ is the cardinality of set $\mathcal{S}$. To characterize asymptotic results, we define the following notation ${ }^{1}$.

Let $f_{N}^{(i)}$ be the $i$-th row vector of $F_{N}$, and $g_{N}^{(i)}$ be the $i$ th row vector of $G_{N}$. Information set $\mathcal{I}=\left\{I_{1}, I_{2}, \ldots, I_{K}\right\}$, where $I_{1}<\ldots<I_{K}$. The number of codewords with Hamming weight $d$ of polar/RM codes is denoted by $N(d)$, and the number of codewords with Hamming weight no larger than $d$ is denoted by $A(d)$. The minimum distance is denoted by $d_{\text {min }}$. The corresponding number of codewords of pre-transformed codes with transformation matrix $T$ is denoted by $N(d, T)$ and $A(d, T)$, respectively. The average number is denoted by $E(N(d, T))$ and $E(A(d, T))$, where the expectation is with respect to random pre-transformation matrix $T$, and we assume $T_{i j}, 1 \leq i<j \leq N$ are i.i.d. Bernoulli( $\frac{1}{2}$ ) r.v..

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    \({ }^{1} f(x) \leq O(g(x))(f(x) \geq \Omega(g(x)))\), if \(\underset{x \rightarrow \infty}{\limsup } \frac{f(x)}{g(x)}<+\infty\)
\(\left(\liminf _{x \rightarrow \infty} \frac{f(x)}{g(x)}>0\right)\), where \(g(x)>0 . f(x)=\Theta(g(x))\), if \(0<\)
\(\liminf _{x \rightarrow \infty}\left|\frac{f(x)}{g(x)}\right| \leq \limsup _{x \rightarrow \infty}\left|\frac{f(x)}{g(x)}\right|<+\infty, f(x)=o(g(x))\), if
\(\lim _{x \rightarrow \infty}\left|\frac{f(x)}{g(x)}\right|=0\).
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Let $P(m, i, d) \stackrel{\text { def }}{=} P\left(w\left(g_{2^{m}}^{(i)}\right)=d\right)$ be the probability that the $i$-th row vector of $G_{N}$ has Hamming weight $d$. By [10, Lemma 2], $P(m, i, d)=0$, if $d<w\left(f_{N}^{(i)}\right)$, i.e., pretransformation does not reduce the Hamming weight of row vectors. According to [10, Lemma 1], the probability that the codeword $c_{1}^{N}=u_{1}^{N} G_{N}$ has Hamming weight $d$ is equal to $P\left(m, I_{j}, d\right)$, as long as $u_{I_{j}}$ is the first non-zero bit in $u_{1}^{N}$, thus we can combine the weight- $d$ codewords induced by these $2^{K-j}$ codewords in $E(N(d, T))$ whose first non-zero bit is $u_{I_{j}}$. Let $N\left(m, I_{j}, d\right) \stackrel{\text { def }}{=} 2^{K-j} P\left(m, I_{j}, d\right)$, according to [10, eq.(7)],

$$
\begin{align*}
E(N(d, T)) & =\sum_{\substack{1 \leq j \leq K \\
w\left(f_{I_{j}}\right) \leq d}} 2^{K-j} P\left(m, I_{j}, d\right) \\
& =\sum_{\substack{1 \leq j \leq K \\
w\left(f_{I_{j}}\right) \leq d}} N\left(m, I_{j}, d\right), \tag{1}
\end{align*}
$$

where $K-j$ is the number of information bits whose indices are greater than $I_{j}$. As explained above, $N\left(m, I_{j}, d\right)$ is the number of weight- $d$ codewords where $u_{I_{j}}$ is the first nonzero bit in the encoded bit sequence $u_{1}^{N}$. We call $N\left(m, I_{j}, d\right)$ the number of weight- $d$ codewords induced by the $I_{j}$-th row vector. From (1), all weight- $d$ codewords are induced by row vectors $f_{N}^{(i)}$ with weight no larger than $d$. Therefore, when analyzing the number of weight- $d$ codewords in pretransformed codes, we only need to consider the row vectors with weight no larger than $d$.

For convenience, we use $P\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, d\right)$ and $N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, d\right)$ instead of $P(m, i, d)$ and $N(m, i, d)$ when $x_{i_{r-1}} \cdots x_{i_{0}}$ represents the $i$-th row vector of $F_{N}$.

## B. Low-weight codewords of RM codes

In this section, we analyze low-weight codewords with Hamming weight within a constant multiple of minimum distance. We provide asymptotic analysis on the number of codewords in $\mathrm{RM}(m, r)$ with Hamming weight no larger than $2^{m-r+k}$, where $k$ is a non-negative integer. The proof idea of Theorem 1 follows from [17] and [18].

Theorem 1. Assume $0<\alpha_{1}<\frac{r}{m}<\alpha_{2}<1$, where $\alpha_{1}$, $\alpha_{2}$ are constants,

$$
\begin{equation*}
\Omega\left(m^{k+1}\right) \leq \log A\left(2^{m-r+k}\right) \leq O\left(m^{k+2}\right) \tag{2}
\end{equation*}
$$

Proof. The proof is in Appendix A
Remark 1. When $k=0, \log A\left(2^{m-r}\right)=\Theta\left(m^{2}\right)$ [11] reaches the upper bound of (2), and when $k=1, \log A\left(2^{m-r+1}\right)=$ $\Theta\left(m^{2}\right)$ [19] reaches the lower bound of (2).

## C. Minimum-weight codewords of pre-transformed RM codes

According to (11), the average number of minimum-weight codewords of pre-transformed $\operatorname{RM}(m, r)$ is

$$
\begin{align*}
& E\left(N\left(2^{m-r}, T\right)\right) \\
& 0 \leq i_{0}<\ldots<i_{r-1} \leq m-1 \tag{3}
\end{align*} N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right) .
$$

Thus we first analyze the number of minimum-weight codewords induced by $x_{i_{r-1}} \cdots x_{i_{0}}$.
Lemma 1. In $R M(m, r)$, the number of information bits whose indices are greater than $I_{x_{i_{r-1}} \cdots x_{i_{0}}}$ is $\sum_{s=0}^{r-1} \sum_{t=0}^{s+1}\binom{i_{s}}{t}$, and

$$
\begin{equation*}
\log P\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)=\sum_{s=0}^{r-1}\left(2^{i_{s}-s}-2^{i_{s}}\right) \tag{4}
\end{equation*}
$$

Thus $\log N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)=$

$$
\begin{equation*}
\sum_{s=0}^{r-1}\left(2^{i_{s}-s}-2^{i_{s}}+\sum_{t=0}^{s+1}\binom{i_{s}}{t}\right) \tag{5}
\end{equation*}
$$

Proof. We prove Lemma 1 via induction on $m$, the proof is in Appendix B

Remark 2. In Lemma 1 we simplify the recursive formulas in [10, Theorem 1] through the monomial representation of the $i$-th row vector of $F_{N}$, this simplified form is convenient for the further theoretical analysis. As seen, $P\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)$ holds for all sub-channel selections, thus (4) will also apply to polar codes.

Based on Lemma 1 we provide asymptotic analysis on $N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)$ as well as $E\left(N\left(2^{m-r}, T\right)\right)$.

## Theorem 2.

$$
\begin{align*}
& \log N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right) \leq \\
& \begin{cases}0 & i_{r-1} \geq r+3, \text { and } \\
& r \text { sufficiently large } ; \\
2 r+3 & i_{r-1}=r+2 \\
3 r & i_{r-1} \leq r+1\end{cases} \tag{6}
\end{align*}
$$

Assume $m-r \geq 2, \frac{r}{m}>\gamma$, where $\gamma>0$ is a constant, then

$$
\begin{equation*}
3 r \leq \log E\left(N\left(2^{m-r}, T\right)\right) \leq 3 r+O(\log r) \tag{7}
\end{equation*}
$$

Proof. The proof is in Appendix C We briefly introduce the proof outline below.

Firstly, we prove (6) by Step 1-2.
Step 1, to further calculate (5), let

$$
\begin{align*}
\mathcal{N}\left(i_{s}, s\right) & =2^{i_{s}-s}-2^{i_{s}}+\sum_{t=0}^{s+1}\binom{i_{s}}{t} \\
& =2^{i_{s}-s}-\sum_{t=0}^{i_{s}-s-2}\binom{i_{s}}{t} \tag{8}
\end{align*}
$$

we prove $\mathcal{N}\left(i_{s}, s\right) \leq 0$ if $s \geq 1, i_{s}-s \geq 3$. We analyze $\mathcal{N}\left(i_{s}, s\right)$ according to the value of $s$, the proof is mainly based on the estimation of combinatorial number.
Step 2, since $\log N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)=\sum_{s=0}^{r-1} \mathcal{N}\left(i_{s}, s\right)$, based on Step 1, we analyze $\log N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)$ with respect to $i_{r-1}$, the proof details can be found in Appendix C

Next, we prove (7) by Step 3.

Step 3, we divide the sum terms in (3) into three parts according to $i_{r-1}: \sum_{i_{r-1} \geq r+3} N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)$, $\sum_{i_{r-1}=r+2} N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)$ and
$\sum_{i_{r-1} \leq r+1} N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)$. Fro $\sum_{i_{r-1}<r+1} N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)$. From Step 2, the first term converges to zero, and the second term is negligible compared to the third term. Thus the minimum-weight codewords are mainly induced by $N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)$ with $i_{r-1} \leq r+1$, so

$$
\begin{align*}
& E\left(N\left(2^{m-r}, T\right)\right) \approx \sum_{i_{r-1} \leq r+1} N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right) \\
\leq & \left|\left\{x_{i_{r-1}} \cdots x_{i_{0}}, i_{r-1} \leq r+1\right\}\right| 2^{3 r}=\binom{r+2}{2} 2^{3 r} . \tag{9}
\end{align*}
$$

Remark 3. Since we can not efficiently calculate the weight spectrum of specific pre-transformed codes, we analyze the average weight spectrum of the code ensemble defined by the random pre-transformation matrix.

The results on the average weight spectrum are significant in two aspects. On the one hand, there exist good codes with minimum-weight codewords no larger than the average. On the other hand, numerical results confirm that, the actual number of minimum-weight codewords is usually very close to the average, i.e., has small variance. In practice, this means that most random pre-transformation matrices are good.
Remark 4. The $\binom{r+2}{2}$ monomials $x_{i_{r-1}} \cdots x_{i_{0}}$ with $i_{r-1} \leq r+1$ induce the majority of miminum-weight codewords of pre-transformed RM codes, which is a tiny part of $\binom{m}{r}$ monomials with degree $r$. It implies that in pre-transformed polar codes, the minimum-weight codewords are mainly induced by a small fraction of monomials. For example, in $\operatorname{RM}(9,2)$, the 498-th row vector $x_{4} x_{3} x_{2}$ satisfies $i_{r-1} \leq r+1$, its corresponding binary representation is $(0,0,0,0,1,1,1,0,0)$. Monomials $x_{i_{r-1}} \cdots x_{i_{0}}$ with $i_{r-1} \leq$ $r+1$ share similar characteristics: they are at the bottom of $F_{N}$ and have high reliability among monomials with degree $r$.

In Fig. 2, we display the number of minimum-weight codewords in RM codes and pre-transformed RM codes on the logarithm domain. The example has code rate $R=0.5$, and the average number of minimum-weight codewords is approximately $2^{3 r}$ in the order sense. In contrast, the number before pre-transformation is $2^{\Theta\left(m^{2}\right)}$. In other words, the logarithm scaling of minimum-weight codewords drops from quadratic growth to linear growth after pre-transformation. The result proves that pre-transformation can reduce minimum-weight codewords significantly, even in the order sense. This also partly explains the gain of PAC codes (a special case of pretransformed RM codes) over RM codes.


Fig. 2. Logarithm scaling of the number of minimum-weight codewords in RM codes and pre-transformed RM codes.

## D. Low-weight codewords of pre-transformed RM codes

In this section, we analyze low-weight codewords with Hamming weight within a constant multiple of minimum distance. We provide asymptotic analysis on the number of codewords in per-transformed $\mathrm{RM}(m, r)$ with Hamming weight no larger than $2^{m-r+k}$, where $k$ is a positive integer.

According to (1), we only need to consider row vectors with weight no larger than $2^{m-r+k}$, or equivalently, monomials with degree at least $r-k$. For monomials with degree $r-q$, where $0 \leq q<k$, their corresponding row vectors have weight $2^{m-r+q}<2^{m-r+k}$, thus they induce codewords with weight from $2^{m-r+q}$ to $2^{m-r+k}$. Let

$$
\begin{align*}
& A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r-k}\right) \\
= & \sum_{d^{\prime}=2^{m-r+q}} N\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, d^{\prime}\right), \tag{10}
\end{align*}
$$

$A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r-k}\right)$ is the number of codewords induced by $x_{i_{r-q-1}} \cdots x_{i_{0}}$ with weight no larger than $2^{m-r+k}$. For monomials with degree $r-k$, their corresponding row vectors have weight exactly $2^{m-r+k}$, thus we only need to consider the number of weight $-2^{m-r+k}$ codewords induced by $x_{i_{r-k-1}} \cdots x_{i_{0}}$. Therefore, we have

$$
\begin{align*}
& E\left(A\left(2^{m-r+k}, T\right)\right) \\
= & \sum_{q=0}^{k-1} \sum_{0 \leq i_{0}<\ldots<i_{r-q-1} \leq m-1} A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \\
& +\sum_{0 \leq i_{0}<\ldots<i_{r-k-1} \leq m-1} N\left(m, x_{i_{r-k-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) . \tag{11}
\end{align*}
$$

Next, we analyze $N\left(m, x_{i_{r-k-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right)$ and $A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right)$, where $0 \leq q<k$, and then provide asymptotic analysis on $E\left(A\left(2^{m-r+k}, T\right)\right)$.

Theorem 3. Let $k$ be a positive integer,

$$
\log N\left(m, x_{i_{r-k-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \leq
$$

$$
\begin{cases}0 & i_{r-k-1} \geq r+3, \text { and }  \tag{12}\\ & r \text { sufficiently large } \\ \left(2^{k+2}-2\right) r+O(1) & i_{r-k-1}=r+2 \\ \left(2^{k+2}-1\right) r & i_{r-k-1} \leq r+1\end{cases}
$$

Assume $m-r \geq 2, \frac{r}{m}>\gamma$, where $\gamma>0$ is a constant. Let $0 \leq q<k$,

$$
\begin{align*}
& \log A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \leq  \tag{13}\\
& \begin{cases}0 & i_{r-q-1} \geq r+3, \text { and } \\
& r \text { sufficiently large } ; \\
\left(2^{k+2}-1\right) r+\log r+O(1) & i_{r-q-1} \leq r+2 .\end{cases}
\end{align*}
$$

Therefore, we have

$$
\begin{align*}
\left(2^{k+2}-1\right)(r-k) & \leq \log E\left(A\left(2^{m-r+k}, T\right)\right) \\
& \leq\left(2^{k+2}-1\right) r+O(\log r) \tag{14}
\end{align*}
$$

Proof. The proof is in Appendix D The method is similar to that in Theorem 2 but due to the sum terms in the recursive formula [10, Theorem 2], the analysis is more complicated. In particular, we derive the upper bound on $A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right)$ when $i_{r-q-1} \leq r+2$ through induction.

In Fig. 3, we display the number of codewords with Hamming weight $2 d_{\text {min }}$ in RM codes and pre-transformed RM codes on the logarithm domain. The example has code rate $R=0.5$, and the average number of codewords is approximately $2^{\left(2^{k+2}-1\right) r}$ in the order sense. Similarly, the logarithm scaling of the weight- $2 d_{\text {min }}$ codewords grows linearly with $m$ under pre-transformation, as opposed to quadratically without pre-transformation. Our approximation is accurate asymptotically, and there is a gap between the true number and approximation when $m$ is small. Note that calculating the accurate number of weight $-2 d_{\min }$ codewords becomes intractable when $m$ is large.


Fig. 3. Logarithm scaling of the number of codewords with Hamming weight $2 d_{\text {min }}$ in RM codes and pre-transformed RM codes.

## E. Minimum-weight codewords of pre-transformed polar codes

In this section, we extend our analysis from RM codes to polar codes. We prove that the average number of minimumweight codewords of pre-transformed polar codes does not
increase after pre-transformation. Unlike RM codes, polar codes do not have a universal sub-channel selection rule. Therefore, their corresponding asymptotic results cannot be obtained as in RM codes. Fortunately, the conclusions in this section are non-asymptotic and apply to arbitrary code lengths.

Let $\mathcal{C}(\mathcal{I})$ be a decreasing polar code, define

$$
\begin{equation*}
\bar{r}=\min \{r \mid \mathcal{C}(\mathcal{I}) \subseteq \operatorname{RM}(m, r)\}, \tag{15}
\end{equation*}
$$

i.e., the largest degree of monomials in $\mathcal{I}$ is $\bar{r}$ and the minimum distance is $2^{m-\bar{r}}$. According to [11, Proposition 7], the number of minimum-weight codewords of $\mathcal{C}(\mathcal{I})$ is

$$
\begin{equation*}
N\left(2^{m-\bar{r}}\right)=\sum_{x_{i_{\bar{r}-1}} \cdots x_{i_{0}} \in \mathcal{I}} 2^{\sum_{s=0}^{\bar{r}-1} i_{s}-s+1} \tag{16}
\end{equation*}
$$

Similarly, we call $2^{\sum_{s=0}^{\bar{r}-1} i_{s}-s+1}$ the number of minimum-weight codewords induced by $x_{i_{\bar{r}-1}} \cdots x_{i_{0}}$ in original polar codes. Let $i^{*}$ be the smallest index in information bits which can be represented by a monomial with degree $\bar{r}$, i.e.

$$
\begin{equation*}
i^{*}=\min \left\{I_{x_{i_{\bar{r}}-1} \cdots x_{i_{0}}} \mid x_{i_{\bar{r}-1}} \cdots x_{i_{0}} \in \mathcal{I}\right\} . \tag{17}
\end{equation*}
$$

Next, we prove that pre-transformation does not increase the average number of minimum-weight codewords.
Theorem 4. If $\mathcal{C}(\mathcal{I})$ is a decreasing polar code, $\bar{r}, i^{*}$ are defined in (15), (17), we have

$$
\begin{equation*}
E\left(N\left(2^{m-\bar{r}}, T\right)\right) \leq N\left(2^{m-\bar{r}}\right) \tag{18}
\end{equation*}
$$

Let the monomial representation of the $i^{*}$-th row vector be $x_{i_{\bar{r}-1}^{*}} \cdots x_{i_{0}^{*}}$. If $\bar{r} \leq 1$, (18) must hold as an equality.
If ${ }_{\bar{T}}^{\bar{r}}>1$, (18) holds as an equality if and only if the following two conditions satisfy:
(*) $i_{\bar{r}-1}^{*} \leq \bar{r}+1$.
$(* *)$ If $I_{f}>i^{*}$ and $\operatorname{deg}(f) \leq \bar{r}$, then $f \in \mathcal{I}$.
Proof. The proof is in Appendix E In fact, the number of minimum-weight codewords induced by every $x_{i_{\bar{r}-1}} \cdots x_{i_{0}} \in$ $\mathcal{I}$ decreases after pre-transformation.

Remark 5. In fact, the number of minimum-weight codewords induced by every $x_{i_{\bar{r}-1}} \cdots x_{i_{0}} \in \mathcal{I}$ decreases after pre-transformation, but the amount of reduction differs. According to [16), the minimum-weight codewords in original polar codes are mainly induced by $x_{i_{\bar{r}-1}} \cdots x_{i_{0}}$ with large $\sum_{s=0}^{\bar{r}-1}\left(i_{s}-s\right)$. Since $i_{s}-s \leq i_{\bar{r}-1}-(\bar{r}-1), 0 \leq s \leq$ $\bar{r}-1$, these monomials also have large $i_{\bar{r}-1}$. As explained in Theorem 2, these codewords are reduced due to pretransformation, which explains why pre-transformation can improve the weight spectrum. Therefore, more $x_{i_{\bar{r}-1}} \cdots x_{i_{0}}$ with large $i_{\bar{r}-1}$ as information bits results in more significant improvement in the weight spectrum. The results prove that pre-transformation can be beneficial for polar codes too.

## IV. CONCLUSION

In this paper, we provide asymptotic analysis on the number of low-weight codewords of the original and pre-transformed RM codes respectively, and prove that pre-transformation can reduce the low-weight codewords significantly. For decreasing polar codes, we prove that pre-transformation does not increase the average number of minimum-weight codewords. The numerical results validate the theoretical analysis and confirm the benefit of pre-transformation.

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## Appendix A

Codewords in polar codes can be regarded as polynomials. Let $f, g$ be two codewords in $\operatorname{RM}(m, r)$, define $d(f, g)$ be the Hamming distance between $f$ and $g$. Define poly $(d)=\{f \in$ $\operatorname{RM}(m, r): \omega(f) \leq d\}$, i.e., polynomials in $\operatorname{RM}(m, r)$ with weight no larger than $d$. Define the degree of a polynomial $f$ to be the maximal degree of monomials in $f$, denoted by $\operatorname{deg}(f)$. Let $f$ be a polynomial and $y \in \mathbf{F}_{2}^{m}$. Define the derivative of $f$ in direction $y$ by

$$
\begin{equation*}
\Delta_{y} f(x)=f(x+y)+f(x) \tag{19}
\end{equation*}
$$

we have $\operatorname{deg}\left(\Delta_{y} f\right) \leq \operatorname{deg}(f)-1$. For example, let $m=2$, $f(x)=x_{0}+x_{0} x_{1}, y=(1,0)$. Then

$$
\begin{align*}
& \Delta_{y} f(x)=f(x+y)+f(x) \\
= & \left(x_{0}+1\right)+\left(x_{0}+1\right) x_{1}+x_{0}+x_{0} x_{1}=x_{1}+1 . \tag{20}
\end{align*}
$$

Here, $\operatorname{deg}\left(\Delta_{y} f\right)=1=\operatorname{deg}(f)-1$. Define the $k$-iterated derivative of $f$ in direction $Y=\left(y_{1}, \ldots, y_{k}\right) \in\left(\mathbf{F}_{2}^{m}\right)^{k}$ by

$$
\begin{equation*}
\Delta_{Y} f(x)=\Delta_{y_{1}} \Delta_{y_{2}} \cdots \Delta_{y_{k}} f(x) \tag{21}
\end{equation*}
$$

Since $\operatorname{deg}\left(\Delta_{y} f\right) \leq \operatorname{deg}(f)-1, \operatorname{deg}\left(\Delta_{Y} f\right) \leq \operatorname{deg}(f)-k$.
Let $\mathcal{S} \subseteq \operatorname{RM}(m, r)$ be a subset of polynomials, we call a subset of polynomials $\mathcal{B}$ is a $\delta$-net for $\mathcal{S}$ if $\forall f \in \mathcal{S}$, there exists $g \in \mathcal{B}$ such that $d(f, g) \leq \delta$.

Lemma 2. 17 Corollary 3.1] Let $t$ be an integer, define

$$
\begin{aligned}
\mathcal{A}_{r-k-1, t}= & \left\{\operatorname{Maj}\left(\Delta_{Y_{1}} f, \ldots, \Delta_{Y_{t}} f\right):\right. \\
& \left.Y_{1}, \ldots, Y_{t} \in\left(\mathbf{F}_{2}^{m}\right)^{r-k-1}, f \in \operatorname{RM}(m, r)\right\}
\end{aligned}
$$

where Maj is the majority function defined in [17]. Then $\mathcal{A}_{r-k-1, t}$ is a $\delta$-net for $\operatorname{poly}\left(2^{m-r+k}\right)$, where $t=$ $\left\lceil 17 \log \left(2^{m} / \delta\right)\right\rceil$.
Proof of Theorem [] Denote $h=m-r+k$. To prove the lower bound, assume $g\left(x_{0}, \ldots, x_{h-1}\right)$ be an arbitrary polynomial with degree $k+1$. Define

$$
f\left(x_{0}, \ldots, x_{m-1}\right)=\left(g\left(x_{0}, \ldots, x_{h-1}\right)+x_{h}\right) x_{h+1} \ldots x_{m-1}
$$

It is clear that $f \in \operatorname{RM}(m, r)$ and $w(f)=2^{h}$. The number of polynomials with $h$ variables and degree $k+1$ is $2^{\binom{h}{k+1}}=$ $2^{\Theta\left(m^{k+1}\right)}$, which implies the lower bound.

To prove the upper bound, let $\delta=2^{m-r-2}$, define $\operatorname{adj}(f)=$ $\left\{g \in \mathcal{A}_{r-k-1, t}: d(f, g) \leq \delta\right\}$, where $t=17(r+2)$. By Lemma 2. $\mathcal{A}_{r-k-1, t}$ is a $\delta$-net for poly $\left(2^{h}\right)$, thus $\forall f \in$ $\operatorname{poly}\left(2^{h}\right), \operatorname{adj}(f) \neq \emptyset$. Next, we prove for any two differernt $f_{1}, f_{2} \in \operatorname{poly}\left(2^{h}\right), \operatorname{adj}\left(f_{1}\right) \bigcap \operatorname{adj}\left(f_{2}\right)=\emptyset$, otherwise there exist $g \in \mathcal{A}_{r-k-1, t}$, such that $d\left(f_{1}, g\right) \leq \delta$ and $d\left(f_{2}, g\right) \leq \delta$. By triangle inequality, $d\left(f_{1}, f_{2}\right) \leq 2^{m-r-1}<d_{\text {min }}=2^{m-r}$, which is a contradiction. Notice that $\operatorname{deg}\left(\Delta_{Y} f\right) \leq k+1$, $\forall Y \in\left(\mathbf{F}_{2}^{m}\right)^{r-k-1}, f \in \operatorname{RM}(m, r)$, we have

$$
\begin{align*}
A\left(2^{h}\right) & \leq \sum_{f \in \operatorname{poly}\left(2^{h}\right)}|\operatorname{adj}(f)|=\left|\bigcup_{f \in \operatorname{poly}\left(2^{h}\right)} \operatorname{adj}(f)\right| \\
& \leq\left|\mathcal{A}_{r-k-1, t}\right| \leq 2^{t \sum_{s=0}^{k+1}\binom{m}{s}}=2^{\Theta\left(m^{k+2}\right)} \tag{22}
\end{align*}
$$

## Appendix B

Proof of Lemma $\square$ We prove Lemma 1 via induction on $m$. Firstly, if $m=1$, Lemma 1 can be proved directly. For the induction step $m-1 \rightarrow m$, we consider two cases according to $i_{r-1}$ :

1) $i_{r-1}=m-1$, i.e., $x_{i_{r-1}} \cdots x_{i_{0}}$ is in the top half of $F_{N}$. The number of information bits in the top half and whose indices are greater than $I_{x_{i_{r-1}} \cdots x_{i_{0}}}$ is equal to the number of information bits whose indices are greater than $I_{x_{i_{r-2}} \cdots x_{i_{0}}}$ in $\mathrm{RM}(m-1, r-1)$, which is $\sum_{s=0}^{r-2} \sum_{t=0}^{s+1}\binom{i_{s}}{t}$ by inductive hypothesis. The number of information bits in the lower half is $\sum_{t=0}^{r}\binom{m-1}{t}=\sum_{t=0}^{r}\binom{i_{r-1}}{t}$, thus the total number of information bits whose indices are greater than $I_{x_{i_{r-1}} \cdots x_{i_{0}}}$ is equal to $\sum_{s=0}^{r-1} \sum_{t=0}^{s+1}\binom{i_{s}}{t}$. According to [10, Theorem 1],

$$
\begin{align*}
& \log P\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right) \\
= & \log P\left(m-1, x_{i_{r-2}} \cdots x_{i_{0}}, 2^{m-r}\right)+2^{m-r}-2^{m-1} \\
= & \log P\left(m-1, x_{i_{r-2}} \cdots x_{i_{0}}, 2^{m-r}\right)+2^{i_{r-1}-(r-1)}-2^{i_{r-1}} \\
= & \sum_{s=0}^{r-1}\left(2^{i_{s}-s}-2^{i_{s}}\right) . \tag{23}
\end{align*}
$$

The last equality is due to inductive hypothesis.
2) $i_{r-1}<m-1$, i.e., $x_{i_{r-1}} \cdots x_{i_{0}}$ is in the lower half of $F_{N}$. The number of information bits whose indices are greater than $I_{x_{i_{r-1}} \cdots x_{i_{0}}}$ is equal to the number of information bits whose indices are greater than $I_{x_{i_{r-1}} \cdots x_{i_{0}}}$ in $\mathrm{RM}(m-1, r)$, which is $\sum_{s=0}^{r-1} \sum_{t=0}^{s+1}\binom{i_{s}}{t}$ by inductive hypothesis. According to [10, Theorem 1],

$$
\begin{aligned}
& \log P\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right) \\
= & \log P\left(m-1, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-1-r}\right) \\
= & \sum_{s=0}^{r-1}\left(2^{i_{s}-s}-2^{i_{s}}\right) .
\end{aligned}
$$

The last equality is due to inductive hypothesis.

## Appendix C

Proof of Theorem [2 Firstly, we analyze $\mathcal{N}\left(i_{s}, s\right)$ with respect to $s$, in fact, if $s \geq 1, i_{s}-s \geq 3, \mathcal{N}\left(i_{s}, s\right) \leq 0$ for sufficiently large $i_{s}$.
case 1: $s=0, \mathcal{N}\left(i_{0}, 0\right)=1+i_{0}$.
case 2: $s=1, \mathcal{N}\left(i_{1}, 1\right)=-2^{i_{1}-1}+1+i_{1}+\binom{i_{1}}{2}$, and we have $\mathcal{N}\left(i_{1}, 1\right) \leq 0$ if $i_{1} \geq 5$.
case 3: $2 \leq s \leq\left\lceil\frac{i_{s}}{2}\right\rceil-2$,

$$
\begin{align*}
\mathcal{N}\left(i_{s}, s\right) & \leq 2^{i_{s}-2}-2^{i_{s}}+\sum_{t=0}^{\left\lceil\frac{i_{s}}{2}\right\rceil-1}\binom{i_{s}}{t} \\
& \leq-\frac{3}{4} 2^{i_{s}}+\frac{1}{2} 2^{i_{s}}=-\frac{1}{4} 2^{i_{s}} \leq 0 . \tag{24}
\end{align*}
$$

case 4: $\left\lceil\frac{i_{s}}{2}\right\rceil-2 \leq s \leq i_{s}-\log \left(i_{s}+16 \sqrt{2 i_{s}}\right)$,

$$
\begin{align*}
& \mathcal{N}\left(i_{s}, s\right)=2^{i_{s}-s}-\sum_{t=0}^{i_{s}-s-2}\binom{i_{s}}{t} \leq 2^{i_{s}-s}-\binom{i_{s}}{i_{s}-s-2} \\
& \stackrel{(a)}{\leq} 2^{i_{s}-s}-\frac{2^{i_{s} h\left(\frac{i_{s}-s-2}{i_{s}}\right)}}{\sqrt{2 i_{s}}} \stackrel{(b)}{\leq} 2^{i_{s}-s}-\frac{2^{2\left(i_{s}-s-2\right)}}{\sqrt{2 i_{s}}} \\
&=\left(1-\frac{2^{i_{s}-s}}{16 \sqrt{2 i_{s}}}\right) 2^{i_{s}-s} \leq\left(1-\frac{i_{s}+16 \sqrt{2 i_{s}}}{16 \sqrt{2 i_{s}}}\right) 2^{i_{s}-s} \\
& \leq-\frac{i_{s}^{\frac{3}{2}}}{16 \sqrt{2}}-i_{s} \leq-\frac{i_{s}^{\frac{3}{2}}}{16 \sqrt{2}} \leq 0 \tag{25}
\end{align*}
$$

where $(a)$ is from [20, problem 5.8], (b) is due to $\frac{i_{s}-s-2}{i_{s}} \leq \frac{1}{2}$ when $s \geq\left\lceil\frac{i_{s}}{2}\right\rceil-2$ and $h(x) \geq 2 x, 0<x \leq \frac{1}{2}$.
case 5: $i_{s}-\log \left(i_{s}+16 \sqrt{2 i_{s}}\right) \leq s \leq i_{s}-4$,

$$
\begin{align*}
\mathcal{N}\left(i_{s}, s\right) & \leq i_{s}+16 \sqrt{2 i_{s}}-\sum_{t=0}^{2}\binom{i_{s}}{t} \\
& =16 \sqrt{2 i_{s}}-1-\frac{i_{s}\left(i_{s}-1\right)}{2} \stackrel{(c)}{\leq} 0 \tag{26}
\end{align*}
$$

(c) holds when $i_{s} \geq 14$.
case 6: $s=i_{s}-3, \mathcal{N}\left(i_{s}, i_{s}-3\right)=7-i_{s} \stackrel{(d)}{\leq} 0,(d)$ holds when $i_{s} \geq 7$.
case 7: $i_{s}-2 \leq s \leq i_{s}, \mathcal{N}\left(i_{s}, s\right) \leq 3$.
We conclude that $\mathcal{N}\left(i_{s}, s\right) \leq 0$ if $i_{s} \geq 14, i_{s}-s \geq 3$, $s \geq 1$ from the discussion above. When $i_{s} \leq 13, s \geq 1$, through compute search, we have $\mathcal{N}\left(i_{s}, s\right) \leq 3$. Therefore,

$$
\begin{equation*}
\mathcal{N}\left(i_{s}, s\right) \leq 3, \text { if } s \geq 1 \tag{27}
\end{equation*}
$$

Next, we analyze $\log N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)=$ $\sum_{s=0}^{r-1} \mathcal{N}\left(i_{s}, s\right)$ with respect tof $i_{r-1}$.

1) $i_{r-1} \leq r+1$, we have $i_{s}-s \leq i_{r-1}-(r-1) \leq 2$, by case 7,

$$
\begin{equation*}
\log N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right) \leq 3 r \tag{28}
\end{equation*}
$$

2) $i_{r-1}=r+2$, we have $i_{0} \leq i_{r-1}-(r-1) \leq 3$, thus $\mathcal{N}\left(i_{0}, 0\right)=1+i_{0} \leq 4$, by (27),

$$
\begin{align*}
& \log N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right) \\
= & \mathcal{N}(r+2, r-1)+\sum_{s=1}^{r-2} \mathcal{N}\left(i_{s}, s\right)+\mathcal{N}\left(i_{0}, 0\right) \\
\leq & 7-(r+2)+3(r-2)+4 \\
= & 2 r+3 \tag{29}
\end{align*}
$$

3) $i_{r-1} \geq r+3$, according to case $3-5$, when $r$ is sufficiently large, we have

$$
\begin{equation*}
\mathcal{N}\left(i_{r-1}, r-1\right) \leq-\frac{i_{r-1}^{\frac{3}{2}}}{16 \sqrt{2}} \leq-\frac{r^{\frac{3}{2}}}{16 \sqrt{2}} \tag{30}
\end{equation*}
$$

Since $i_{0} \leq i_{1}-1$,

$$
\begin{align*}
& \mathcal{N}\left(i_{1}, 1\right)+\mathcal{N}\left(i_{0}, 0\right)=-2^{i_{1}-1}+2+i_{1}+i_{0}+\binom{i_{1}}{2} \\
\leq & -2^{i_{1}-1}+1+2 i_{1}+\binom{i_{1}}{2} \leq 7 \tag{31}
\end{align*}
$$

we prove the last inequality through computer search. Thus

$$
\begin{align*}
& \log N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right) \\
\leq & -\frac{r^{\frac{3}{2}}}{16 \sqrt{2}}+3(r-3)+7 \stackrel{(e)}{\leq}-\frac{r^{\frac{3}{2}}}{32} \leq 0 \tag{32}
\end{align*}
$$

(e) holds when $r$ is sufficiently large. Thus (6) is proved from (28) (29) (32).

Now we are ready to prove (7). On the one hand, $x_{r+1} \cdots x_{2} \in \operatorname{RM}(m, r)$ when $m-r \geq 2$, therefore

$$
\begin{equation*}
\log E\left(N\left(2^{m-r}, T\right)\right) \geq \log N\left(m, x_{r+1} \cdots x_{2}, 2^{m-r}\right)=3 r \tag{33}
\end{equation*}
$$

On the other hand,

$$
\begin{align*}
& \quad \sum_{i_{r-1} \leq r+1} N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right) \\
& \leq\left|\left\{x_{i_{r-1}} \cdots x_{i_{0}}, i_{r-1} \leq r+1\right\}\right| 2^{3 r}=\binom{r+2}{2} 2^{3 r} .  \tag{34}\\
& \sum_{i_{r-1}=r+2} N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right)  \tag{35}\\
& \leq\left|\left\{x_{i_{r-1}} \cdots x_{i_{0}}, i_{r-1}=r+2\right\}\right| 2^{2 r+3}=\binom{r+2}{3} 2^{2 r+3} . \\
& \quad \sum_{i_{r-1} \geq r+3} N\left(m, x_{i_{r-1}} \cdots x_{i_{0}}, 2^{m-r}\right) \\
& \quad(f) \\
& \quad \left\lvert\,\left\{\left\{x_{i_{r-1}} \cdots x_{i_{0}}, i_{r-1} \geq r+3\right\} \left\lvert\, 2^{-\frac{r^{\frac{3}{2}}}{32}} \leq 2^{m-\frac{r^{\frac{3}{2}}}{32}}\right.\right.\right.  \tag{36}\\
& \leq 2^{\frac{r}{\gamma}-\frac{r^{2}}{32}} \stackrel{(g)}{<} 1,
\end{align*}
$$

where $(f)$ holds if (32) holds, $(g)$ holds if $r \geq\left(\frac{32}{\gamma}\right)^{2}$. Divide the sum terms in (3) into three parts according to $i_{r-1}$, by (34)-(36),

$$
\begin{align*}
& E\left(N\left(2^{m-r}, T\right)\right) \\
\leq & \binom{r+2}{2} 2^{3 r}+\binom{r+2}{3} 2^{2 r+3}+1 \\
= & \binom{r+2}{2} 2^{3 r}(1+o(1)),  \tag{37}\\
& \log E\left(N\left(2^{m-r}, T\right)\right) \leq 3 r+\log \binom{r+2}{2}+o(1) . \tag{38}
\end{align*}
$$

Combine (33) and (38), we complete the proof of (7).

## Appendix D

## Proof outline:

Firstly, we prove (12) by Step 1-2, the proof of (12) is similar to that of (6), and is omitted due to space limitation. Step 1, let

$$
\begin{align*}
\mathcal{N}_{k}\left(i_{s}, s\right) & =2^{i_{s}-s}-2^{i_{s}}+\sum_{t=0}^{s+k+1}\binom{i_{s}}{t} \\
& =2^{i_{s}-s}-\sum_{t=0}^{i_{s}-s-k-2}\binom{i_{s}}{t} \tag{39}
\end{align*}
$$

we prove $\mathcal{N}_{k}\left(i_{s}, s\right) \leq 0$ if $s \geq 1, i_{s}-s \geq k+3$.
Step 2, similar to (5), we have

$$
\begin{equation*}
\log N\left(m, x_{i_{r-k-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right)=\sum_{s=0}^{r-k-1} \mathcal{N}_{k}\left(i_{s}, s\right), \tag{40}
\end{equation*}
$$

and we analyze $\log N\left(m, x_{i_{r-k-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right)$ with respect to $i_{r-k-1}$.

Next, we prove (13) by Step 3-5.
Step 3, let

$$
\begin{align*}
& \mathcal{N}_{k, q}\left(i_{s}, s\right) \stackrel{\text { def }}{=} \\
& \max _{2^{i_{s}-s} \leq d^{\prime} \leq 2^{i_{s}-s+k-q}}\left(2^{i_{s}}-d^{\prime}\right)\left(h\left(\frac{2^{i_{s}-s+k-q}-d^{\prime}}{2\left(2^{i_{s}}-d^{\prime}\right)}\right)-1\right) \\
& +\sum_{t=0}^{q+s+1}\binom{i_{s}}{t}+i_{s}-s+k-q \tag{41}
\end{align*}
$$

where $s \geq k-q+1$, by (45) and (46), we prove

$$
\begin{align*}
& \log A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \\
\leq & m+\sum_{s=0}^{k-q}\left(\sum_{t=0}^{q+s+1}\binom{i_{s}}{t}+i_{s}\right)+\sum_{s=k-q+1}^{r-q-1} \mathcal{N}_{k, q}\left(i_{s}, s\right) \tag{42}
\end{align*}
$$

via induction on $m$.
Step 4, we prove $\mathcal{N}_{k, s}\left(i_{s}, s\right) \leq 0$ when $i_{s}-s$ is large. Therefore, if $i_{r-q-1} \geq r+3$, we have $\log A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \leq 0$.

The proof of Step 3-4 is omitted due to space limitation.
Step 5, when $i_{r-q-1} \leq r+2$, we prove

$$
\begin{aligned}
& \log A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \\
\leq & \left(2^{k+2}-1\right) r+\log r+O(1)
\end{aligned}
$$

via induction.
Finally, we prove (14) by Step 6.
Step 6, based on Step 4-5, we have

$$
\begin{align*}
& E\left(A\left(2^{m-r+k}, T\right)\right) \\
\approx & \sum_{q=0}^{k-1} \sum_{i_{r-q-1} \leq r+2} A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \\
& +\sum_{i_{r-k-1} \leq r+2} N\left(m, x_{i_{r-k-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) . \tag{43}
\end{align*}
$$

Combine (12) and (13), we complete the proof of (14).
Proof of Theorem $3 \forall 0 \leq q<k$,
$N\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, d\right)$ is the number of weight- $d$ codewords induced by $x_{i_{r-q-1}} \cdots x_{i_{0}}$ in $\operatorname{RM}(m, r)$, we have

$$
\begin{align*}
& \log N\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, d\right) \\
= & \log P\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, d\right)+\sum_{s=0}^{r-q-1} \sum_{t=0}^{s+q+1}\binom{i_{s}}{t} . \tag{44}
\end{align*}
$$

According to [10, Theorem 2], if $i_{r-q-1}=m-1$,

$$
\begin{align*}
& N\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, d\right) \\
= & \sum_{\substack{d^{\prime}=2^{m-r+q} \\
d-d^{\prime} i_{\text {even }}}}^{d}\left(N\left(m-1, x_{i_{r-q-2}} \cdots x_{i_{0}}, d^{\prime}\right) *\right. \\
& \left.2^{\left(d^{\prime}-\sum_{t=0}^{m-r-2}\binom{m-1}{t}\right)} *\binom{2^{m-1}-d^{\prime}}{\frac{d-d^{\prime}}{2}}\right) . \tag{45}
\end{align*}
$$

If $i_{r-q-1}<m-1$,

$$
\begin{equation*}
N\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, d\right)=N\left(m-1, x_{i_{r-q-1}} \cdots x_{i_{0}}, \frac{d}{2}\right) . \tag{46}
\end{equation*}
$$

We are going to prove $\log A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \leq$ $\left(2^{k+2}-1\right) r+\log r+O(1)$ when $i_{r-q-1} \leq r+2$.
Let $i_{r-q-1}=r-q-1+\ell$, where $0 \leq \ell \leq q+3$, apply (46) $m-r+q-\ell$ times repeatedly, we have

$$
\begin{align*}
& N\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, d\right) \\
= & N\left(r-q+\ell, x_{i_{r-q-1}} \cdots x_{i_{0}}, \frac{d}{2^{m-r+q-\ell}}\right) \tag{47}
\end{align*}
$$

where $2^{m-r+q} \leq d \leq 2^{m-r+k}$. (47) must be zero unless $\frac{d}{2^{m-r+q-\ell}}=2^{\ell}+2 v$, where $v$ is a non-negtive integer, therefore $N\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, d\right)=$

$$
\left\{\begin{array}{l}
N\left(r-q+\ell, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{\ell}+2 v\right)  \tag{48}\\
d=\left(2^{\ell}+2 v\right) 2^{m-r+q-\ell}, v \geq 0 \\
0 \quad \text { otherwise }
\end{array}\right.
$$

Next, let $C_{0}=1, C_{v}=2^{v-1}, v \geq 1$, we are going to prove that if $i_{r-q-1}=r-q-1+\ell, 0 \leq \ell \leq q+1$,

$$
\begin{align*}
& N\left(r-q+\ell, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{\ell}+2 v\right) \\
\leq & C_{v}(r-q+\ell) 2^{\left(2^{\ell}+2 v\right)(r-q+\ell)} \tag{49}
\end{align*}
$$

via induction on $\ell, v$ and $r-q$, the degree of the monomial $x_{i_{r-q-1}} \cdots x_{i_{0}}$.
When $\ell=0$, we prove (49) holds via induction on $v$ and $r-q$. When $\ell \geq 1$, in addition to induction on $v$ and $r-q$, we also use the inductive hypothesis that (49) holds from 0 to $\ell-1, \forall v \geq 0, r-q \geq 0$.

If $v=0$, by 40),

$$
\begin{align*}
& \log N\left(r-q+\ell, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{\ell}\right)=\sum_{s=0}^{r-q-1} \mathcal{N}_{q}\left(i_{s}, s\right) \\
= & \sum_{s=0}^{r-q-1}\left(2^{i_{s}-s}-\sum_{t=0}^{i_{s}-s-q-2}\binom{i_{s}}{t}\right) \leq 2^{\ell}(r-q) \tag{50}
\end{align*}
$$

where the last inequality is due to $i_{s}-s \leq i_{r-q-1}-(r-q-$ $1)=\ell$. Therefore,

$$
\begin{equation*}
N\left(r-q+\ell, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{\ell}\right) \leq C_{0}(r-q+\ell) 2^{2^{\ell}(r-q+\ell)} \tag{51}
\end{equation*}
$$

For the induction step $v-1 \rightarrow v$, denote $r-q=n$ for convience, we complete the induction step via induction on $n$.

When $n=0, v \geq 1$, no codeword has Hamming weight $2^{\ell}+2 v$ which is larger than the code length $2^{\ell}$, thus

$$
\begin{equation*}
N\left(\ell, \mathbf{1}, 2^{\ell}+2 v\right)=0 \leq C_{v} \ell 2^{\left(2^{\ell}+2 v\right) \ell} \tag{52}
\end{equation*}
$$

where 1 represents the monomial with degree 0 .
For the induction step $n-1 \rightarrow n$, by (45),

$$
\begin{align*}
& N\left(n+\ell, x_{i_{n-1}} \cdots x_{i_{0}}, 2^{\ell}+2 v\right) \\
= & \sum_{\mu=0}^{v}\left(N\left(n-1+\ell, x_{i_{n-2}} \cdots x_{i_{0}}, 2^{\ell}+2 \mu\right) *\right. \\
& \left.2^{\left(2^{\ell}+2 \mu-\sum_{t=0}^{\ell-q-2}\binom{n-1+\ell}{t}\right)} *\binom{2^{n-1+\ell}-\left(2^{\ell}+2 \mu\right)}{v-\mu}\right) \tag{53}
\end{align*}
$$

$$
\begin{align*}
\stackrel{(h)}{\leq} & \sum_{\mu=0}^{v}\left(N\left(n-1+\ell, x_{i_{n-2}} \cdots x_{i_{0}}, 2^{\ell}+2 \mu\right) *\right. \\
& \left.2^{2^{\ell}+2 \mu+(v-\mu)(n+\ell)}\right), \tag{54}
\end{align*}
$$

where $(h)$ is from $\binom{2^{n-1+\ell}-\left(2^{\ell}+2 \mu\right)}{v-\mu} \leq 2^{(v-\mu)(n+\ell)}$, and $\sum_{t=0}^{\ell-q-2}\binom{n-1+\ell}{t}=0$ since $\ell \leq q+1$.
If $i_{n-2}=n-2+\ell$, by inductive hypo

If $i_{n-2}=n-2+\ell$, by inductive hypothesis on $n-1$,

$$
\begin{align*}
& N\left(n-1+\ell, x_{i_{n-2}} \cdots x_{i_{0}}, 2^{\ell}+2 \mu\right) \\
\leq & C_{\mu}(n-1+\ell) 2^{\left(2^{\ell}+2 \mu\right)(n-1+\ell)} . \tag{55}
\end{align*}
$$

If $i_{n-2}=n-2+\bar{\ell}, \bar{\ell}<\ell$, apply (46) $\ell-\bar{\ell}$ times repeatedly,

$$
\begin{align*}
& N\left(n-1+\ell, x_{i_{n-2}} \cdots x_{i_{0}}, 2^{\ell}+2 \mu\right) \\
= & N\left(n-1+\bar{\ell}, x_{i_{n-2}} \cdots x_{i_{0}}, 2^{\bar{\ell}}+2 \frac{\mu}{2^{\ell-\bar{\ell}}}\right) . \tag{56}
\end{align*}
$$

If $\frac{\mu}{2^{\ell-\bar{\ell}}}$ is an integer, by inductive hypothesis on $\bar{\ell}$,

$$
\begin{align*}
& N\left(n-1+\bar{\ell}, x_{i_{n-2}} \cdots x_{i_{0}}, 2^{\bar{\ell}}+2 \frac{\mu}{2^{\ell-\bar{\ell}}}\right) \\
\leq & C \frac{\mu}{2^{\ell-\bar{\ell}}}(n-1+\bar{\ell}) 2^{\left(2^{\bar{\ell}}+2 \frac{\mu}{2^{\ell-\bar{\ell}}}\right)(n-1+\bar{\ell})} \\
\leq & C_{\mu}(n-1+\ell) 2^{\left(2^{\ell}+2 \mu\right)(n-1+\ell)}, \tag{57}
\end{align*}
$$

otherwise

$$
\begin{equation*}
N\left(n-1+\bar{\ell}, x_{i_{n-2}} \cdots x_{i_{0}}, 2^{\bar{\ell}}+2 \frac{\mu}{2^{\ell-\bar{\ell}}}\right)=0 \tag{58}
\end{equation*}
$$

Combine (55)-(58), we have

$$
\begin{align*}
& N\left(n-1+\ell, x_{i_{n-2}} \cdots x_{i_{0}}, 2^{\ell}+2 \mu\right) \\
\leq & C_{\mu}(n-1+\ell) 2^{\left(2^{\ell}+2 \mu\right)(n-1+\ell)} . \tag{59}
\end{align*}
$$

Continue the proof in (54), we have

$$
\begin{align*}
& N\left(n+\ell, x_{i_{n-1}} \cdots x_{i_{0}}, 2^{\ell}+2 v\right) \\
\leq & \sum_{\mu=0}^{v-1} C_{\mu}(n-1+\ell) 2^{\left(2^{\ell}+v+\mu\right)(n+\ell)} \\
& +C_{v}(n-1+\ell) 2^{\left(2^{\ell}+2 v\right)(n+\ell)} \\
= & \left(\sum_{\mu=0}^{v-1} C_{\mu} \frac{n-1+\ell}{2^{(v-\mu)(n+\ell)}}+C_{v}(n-1+\ell)\right) 2^{\left(2^{\ell}+2 v\right)(n+\ell)} \\
\leq & \left(\sum_{\mu=0}^{v-1} C_{\mu} \frac{n+\ell}{2^{n+\ell}}+C_{v}(n-1+\ell)\right) 2^{\left(2^{\ell}+2 v\right)(n+\ell)} \\
\leq & \left(\sum_{\mu=0}^{v-1} C_{\mu}+C_{v}(n-1+\ell)\right) 2^{\left(2^{\ell}+2 v\right)(n+\ell)} \\
= & \left(1+\sum_{\mu=1}^{v-1} 2^{\mu-1}+2^{v-1}(n-1+\ell)\right) 2^{\left(2^{\ell}+2 v\right)(n+\ell)} \\
= & C_{v}(n+\ell) 2^{\left(2^{\ell}+2 v\right)(n+\ell)}, \tag{60}
\end{align*}
$$

the induction step $n-1 \rightarrow n$ holds, thus we complete the proof of (49).

Therefore, when $i_{r-q-1}=r-q-1+\ell, \ell \leq q+1$,

$$
\begin{align*}
& A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \\
&= \sum_{d=2^{m-r+q}}^{2^{m-r+k}} N\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, d\right) \\
& \stackrel{(i)}{2^{k-q+\ell-1}-\left\lceil 2^{\ell-1}\right\rceil} \sum_{v=0} N\left(r-q+\ell, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{\ell}+2 v\right) \\
& \leq \sum_{v=0}^{2^{k-q+\ell-1}-\left\lceil 2^{\ell-1}\right\rceil} C_{v}(r-q+\ell) 2^{\left(2^{\ell}+2 v\right)(r-q+\ell)} \tag{61}
\end{align*}
$$

$$
\begin{align*}
& \stackrel{(j)}{\leq}\left(2^{k-q+\ell-1}-\left\lceil 2^{\ell-1}\right\rceil+1\right) * \\
& \quad C_{2^{k-q+\ell-1}}(r-q+\ell) 2^{2^{k-q+\ell}(r-q+\ell)}  \tag{62}\\
& \stackrel{(k)}{\leq} 2^{k+2} C_{2^{k+2}}(r+3) 2^{\left(2^{k+2}-1\right)(r+3)}, \tag{63}
\end{align*}
$$

where $(i)$ is due to (48). In $(j)$, since the sum terms in 61) are increasing with respect to $v, \forall v \leq 2^{k-q+\ell-1}-\left\lceil 2^{\ell-1}\right\rceil$,

$$
\begin{align*}
& C_{v}(r-q+\ell) 2^{\left(2^{\ell}+2 v\right)(r-q+\ell)} \\
\leq & C_{2^{k-q+\ell-1}-\left\lceil 2^{\ell-1}\right\rceil}(r-q+\ell) 2^{2^{k-q+\ell+2^{\ell}-2\left\lceil 2^{\ell-1}\right.}(r-q+\ell)} \\
\leq & C_{2^{k-q+\ell-1}}(r-q+\ell) 2^{2^{k-q+\ell}(r-q+\ell)} . \tag{64}
\end{align*}
$$

$(k)$ is due to $\ell \leq q+1$, and we take (63) as an upper bound on (62) independent of $\ell$ for the convenience of the following analysis.

If $i_{r-q-1}=r-q-1+\ell, q+2 \leq \ell \leq q+3$, similar results can be proved via induction, due to space limitation, we only provide inductive hypothesis when $q+2 \leq \ell \leq q+3$ without proof.

$$
\text { If } \ell=q+2, i_{r-q-1}=r+1
$$

$$
\begin{align*}
& N\left(r+2, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{q+2}+2 v\right) \\
\leq & C_{v}(r+2) 2^{\left(2^{q+2}+2 v-1\right)(r+2)} . \tag{65}
\end{align*}
$$

The only difference between the proof of (65) and (49) is that in (53), $\sum_{t=0}^{\ell-q-2}\binom{n-1+\ell}{t}=1$ when $\ell=q+2$.

Similar to (63), we have

$$
\begin{align*}
& A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \\
\leq & 2^{k+1} C_{2^{k+1}-2^{q+1}}(r+2) 2^{\left(2^{k+2}-1\right)(r+2)} \\
\leq & 2^{k+2} C_{2^{k+2}}(r+3) 2^{\left(2^{k+2}-1\right)(r+3)} . \tag{66}
\end{align*}
$$

If $\ell=q+3, i_{r-q-1}=r+2$,

$$
\begin{align*}
& N\left(r+3, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{q+3}+2 v\right) \\
\leq & C_{v}(r+3) 2^{\left(2^{q+2}+v-1\right)(r+3)} . \tag{67}
\end{align*}
$$

The only difference between the proof of (67) and (49) is that in (53), $\sum_{t=0}^{\ell-q-2}\binom{n-1+\ell}{t}=n+q+3$ when $\ell=q+3$.

Similar to (63), we have

$$
\begin{align*}
& A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \\
\leq & 2^{k+2} C_{2^{k+2}-2^{q+2}}(r+3) 2^{\left(2^{k+2}-1\right)(r+3)} \\
\leq & 2^{k+2} C_{2^{k+2}}(r+3) 2^{\left(2^{k+2}-1\right)(r+3)} . \tag{68}
\end{align*}
$$

Combine (63) (66) (68), if $i_{r-q-1} \leq r+2$,

$$
\begin{align*}
& \log A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \\
\leq & \left(2^{k+2}-1\right) r+\log r+O(1) \tag{69}
\end{align*}
$$

Now we are ready to prove (14). On the one hand, $x_{r+1} \cdots x_{k+2} \in \operatorname{RM}(m, r)$ when $m-r \geq 2$, by (40),
$\log E\left(A\left(2^{m-r+k}, T\right)\right) \geq \log N\left(m, x_{r+1} \cdots x_{k+2}, 2^{m-r+k}\right)$

$$
\begin{equation*}
=\left(2^{k+2}-1\right)(r-k) \tag{70}
\end{equation*}
$$

On the other hand, by (12) and (13), similar to (36), $\sum_{q=0}^{k-1} \sum_{i_{r-q-1} \geq r+3} A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right)$ and $\sum_{i_{r-k-1} \geq r+3} N\left(m, x_{i_{r-k-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right)$ are negligible,

$$
\begin{align*}
& E\left(A\left(2^{m-r+k}, T\right)\right) \\
\leq & \sum_{q=0}^{k-1} \sum_{i_{r-q-1} \leq r+2} A\left(m, x_{i_{r-q-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right) \\
& +\sum_{i_{r-k-1} \leq r+2} N\left(m, x_{i_{r-k-1}} \cdots x_{i_{0}}, 2^{m-r+k}\right)+O(1) \\
\leq & \sum_{q=0}^{k-1}\binom{r+3}{q+3} 2^{k+2} C_{2^{k+2}}(r+3) 2^{\left(2^{k+2}-1\right)(r+3)} \\
& +\binom{r+3}{k+3} 2^{\left(2^{k+2}-1\right)(r-k)}+O(1) \tag{71}
\end{align*}
$$

Combine (70) and (71), we have

$$
\begin{align*}
\left(2^{k+2}-1\right)(r-k) & \leq \log E\left(A\left(2^{m-r+k}, T\right)\right) \\
& \leq\left(2^{k+2}-1\right) r+O(\log r) \tag{72}
\end{align*}
$$

## Appendix E

Proof of Theorem 4 Firstly, we prove

$$
\begin{equation*}
\mathcal{N}\left(i_{s}, s\right)=2^{i_{s}-s}-2^{i_{s}}+\sum_{t=0}^{s+1}\binom{i_{s}}{t} \leq i_{s}-s+1 \tag{73}
\end{equation*}
$$

1) If $s=0$ or $i_{s}-s \leq 2$, (73) holds as an equality.
2) If $s \geq 1$ and $i_{s}-s \geq 3$, by (27),

$$
\begin{equation*}
\mathcal{N}\left(i_{s}, s\right) \leq 3<i_{s}-s+1 \tag{74}
\end{equation*}
$$

Therefore, if $s \geq 1$ and $i_{s}-s \geq 3$,

$$
\begin{equation*}
2^{i_{s}-s}-2^{i_{s}}+\sum_{t=0}^{s+1}\binom{i_{s}}{t}<i_{s}-s+1 \tag{75}
\end{equation*}
$$

Now we are ready to prove (18),

$$
\begin{align*}
& E\left(N\left(2^{m-\bar{r}}, T\right)\right) \\
& =\sum_{x_{i_{\bar{r}}-1} \cdots x_{i_{0}} \in \mathcal{I}} 2^{\sum_{s=0}^{\bar{r}-1}\left(2^{i_{s}-s}-2^{i_{s}}\right)+\left|\left\{I_{f}>I_{x_{i_{\bar{r}}-1} \cdots x_{i_{0}}} \mid f \in \mathcal{I}\right\}\right|} \\
& \left.\leq \sum_{x_{i_{\bar{r}}-1} \cdots x_{i_{0}} \in \mathcal{I}} 2^{\sum_{s=0}^{\bar{r}-1}\left(2^{i_{s}-s}-2^{i_{s}}\right)+\mid\left\{I_{f}>I_{x_{i_{\bar{r}}-1}} \cdots x_{i_{0}}\right.} \mid \operatorname{deg}(f) \leq \bar{r}\right\} \mid \\
& =\sum_{x_{i_{\bar{r}-1}} \cdots x_{i_{0}} \in \mathcal{I}} \sum_{2^{s=0}}^{\bar{r}_{-1}}\left(2^{i_{s}-s}-2^{i_{s}}+\sum_{t=0}^{s+1}\binom{i_{s}}{t}\right) \\
& \leq \sum_{x_{i_{\bar{r}}-1} \cdots x_{i_{0}} \in \mathcal{I}} 2^{\sum_{s=0}^{\bar{r}-1}\left(i_{s}-s+1\right)} \\
& =N\left(2^{m-\bar{r}}\right) \text {, } \tag{76}
\end{align*}
$$

where the first inequality holds since $\bar{r}$ is the largest degree of monomials in $\mathcal{I}$.

When $\bar{r} \leq 1$, the above two inequalities must be equalities, thus (18) holds as an equality.

When $\bar{r}>1$,

$$
\sum_{s=0}^{\bar{r}-1}\left(i_{s}-s+1\right)=\sum_{s=0}^{\bar{r}-1}\left(2^{i_{s}-s}-2^{i_{s}}+\sum_{t=0}^{s+1}\binom{i_{s}}{t}\right)
$$

if and only if $i_{\bar{r}-1} \leq \bar{r}+1$, we conclude that (18) holds as an equality if and only if $\forall x_{i_{\bar{r}-1}} \cdots x_{i_{0}} \in \mathcal{I}$

$$
\begin{aligned}
& \left(*^{\prime}\right) \quad i_{\bar{r}-1} \leq \bar{r}+1 . \\
& \left(* *^{\prime}\right) \text { If } I_{f}>I_{x_{i_{\bar{r}-1}} \cdots x_{i_{0}}} \text { and } \operatorname{deg}(f) \leq \bar{r} \text {, then } f \in \mathcal{I} \text {. }
\end{aligned}
$$

Apparently, $\left(*^{\prime}\right)$ is equivalent to $(*)$, and $\left(* *^{\prime}\right)$ is equivalent to $(* *)$, thus the proof is completed.

