# On the Weight Spectrum Improvement of Pre-transformed Reed-Muller Codes and Polar Codes

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*Abstract*—Pre-transformation with an upper-triangular matrix (including cyclic redundancy check (CRC), parity-check (PC) and polarization-adjusted convolutional (PAC) codes) improves the weight spectrum of Reed-Muller (RM) codes and polar codes significantly. However, a theoretical analysis to quantify the improvement is missing. In this paper, we provide asymptotic analysis on the number of low-weight codewords of the original and pre-transformed RM codes respectively, and prove that pretransformation significantly reduces low-weight codewords, even in the order sense. For polar codes, we prove that the average number of minimum-weight codewords does not increase after pre-transformation. Both results confirm the advantages of pretransformation.

# I. INTRODUCTION

Polar codes [1], invented by Arıkan, are a great breakthrough in coding theory. As code length approaches infinity, the synthesized channels become either noiseless or purenoise. Channel polarization occurs under successive cancellation (SC) decoding, which has a low complexity. However, the performance of polar codes under SC decoding is poor at short to moderate block lengths.

To boost finited-length performance, a successive cancellation list (SCL) decoding algorithm was proposed [2]. As list size L increases, the performance of SCL decoding approaches that of maximum-likehood (ML) decoding. But the ML performance of polar codes is still inferior due to low minimum distance. Consequently, concatenation of polar codes with CRC [3] and PC [4] were proposed to improve weight spectrum. In Arıkan's PAC codes [5], convolutional precoding and RM rate-profiling were applied to approach binary input additive white Gaussian noise (BIAWGN) dispersion bound [6] under large list decoding [7].

CRC-Aided (CA) polar, PC-polar, and PAC codes can be viewed as pre-transformed polar codes with upper-triangular transformation matrices [8]. In polar codes, frozen bits are all zeros, while in pre-transformed polar codes, traditional frozen bits are replaced by dynamically frozen bits [9], whose value depends on previous bits. It is proved that any upper-triangular pre-transformation does not reduce minimum distance [8]. In [10], efficient recursive formulas were proposed to calculate the average weight spectrum of pre-transformed polar codes with polynomial complexity rather than exponential complexity.

In this paper, we simplify the recursive formulas in [10] through the monomial representation of row vectors. From [8] [10], low-weight codewords are induced by low-weight row vectors. We further prove that, low-weight codewords are mainly induced by a small fraction of low-weight row vectors. Based on this discovery, we provide asymptotic analysis on the number of low-weight codewords of pre-transformed codes, and quantitatively analyze the improvement of weight spectrum.

This paper is organized as follows. In section II, we review polar codes and pre-transformed polar codes. In section III, we analyze the number of low-weight codewords of the original and pre-transformed RM codes respectively. Asymptotic analysis shows that low-weight codewords reduce significantly after pre-transformation. For polar codes, we prove that the average number of minimum-weight codewords does not increase after pre-transformation, as long as the code is decreasing [11]. Finally we draw some conclusions in section IV.

# II. BACKGROUND

## A. Polar Codes as Monomial Codes

Let  $F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ ,  $N = 2^m$ , and  $F_N = F^{\otimes m}$ . Starting from N binary-input discrete memoryless channels (B-DMC) W, we obtain N synthetic channels  $W_N^{(i)}$  after polarization. Polar codes can be constructed by selecting the indices of K most reliable information sub-channels, i.e., K row vectors of  $F_N$ , as information set  $\mathcal{I}$ . Density evolution (DE) algorithm [12], Gaussian approximation (GA) algorithm [13] and the channel-independent polarization weight (PW) construction method [14] are efficient methods to find reliable sub-channels.

After determining the information set  $\mathcal{I}$ , its complement set is called the frozen set  $\mathcal{F}$ . Let  $u_1^N = (u_1, u_2, \dots, u_N)$  be the bit sequence to be encoded. K bits are inserted into  $u_{\mathcal{I}}$ , and all zeros are filled into  $u_{\mathcal{F}}$ . Then the codeword  $c_1^N$  is obtained by  $c_1^N = u_1^N F_N$ .

Polar codes can also be expressed as monomial codes [11] and the monomial set is denoted by

$$\mathcal{M}_{m} \stackrel{def}{=} \{ x_{m-1}^{a_{m-1}} \cdots x_{0}^{a_{0}} | (a_{m-1}, \dots, a_{0}) \in \mathbf{F}_{2}^{m} \}.$$

From this point of view, each row vector of  $F_N$  corresponds to a monomial represented by m binary variables  $\{x_i\}, 0 \le i \le m-1$ . For instance, a monomial  $f = x_{i_{r-1}} \cdots x_{i_0}$ , where  $0 \le i_0 < \ldots < i_{r-1} \le m-1$ , the degree of f is r, denoted by deg(f). Denote  $I_f = i$ , if the monomial f represents the i-th row vector of  $F_N$ .

To be specific, let  $(a_{m-1}, ..., a_0)$  be the binary representation of N - i, i.e.,  $N - i = \sum_{j=0}^{m-1} 2^j a_j$ , then the *i*-th row vector of  $F_N$  can be represented by monomial  $x_{m-1}^{a_{m-1}} \cdots x_0^{a_0}$ . For example, the monomial representation of  $F_8$  is shown in Fig. 1.

|         |    |   |   |   |   |   |   |   | $a_2$ | $a_1$ | $a_0$ |               |
|---------|----|---|---|---|---|---|---|---|-------|-------|-------|---------------|
| $F_8 =$ | [1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1     | 1     | 1     | $x_2 x_1 x_0$ |
|         | 1  | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1     | 1     | 0     | $x_{2}x_{1}$  |
|         | 1  | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1     | 0     | 1     | $x_2x_0$      |
|         | 1  | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1     | 0     | 0     | $x_2$         |
|         | 1  | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0     | 1     | 1     | $x_1x_0$      |
|         | 1  | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0     | 1     | 0     | $x_1$         |
|         | 1  | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0     | 0     | 1     | $x_0$         |
|         | 1  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0     | 0     | 0     | 1             |
| L _     |    |   |   |   |   |   |   |   |       |       |       |               |

Fig. 1. The monomial representation of  $F_8$ .

From now on, the *i*-th row vector of  $F_N$  and the corresponding monomial  $x_{m-1}^{a_{m-1}} \cdots x_0^{a_0}$  are used interchangeably because they refer to the same thing.

## B. Decreasing Monomial Codes

It was revealed in [11] and [15] that the reliability of synthetic channels follows a partial order " $\leq$ ". If  $f, g \in \mathcal{M}_m$ ,  $g \leq f$  means g is universally more reliable than f. For monomials of the same degree, partial order is defined as

$$x_{i_{r-1}}\cdots x_{i_0} \preceq x_{j_{r-1}}\cdots x_{j_0} \iff i_s \le j_s, \ 0 \le s \le r-1,$$

and for monomials of different degrees

$$g \preceq f \iff \exists f^* \mid f, \ deg(f^*) = deg(g), \ \text{and} \ g \preceq f^*.$$

Denote the monomial code with information set  $\mathcal{I}$  by  $\mathcal{C}(\mathcal{I})$ ,  $\mathcal{C}(\mathcal{I})$  is a decreasing monomial code if  $\mathcal{I}$  satisfies partial order, i.e.,

$$\forall f \in \mathcal{I} \text{ and } g \in \mathcal{M}_m, \text{ if } g \preceq f \text{ then } g \in \mathcal{I}$$

For example, the information set of RM(m, r) consists of monomials with degree no larger than r. By definition, RM codes are decreasing monomial codes.

$$T = \begin{bmatrix} 1 & T_{12} & \cdots & T_{1N} \\ 0 & 1 & \cdots & T_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Let  $G_N = TF_N$  be the generator matrix of pre-transformed polar codes, where T is an upper triangular pre-transformation matrix defined above. The codeword of the pre-transformed polar codes is given by  $c_1^N = u_1^N G_N = u_1^N TF_N$ . In PAC codes, the pre-transformation matrix T is a Toeplitz matrix.

In [16], a deterministic algorithm for computing the weight spectrum of any given pre-transformed polar code was proposed. However, the computational complexity is still exponential. In [10], an efficient algorithm was proposed to compute the average weight spectrum of pre-transformed polar codes with polynomial complexity. The code ensemble assumes that  $T_{ij}$ ,  $1 \le i < j \le N$  are *i.i.d.*  $Bernoulli(\frac{1}{2})$  r.v..

### III. THE NUMBER OF LOW-WEIGHT CODEWORDS

In this section, we provide asymptotic analysis on the number of low-weight codewords of the original and pretransformed RM codes respectively. We prove that, for decreasing polar codes, the average number of minimum-weight codewords after pre-transformation is no larger than that of the original codes.

#### A. Notations and definitions

In this paper,  $\log x$  is the base-2 logarithm of x,  $\lceil x \rceil$  is the ceiling function of x, and w(x) is the Hamming weight of x. The entropy function  $h(x) = -x \log x - (1-x) \log(1-x)$ , 0 < x < 1, and |S| is the cardinality of set S. To characterize asymptotic results, we define the following notations<sup>1</sup>.

Let  $f_N^{(i)}$  be the *i*-th row vector of  $F_N$ , and  $g_N^{(i)}$  be the *i*-th row vector of  $G_N$ . Information set  $\mathcal{I} = \{I_1, I_2, \ldots, I_K\}$ , where  $I_1 < \ldots < I_K$ . The number of codewords with Hamming weight d of polar/RM codes is denoted by N(d), and the number of codewords with Hamming weight no larger than d is denoted by A(d). The minimum distance is denoted by  $d_{min}$ . The corresponding number of codewords of pre-transformed codes with transformation matrix T is denoted by N(d,T) and A(d,T), respectively. The average number is denoted by E(N(d,T)) and E(A(d,T)), where the expectation is with respect to random pre-transformation matrix T, and we assume  $T_{ij}$ ,  $1 \leq i < j \leq N$  are *i.i.d.* Bernoulli $(\frac{1}{2})$  r.v..

$$\begin{array}{lll} {}^{1}f(x) & \leq & O(g(x)) \ (f(x) & \geq & \Omega(g(x))), \ \text{if } \limsup_{x \to \infty} \frac{f(x)}{g(x)} & < & +\infty \\ \left(\liminf_{x \to \infty} \frac{f(x)}{g(x)} > 0\right), \ \text{where } g(x) & > & 0. \ f(x) & = & \Theta(g(x)), \ \text{if } 0 & < \\ \liminf_{x \to \infty} |\frac{f(x)}{g(x)}| & \leq & \limsup_{x \to \infty} |\frac{f(x)}{g(x)}| & < & +\infty, \ f(x) & = & o(g(x)), \ \text{if } \\ \lim_{x \to \infty} |\frac{f(x)}{g(x)}| & = 0. \end{array}$$

Let  $P(m, i, d) \stackrel{def}{=} P\left(w\left(g_{2^m}^{(i)}\right) = d\right)$  be the probability that the *i*-th row vector of  $G_N$  has Hamming weight d. By [10, Lemma 2], P(m, i, d) = 0, if  $d < w(f_N^{(i)})$ , i.e., pretransformation does not reduce the Hamming weight of row vectors. According to [10, Lemma 1], the probability that the codeword  $c_1^N = u_1^N G_N$  has Hamming weight d is equal to  $P(m, I_j, d)$ , as long as  $u_{I_j}$  is the first non-zero bit in  $u_1^N$ , thus we can combine the weight-d codewords induced by these  $2^{K-j}$  codewords in E(N(d,T)) whose first non-zero bit is  $u_{I_j}$ . Let  $N(m, I_j, d) \stackrel{def}{=} 2^{K-j} P(m, I_j, d)$ , according to [10, eq.(7)],

$$E(N(d,T)) = \sum_{\substack{1 \le j \le K \\ w(f_{I_j}) \le d}} 2^{K-j} P(m,I_j,d)$$
  
= 
$$\sum_{\substack{1 \le j \le K \\ w(f_{I_j}) \le d}} N(m,I_j,d),$$
(1)

where K - j is the number of information bits whose indices are greater than  $I_j$ . As explained above,  $N(m, I_j, d)$  is the number of weight-*d* codewords where  $u_{I_j}$  is the first nonzero bit in the encoded bit sequence  $u_1^N$ . We call  $N(m, I_j, d)$ the number of weight-*d* codewords induced by the  $I_j$ -th row vector. From (1), all weight-*d* codewords are induced by row vectors  $f_N^{(i)}$  with weight no larger than *d*. Therefore, when analyzing the number of weight-*d* codewords in pretransformed codes, we only need to consider the row vectors with weight no larger than *d*.

For convenience, we use  $P(m, x_{i_{r-1}} \cdots x_{i_0}, d)$  and  $N(m, x_{i_{r-1}} \cdots x_{i_0}, d)$  instead of P(m, i, d) and N(m, i, d) when  $x_{i_{r-1}} \cdots x_{i_0}$  represents the *i*-th row vector of  $F_N$ .

# B. Low-weight codewords of RM codes

In this section, we analyze low-weight codewords with Hamming weight within a constant multiple of minimum distance. We provide asymptotic analysis on the number of codewords in RM(m, r) with Hamming weight no larger than  $2^{m-r+k}$ , where k is a non-negative integer. The proof idea of Theorem 1 follows from [17] and [18].

**Theorem 1.** Assume  $0 < \alpha_1 < \frac{r}{m} < \alpha_2 < 1$ , where  $\alpha_1, \alpha_2$  are constants,

$$\Omega(m^{k+1}) \le \log A(2^{m-r+k}) \le O(m^{k+2}).$$
(2)

*Proof.* The proof is in Appendix A.

**Remark 1.** When k = 0,  $\log A(2^{m-r}) = \Theta(m^2)$  [11] reaches the upper bound of (2), and when k = 1,  $\log A(2^{m-r+1}) = \Theta(m^2)$  [19] reaches the lower bound of (2).

# C. Minimum-weight codewords of pre-transformed RM codes

According to (1), the average number of minimum-weight codewords of pre-transformed RM(m, r) is

$$E(N(2^{m-r},T)) = \sum_{0 \le i_0 < \dots < i_{r-1} \le m-1} N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r}).$$
(3)

Thus we first analyze the number of minimum-weight codewords induced by  $x_{i_{r-1}} \cdots x_{i_0}$ .

**Lemma 1.** In RM(m, r), the number of information bits whose indices are greater than  $I_{x_{i_{r-1}}\cdots x_{i_0}}$  is  $\sum_{s=0}^{r-1} \sum_{t=0}^{s+1} {i_s \choose t}$ , and

$$\log P(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r}) = \sum_{s=0}^{r-1} (2^{i_s - s} - 2^{i_s}).$$
(4)

Thus  $\log N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r}) =$ 

$$\sum_{s=0}^{r-1} \left( 2^{i_s - s} - 2^{i_s} + \sum_{t=0}^{s+1} \binom{i_s}{t} \right).$$
 (5)

*Proof.* We prove Lemma 1 via induction on m, the proof is in Appendix B.

**Remark 2.** In Lemma 1, we simplify the recursive formulas in [10, Theorem 1] through the monomial representation of the *i*-th row vector of  $F_N$ , this simplified form is convenient for the further theoretical analysis. As seen,  $P(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r})$  holds for all sub-channel selections, thus (4) will also apply to polar codes.

Based on Lemma 1, we provide asymptotic analysis on  $N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r})$  as well as  $E(N(2^{m-r}, T))$ .

# Theorem 2.

$$\log N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r}) \leq \begin{cases} 0 & i_{r-1} \geq r+3, and \\ & r \ sufficiently \ large; \\ 2r+3 & i_{r-1} = r+2; \\ 3r & i_{r-1} \leq r+1. \end{cases}$$
(6)

Assume  $m-r \ge 2$ ,  $\frac{r}{m} > \gamma$ , where  $\gamma > 0$  is a constant, then

$$3r \le \log E\left(N(2^{m-r}, T)\right) \le 3r + O(\log r). \tag{7}$$

*Proof.* The proof is in Appendix C. We briefly introduce the proof outline below.

Firstly, we prove (6) by *Step 1-2*. *Step 1*, to further calculate (5), let

$$\mathcal{N}(i_s, s) = 2^{i_s - s} - 2^{i_s} + \sum_{t=0}^{s+1} \binom{i_s}{t}$$
$$= 2^{i_s - s} - \sum_{t=0}^{i_s - s - 2} \binom{i_s}{t}, \tag{8}$$

we prove  $\mathcal{N}(i_s, s) \leq 0$  if  $s \geq 1$ ,  $i_s - s \geq 3$ . We analyze  $\mathcal{N}(i_s, s)$  according to the value of s, the proof is mainly based on the estimation of combinatorial number.

Step 2, since  $\log N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r}) = \sum_{s=0}^{r-1} \mathcal{N}(i_s, s)$ , based on Step 1, we analyze  $\log N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r})$  with respect to  $i_{r-1}$ , the proof details can be found in Appendix C.

Next, we prove (7) by Step 3.

Step 3, we divide the sum terms in (3) into three parts according to  $i_{r-1}$ :  $\sum_{i_{n-1} \ge r+3} N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r}),$ 

$$\sum_{\substack{i_{r-1}=r+2\\i_{r-1}\leq r+1}}^{N(m,x_{i_{r-1}}\cdots x_{i_0},2^{m-r})} \text{ and}$$

$$\sum_{i_{r-1}\leq r+1}^{N(m,x_{i_{r-1}}\cdots x_{i_0},2^{m-r})} N(m,x_{i_{r-1}}\cdots x_{i_0},2^{m-r}). \text{ From } Step 2, \text{ the first}$$

term converges to zero, and the second term is negligible compared to the third term. Thus the minimum-weight codewords are mainly induced by  $N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r})$  with  $i_{r-1} \leq r+1$ , so

$$E(N(2^{m-r},T)) \approx \sum_{i_{r-1} \le r+1} N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r})$$
$$\le |\{x_{i_{r-1}} \cdots x_{i_0}, i_{r-1} \le r+1\}| 2^{3r} = \binom{r+2}{2} 2^{3r}.$$
(9)

**Remark 3.** Since we can not efficiently calculate the weight spectrum of specific pre-transformed codes, we analyze the average weight spectrum of the code ensemble defined by the random pre-transformation matrix.

The results on the average weight spectrum are significant in two aspects. On the one hand, there exist good codes with minimum-weight codewords no larger than the average. On the other hand, numerical results confirm that, the actual number of minimum-weight codewords is usually very close to the average, i.e., has small variance. In practice, this means that most random pre-transformation matrices are good.

**Remark 4.** The  $\binom{r+2}{2}$  monomials  $x_{i_{r-1}} \cdots x_{i_0}$  with  $i_{r-1} \leq r+1$  induce the majority of miminum-weight codewords of pre-transformed RM codes, which is a tiny part of  $\binom{m}{r}$  monomials with degree r. It implies that in pre-transformed polar codes, the minimum-weight codewords are mainly induced by a small fraction of monomials. For example, in RM(9,2), the 498-th row vector  $x_4x_3x_2$  satisfies  $i_{r-1} \leq r+1$ , its corresponding binary representation is (0,0,0,0,1,1,1,0,0). Monomials  $x_{i_{r-1}} \cdots x_{i_0}$  with  $i_{r-1} \leq r+1$  share similar characteristics: they are at the bottom of  $F_N$  and have high reliability among monomials with degree r.

In Fig. 2, we display the number of minimum-weight codewords in RM codes and pre-transformed RM codes on the logarithm domain. The example has code rate R = 0.5, and the average number of minimum-weight codewords is approximately  $2^{3r}$  in the order sense. In contrast, the number before pre-transformation is  $2^{\Theta(m^2)}$ . In other words, the logarithm scaling of minimum-weight codewords drops from quadratic growth to linear growth after pre-transformation. The result proves that pre-transformation can reduce minimum-weight codewords significantly, even in the order sense. This also partly explains the gain of PAC codes (a special case of pre-transformed RM codes) over RM codes.



Fig. 2. Logarithm scaling of the number of minimum-weight codewords in RM codes and pre-transformed RM codes.

## D. Low-weight codewords of pre-transformed RM codes

In this section, we analyze low-weight codewords with Hamming weight within a constant multiple of minimum distance. We provide asymptotic analysis on the number of codewords in per-transformed RM(m, r) with Hamming weight no larger than  $2^{m-r+k}$ , where k is a positive integer.

According to (1), we only need to consider row vectors with weight no larger than  $2^{m-r+k}$ , or equivalently, monomials with degree at least r-k. For monomials with degree r-q, where  $0 \le q < k$ , their corresponding row vectors have weight  $2^{m-r+q} < 2^{m-r+k}$ , thus they induce codewords with weight from  $2^{m-r+q}$  to  $2^{m-r+k}$ . Let

$$A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r-\kappa}) = \sum_{d'=2^{m-r+q}}^{2^{m-r+\kappa}} N(m, x_{i_{r-q-1}} \cdots x_{i_0}, d'),$$
(10)

 $A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r-k})$  is the number of codewords induced by  $x_{i_{r-q-1}} \cdots x_{i_0}$  with weight no larger than  $2^{m-r+k}$ . For monomials with degree r - k, their corresponding row vectors have weight exactly  $2^{m-r+k}$ , thus we only need to consider the number of weight- $2^{m-r+k}$  codewords induced by  $x_{i_{r-k-1}} \cdots x_{i_0}$ . Therefore, we have

$$E(A(2^{m-r+k},T))$$

$$=\sum_{q=0}^{k-1}\sum_{0\leq i_0<\ldots< i_{r-q-1}\leq m-1}A(m,x_{i_{r-q-1}}\cdots x_{i_0},2^{m-r+k})$$

$$+\sum_{0\leq i_0<\ldots< i_{r-k-1}\leq m-1}N(m,x_{i_{r-k-1}}\cdots x_{i_0},2^{m-r+k}).$$
(11)

Next, we analyze  $N(m, x_{i_{r-k-1}} \cdots x_{i_0}, 2^{m-r+k})$  and  $A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k})$ , where  $0 \le q < k$ , and then provide asymptotic analysis on  $E(A(2^{m-r+k}, T))$ .

**Theorem 3.** Let k be a positive integer,

$$\log N(m, x_{i_{r-k-1}} \cdots x_{i_0}, 2^{m-r+k}) \leq$$

$$\begin{cases} 0 & i_{r-k-1} \ge r+3, and \\ r \text{ sufficiently large;} \\ (2^{k+2}-2)r+O(1) & i_{r-k-1}=r+2; \\ (2^{k+2}-1)r & i_{r-k-1} \le r+1. \end{cases}$$
(12)

Assume  $m-r \ge 2$ ,  $\frac{r}{m} > \gamma$ , where  $\gamma > 0$  is a constant. Let  $0 \leq q < k$ ,

$$\log A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k}) \leq \\ \begin{cases} 0 & i_{r-q-1} \geq r+3, and \\ r \text{ sufficiently large;} & (13) \\ (2^{k+2}-1)r + \log r + O(1) & i_{r-q-1} \leq r+2. \end{cases}$$

Therefore, we have

$$(2^{k+2} - 1)(r - k) \le \log E \left( A(2^{m-r+k}, T) \right) \le (2^{k+2} - 1)r + O(\log r).$$
(14)

*Proof.* The proof is in Appendix D. The method is similar to that in Theorem 2, but due to the sum terms in the recursive formula [10, Theorem 2], the analysis is more complicated. In particular, we derive the upper bound on  $A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k})$  when  $i_{r-q-1} \leq r+2$  through induction. 

In Fig. 3, we display the number of codewords with Hamming weight  $2d_{min}$  in RM codes and pre-transformed RM codes on the logarithm domain. The example has code rate R = 0.5, and the average number of codewords is approximately  $2^{(2^{k+2}-1)r}$  in the order sense. Similarly, the logarithm scaling of the weight- $2d_{min}$  codewords grows linearly with m under pre-transformation, as opposed to quadratically without pre-transformation. Our approximation is accurate asymptotically, and there is a gap between the true number and approximation when m is small. Note that calculating the accurate number of weight- $2d_{min}$  codewords becomes intractable when m is large.



Fig. 3. Logarithm scaling of the number of codewords with Hamming weight  $2d_{min}$  in RM codes and pre-transformed RM codes.

E. Minimum-weight codewords of pre-transformed polar codes

In this section, we extend our analysis from RM codes to polar codes. We prove that the average number of minimumweight codewords of pre-transformed polar codes does not increase after pre-transformation. Unlike RM codes, polar codes do not have a universal sub-channel selection rule. Therefore, their corresponding asymptotic results cannot be obtained as in RM codes. Fortunately, the conclusions in this section are non-asymptotic and apply to arbitrary code lengths. Let  $\mathcal{C}(\mathcal{I})$  be a decreasing polar code, define

$$\overline{r} = \min\{r \mid \mathcal{C}(\mathcal{I}) \subseteq \mathbf{RM}(m, r)\},\tag{15}$$

i.e., the largest degree of monomials in  $\mathcal{I}$  is  $\overline{r}$  and the minimum distance is  $2^{m-\overline{r}}$ . According to [11, Proposition 7], the number of minimum-weight codewords of  $\mathcal{C}(\mathcal{I})$  is

$$N(2^{m-\overline{r}}) = \sum_{x_{i_{\overline{r}-1}}\cdots x_{i_0}\in\mathcal{I}} 2^{\sum_{s=0}^{\overline{r}-1}i_s-s+1}.$$
 (16)

Similarly, we call  $2^{rac{\overline{r}-1}{\sum_{s=0}^{i_s-s+1}}}$  the number of minimum-weight codewords induced by  $x_{i_{\overline{r}-1}}\cdots x_{i_0}$  in original polar codes. Let  $i^*$  be the smallest index in information bits which can be represented by a monomial with degree  $\overline{r}$ , i.e.

$$i^* = \min\{I_{x_{i_{\overline{r}-1}}\cdots x_{i_0}} | x_{i_{\overline{r}-1}}\cdots x_{i_0} \in \mathcal{I}\}.$$
 (17)

Next, we prove that pre-transformation does not increase the average number of minimum-weight codewords.

**Theorem 4.** If  $C(\mathcal{I})$  is a decreasing polar code,  $\overline{r}$ ,  $i^*$  are defined in (15), (17), we have

$$E(N(2^{m-\overline{r}},T)) \le N(2^{m-\overline{r}}).$$
(18)

Let the monomial representation of the  $i^*$ -th row vector be  $x_{i_{\overline{\tau}-1}}\cdots x_{i_0}$ . If  $\overline{\tau} \leq 1$ , (18) must hold as an equality. If  $\overline{r} > 1$ , (18) holds as an equality if and only if the following two conditions satisfy:

(\*) 
$$i_{\overline{r}-1}^* \leq \overline{r} + 1.$$
  
(\*\*) If  $I_f > i^*$  and  $deg(f) \leq \overline{r}$ , then  $f \in \mathcal{I}$ .

Proof. The proof is in Appendix E. In fact, the number of minimum-weight codewords induced by every  $x_{i_{\overline{r}-1}} \cdots x_{i_0} \in$  $\mathcal{I}$  decreases after pre-transformation.

Remark 5. In fact, the number of minimum-weight codewords induced by every  $x_{i_{\overline{r}-1}}\cdots x_{i_0} \in \mathcal{I}$  decreases after pre-transformation, but the amount of reduction differs. According to (16), the minimum-weight codewords in original polar codes are mainly induced by  $x_{i_{\overline{r}-1}}\cdots x_{i_0}$  with large  $\sum_{s=0}^{\overline{r}-1} (i_s - s). \text{ Since } i_s - s \leq i_{\overline{r}-1} - (\overline{r} - 1), \ 0 \leq s \leq s$  $\overline{r}$  – 1, these monomials also have large  $i_{\overline{r}-1}$ . As explained in Theorem 2, these codewords are reduced due to pretransformation, which explains why pre-transformation can improve the weight spectrum. Therefore, more  $x_{i_{\overline{r}-1}} \cdots x_{i_0}$ with large  $i_{\overline{r}-1}$  as information bits results in more significant improvement in the weight spectrum. The results prove that pre-transformation can be beneficial for polar codes too.

## IV. CONCLUSION

In this paper, we provide asymptotic analysis on the number of low-weight codewords of the original and pre-transformed RM codes respectively, and prove that pre-transformation can reduce the low-weight codewords significantly. For decreasing polar codes, we prove that pre-transformation does not increase the average number of minimum-weight codewords. The numerical results validate the theoretical analysis and confirm the benefit of pre-transformation.

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## APPENDIX A

Codewords in polar codes can be regarded as polynomials. Let f, g be two codewords in RM(m, r), define d(f, g) be the Hamming distance between f and g. Define  $\text{poly}(d) = \{f \in \text{RM}(m, r) : \omega(f) \leq d\}$ , i.e., polynomials in RM(m, r) with weight no larger than d. Define the degree of a polynomial f to be the maximal degree of monomials in f, denoted by deg(f). Let f be a polynomial and  $y \in \mathbf{F}_2^m$ . Define the derivative of f in direction y by

$$\Delta_y f(x) = f(x+y) + f(x), \tag{19}$$

we have  $deg(\Delta_y f) \leq deg(f) - 1$ . For example, let m = 2,  $f(x) = x_0 + x_0 x_1$ , y = (1, 0). Then

$$\Delta_y f(x) = f(x+y) + f(x)$$
  
=(x<sub>0</sub>+1) + (x<sub>0</sub>+1)x<sub>1</sub> + x<sub>0</sub> + x<sub>0</sub>x<sub>1</sub> = x<sub>1</sub> + 1. (20)

Here,  $deg(\Delta_y f) = 1 = deg(f) - 1$ . Define the k-iterated derivative of f in direction  $Y = (y_1, \ldots, y_k) \in (\mathbf{F}_2^m)^k$  by

$$\Delta_Y f(x) = \Delta_{y_1} \Delta_{y_2} \cdots \Delta_{y_k} f(x).$$
(21)

Since  $deg(\Delta_y f) \leq deg(f) - 1$ ,  $deg(\Delta_Y f) \leq deg(f) - k$ .

Let  $S \subseteq \text{RM}(m, r)$  be a subset of polynomials, we call a subset of polynomials  $\mathcal{B}$  is a  $\delta$ -net for S if  $\forall f \in S$ , there exists  $g \in \mathcal{B}$  such that  $d(f,g) \leq \delta$ .

Lemma 2. [17, Corollary 3.1] Let t be an integer, define

$$\mathcal{A}_{r-k-1,t} = \{ \operatorname{Maj}(\Delta_{Y_1}f, \dots, \Delta_{Y_t}f) : \\ Y_1, \dots, Y_t \in (\mathbf{F}_2^m)^{r-k-1}, f \in \operatorname{RM}(m,r) \},$$

where Maj is the majority function defined in [17]. Then  $\mathcal{A}_{r-k-1,t}$  is a  $\delta$ -net for  $\operatorname{poly}(2^{m-r+k})$ , where  $t = \lceil 17 \log(2^m/\delta) \rceil$ .

*Proof of Theorem 1.* Denote h = m - r + k. To prove the lower bound, assume  $g(x_0, ..., x_{h-1})$  be an arbitrary polynomial with degree k + 1. Define

$$f(x_0, \dots, x_{m-1}) = (g(x_0, \dots, x_{h-1}) + x_h)x_{h+1} \dots x_{m-1}.$$

It is clear that  $f \in \text{RM}(m, r)$  and  $w(f) = 2^h$ . The number of polynomials with h variables and degree k + 1 is  $2^{\binom{h}{k+1}} = 2^{\Theta(m^{k+1})}$ , which implies the lower bound.

To prove the upper bound, let  $\delta = 2^{m-r-2}$ , define  $\operatorname{adj}(f) = \{g \in \mathcal{A}_{r-k-1,t} : d(f,g) \leq \delta\}$ , where t = 17(r+2). By Lemma 2,  $\mathcal{A}_{r-k-1,t}$  is a  $\delta$ -net for  $\operatorname{poly}(2^h)$ , thus  $\forall f \in \operatorname{poly}(2^h)$ ,  $\operatorname{adj}(f) \neq \emptyset$ . Next, we prove for any two different  $f_1, f_2 \in \operatorname{poly}(2^h)$ ,  $\operatorname{adj}(f_1) \cap \operatorname{adj}(f_2) = \emptyset$ , otherwise there exist  $g \in \mathcal{A}_{r-k-1,t}$ , such that  $d(f_1,g) \leq \delta$  and  $d(f_2,g) \leq \delta$ . By triangle inequality,  $d(f_1, f_2) \leq 2^{m-r-1} < d_{min} = 2^{m-r}$ , which is a contradiction. Notice that  $deg(\Delta_Y f) \leq k+1$ ,  $\forall Y \in (\mathbf{F}_2^m)^{r-k-1}$ ,  $f \in \operatorname{RM}(m, r)$ , we have

$$4(2^{h}) \leq \sum_{f \in \text{poly}(2^{h})} |\operatorname{adj}(f)| = |\bigcup_{f \in \text{poly}(2^{h})} \operatorname{adj}(f)|$$
$$\leq |\mathcal{A}_{r-k-1,t}| \leq 2^{t} \sum_{s=0}^{k+1} {m \choose s} = 2^{\Theta(m^{k+2})}.$$
(22)

## APPENDIX B

*Proof of Lemma 1.* We prove Lemma 1 via induction on m. Firstly, if m = 1, Lemma 1 can be proved directly. For the induction step  $m - 1 \rightarrow m$ , we consider two cases according to  $i_{r-1}$ :

1)  $i_{r-1} = m - 1$ , i.e.,  $x_{i_{r-1}} \cdots x_{i_0}$  is in the top half of  $F_N$ . The number of information bits in the top half and whose indices are greater than  $I_{x_{i_{r-1}}\cdots x_{i_0}}$  is equal to the number of information bits whose indices are greater than  $I_{x_{i_{r-2}}\cdots x_{i_0}}$  in  $\operatorname{RM}(m-1,r-1)$ , which is  $\sum_{s=0}^{r-2} \sum_{t=0}^{s+1} {i_s \choose t}$  by inductive hypothesis. The number of information bits in the lower half is  $\sum_{t=0}^{r} {m-1 \choose t} = \sum_{t=0}^{r} {i_{r-1} \choose t}$ , thus the total number of information bits whose indices are greater than  $I_{x_{i_{r-1}}\cdots x_{i_0}}$  is equal to  $\sum_{s=0}^{r} \sum_{t=0}^{s+1} {i_s \choose t}$ . According to [10, Theorem 1],

$$\log P(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r}) = \log P(m-1, x_{i_{r-2}} \cdots x_{i_0}, 2^{m-r}) + 2^{m-r} - 2^{m-1} = \log P(m-1, x_{i_{r-2}} \cdots x_{i_0}, 2^{m-r}) + 2^{i_{r-1}-(r-1)} - 2^{i_{r-1}} = \sum_{s=0}^{r-1} (2^{i_s-s} - 2^{i_s}).$$
(23)

The last equality is due to inductive hypothesis.

2)  $i_{r-1} < m-1$ , i.e.,  $x_{i_{r-1}} \cdots x_{i_0}$  is in the lower half of  $F_N$ . The number of information bits whose indices are greater than  $I_{x_{i_{r-1}} \cdots x_{i_0}}$  is equal to the number of information bits whose indices are greater than  $I_{x_{i_{r-1}} \cdots x_{i_0}}$  in RM(m-1, r), which is  $\sum_{s=0}^{r-1} \sum_{t=0}^{s+1} {i_s \choose t}$  by inductive hypothesis. According to [10, Theorem 1],

$$\log P(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r}) = \log P(m-1, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-1-r}) = \sum_{s=0}^{r-1} (2^{i_s-s} - 2^{i_s}).$$

The last equality is due to inductive hypothesis.

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#### APPENDIX C

*Proof of Theorem 2.* Firstly, we analyze  $\mathcal{N}(i_s, s)$  with respect to s, in fact, if  $s \ge 1$ ,  $i_s - s \ge 3$ ,  $\mathcal{N}(i_s, s) \le 0$  for sufficiently large  $i_s$ .

case 1: 
$$s = 0$$
,  $\mathcal{N}(i_0, 0) = 1 + i_0$ .  
case 2:  $s = 1$ ,  $\mathcal{N}(i_1, 1) = -2^{i_1-1} + 1 + i_1 + {i_1 \choose 2}$ , and we  
have  $\mathcal{N}(i_1, 1) \le 0$  if  $i_1 \ge 5$ .  
case 3:  $2 \le s \le \lceil \frac{i_2}{2} \rceil - 2$ .

$$\mathcal{N}(i_s, s) \le 2^{i_s - 2} - 2^{i_s} + \sum_{t=0}^{\lceil \frac{i_s}{2} \rceil - 1} {i_s \choose t} \\ \le -\frac{3}{4} 2^{i_s} + \frac{1}{2} 2^{i_s} = -\frac{1}{4} 2^{i_s} \le 0.$$
(24)  
e 4:  $\lceil \frac{i_s}{2} \rceil - 2 \le s \le i_s - \log(i_s + 16\sqrt{2i_s}),$ 

$$\mathcal{N}(i_{s},s) = 2^{i_{s}-s} - \sum_{t=0}^{i_{s}-s-2} {i_{s} \choose t} \le 2^{i_{s}-s} - {i_{s} \choose i_{s}-s-2}$$

$$\stackrel{(a)}{\leq} 2^{i_{s}-s} - \frac{2^{i_{s}h(\frac{i_{s}-s-2}{i_{s}})}}{\sqrt{2i_{s}}} \stackrel{(b)}{\leq} 2^{i_{s}-s} - \frac{2^{2(i_{s}-s-2)}}{\sqrt{2i_{s}}}$$

$$= (1 - \frac{2^{i_{s}-s}}{16\sqrt{2i_{s}}})2^{i_{s}-s} \le (1 - \frac{i_{s}+16\sqrt{2i_{s}}}{16\sqrt{2i_{s}}})2^{i_{s}-s}$$

$$\le -\frac{i_{s}^{\frac{3}{2}}}{16\sqrt{2}} - i_{s} \le -\frac{i_{s}^{\frac{3}{2}}}{16\sqrt{2}} \le 0,$$
(25)

where (a) is from [20, problem 5.8], (b) is due to  $\frac{i_s - s - 2}{i_s} \leq \frac{1}{2}$ when  $s \geq \lfloor \frac{i_s}{2} \rfloor - 2$  and  $h(x) \geq 2x$ ,  $0 < x \leq \frac{1}{2}$ . case 5:  $i_s - \log(i_s + 16\sqrt{2i_s}) \leq s \leq i_s - 4$ ,

$$\mathcal{N}(i_s, s) \le i_s + 16\sqrt{2i_s} - \sum_{t=0}^2 \binom{i_s}{t}$$
  
=  $16\sqrt{2i_s} - 1 - \frac{i_s(i_s - 1)}{2} \stackrel{(c)}{\le} 0,$  (26)

(c) holds when  $i_s \ge 14$ .

case 6:  $s = i_s - 3$ ,  $\mathcal{N}(i_s, i_s - 3) = 7 - i_s \stackrel{(d)}{\leq} 0$ , (d) holds when  $i_s \geq 7$ .

case 7:  $i_s - 2 \le s \le i_s$ ,  $\mathcal{N}(i_s, s) \le 3$ .

We conclude that  $\mathcal{N}(i_s, s) \leq 0$  if  $i_s \geq 14$ ,  $i_s - s \geq 3$ ,  $s \geq 1$  from the discussion above. When  $i_s \leq 13$ ,  $s \geq 1$ , through compute search, we have  $\mathcal{N}(i_s, s) \leq 3$ . Therefore,

$$\mathcal{N}(i_s, s) \le 3, \text{ if } s \ge 1.$$

Next, we analyze  $\log N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r}) = \sum_{s=0}^{r-1} \mathcal{N}(i_s, s)$  with respect tof  $i_{r-1}$ . 1)  $i_{r-1} \leq r+1$ , we have  $i_s - s \leq i_{r-1} - (r-1) \leq 2$ , by case 7,

$$\log N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r}) \le 3r.$$
(28)

2)  $i_{r-1} = r+2$ , we have  $i_0 \leq i_{r-1} - (r-1) \leq 3$ , thus  $\mathcal{N}(i_0, 0) = 1 + i_0 \leq 4$ , by (27),

$$\log N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r})$$
  
= $\mathcal{N}(r+2, r-1) + \sum_{s=1}^{r-2} \mathcal{N}(i_s, s) + \mathcal{N}(i_0, 0)$   
 $\leq 7 - (r+2) + 3(r-2) + 4$   
= $2r + 3.$  (29)

3)  $i_{r-1} \ge r+3$ , according to *case 3-5*, when r is sufficiently large, we have

$$\mathcal{N}(i_{r-1}, r-1) \le -\frac{i_{r-1}^{\frac{3}{2}}}{16\sqrt{2}} \le -\frac{r^{\frac{3}{2}}}{16\sqrt{2}}.$$
 (30)

Since  $i_0 \le i_1 - 1$ ,

$$\mathcal{N}(i_1, 1) + \mathcal{N}(i_0, 0) = -2^{i_1 - 1} + 2 + i_1 + i_0 + \binom{i_1}{2}$$
  
$$\leq -2^{i_1 - 1} + 1 + 2i_1 + \binom{i_1}{2} \leq 7,$$
(31)

we prove the last inequality through computer search. Thus

$$\log N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r}) \\ \leq -\frac{r^{\frac{3}{2}}}{16\sqrt{2}} + 3(r-3) + 7 \stackrel{(e)}{\leq} -\frac{r^{\frac{3}{2}}}{32} \leq 0,$$
(32)

(e) holds when r is sufficiently large. Thus (6) is proved from (28) (29) (32).

Now we are ready to prove (7). On the one hand,  $x_{r+1} \cdots x_2 \in \text{RM}(m, r)$  when  $m - r \ge 2$ , therefore

$$\log E(N(2^{m-r},T)) \ge \log N(m, x_{r+1} \cdots x_2, 2^{m-r}) = 3r.$$
(33)

On the other hand,

$$\sum_{i_{r-1} \le r+1} N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r})$$
  
$$\le |\{x_{i_{r-1}} \cdots x_{i_0}, i_{r-1} \le r+1\}| 2^{3r} = \binom{r+2}{2} 2^{3r}. \quad (34)$$

$$\sum_{i_{r-1}=r+2} N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r})$$
(35)

$$\sum_{\substack{i_{r-1} \ge r+3 \\ \leq |\{x_{i_{r-1}} \cdots x_{i_0}, i_{r-1} \ge r+2\}|^2} N(m, x_{i_{r-1}} \cdots x_{i_0}, 2^{m-r})$$

$$\sum_{\substack{(f) \\ \leq |\{\{x_{i_{r-1}} \cdots x_{i_0}, i_{r-1} \ge r+3\}|2^{-\frac{r^2}{32}} \le 2^{m-\frac{r^2}{32}}} \le 2^{m-\frac{r^2}{32}}$$

$$\le 2^{\frac{r}{\gamma} - \frac{r^2}{32}} \le 1, \qquad (36)$$

where (f) holds if (32) holds, (g) holds if  $r \ge \left(\frac{32}{\gamma}\right)^2$ . Divide the sum terms in (3) into three parts according to  $i_{r-1}$ , by (34)-(36),

$$E(N(2^{m-r},T)) \leq {\binom{r+2}{2}} 2^{3r} + {\binom{r+2}{3}} 2^{2r+3} + 1 = {\binom{r+2}{2}} 2^{3r} (1+o(1)),$$
(37)

$$\log E(N(2^{m-r},T)) \le 3r + \log\binom{r+2}{2} + o(1).$$
(38)

Combine (33) and (38), we complete the proof of (7).  $\Box$ 

## APPENDIX D

Proof outline:

Firstly, we prove (12) by *Step 1-2*, the proof of (12) is similar to that of (6), and is omitted due to space limitation. *Step 1*, let

$$\mathcal{N}_{k}(i_{s},s) = 2^{i_{s}-s} - 2^{i_{s}} + \sum_{t=0}^{s+k+1} \binom{i_{s}}{t}$$
$$= 2^{i_{s}-s} - \sum_{t=0}^{i_{s}-s-k-2} \binom{i_{s}}{t}, \qquad (39)$$

we prove  $\mathcal{N}_k(i_s, s) \leq 0$  if  $s \geq 1$ ,  $i_s - s \geq k + 3$ . Step 2, similar to (5), we have

$$\log N(m, x_{i_{r-k-1}} \cdots x_{i_0}, 2^{m-r+k}) = \sum_{s=0}^{r-k-1} \mathcal{N}_k(i_s, s), \quad (40)$$

and we analyze  $\log N(m, x_{i_{r-k-1}} \cdots x_{i_0}, 2^{m-r+k})$  with respect to  $i_{r-k-1}$ .

Next, we prove (13) by *Step 3-5*. *Step 3*, let

$$\mathcal{N}_{k,q}(i_s,s) \stackrel{def}{=} \\ \max_{\substack{2^{i_s-s} \le d' \le 2^{i_s-s+k-q} \\ t = 0}} (2^{i_s} - d')(h(\frac{2^{i_s-s+k-q} - d'}{2(2^{i_s} - d')}) - 1) \\ + \sum_{t=0}^{q+s+1} {i_s \atop t = 0} + i_s - s + k - q,$$
(41)

where  $s \ge k - q + 1$ , by (45) and (46), we prove

$$\log A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k}) \\ \leq m + \sum_{s=0}^{k-q} \left( \sum_{t=0}^{q+s+1} {i_s \choose t} + i_s \right) + \sum_{s=k-q+1}^{r-q-1} \mathcal{N}_{k,q}(i_s, s)$$
(42)

via induction on m.

Step 4, we prove  $\mathcal{N}_{k,s}(i_s,s) \leq 0$  when  $i_s - s$  is large. Therefore, if  $i_{r-q-1} \geq r+3$ , we have  $\log A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k}) \leq 0$ .

The proof of *Step 3-4* is omitted due to space limitation. Step 5, when  $i_{r-q-1} \leq r+2$ , we prove

$$\log A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k}) \le (2^{k+2} - 1)r + \log r + O(1)$$

via induction.

Finally, we prove (14) by Step 6.

Step 6, based on Step 4-5, we have

$$E(A(2^{m-r+k},T))$$

$$\approx \sum_{q=0}^{k-1} \sum_{i_{r-q-1} \le r+2} A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k})$$

$$+ \sum_{i_{r-k-1} \le r+2} N(m, x_{i_{r-k-1}} \cdots x_{i_0}, 2^{m-r+k}). \quad (43)$$

Combine (12) and (13), we complete the proof of (14).

Proof of Theorem 3.  $\forall 0 \le q < k$ ,  $N(m, x_{i_{r-q-1}} \cdots x_{i_0}, d)$  is the number of weight-*d* codewords induced by  $x_{i_{r-q-1}} \cdots x_{i_0}$  in RM(m, r), we have

$$\log N(m, x_{i_{r-q-1}} \cdots x_{i_0}, d) = \log P(m, x_{i_{r-q-1}} \cdots x_{i_0}, d) + \sum_{s=0}^{r-q-1} \sum_{t=0}^{s+q+1} \binom{i_s}{t}.$$
 (44)

According to [10, Theorem 2], if  $i_{r-q-1} = m - 1$ ,

$$N(m, x_{i_{r-q-1}} \cdots x_{i_0}, d) = \sum_{\substack{d' = 2^{m-r+q} \\ d - d' is even}}^{d} \left( N(m-1, x_{i_{r-q-2}} \cdots x_{i_0}, d') * \right)_{2^{d'-\sum_{t=0}^{m-r-2} \binom{m-1}{t}} * \binom{2^{m-1} - d'}{\frac{d-d'}{2}} .$$
(45)

If  $i_{r-q-1} < m-1$ ,

....

$$N(m, x_{i_{r-q-1}} \cdots x_{i_0}, d) = N(m-1, x_{i_{r-q-1}} \cdots x_{i_0}, \frac{d}{2}).$$
(46)

We are going to prove  $\log A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k}) \le (2^{k+2}-1)r + \log r + O(1)$  when  $i_{r-q-1} \le r+2$ .

Let  $i_{r-q-1} = r - q - 1 + \ell$ , where  $0 \le \ell \le q + 3$ , apply (46)  $m - r + q - \ell$  times repeatedly, we have

$$N(m, x_{i_{r-q-1}} \cdots x_{i_0}, d) = N(r - q + \ell, x_{i_{r-q-1}} \cdots x_{i_0}, \frac{d}{2^{m-r+q-\ell}}), \qquad (47)$$

where  $2^{m-r+q} \leq d \leq 2^{m-r+k}$ . (47) must be zero unless  $\frac{d}{2^{m-r+q-\ell}} = 2^{\ell} + 2v$ , where v is a non-negtive integer, therefore  $N(m, x_{i_{r-q-1}} \cdots x_{i_0}, d) =$ 

$$\begin{cases} N(r-q+\ell, x_{i_{r-q-1}}\cdots x_{i_0}, 2^{\ell}+2v) \\ d = (2^{\ell}+2v)2^{m-r+q-\ell}, v \ge 0; \\ 0 & otherwise. \end{cases}$$
(48)

Next, let  $C_0 = 1, C_v = 2^{v-1}, v \ge 1$ , we are going to prove that if  $i_{r-q-1} = r - q - 1 + \ell$ ,  $0 \le \ell \le q + 1$ ,

$$N(r - q + \ell, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{\ell} + 2v) \le C_v(r - q + \ell) 2^{(2^{\ell} + 2v)(r - q + \ell)}$$
(49)

via induction on  $\ell$ , v and r - q, the degree of the monomial  $x_{i_{r-q-1}} \cdots x_{i_0}$ .

When  $\ell = 0$ , we prove (49) holds via induction on v and r - q. When  $\ell \ge 1$ , in addition to induction on v and r - q, we also use the inductive hypothesis that (49) holds from 0 to  $\ell - 1$ ,  $\forall v \ge 0, r - q \ge 0$ . If v = 0, by (40),

$$\log N(r - q + \ell, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{\ell}) = \sum_{s=0}^{r-q-1} \mathcal{N}_q(i_s, s)$$
$$= \sum_{s=0}^{r-q-1} \left( 2^{i_s - s} - \sum_{t=0}^{i_s - s - q - 2} {i_s \choose t} \right) \le 2^{\ell} (r - q), \tag{50}$$

where the last inequality is due to  $i_s - s \leq i_{r-q-1} - (r-q-1) = \ell$ . Therefore,

$$N(r-q+\ell, x_{i_{r-q-1}}\cdots x_{i_0}, 2^{\ell}) \le C_0(r-q+\ell)2^{2^{\ell}(r-q+\ell)}.$$
(51)

For the induction step  $v - 1 \rightarrow v$ , denote r - q = n for convience, we complete the induction step via induction on n. When  $n = 0, v \ge 1$ , no codeword has Hamming weight  $2^{\ell} + 2v$  which is larger than the code length  $2^{\ell}$ , thus

$$N(\ell, \mathbf{1}, 2^{\ell} + 2v) = 0 \le C_v \ell 2^{(2^{\ell} + 2v)\ell},$$
(52)

where 1 represents the monomial with degree 0. For the induction step  $n - 1 \rightarrow n$ , by (45),

$$N(n + \ell, x_{i_{n-1}} \cdots x_{i_0}, 2^{\ell} + 2v) = \sum_{\mu=0}^{v} \left( N(n - 1 + \ell, x_{i_{n-2}} \cdots x_{i_0}, 2^{\ell} + 2\mu) * \right)_{2} \left( 2^{\ell} + 2\mu - \sum_{t=0}^{\ell-q-2} \binom{n - 1 + \ell}{t} \right)_{*} \left( 2^{n-1+\ell} - (2^{\ell} + 2\mu) \right)_{(53)}$$

$$\stackrel{(h)}{\leq} \sum_{\mu=0}^{v} \left( N(n - 1 + \ell, x_{i_{n-2}} \cdots x_{i_0}, 2^{\ell} + 2\mu) * \right)_{*}$$

$$2^{2^{\ell}+2\mu+(\nu-\mu)(n+\ell)}\bigg),$$
(54)

where (h) is from  $\binom{2^{n-1+\ell}-(2^{\ell}+2\mu)}{v-\mu} \leq 2^{(v-\mu)(n+\ell)}$ , and  $\sum_{t=0}^{\ell-q-2} \binom{n-1+\ell}{t} = 0$  since  $\ell \leq q+1$ . If  $i_{n-2} = n-2+\ell$ , by inductive hypothesis on n-1,

$$N(n-1+\ell, x_{i_{n-2}}\cdots x_{i_0}, 2^{\ell}+2\mu) \le C_{\mu}(n-1+\ell)2^{(2^{\ell}+2\mu)(n-1+\ell)}.$$
(55)

If  $i_{n-2} = n - 2 + \overline{\ell}$ ,  $\overline{\ell} < \ell$ , apply (46)  $\ell - \overline{\ell}$  times repeatedly,

$$N(n-1+\ell, x_{i_{n-2}}\cdots x_{i_0}, 2^{\ell}+2\mu) = N(n-1+\overline{\ell}, x_{i_{n-2}}\cdots x_{i_0}, 2^{\overline{\ell}}+2\frac{\mu}{2^{\ell-\overline{\ell}}}).$$
(56)

If  $\frac{\mu}{2\ell-\overline{\ell}}$  is an integer, by inductive hypothesis on  $\overline{\ell}$ ,

$$N(n-1+\overline{\ell}, x_{i_{n-2}}\cdots x_{i_{0}}, 2^{\overline{\ell}}+2\frac{\mu}{2^{\ell-\overline{\ell}}})$$

$$\leq C_{\frac{\mu}{2^{\ell-\overline{\ell}}}}(n-1+\overline{\ell})2^{(2^{\overline{\ell}}+2\frac{\mu}{2^{\ell-\overline{\ell}}})(n-1+\overline{\ell})}$$

$$\leq C_{\mu}(n-1+\ell)2^{(2^{\ell}+2\mu)(n-1+\ell)}, \qquad (57)$$

otherwise

$$N(n-1+\bar{\ell}, x_{i_{n-2}}\cdots x_{i_0}, 2^{\bar{\ell}}+2\frac{\mu}{2^{\ell-\bar{\ell}}})=0.$$
 (58)

Combine (55)-(58), we have

$$N(n-1+\ell, x_{i_{n-2}}\cdots x_{i_0}, 2^{\ell}+2\mu) \le C_{\mu}(n-1+\ell)2^{(2^{\ell}+2\mu)(n-1+\ell)}.$$
(59)

Continue the proof in (54), we have

$$N(n + \ell, x_{i_{n-1}} \cdots x_{i_0}, 2^{\ell} + 2v)$$

$$\leq \sum_{\mu=0}^{\nu-1} C_{\mu}(n - 1 + \ell) 2^{(2^{\ell} + \nu + \mu)(n + \ell)}$$

$$+ C_{\nu}(n - 1 + \ell) 2^{(2^{\ell} + 2\nu)(n + \ell)}$$

$$= \left(\sum_{\mu=0}^{\nu-1} C_{\mu} \frac{n - 1 + \ell}{2^{(\nu-\mu)(n+\ell)}} + C_{\nu}(n - 1 + \ell)\right) 2^{(2^{\ell} + 2\nu)(n + \ell)}$$

$$\leq \left(\sum_{\mu=0}^{\nu-1} C_{\mu} \frac{n + \ell}{2^{n+\ell}} + C_{\nu}(n - 1 + \ell)\right) 2^{(2^{\ell} + 2\nu)(n + \ell)}$$

$$= \left(1 + \sum_{\mu=1}^{\nu-1} 2^{\mu-1} + 2^{\nu-1}(n - 1 + \ell)\right) 2^{(2^{\ell} + 2\nu)(n + \ell)}$$

$$= C_{\nu}(n + \ell) 2^{(2^{\ell} + 2\nu)(n + \ell)}, \qquad (60)$$

the induction step  $n-1 \rightarrow n$  holds, thus we complete the proof of (49).

Therefore, when  $i_{r-q-1} = r - q - 1 + \ell$ ,  $\ell \leq q + 1$ ,

$$A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k}) = \sum_{\substack{d=2^{m-r+k}\\ 2^{k-q+\ell-1} - \lceil 2^{\ell-1} \rceil\\ v=0}}^{2^{m-r+k}} N(m, x_{i_{r-q-1}} \cdots x_{i_0}, d)$$

$$\stackrel{(i)}{=} \sum_{\substack{v=0\\ 2^{k-q+\ell-1} - \lceil 2^{\ell-1} \rceil\\ v=0}}^{2^{k-q+\ell-1} - \lceil 2^{\ell-1} \rceil} N(r-q+\ell, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{\ell} + 2v)$$

$$\leq \sum_{\substack{v=0\\ v=0}}^{2^{k-q+\ell-1} - \lceil 2^{\ell-1} \rceil} C_v(r-q+\ell) 2^{(2^{\ell}+2v)(r-q+\ell)}$$
(61)

$$\leq^{(j)} (2^{k-q+\ell-1} - \lceil 2^{\ell-1} \rceil + 1) *$$

$$C_{2^{k-q+\ell-1}}(r-q+\ell) 2^{2^{k-q+\ell}(r-q+\ell)}$$
(62)

$$\stackrel{(k)}{\leq} 2^{k+2} C_{2^{k+2}}(r+3) 2^{(2^{k+2}-1)(r+3)},\tag{63}$$

where (i) is due to (48). In (j), since the sum terms in (61) are increasing with respect to  $v, \forall v \leq 2^{k-q+\ell-1} - \lceil 2^{\ell-1} \rceil$ ,

$$C_{v}(r-q+\ell)2^{(2^{\ell}+2v)(r-q+\ell)} \leq C_{2^{k-q+\ell-1}-\lceil 2^{\ell-1}\rceil}(r-q+\ell)2^{2^{k-q+\ell+2^{\ell}-2\lceil 2^{\ell-1}\rceil}(r-q+\ell)} \leq C_{2^{k-q+\ell-1}}(r-q+\ell)2^{2^{k-q+\ell}(r-q+\ell)}.$$
(64)

(k) is due to  $\ell \leq q + 1$ , and we take (63) as an upper bound on (62) independent of  $\ell$  for the convenience of the following analysis.

If  $i_{r-q-1} = r-q-1+\ell$ ,  $q+2 \le \ell \le q+3$ , similar results can be proved via induction, due to space limitation, we only provide inductive hypothesis when  $q+2 \le \ell \le q+3$  without proof.

If 
$$\ell = q + 2$$
,  $i_{r-q-1} = r + 1$ ,  
 $N(r + 2, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{q+2} + 2v)$   
 $\leq C_v(r+2)2^{(2^{q+2}+2v-1)(r+2)}$ . (65)

The only difference between the proof of (65) and (49) is that

in (53), 
$$\sum_{t=0}^{t-q-2} \binom{n-1+\ell}{t} = 1$$
 when  $\ell = q+2$ .  
Similar to (63), we have

$$A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k})$$

$$\leq 2^{k+1} C_{2^{k+1}-2^{q+1}}(r+2) 2^{(2^{k+2}-1)(r+2)}$$

$$\leq 2^{k+2} C_{2^{k+2}}(r+3) 2^{(2^{k+2}-1)(r+3)}.$$
(66)

If 
$$\ell = q + 3$$
,  $i_{r-q-1} = r + 2$ ,  
 $N(r+3, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{q+3} + 2v)$   
 $\leq C_v(r+3)2^{(2^{q+2}+v-1)(r+3)}.$  (67)

The only difference between the proof of (67) and (49) is that in (53),  $\sum_{t=0}^{\ell-q-2} \binom{n-1+\ell}{t} = n+q+3$  when  $\ell = q+3$ . Similar to (63), we have

 $A(m, r_i, \dots, r_i, 2^{m-r+k})$ 

$$\leq 2^{k+2}C_{2^{k+2}-2^{q+2}}(r+3)2^{(2^{k+2}-1)(r+3)} \leq 2^{k+2}C_{2^{k+2}-2^{q+2}}(r+3)2^{(2^{k+2}-1)(r+3)}.$$
(68)

Combine (63) (66) (68), if  $i_{r-q-1} \le r+2$ ,

$$\log A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k}) \le (2^{k+2} - 1)r + \log r + O(1).$$
(69)

Now we are ready to prove (14). On the one hand,  $x_{r+1} \cdots x_{k+2} \in \text{RM}(m, r)$  when  $m - r \ge 2$ , by (40),

$$\log E(A(2^{m-r+k}, T)) \ge \log N(m, x_{r+1} \cdots x_{k+2}, 2^{m-r+k})$$
$$= (2^{k+2} - 1)(r-k).$$
(70)

On the other hand, by (12) and (13), similar to (36),  $\sum_{q=0}^{k-1} \sum_{i_{r-q-1} \ge r+3} A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k}) \text{ and } \sum_{i_{r-k-1} \ge r+3} N(m, x_{i_{r-k-1}} \cdots x_{i_0}, 2^{m-r+k}) \text{ are negligible,}$ 

$$E(A(2^{m-r+k},T))$$

$$\leq \sum_{q=0}^{k-1} \sum_{i_{r-q-1} \leq r+2} A(m, x_{i_{r-q-1}} \cdots x_{i_0}, 2^{m-r+k}) + \sum_{i_{r-k-1} \leq r+2} N(m, x_{i_{r-k-1}} \cdots x_{i_0}, 2^{m-r+k}) + O(1)$$

$$\leq \sum_{q=0}^{k-1} \binom{r+3}{q+3} 2^{k+2} C_{2^{k+2}}(r+3) 2^{(2^{k+2}-1)(r+3)} + \binom{r+3}{k+3} 2^{(2^{k+2}-1)(r-k)} + O(1).$$
(71)

Combine (70) and (71), we have

$$(2^{k+2} - 1)(r - k) \le \log E \left( A(2^{m-r+k}, T) \right) \le (2^{k+2} - 1)r + O(\log r).$$
(72)

# APPENDIX E

Proof of Theorem 4. Firstly, we prove

$$\mathcal{N}(i_s, s) = 2^{i_s - s} - 2^{i_s} + \sum_{t=0}^{s+1} \binom{i_s}{t} \le i_s - s + 1.$$
(73)

If s = 0 or i<sub>s</sub> - s ≤ 2, (73) holds as an equality.
 If s ≥ 1 and i<sub>s</sub> - s ≥ 3, by (27),

$$\mathcal{N}(i_s, s) \le 3 < i_s - s + 1.$$
 (74)

Therefore, if  $s \ge 1$  and  $i_s - s \ge 3$ ,

$$2^{i_s - s} - 2^{i_s} + \sum_{t=0}^{s+1} \binom{i_s}{t} < i_s - s + 1.$$
(75)

Now we are ready to prove (18),

$$E(N(2^{m-\overline{r}}, T))$$

$$= \sum_{x_{i_{\overline{r}-1}}\cdots x_{i_{0}}\in\mathcal{I}} 2^{\sum_{s=0}^{\overline{r}-1} (2^{i_{s}-s}-2^{i_{s}})+|\{I_{f}>I_{x_{i_{\overline{r}-1}}\cdots x_{i_{0}}}|f\in\mathcal{I}\}|}$$

$$\leq \sum_{x_{i_{\overline{r}-1}}\cdots x_{i_{0}}\in\mathcal{I}} 2^{\sum_{s=0}^{\overline{r}-1} (2^{i_{s}-s}-2^{i_{s}})+|\{I_{f}>I_{x_{i_{\overline{r}-1}}\cdots x_{i_{0}}}|deg(f)\leq\overline{r}\}|}$$

$$= \sum_{x_{i_{\overline{r}-1}}\cdots x_{i_{0}}\in\mathcal{I}} 2^{\sum_{s=0}^{\overline{r}-1} (2^{i_{s}-s}-2^{i_{s}}+\sum_{t=0}^{s+1} (i_{s}))}$$

$$\leq \sum_{x_{i_{\overline{r}-1}}\cdots x_{i_{0}}\in\mathcal{I}} 2^{\sum_{s=0}^{\overline{r}-1} (i_{s}-s+1)}$$

$$= N(2^{m-\overline{r}}), \qquad (76)$$

where the first inequality holds since  $\overline{r}$  is the largest degree of monomials in  $\mathcal{I}$ .

When  $\overline{r} \leq 1$ , the above two inequalities must be equalities, thus (18) holds as an equality.

When  $\overline{r} > 1$ ,

$$\sum_{s=0}^{\overline{r}-1} (i_s - s + 1) = \sum_{s=0}^{\overline{r}-1} \left( 2^{i_s - s} - 2^{i_s} + \sum_{t=0}^{s+1} \binom{i_s}{t} \right)$$

if and only if  $i_{\overline{r}-1} \leq \overline{r}+1$ , we conclude that (18) holds as an equality if and only if  $\forall x_{i_{\overline{r}-1}} \cdots x_{i_0} \in \mathcal{I}$ 

$$\begin{array}{l} (*') \quad i_{\overline{r}-1} \leq \overline{r}+1. \\ (**') \text{ If } I_f > I_{x_{i_{\overline{r}-1}}} \cdots x_{i_0} \text{ and } deg(f) \leq \overline{r}, \text{then } f \in \mathcal{I}. \end{array}$$

Apparently, (\*') is equivalent to (\*), and (\*\*') is equivalent to (\*\*), thus the proof is completed.