Two-Dimensional Z-Complementary Array Quads with Low Column Sequence PMEPRs

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Abstract—In this paper, we first propose a new design strategy of 2D Z-complementary array quads (2D-ZCAQs) with feasible array sizes. A 2D-ZCAQ consists of four distinct unimodular arrays satisfying zero 2D auto-correlation sums for non-trivial 2D time-shifts within certain zone. Then, we obtain the upper bounds on the column sequence peak-to-mean envelope power ratio (PMEPR) of the constructed 2D-ZCAQs by using specific auto-correlation properties of some seed sequences. The constructed 2D-ZCAQs with bounded column sequence PMEPR can be used as a potential alternative to 2D Golay complementary array sets for practical applications.

Index Terms—Z-Complementary Pairs (ZCPs), Z-Complementary Array Quads (ZCAQs), Peak-to-Mean Envelope Power Ratio (PMEPR).

I. INTRODUCTION

A Golay complementary pair (GCP) represents a pair of sequences satisfying zero out-of-phase auto-correlation sums [1]. Due to the preferable correlation property, GCPs have found numerous engineering applications, including peak-to-mean envelope power ratio (PMEPR) control in coded orthogonal frequency-division multiplexing (OFDM) systems [2], [3], and channel estimation [4]. There have been many construction methods for binary and complex GCPs with various sequence lengths [2], [5]. To obtain the admissible sequence lengths of GCPs, Fan et al. proposed the concept of Z-complementary pairs (ZCPs) in [6]. A ZCP satisfies the zero auto-correlation sums within a zone around the zero time-shift. This region is said to be a zerocorrelation zone (ZCZ). In [7], it has been proved that binary ZCPs can exist for all lengths. Such GCPs and ZCPs fall into the categories of one-dimensional GCPs (1D-GCPs) and 1D-ZCPs, respectively. There have been many research activities to design complementary (and Z-complementary) sequences [8]-[21].

The idea of 1D-GCPs is extended to 2D array sets, called 2D Golay complementary array sets (2D-GCASs) [22]-[24]. A 2D Golay complementary array quad (2D-GCAQ) refers to a 2D-GCAS containing four distinct 2D arrays and satisfying zero 2D auto-correlation sums for any non-trivial 2D time-shifts. An array quad is a set of four distinct arrays with equal size. Since signal processing of 2D-GCASs uses exponential smoothing, there is a considerable unbalance among distinct phases when the number of phases increases [25]. Thus, due to the smaller alphabet size, 2D-GCAQs with

four-phases can be used to distinguish the signal levels at the receiver [25], [26]. However, 2D-GCAQs can exist only for limited array sizes. For instance, a 4-ary 2D-GCAQ with array size 7×3 may not exist due to the unavailability of 2ary and 4-ary 1D-GCP of length 7 [27]. This motivates us to investigate 2D Z-complementary array quads (2D-ZCAQs) with varieties of array sizes and ZCZ widths.

Owing to their perfect auto-correlation properties, 2D-GCASs and 2D-ZCAQs can also be employed as spreading sequences in the ultra wide band (UWB) multi-carrier CDMA (MC-CDMA) systems [28]-[31]. In a 2D-ZCAQbased UWB MC-CDMA system, the data bits can be transmitted over the time slots by adopting the "shift and add" operation to obtain an interference-free communication. At the same time, a high PMEPR issue exists as it is based on the OFDM technique. It has been realized that the PMEPR value is connected only with the column sequences of 2D-ZCAQ/2D-GCAS [28]. In the second dimension (i.e., along y-axis), the column sequences of 2D-ZCAQ can take the role of controlling the PMEPR value. That is, 2D-ZCAQs can offer bandwidth efficiency and reduced PMEPR value along x-axis and y-axis, respectively, in the UWB MC-CDMA system [28]. Thus, it is important to construct 2D-ZCAQs with substantially lower column sequence PMEPR values. So far, most of the works have studied only the design of 2D Z-complementary array pairs (2D-ZCAPs) [32]-[37]. A 2D-ZCAP consists of two distinct arrays satisfying zero 2D auto-correlation sums within the 2D-ZCZ width [33].

In this paper, the main contributions are as follows.

- We present a novel design strategy of 2D-ZCAQs with array size $L \times N$ and 2D-ZCZ width $Z \times N$ by using an arbitrary 1D-GCP of length N and a 1D (L, Z)-ZCPs. The proposed construction method is based on generating 2D polynomials in z-domain. We show that the flexible sequence lengths of complex 1D-GCPs and 1D-ZCPs offer many new array sizes and ZCZ widths of 2D-ZCAQs.
- We derive the upper bounds on the column sequence PMEPR of the constructed 2D-ZCAQs by using specific auto-correlation properties of some existing seed 1D-ZCPs.
- We show that the proposed 2D-ZCAQ has the column sequence PMEPRs at most 3.33, 4.00 and 3.714 when

we consider the seed 1D-ZCPs from [9], [14] and [13, Th. 1], respectively.

II. BACKGROUND

In this section, basic definitions and notations have been presented.

A 2D array $\hat{X} = [\hat{x}_{i,j}]_{i=0,j=0}^{N_1-1,N_2-1}$ of size $N_1 \times N_2$ is said to be a q-ary array if $\hat{x}_{i,j} \in \mathbb{Z}_q$ for all $0 \le i \le N_1 - 1$ and $0 \le j \le N_2 - 1$. Let $\xi_q = e^{-2\pi\sqrt{-1}/q}$ be the q-th root of unity. The corresponding complex-valued array of \hat{X} is denoted by $X = [x_{i,j}]_{i=0,j=0}^{N_1-1,N_2-1}$ modulated by q-phase-shift keying (q-PSK), where $x_{i,j} = \xi_q^{\hat{x}_{i,j}}$.

A. 2D Z-Complementary Array Quads

Given two complex-valued arrays X and Y of size $N_1 \times N_2$, at 2-D time-shift (τ_1, τ_2) , the 2-D aperiodic correlation function [35] of two arrays X and Y is defined by

$$\rho_{X,Y}(\tau_1,\tau_2) = \sum_{i=0}^{N_1 - 1 - \tau_1} \sum_{j=0}^{N_2 - 1 - \tau_2} x_{i,j} y_{i+\tau_1,j+\tau_2}^*, \quad (1)$$

where $0 \leq \tau_1 < N_1$, $0 \leq \tau_2 < N_2$ and $(\cdot)^*$ refers to the complex conjugate. If X = Y, then $\rho_{X,Y}(\tau_1, \tau_2)$ is said to be 2-D aperiodic auto-correlation function (2D-AACF); otherwise, it is 2-D aperiodic cross-correlation function (2D-ACCF). The 2D-AACF of X is denoted as $\rho_X(\tau_1, \tau_2)$.

When $N_1 = 1$, two 2D arrays X and Y become 1D complex-valued sequences $\mathbf{x} = (x_j)_{j=0}^{N_2-1}$ and $\mathbf{y} = (y_j)_{j=0}^{N_2-1}$, respectively, with the corresponding 1D-ACCF given by

$$\rho_{\mathbf{x},\mathbf{y}}(\tau) = \sum_{j=0}^{N_2 - 1 - \tau} x_j y_{j+\tau}^*.$$
 (2)

The 1D-AACF of \boldsymbol{x} is simply denoted by $\rho_{\boldsymbol{x}}(\tau)$.

Definition 1 (Golay Complementary Pairs (GCPs)): A pair (x, y) of 1D sequences with length N is called a Golay complementary pair (GCP) [2] if

$$\rho_{\mathbf{x}}(\tau) + \rho_{\mathbf{y}}(\tau) = \begin{cases} 2N, \text{ if } \tau = 0\\ 0, \text{ if } 0 < |\tau| \le N - 1. \end{cases}$$
(3)

Note that binary GCPs exist only for lengths $2^{\alpha}10^{\beta}26^{\gamma}$ with $\alpha, \beta, \gamma \ge 0$ and $\alpha + \gamma \ge 1 - \beta$. However, complex GCPs allow more flexible lengths of the form $2^{a+u}3^{b}5^{c}11^{d}13^{e}$ with $e, u, a, b, c, d \ge 0, u - c \le e$ and $(b + d) + (c + e) \le a + 1 + 2u$ [27].

Definition 2 (Z-Complementary Pairs (ZCPs)): A pair (x, y) of sequences with length N is called a Z-complementary pair (ZCP) [6] if

$$\rho_{\mathbf{x}}(\tau) + \rho_{\mathbf{y}}(\tau) = \begin{cases} 2N; \ \tau = 0\\ 0; \ 0 < |\tau| \le Z - 1, \end{cases}$$
(4)

where $Z \leq N$. In this case, Z refers to the ZCZ width.

Note that a ZCP becomes conventional GCP if Z = N. A ZCP with sequence length N and ZCZ width Z will be denoted by (N, Z)-ZCP. Definition 3 (2D Z-Complementary Array Quads): A set of 4 unimodular arrays $\{X_1, X_2, X_3, X_4\}$ with size $N_1 \times N_2$ is said to be a 2D Z-complementary array quad (2D-ZCAQ) if

$$\rho_{X_1}(\tau_1, \tau_2) + \rho_{X_2}(\tau_1, \tau_2) + \rho_{X_3}(\tau_1, \tau_2) + \rho_{X_4}(\tau_1, \tau_2) \\
= \begin{cases}
4N_1N_2; \ \tau_1 = \tau_2 = 0 \\
0; \ 0 \le |\tau_1| < Z_1, 0 \le |\tau_2| < Z_2, (\tau_1, \tau_2) \ne (0, 0).
\end{cases}$$
(5)

B. Column Sequence PMEPR in MC-CDMA Systems

For a given q-PSK modulated array $X = \left[\xi_q^{\hat{x}_{i,j}}\right]_{i=0,j=0}^{N_1-1,N_2-1}$, let $X_j^T = \left[X_{0,j}, X_{1,j}, \cdots, X_{N_1-1,j}\right]^T$ be the *j*-th column sequence of X, where $0 \le j \le N_2 - 1$. Then, the time-domain complex baseband OFDM signal can be written by

$$S_{\mathbf{X}_{j}^{T}}(t) = \sum_{i=0}^{N_{1}-1} \xi_{q}^{\hat{x}_{i,j}} e^{2\pi\sqrt{-1}(f_{c}+i\Delta f)t}, \ 0 \le t \le \frac{1}{\Delta f}, \quad (6)$$

where f_c denotes the carrier frequency, Δf refers to the subcarrier spacing, and N_1 denotes the number of subcarriers in OFDM system. The instantaneous envelope power ratio (IEPR) of the column sequence X_j^T is given by $P_{X_j^T}(t) = \frac{\left|S_{X_j^T}(t)\right|^2}{N_1}$. Note that N_1 is the mean envelope power for q-PSK modulated sequences. Then, the peak-to-

power for q-PSK modulated sequences. Then, the peak-tomean envelope power ratio (PMEPR) [2], [3] of the column sequence X_j^T can be defined by

$$PMEPR\left(X_{j}^{T}\right) = \sup_{0 \le t \le \frac{1}{\Delta f}} P_{X_{j}^{T}}(t).$$

$$(7)$$

In MC-CDMA system, the column sequence PMEPR value of X_j^T should be as low as possible. In this case, the column sequence X_j^T is spreaded over the N_1 subcarriers in the *j*-th chip-slot [38]. That is, $X_{i,j}$ is the modulated signal in the *i*-th subcarrier for $i = 0, 1, \dots, N_1 - 1$.

C. Generating Polynomials

The generating polynomial [36], [39] of 2D array X of size $N_1 \times N_2$ over two variables z_1^{-1} and z_2^{-1} is given by

$$X(z_1, z_2) = \sum_{i=0}^{N_1 - 1} \sum_{j=0}^{N_2 - 1} x_{i,j} z_1^{-i} z_2^{-j}.$$
 (8)

Note that $X(z_1, z_2)$ can also be considered as 2D *z*-transform of the array X [36]. The 2D polynomial (i.e., *z*-transform) of 2D-AACF $\rho_X(\tau_1, \tau_2)$ is given by

$$\rho_X(z_1, z_2) = X(z_1, z_2) \cdot X^* \left(z_1^{-1}, z_2^{-1} \right). \tag{9}$$

The polynomial representation of 1D-AACF $\rho_{x,x}(\tau)$ of x can be expressed by $\rho_{x,x}(z) = x(z) \cdot x^*(z^{-1})$, where x(z) denotes the polynomial of the 1D sequence x. The polynomial representation of 1D-AACF $\rho_x(\tau)$ is simply denoted as $\rho_x(z) = x(z) \cdot x^*(z^{-1})$. Thus, x and y form an (N, Z)-ZCP if

$$\rho_{\mathbf{x}}(z) + \rho_{\mathbf{y}}(z) = 2N + \sum_{Z \le |\tau| < N} (\rho_{\mathbf{x}}(\tau) + \rho_{\mathbf{y}}(\tau)) z^{-\tau}.$$
 (10)

We will use (·) to denote both conjugate and reverseordering of a sequence. That is, $\tilde{\mathbf{x}} = [x^*(N-1), x^*(N-2), \cdots, x^*(1), x^*(0)].$

III. NEW DESIGN METHOD OF TWO-DIMENSIONAL Z-COMPLEMENTARY ARRAY QUADS

In this section, we first propose a novel construction method for 2D-ZCAQs. Then, we compare the proposed 2D-ZCAQ parameters with some existing works.

Theorem 1: Given a 1D-GCP (x(z), y(z)) of length N and a 1D (L, Z)-ZCP (a(z), b(z)), we construct four arrays of equal size $L \times N$ in z-domain as follows:

$$X_1(z_1, z_2) = a(z_1)x(z_2); X_2(z_1, z_2) = b(z_1)y(z_2)$$

$$X_3(z_1, z_2) = -a(z_1)\tilde{y}(z_2); X_4(z_1, z_2) = b(z_1)\tilde{x}(z_2).$$
(11)

Then, the set $\{X_1, X_2, X_3, X_4\}$ forms a 2D-ZCAQ with array size $L \times N$ and 2D-ZCZ width $Z \times N$.

Proof: According to (11), it is clear that each of the arrays is unimodular array of size $L \times N$. Since (a, b) is an (L, Z)-ZCP and (x, y) is a GCP of length N, we have

$$\rho_{\mathbf{x}}(z) + \rho_{\mathbf{y}}(z) = 2N,\tag{12}$$

$$\rho_{a}(z) + \rho_{b}(z) = 2L + \sum_{Z \le |\tau| < L} (\rho_{a}(\tau) + \rho_{b}(\tau)) z^{-\tau}.$$
 (13)

One can notice that $\rho_{\tilde{y}}(z) = \rho_y(z)$ and $\rho_{\tilde{x}}(z) = \rho_x(z)$. By using (12) and (13), we have

$$\rho_{X_1}(z_1, z_2) + \rho_{X_2}(z_1, z_2) + \rho_{X_3}(z_1, z_2) + \rho_{X_4}(z_1, z_2)$$

= 2N(\rho_a(z_1) + \rho_b(z_1))
= 4NL + \sum_{Z \leq |\tau_1| < L} 2N(\rho_a(\tau_1) + \rho_b(\tau_1))z_1^{-\tau_1}. (14)

The 2D-ZCZ width is $Z_1 \times Z_2$, where $Z_1 = Z$ and $Z_2 = N$. Thus, the set $\{X_1, X_2, X_3, X_4\}$ is a 2D-ZCAQ with array size $L \times N$ and 2D-ZCZ width $Z \times N$.

Our design method uses a combination of seed sequences formed by 1D-GCPs and 1D-ZCPs. The previous design [22, Th. 3.5] does not admit such a possibility leading to unimodular 2D-ZCAQs. We compare the set sizes, array sizes and 2D-ZCZ widths for different existing construction methods of 2D-ZCAPs in Table I.

Remark 1: The number of phases of the constructed 2D-ZCAQs is given by $q = \text{lcm}\{q_0, q_1\}$, where q_0 and q_1 denote the number of phases for the seed 1D-GCP and 1D-ZCP, respectively.

Example 1: Let N = 3 and L = 7 with Z = 4. We consider a length-3 four-phase GCP (\mathbf{x}, \mathbf{y}) with $\mathbf{x} = [+ + -]$ and $\mathbf{y} = [+ j +]$, and a binary (7, 4)-ZCP (\mathbf{a}, \mathbf{b}) with $\mathbf{a} = [+ + + + - - +]$ and $\mathbf{b} = [+ + - + - + +]$, where + and - represent +1 and -1, respectively. We have four 7×3 arrays X_1, X_2, X_3 and X_4 given by

Fig. 1, we show the absolute values of 2D-AACF sums for four arrays X_1 , X_2 , X_3 and X_4 . The number of phases of the constructed array is $q = \text{lcm}\{2,4\} = 4$. Thus, the set $\{X_1, X_2, X_3, X_4\}$ forms a 4-ary 2D-ZCAQ with array size 7×3 and 2D-ZCZ width 4×3 .



Fig. 1: The sum of 2D-AACFs from Example 1

IV. THE COLUMN SEQUENCE PMEPR VALUES OF 2D Z-COMPLEMENTARY ARRAY QUADS

For any sequence pair (p, q) of length L, there is a wellknown connection between the auto-correlation properties of p and q and their PMEPR values given by [8]

$$PMEPR(\boldsymbol{p}) \le 2 + \frac{2}{L} \sum_{\tau=1}^{L-1} \left| \rho_{\boldsymbol{p}}(\tau) + \rho_{\boldsymbol{q}}(\tau) \right|.$$
(15)

In what follows, we provide a theorem to show that the column sequence PMEPR values of the constructed 2D-ZCAQs from *Theorem 1* is connected only to the specific auto-correlation properties of the seed (L, Z)-ZCP (a, b).

Theorem 2: The column sequence PMEPR values of the constructed 2D-ZCAQs from *Theorem 1* are given by

$$PMEPR(X_{m,j}^{T}) \le 2 + \frac{2}{L} \sum_{\tau=1}^{L-1} \left| \rho_{a}(\tau) + \rho_{b}(\tau) \right|, \quad (16)$$

where $X_{m,j}^T$ is the *j*-th column sequence of the array X_m with m = 1, 2, 3, 4 and $j = 0, 1, \dots, N-1$.

Proof: According to (11), the array X_1 of size $L \times N$ can be written in time-domain as follows:

$$\mathbf{X}_{1} = \begin{bmatrix} x_{0} \boldsymbol{a}^{T} & x_{1} \boldsymbol{a}^{T} & \cdots & x_{N-1} \boldsymbol{a}^{T} \end{bmatrix}$$

= $[X_{1,0}, X_{1,1}, \cdots, X_{1,N-1}]$ (17)

Therefore, the *j*-th column sequence $X_{1,j}^T$ of the array X_1 is given by $X_{1,j}^T = x_j a$ for $j = 0, 1, \dots, N-1$. Similarly, the *j*-th column sequence $X_{2,j}^T, X_{3,j}^T$ and $X_{4,j}^T$ of the array X_2, X_3 , and X_4 , respectively, can be written by

$$X_{2,j}^T = y_j \boldsymbol{b}; \ X_{3,j}^T = -y_{N-1-j}^* \boldsymbol{a} \text{ and } X_{4,j}^T = x_{N-1-j}^* \boldsymbol{b}.$$
(18)

According to (15), the column sequence PMEPR value for the sequence pair $(X_{1,i}^T, X_{4,N-1-i}^T)$ of length L is given by

$$PMEPR(X_{1,j}^{T}) \leq 2 + \frac{2}{L} \sum_{\tau=1}^{L-1} \left| x_{j}^{2}(\rho_{a}(\tau) + \rho_{b}(\tau)) \right|$$
$$\leq 2 + \frac{2}{L} \sum_{\tau=1}^{L-1} \left| \rho_{a}(\tau) + \rho_{b}(\tau) \right|, \quad (19)$$

Table I: A Summary of Existing Works

Refs.	Phases	Set Size	Array Size	2D-ZCZ Width	Constraints	Based On
[33]	2	2	$L_1 \times L_2$	$Z_1 \times Z_2$	$L_1, L_2 \ge 2$	(L_1, Z_1) -ZCP and (L_2, Z_2) -ZCP
[34]	q	2	$(2^{m_1-1}+2^{n+1}) \times (2^{m_2}+4)$	$(2^{\pi(n+1)} + 2^{n+1}) \times (2^{m_2-2} + 2^{\phi(m_2-3)} + 1)$	$2 q; m_1 \ge 2; n \le m_1 - 3; m_2 \ge 4$	Generalized Boolean Functions
[35]	q	2	$2^n \times (2^{m-1} + \sum_{\alpha=t+1}^{m-1} d_\alpha 2^{\alpha-1} + 2^{\nu})$	$2^n \times (2^{t-1} + 2^{\nu})$	$2 q; m, n, t, \nu \ge 0; \nu < t < m; d_{\alpha} \in \{0, 1\}$	2D Generalized Boolean Functions
[37, Th. 2]	q	2	$14 \cdot 2^n \times 2^{m-n}$	$12 \cdot 2^n \times 2^{m-n}$	$2 q; 0 \le n \le m$	2D Generalized Boolean Functions
[37, Lem. 5]	q	2	$2^n \times (2^{m-1} + \sum_{\alpha=t+1}^{m-1} d_\alpha 2^{\alpha-1} + 2^{\nu})$	$2^n \times (2^{t-1} + 2^{\nu})$	$2 q; m, n, t, \nu \ge 0; \nu < t < m; d_{\alpha} \in \{0, 1\}$	Generalized Boolean Functions
Proposed	q	4	$L \times 2^{\alpha} 10^{\beta} 26^{\gamma}$	$Z \times 2^{\alpha} 10^{\beta} 26^{\gamma}$	$q = lcm\{2, q_1\}, q_1 \ge 2$	Binary GCP and (L, Z) -ZCP
	q	4	$L \times 2^{a+u} 3^b 5^c 11^d 13^e$	$Z \times 2^{a+u} 3^b 5^c 11^d 13^e$	$q = \operatorname{lcm}\{q_0, q_1\}, q_0, q_1 \ge 2$	Complex GCP and (L, Z) -ZCP

where $|x_j| = 1$. Similarly, the column sequence PMEPR value for the sequence pair $(X_{2,j}^T, X_{3,N-1-j}^T)$ of length L is given by

$$PMEPR(X_{2,j}^{T}) \le 2 + \frac{2}{L} \sum_{\tau=1}^{L-1} \left| \rho_{a}(\tau) + \rho_{b}(\tau) \right|, \qquad (20)$$

where $|y_i| = 1$. This completes the proof.

Remark 2: Based on *Theorem 2*, the PMEPR values are calculated by using only unimodular seed 1D-GCPs and 1D-ZCPs in *Theorem 1*.

Proposition 1: The column sequence PMEPR of the constructed 2D-ZCAQs by *Theorem 1* is upper bounded by 3.33 when a seed (L, Z)-ZCP from [9] is used with $L = 2^{n+1} + 2^n$ and $Z = 2^{n+1}$.

Proof: In [9, Th. 3], it has been shown that the constructed (L, Z)-ZCP (a, b) from [9] has the following auto-correlation property given by

$$\rho_{a}(\tau) + \rho_{b}(\tau) = \begin{cases} 2^{n+2} + 2^{n+1}, & \text{if } \tau = 0, \\ \pm 2^{n+1}, & \text{if } \tau = 2^{n+1}, \\ 0, & \text{otherwise.} \end{cases}$$
(21)

By using (19), the *j*-th column sequence $X_{1,j}^T$ of the array X_1 is given by

$$\mathsf{PMEPR}(X_{1,j}^T) \le 2 + \frac{2}{2^{n+1} + 2^n} \cdot 2^{n+1} = 2 + \frac{4}{3} \approx 3.33$$
(22)

for $j = 0, 1, \dots, N - 1$. Similarly, we can show that $\text{PMEPR}(X_{2,j}^T) \leq 3.33$, $\text{PMEPR}(X_{3,N-1-j}^T) \leq 3.33$ and $\text{PMEPR}(X_{4,N-1-j}^T) \leq 3.33$. This completes the proof.

Example 2: Let n = 3 with L = 24, Z = 16 and N = 32. Let us consider a (24, 16)-ZCP (a, b) constructed from [9] with a = [+-+-++-----] and b = [-++----++--+---]. We take a GCP (x, y) of length 32 with x = [+++++---] and y = [-----++++--+--+----+--] and y = [-----++++---++---+---+--]. Based on *Theorem 1*, we construct four arrays X_1, X_2, X_3 and X_4 of equal size 24×32 . We show the absolute values of 2D-AACF sums for four arrays X_1, X_2, X_3 and X_4 in Fig. 2. We can see that the set



Fig. 2: The sum of 2D-AACFs from Example 2



Fig. 3: The IEPRs of column sequences for X_1, X_2, X_3 and X_4 given in *Example 2*

 ${X_1, X_2, X_3, X_4}$ forms a binary 2D-ZCAQ with array size 24 × 32 and 2D-ZCZ width 16 × 32. The maximum column sequence PMEPR of each column of X_1 and X_3 is 3.197. The maximum column sequence PMEPR of each column of X_2 and X_4 is 2.851. For example, we consider the column sequence IEPR of the first column of X_2 as illustrated in Fig. 3. Note that the sub-carrier spacing is normalized to 1 for Fig. 3. According to Fig. 3, we have

$$PMEPR(X_{2,0}^T) = \sup_{0 \le t \le 1} IEPR_{X_{2,0}^T}(t) = 2.851, \quad (23)$$

whereas the theoretical PMEPR upper bound is 3.33 (given by (22)) from *Proposition 1*. Similarly, the maximum column sequence PMEPR of each column of X_1 is 3.197 as observed in Fig. 3. Proposition 2: The column sequence PMEPR of the constructed 2D-ZCAQs by *Theorem 1* is upper bounded by 4 when a seed (L, Z)-ZCP from [14] is used with L = 2N+2 and Z = 3N/2 + 1.

Proof: By using [14, (16), (17) and (18)], the constructed (L, Z)-ZCP (a, b) from [14] has the following auto-correlation property given by

$$\rho_{a}(\tau) + \rho_{b}(\tau) = \begin{cases}
0, & \text{if } 0 < \tau \le 3N/2 \\
\pm 4, & \text{if } 3N/2 < \tau \le 2N, \\
0, & \text{if } \tau = 2N + 1,
\end{cases}$$
(24)

Based on (19), the *j*-th column sequence $X_{1,j}^T$ of the array X_1 is given by

$$\mathsf{PMEPR}(X_{1,j}^T) \le 2 + \frac{2}{2N+2} \sum_{\tau=3N/2+1}^{2N} 4 < 4 \qquad (25)$$

for $j = 0, 1, \dots, N - 1$. Similarly, we can show that $\text{PMEPR}(X_{2,j}^T) \leq 4$, $\text{PMEPR}(X_{3,N-1-j}^T) \leq 4$ and $\text{PMEPR}(X_{4,N-1-j}^T) \leq 4$. This completes the proof.

Example 3: Let L = 18, Z = 13 and N = 26. Let us take a binary (18, 13)-ZCP (a, b) constructed from [14] with a = [-+++-++-+-+-] and b = (-+++)[++++---+--]. We consider a binary GCP (x, y) of length 26 with x = [++++-++--+-+-+- and y = [++++-+]+--++++++---]. According to Theorem 1, we construct four binary arrays X_1, X_2, X_3 and X_4 of equal size 18×26 . The absolute values of 2D-AACF sums for four arrays X_1, X_2, X_3 and X_4 are shown in Fig. 4. We can observe that the set $\{X_1, X_2, X_3, X_4\}$ forms a binary 2D-ZCAQ with array size 18×26 and 2D-ZCZ width 13×26 . The maximum column sequence PMEPR of each column of X_1 and X_3 is 2.797. The maximum column sequence PMEPR of each column of X_2 and X_4 is 2.706. For example, we consider the column sequence IEPR of the first column of X_1 as illustrated in Fig. 5. Based on Fig. 5, we have

$$PMEPR(X_{1,0}^T) = \sup_{0 \le t \le 1} IEPR_{X_{1,0}^T}(t) = 2.797, \quad (26)$$

whereas the theoretical PMEPR upper bound is 4 from *Proposition 2*. Similarly, the maximum column sequence PMEPR of each column of X_2 is 2.706 as depicted in Fig. 5.

Proposition 3: The column sequence PMEPR of the constructed 2D-ZCAQs by *Theorem 1* is upper bounded by 3.714 when a seed (L, Z)-ZCP from [13, Th. 1] is used with $L = 2^{n+3} + 2^{n+2} + 2^{n+1}$ and $Z = 2^{n+3}$.

Proof: The constructed (L, Z)-ZCP (a, b) from [13, Th. 1] has the following auto-correlation property [17] given by

$$\begin{split} \rho_{a}(\tau) &+ \rho_{b}(\tau) \\ &= \begin{cases} 2^{n+4} + 2^{n+3} + 2^{n+2}, \text{ when } \tau = 0, \\ \pm 2^{n+2}, \text{ when } \tau = 2^{n+3} + l \cdot 2^{n+1} \text{ for } l = 0, 1, 2, \\ 0, \text{ otherwise.} \end{cases} \end{split}$$

(27)



Fig. 4: The sum of 2D-AACFs from Example 3



Fig. 5: The IEPRs of column sequences for X_1, X_2, X_3 and X_4 given in *Example 3*

According to (19), the *j*-th column sequence $X_{1,j}^T$ of the array X_1 is given by

$$PMEPR(X_{1,j}^T) \le 2 + \frac{2}{2^{n+3} + 2^{n+2} + 2^{n+1}} \cdot 3 \cdot 2^{n+2}$$
$$= 2 + \frac{12}{7} \approx 3.714$$
(28)

for $j = 0, 1, \dots, N - 1$. Similarly, we can show that $\text{PMEPR}(X_{2,j}^T) \leq 3.714$, $\text{PMEPR}(X_{3,N-1-j}^T) \leq 3.714$ and $\text{PMEPR}(X_{4,N-1-j}^T) \leq 3.714$.

V. CONCLUSION

In this paper, we have investigated a new construction for 2D-ZCAQs with various array sizes $L \times N$, where $N = 2^{\alpha}10^{\beta}26^{\gamma}$ or $2^{a+u}3^{b}5^{c}11^{d}13^{e}$ and L is any positive integer ≥ 2 . The admissible lengths of 1D-GCPs and 1D-ZCPs allow us to offer more feasible array sizes of 2D-ZCAQs. Furthermore, we have shown that the column sequence PMEPR of the proposed 2D-ZCAQ is upper bounded by 4 when we use seed 1D-ZCPs from [9], [14] and [13, Th. 1]. The proposed 2D-ZCAQs with low column sequence PMEPRs can be utilized as alternatives to 2D-GCASs in UWB MC-CDMA systems.

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