# Two-Dimensional Z-Complementary Array Quads with Low Column Sequence PMEPRs 

Shibsankar Das*, Adrish Banerjee* and Udaya Parampalli ${ }^{\dagger}$<br>*Department of Electrical Engineering, Indian Institute of Technology Kanpur, Kanpur, India<br>${ }^{\dagger}$ School of Computing and Information Systems, University of Melbourne, Victoria, Australia<br>Email:\{shibsankar, adrish\}@iitk.ac.in, udaya@unimelb.edu.au.


#### Abstract

In this paper, we first propose a new design strategy of 2D $Z$-complementary array quads (2D-ZCAQs) with feasible array sizes. A 2D-ZCAQ consists of four distinct unimodular arrays satisfying zero 2D auto-correlation sums for non-trivial 2D time-shifts within certain zone. Then, we obtain the upper bounds on the column sequence peak-to-mean envelope power ratio (PMEPR) of the constructed 2D-ZCAQs by using specific auto-correlation properties of some seed sequences. The constructed 2D-ZCAQs with bounded column sequence PMEPR can be used as a potential alternative to 2D Golay complementary array sets for practical applications.


Index Terms-Z-Complementary Pairs (ZCPs), ZComplementary Array Quads (ZCAQs), Peak-to-Mean Envelope Power Ratio (PMEPR).

## I. Introduction

A Golay complementary pair (GCP) represents a pair of sequences satisfying zero out-of-phase auto-correlation sums [1]. Due to the preferable correlation property, GCPs have found numerous engineering applications, including peak-to-mean envelope power ratio (PMEPR) control in coded orthogonal frequency-division multiplexing (OFDM) systems [2], [3], and channel estimation [4]. There have been many construction methods for binary and complex GCPs with various sequence lengths [2], [5]. To obtain the admissible sequence lengths of GCPs, Fan et al. proposed the concept of $Z$-complementary pairs (ZCPs) in [6]. A ZCP satisfies the zero auto-correlation sums within a zone around the zero time-shift. This region is said to be a zerocorrelation zone (ZCZ). In [7], it has been proved that binary ZCPs can exist for all lengths. Such GCPs and ZCPs fall into the categories of one-dimensional GCPs (1D-GCPs) and 1DZCPs, respectively. There have been many research activities to design complementary (and $Z$-complementary) sequences [8]-[21].

The idea of 1D-GCPs is extended to 2D array sets, called 2D Golay complementary array sets (2D-GCASs) [22][24]. A 2D Golay complementary array quad (2D-GCAQ) refers to a 2D-GCAS containing four distinct 2D arrays and satisfying zero 2D auto-correlation sums for any non-trivial 2D time-shifts. An array quad is a set of four distinct arrays with equal size. Since signal processing of 2D-GCASs uses exponential smoothing, there is a considerable unbalance among distinct phases when the number of phases increases [25]. Thus, due to the smaller alphabet size, 2D-GCAQs with
four-phases can be used to distinguish the signal levels at the receiver [25], [26]. However, 2D-GCAQs can exist only for limited array sizes. For instance, a 4-ary 2D-GCAQ with array size $7 \times 3$ may not exist due to the unavailability of 2 ary and 4 -ary 1D-GCP of length 7 [27]. This motivates us to investigate 2D $Z$-complementary array quads (2D-ZCAQs) with varieties of array sizes and ZCZ widths.

Owing to their perfect auto-correlation properties, 2DGCASs and 2D-ZCAQs can also be employed as spreading sequences in the ultra wide band (UWB) multi-carrier CDMA (MC-CDMA) systems [28]-[31]. In a 2D-ZCAQbased UWB MC-CDMA system, the data bits can be transmitted over the time slots by adopting the "shift and add" operation to obtain an interference-free communication. At the same time, a high PMEPR issue exists as it is based on the OFDM technique. It has been realized that the PMEPR value is connected only with the column sequences of 2 D -ZCAQ/2D-GCAS [28]. In the second dimension (i.e., along $y$-axis), the column sequences of 2D-ZCAQ can take the role of controlling the PMEPR value. That is, 2D-ZCAQs can offer bandwidth efficiency and reduced PMEPR value along $x$-axis and $y$-axis, respectively, in the UWB MC-CDMA system [28]. Thus, it is important to construct 2D-ZCAQs with substantially lower column sequence PMEPR values. So far, most of the works have studied only the design of 2D $Z$-complementary array pairs (2D-ZCAPs) [32]-[37]. A 2D-ZCAP consists of two distinct arrays satisfying zero 2D auto-correlation sums within the 2D-ZCZ width [33].

In this paper, the main contributions are as follows.

- We present a novel design strategy of 2D-ZCAQs with array size $L \times N$ and $2 \mathrm{D}-\mathrm{ZCZ}$ width $Z \times N$ by using an arbitrary 1D-GCP of length $N$ and a 1D $(L, Z)$ ZCPs. The proposed construction method is based on generating 2D polynomials in $z$-domain. We show that the flexible sequence lengths of complex 1D-GCPs and 1D-ZCPs offer many new array sizes and ZCZ widths of 2D-ZCAQs.
- We derive the upper bounds on the column sequence PMEPR of the constructed 2D-ZCAQs by using specific auto-correlation properties of some existing seed 1D-ZCPs.
- We show that the proposed 2D-ZCAQ has the column sequence PMEPRs at most $3.33,4.00$ and 3.714 when
we consider the seed 1D-ZCPs from [9], [14] and [13, Th. 1], respectively.


## II. Background

In this section, basic definitions and notations have been presented.

A 2D array $\hat{\boldsymbol{X}}=\left[\hat{x}_{i, j}\right]_{i=0, j=0}^{N_{1}-1, N_{2}-1}$ of size $N_{1} \times N_{2}$ is said to be a $q$-ary array if $\hat{x}_{i, j} \in \mathbb{Z}_{q}$ for all $0 \leq i \leq N_{1}-1$ and $0 \leq j \leq N_{2}-1$. Let $\xi_{q}=e^{-2 \pi \sqrt{-1} / q}$ be the $q$-th root of unity. The corresponding complex-valued array of $\hat{\boldsymbol{X}}$ is denoted by $\boldsymbol{X}=\left[x_{i, j}\right]_{i=0, j=0}^{N_{1}-1, N_{2}-1}$ modulated by $q$-phase-shift keying ( $q$-PSK), where $x_{i, j}=\xi_{q}^{\hat{x}_{i, j}}$.

## A. 2D Z-Complementary Array Quads

Given two complex-valued arrays $\boldsymbol{X}$ and $\boldsymbol{Y}$ of size $N_{1} \times$ $N_{2}$, at 2-D time-shift $\left(\tau_{1}, \tau_{2}\right)$, the 2-D aperiodic correlation function [35] of two arrays $\boldsymbol{X}$ and $\boldsymbol{Y}$ is defined by

$$
\begin{equation*}
\rho_{\boldsymbol{X}, \boldsymbol{Y}}\left(\tau_{1}, \tau_{2}\right)=\sum_{i=0}^{N_{1}-1-\tau_{1}} \sum_{j=0}^{N_{2}-1-\tau_{2}} x_{i, j} y_{i+\tau_{1}, j+\tau_{2}}^{*} \tag{1}
\end{equation*}
$$

where $0 \leq \tau_{1}<N_{1}, 0 \leq \tau_{2}<N_{2}$ and $(\cdot)^{*}$ refers to the complex conjugate. If $\boldsymbol{X}=\boldsymbol{Y}$, then $\rho_{\boldsymbol{X}, \boldsymbol{Y}}\left(\tau_{1}, \tau_{2}\right)$ is said to be 2-D aperiodic auto-correlation function (2D-AACF); otherwise, it is 2-D aperiodic cross-correlation function (2DACCF). The 2D-AACF of $\boldsymbol{X}$ is denoted as $\rho_{\boldsymbol{X}}\left(\tau_{1}, \tau_{2}\right)$.

When $N_{1}=1$, two 2 D arrays $\boldsymbol{X}$ and $\boldsymbol{Y}$ become 1D complex-valued sequences $\boldsymbol{x}=\left(x_{j}\right)_{j=0}^{N_{2}-1}$ and $\boldsymbol{y}=$ $\left(y_{j}\right)_{j=0}^{N_{2}-1}$, respectively, with the corresponding 1D-ACCF given by

$$
\begin{equation*}
\rho_{x, y}(\tau)=\sum_{j=0}^{N_{2}-1-\tau} x_{j} y_{j+\tau}^{*} \tag{2}
\end{equation*}
$$

The 1D-AACF of $\boldsymbol{x}$ is simply denoted by $\rho_{\boldsymbol{x}}(\tau)$.
Definition 1 (Golay Complementary Pairs (GCPs)): A pair $(\boldsymbol{x}, \boldsymbol{y})$ of 1D sequences with length $N$ is called a Golay complementary pair (GCP) [2] if

$$
\rho_{x}(\tau)+\rho_{y}(\tau)= \begin{cases}2 N, & \text { if } \tau=0  \tag{3}\\ 0, & \text { if } 0<|\tau| \leq N-1\end{cases}
$$

Note that binary GCPs exist only for lengths $2^{\alpha} 10^{\beta} 26^{\gamma}$ with $\alpha, \beta, \gamma \geq 0$ and $\alpha+\gamma \geq 1-\beta$. However, complex GCPs allow more flexible lengths of the form $2^{a+u} 3^{b} 5^{c} 11^{d} 13^{e}$ with $e, u, a, b, c, d \geq 0, u-c \leq e$ and $(b+d)+(c+e) \leq$ $a+1+2 u$ [27].

Definition 2 ( $Z$-Complementary Pairs (ZCPs)): A pair $(\boldsymbol{x}, \boldsymbol{y})$ of sequences with length $N$ is called a $Z$ complementary pair (ZCP) [6] if

$$
\rho_{x}(\tau)+\rho_{y}(\tau)= \begin{cases}2 N ; & \tau=0  \tag{4}\\ 0 ; & 0<|\tau| \leq Z-1\end{cases}
$$

where $Z \leq N$. In this case, $Z$ refers to the ZCZ width. Note that a ZCP becomes conventional GCP if $Z=N$. A ZCP with sequence length $N$ and ZCZ width $Z$ will be denoted by $(N, Z)-Z C P$.

Definition 3 (2D Z-Complementary Array Quads): A set of 4 unimodular arrays $\left\{\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}, \boldsymbol{X}_{4}\right\}$ with size $N_{1} \times N_{2}$ is said to be a 2D $Z$-complementary array quad (2D-ZCAQ) if

$$
\begin{align*}
& \rho_{\boldsymbol{X}_{1}}\left(\tau_{1}, \tau_{2}\right)+\rho_{\boldsymbol{X}_{2}}\left(\tau_{1}, \tau_{2}\right)+\rho_{\boldsymbol{X}_{3}}\left(\tau_{1}, \tau_{2}\right)+\rho_{\boldsymbol{X}_{4}}\left(\tau_{1}, \tau_{2}\right) \\
& =\left\{\begin{array}{l}
4 N_{1} N_{2} ; \tau_{1}=\tau_{2}=0 \\
0 ; 0 \leq\left|\tau_{1}\right|<Z_{1}, 0 \leq\left|\tau_{2}\right|<Z_{2},\left(\tau_{1}, \tau_{2}\right) \neq(0,0) .
\end{array}\right. \tag{5}
\end{align*}
$$

## B. Column Sequence PMEPR in MC-CDMA Systems

For a given $q$-PSK modulated array $\boldsymbol{X}=$ $\left[\xi_{q}^{\hat{x}_{i, j}}\right]_{i=0, j=0}^{N_{1}-1, N_{2}-1}$, let $\boldsymbol{X}_{j}^{T}=\left[X_{0, j}, X_{1, j}, \cdots, X_{N_{1}-1, j}\right]^{T}$ be the $j$-th column sequence of $\boldsymbol{X}$, where $0 \leq j \leq N_{2}-1$. Then, the time-domain complex baseband OFDM signal can be written by

$$
\begin{equation*}
S_{X_{j}^{T}}(t)=\sum_{i=0}^{N_{1}-1} \xi_{q}^{\hat{x}_{i, j}} e^{2 \pi \sqrt{-1}\left(f_{c}+i \Delta f\right) t}, \quad 0 \leq t \leq \frac{1}{\Delta f} \tag{6}
\end{equation*}
$$

where $f_{c}$ denotes the carrier frequency, $\Delta f$ refers to the subcarrier spacing, and $N_{1}$ denotes the number of subcarriers in OFDM system. The instantaneous envelope power ratio (IEPR) of the column sequence $\boldsymbol{X}_{j}^{T}$ is given by $P_{\boldsymbol{X}_{j}^{T}}(t)=\frac{\left|S_{X_{j}^{T}}(t)\right|^{2}}{N_{1}}$. Note that $N_{1}$ is the mean envelope power for $q$-PSK modulated sequences. Then, the peak-tomean envelope power ratio (PMEPR) [2], [3] of the column sequence $\boldsymbol{X}_{j}^{T}$ can be defined by

$$
\begin{equation*}
\operatorname{PMEPR}\left(\boldsymbol{X}_{j}^{T}\right)=\sup _{0 \leq t \leq \frac{1}{\Delta f}} P_{\boldsymbol{X}_{j}^{T}}(t) \tag{7}
\end{equation*}
$$

In MC-CDMA system, the column sequence PMEPR value of $\boldsymbol{X}_{j}^{T}$ should be as low as possible. In this case, the column sequence $\boldsymbol{X}_{j}^{T}$ is spreaded over the $N_{1}$ subcarriers in the $j$-th chip-slot [38]. That is, $X_{i, j}$ is the modulated signal in the $i$-th subcarrier for $i=0,1, \cdots, N_{1}-1$.

## C. Generating Polynomials

The generating polynomial [36], [39] of 2D array $\boldsymbol{X}$ of size $N_{1} \times N_{2}$ over two variables $z_{1}^{-1}$ and $z_{2}^{-1}$ is given by

$$
\begin{equation*}
X\left(z_{1}, z_{2}\right)=\sum_{i=0}^{N_{1}-1} \sum_{j=0}^{N_{2}-1} x_{i, j} z_{1}^{-i} z_{2}^{-j} \tag{8}
\end{equation*}
$$

Note that $X\left(z_{1}, z_{2}\right)$ can also be considered as 2D $z-$ transform of the array $\boldsymbol{X}$ [36]. The 2D polynomial (i.e., $z$ transform) of $2 \mathrm{D}-\mathrm{AACF} \rho_{\boldsymbol{X}}\left(\tau_{1}, \tau_{2}\right)$ is given by

$$
\begin{equation*}
\rho_{\boldsymbol{X}}\left(z_{1}, z_{2}\right)=X\left(z_{1}, z_{2}\right) \cdot X^{*}\left(z_{1}^{-1}, z_{2}^{-1}\right) . \tag{9}
\end{equation*}
$$

The polynomial representation of 1D-AACF $\rho_{x, \boldsymbol{x}}(\tau)$ of $\boldsymbol{x}$ can be expressed by $\rho_{\boldsymbol{x}, \boldsymbol{x}}(z)=x(z) \cdot x^{*}\left(z^{-1}\right)$, where $x(z)$ denotes the polynomial of the 1D sequence $\boldsymbol{x}$. The polynomial representation of 1D-AACF $\rho_{x}(\tau)$ is simply denoted as $\rho_{\boldsymbol{x}}(z)=x(z) \cdot x^{*}\left(z^{-1}\right)$. Thus, $\boldsymbol{x}$ and $\boldsymbol{y}$ form an $(N, Z)$-ZCP if

$$
\begin{equation*}
\rho_{x}(z)+\rho_{y}(z)=2 N+\sum_{z \leq|\tau|<N}\left(\rho_{x}(\tau)+\rho_{y}(\tau)\right) z^{-\tau} . \tag{10}
\end{equation*}
$$

We will use $(\tilde{\cdot})$ to denote both conjugate and reverseordering of a sequence. That is, $\tilde{\boldsymbol{x}}=\left[x^{*}(N-1), x^{*}(N-\right.$ $\left.2), \cdots, x^{*}(1), x^{*}(0)\right]$.

## III. New Design Method of Two-Dimensional $Z$-Complementary Array Quads

In this section, we first propose a novel construction method for 2D-ZCAQs. Then, we compare the proposed 2D-ZCAQ parameters with some existing works.

Theorem 1: Given a 1D-GCP $(x(z), y(z))$ of length $N$ and a $1 \mathrm{D}(L, Z)$-ZCP $(a(z), b(z))$, we construct four arrays of equal size $L \times N$ in $z$-domain as follows:

$$
\begin{align*}
& X_{1}\left(z_{1}, z_{2}\right)=a\left(z_{1}\right) x\left(z_{2}\right) ; X_{2}\left(z_{1}, z_{2}\right)=b\left(z_{1}\right) y\left(z_{2}\right) \\
& X_{3}\left(z_{1}, z_{2}\right)=-a\left(z_{1}\right) \tilde{y}\left(z_{2}\right) ; X_{4}\left(z_{1}, z_{2}\right)=b\left(z_{1}\right) \tilde{x}\left(z_{2}\right) . \tag{11}
\end{align*}
$$

Then, the set $\left\{\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}, \boldsymbol{X}_{4}\right\}$ forms a $2 \mathrm{D}-\mathrm{ZCAQ}$ with array size $L \times N$ and $2 \mathrm{D}-\mathrm{ZCZ}$ width $Z \times N$.

Proof: According to (11), it is clear that each of the arrays is unimodular array of size $L \times N$. Since $(\boldsymbol{a}, \boldsymbol{b})$ is an $(L, Z)$-ZCP and $(\boldsymbol{x}, \boldsymbol{y})$ is a GCP of length $N$, we have

$$
\begin{align*}
& \rho_{\boldsymbol{x}}(z)+\rho_{\boldsymbol{y}}(z)=2 N,  \tag{12}\\
& \rho_{\boldsymbol{a}}(z)+\rho_{\boldsymbol{b}}(z)=2 L+\sum_{Z \leq|\tau|<L}\left(\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau)\right) z^{-\tau} . \tag{13}
\end{align*}
$$

One can notice that $\rho_{\tilde{y}}(z)=\rho_{\boldsymbol{y}}(z)$ and $\rho_{\tilde{\boldsymbol{x}}}(z)=\rho_{\boldsymbol{x}}(z)$. By using (12) and (13), we have

$$
\begin{align*}
& \rho_{\boldsymbol{X}_{1}}\left(z_{1}, z_{2}\right)+\rho_{\boldsymbol{X}_{2}}\left(z_{1}, z_{2}\right)+\rho_{\boldsymbol{X}_{3}}\left(z_{1}, z_{2}\right)+\rho_{\boldsymbol{X}_{4}}\left(z_{1}, z_{2}\right) \\
& =2 N\left(\rho_{\boldsymbol{a}}\left(z_{1}\right)+\rho_{\boldsymbol{b}}\left(z_{1}\right)\right) \\
& =4 \leq\left|\tau_{1}\right|<L \tag{14}
\end{align*}
$$

The 2D-ZCZ width is $Z_{1} \times Z_{2}$, where $Z_{1}=Z$ and $Z_{2}=N$. Thus, the set $\left\{\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}, \boldsymbol{X}_{4}\right\}$ is a $2 \mathrm{D}-\mathrm{ZCAQ}$ with array size $L \times N$ and $2 \mathrm{D}-\mathrm{ZCZ}$ width $Z \times N$.

Our design method uses a combination of seed sequences formed by 1D-GCPs and 1D-ZCPs. The previous design [22, Th. 3.5] does not admit such a possibility leading to unimodular 2D-ZCAQs. We compare the set sizes, array sizes and 2D-ZCZ widths for different existing construction methods of 2D-ZCAPs in Table I.

Remark 1: The number of phases of the constructed 2DZCAQs is given by $q=\operatorname{lcm}\left\{q_{0}, q_{1}\right\}$, where $q_{0}$ and $q_{1}$ denote the number of phases for the seed 1D-GCP and 1D-ZCP, respectively.

Example 1: Let $N=3$ and $L=7$ with $Z=4$. We consider a length-3 four-phase GCP $(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{x}=[++-]$ and $\boldsymbol{y}=[+j+]$, and a binary $(7,4)-\mathrm{ZCP}(\boldsymbol{a}, \boldsymbol{b})$ with $\boldsymbol{a}=[++++--+]$ and $\boldsymbol{b}=[++-+-++]$, where + and - represent +1 and -1 , respectively. We have four $7 \times 3$ arrays $\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}$ and $\boldsymbol{X}_{4}$ given by
$\boldsymbol{X}_{1}^{T}=\left[\begin{array}{l}++++--+ \\ ++++--+ \\ ----++-\end{array}\right], \boldsymbol{X}_{2}^{T}=\left[\begin{array}{ccccccc}+ & + & - & + & - & + & + \\ j & j & -j & j & -j & j & j \\ + & + & - & + & - & + & +\end{array}\right]$,
$\boldsymbol{X}_{3}^{T}=\left[\begin{array}{cccccc}- & - & - & - & + & + \\ j & j & j & j & -j & -j \\ j \\ - & - & - & - & + & + \\ -\end{array}\right], \boldsymbol{X}_{4}^{T}=\left[\begin{array}{l}--+-+-- \\ ++-+-++ \\ ++-+-++\end{array}\right]$. In
Fig. 1, we show the absolute values of 2D-AACF sums for four arrays $\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}$ and $\boldsymbol{X}_{4}$. The number of phases of the constructed array is $q=\operatorname{lcm}\{2,4\}=4$. Thus, the set $\left\{\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}, \boldsymbol{X}_{4}\right\}$ forms a 4 -ary $2 \mathrm{D}-\mathrm{ZCAQ}$ with array size $7 \times 3$ and $2 \mathrm{D}-\mathrm{ZCZ}$ width $4 \times 3$.


Fig. 1: The sum of 2D-AACFs from Example 1

## IV. The Column Sequence PMEPR Values of 2D $Z$-Complementary Array Quads

For any sequence pair $(\boldsymbol{p}, \boldsymbol{q})$ of length $L$, there is a wellknown connection between the auto-correlation properties of $\boldsymbol{p}$ and $\boldsymbol{q}$ and their PMEPR values given by [8]

$$
\begin{equation*}
\operatorname{PMEPR}(\boldsymbol{p}) \leq 2+\frac{2}{L} \sum_{\tau=1}^{L-1}\left|\rho_{\boldsymbol{p}}(\tau)+\rho_{\boldsymbol{q}}(\tau)\right| \tag{15}
\end{equation*}
$$

In what follows, we provide a theorem to show that the column sequence PMEPR values of the constructed 2DZCAQs from Theorem 1 is connected only to the specific auto-correlation properties of the seed $(L, Z)$-ZCP $(\boldsymbol{a}, \boldsymbol{b})$.

Theorem 2: The column sequence PMEPR values of the constructed 2D-ZCAQs from Theorem 1 are given by

$$
\begin{equation*}
\operatorname{PMEPR}\left(X_{m, j}^{T}\right) \leq 2+\frac{2}{L} \sum_{\tau=1}^{L-1}\left|\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau)\right| \tag{16}
\end{equation*}
$$

where $X_{m, j}^{T}$ is the $j$-th column sequence of the array $\boldsymbol{X}_{m}$ with $m=1,2,3,4$ and $j=0,1, \cdots, N-1$.

Proof: According to (11), the array $\boldsymbol{X}_{1}$ of size $L \times N$ can be written in time-domain as follows:

$$
\begin{align*}
\boldsymbol{X}_{1} & =\left[\begin{array}{llll}
x_{0} \boldsymbol{a}^{T} & x_{1} \boldsymbol{a}^{T} & \cdots & x_{N-1} \boldsymbol{a}^{T}
\end{array}\right] \\
& =\left[\begin{array}{ll}
X_{1,0}, X_{1,1}, \cdots, X_{1, N-1}
\end{array}\right] \tag{17}
\end{align*}
$$

Therefore, the $j$-th column sequence $X_{1, j}^{T}$ of the array $\boldsymbol{X}_{1}$ is given by $X_{1, j}^{T}=x_{j} \boldsymbol{a}$ for $j=0,1, \cdots, N-1$. Similarly, the $j$-th column sequence $X_{2, j}^{T}, X_{3, j}^{T}$ and $X_{4, j}^{T}$ of the array $\boldsymbol{X}_{2}, \boldsymbol{X}_{3}$, and $\boldsymbol{X}_{4}$, respectively, can be written by

$$
\begin{equation*}
X_{2, j}^{T}=y_{j} \boldsymbol{b} ; \quad X_{3, j}^{T}=-y_{N-1-j}^{*} \boldsymbol{a} \text { and } X_{4, j}^{T}=x_{N-1-j}^{*} \boldsymbol{b} . \tag{18}
\end{equation*}
$$

According to (15), the column sequence PMEPR value for the sequence pair $\left(X_{1, j}^{T}, X_{4, N-1-j}^{T}\right)$ of length $L$ is given by

$$
\begin{align*}
\operatorname{PMEPR}\left(X_{1, j}^{T}\right) & \leq 2+\frac{2}{L} \sum_{\tau=1}^{L-1}\left|x_{j}^{2}\left(\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau)\right)\right| \\
& \leq 2+\frac{2}{L} \sum_{\tau=1}^{L-1}\left|\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau)\right| \tag{19}
\end{align*}
$$

Table I: A Summary of Existing Works

| Refs. | Phases | Set Size | Array Size | 2D-ZCZ Width | Constraints | Based On |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| [33] | 2 | 2 | $L_{1} \times L_{2}$ | $Z_{1} \times Z_{2}$ | $L_{1}, L_{2} \geq 2$ | ( $L_{1}, Z_{1}$ )-ZCP and ( $L_{2}, Z_{2}$ )-ZCP |
| [34] | $q$ | 2 | $\left(2^{m_{1}-1}+2^{n+1}\right) \times\left(2^{m_{2}}+4\right)$ | $\left(2^{\pi(n+1)}+2^{n+1}\right) \times\left(2^{m_{2}-2}+2^{\phi\left(m_{2}-3\right)}+1\right)$ | $2 \mid q ; m_{1} \geq 2 ; n \leq m_{1}-3 ; m_{2} \geq 4$ | Generalized Boolean Functions |
| [35] | $q$ | 2 | $2^{n} \times\left(2^{m-1}+\sum_{\alpha=t+1}^{m-1} d_{\alpha} 2^{\alpha-1}+2^{\nu}\right)$ | $2^{n} \times\left(2^{t-1}+2^{\nu}\right)$ | $2 \mid q ; m, n, t, \nu \geq 0 ; \nu<t<m ; d_{\alpha} \in\{0,1\}$ | 2D Generalized Boolean Functions |
| [37, Th. 2] | $q$ | 2 | $14 \cdot 2^{n} \times 2^{m-n}$ | $12 \cdot 2^{n} \times 2^{m-n}$ | $2 \mid q ; 0 \leq n \leq m$ | 2D Generalized Boolean Functions |
| [37, Lem. 5] | $q$ | 2 | $2^{n} \times\left(2^{m-1}+\sum_{\alpha=t+1}^{m-1} d_{\alpha} 2^{\alpha-1}+2^{\nu}\right)$ | $2^{n} \times\left(2^{t-1}+2^{\nu}\right)$ | $2 \mid q ; m, n, t, \nu \geq 0 ; \nu<t<m ; d_{\alpha} \in\{0,1\}$ | Generalized Boolean Functions |
| Proposed | $q$ | 4 | $L \times 2^{\alpha} 10^{\beta} 26^{\gamma}$ | $Z \times 2^{\alpha} 10^{\beta} 26^{\gamma}$ | $q=\operatorname{lcm}\left\{2, q_{1}\right\}, q_{1} \geq 2$ | Binary GCP and (L,Z)-ZCP |
|  | $q$ | 4 | $L \times 2^{a+u} 3^{b} 5^{c} 11^{d} 13^{e}$ | $Z \times 2^{a+u} 3^{b} 5^{c} 11^{d} 13^{e}$ | $q=\operatorname{lcm}\left\{q_{0}, q_{1}\right\}, q_{0}, q_{1} \geq 2$ | Complex GCP and (L,Z)-ZCP |

where $\left|x_{j}\right|=1$. Similarly, the column sequence PMEPR value for the sequence pair $\left(X_{2, j}^{T}, X_{3, N-1-j}^{T}\right)$ of length $L$ is given by

$$
\begin{equation*}
\operatorname{PMEPR}\left(X_{2, j}^{T}\right) \leq 2+\frac{2}{L} \sum_{\tau=1}^{L-1}\left|\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau)\right| \tag{20}
\end{equation*}
$$

where $\left|y_{j}\right|=1$. This completes the proof.
Remark 2: Based on Theorem 2, the PMEPR values are calculated by using only unimodular seed 1D-GCPs and 1DZCPs in Theorem 1.

Proposition 1: The column sequence PMEPR of the constructed 2D-ZCAQs by Theorem 1 is upper bounded by 3.33 when a seed $(L, Z)$-ZCP from [9] is used with $L=2^{n+1}+2^{n}$ and $Z=2^{n+1}$.

Proof: In [9, Th. 3], it has been shown that the constructed $(L, Z)$-ZCP $(\boldsymbol{a}, \boldsymbol{b})$ from [9] has the following auto-correlation property given by

$$
\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau)=\left\{\begin{array}{l}
2^{n+2}+2^{n+1}, \text { if } \tau=0  \tag{21}\\
\pm 2^{n+1}, \text { if } \tau=2^{n+1} \\
0, \text { otherwise }
\end{array}\right.
$$

By using (19), the $j$-th column sequence $X_{1, j}^{T}$ of the array $\boldsymbol{X}_{1}$ is given by

$$
\begin{equation*}
\operatorname{PMEPR}\left(X_{1, j}^{T}\right) \leq 2+\frac{2}{2^{n+1}+2^{n}} \cdot 2^{n+1}=2+\frac{4}{3} \approx 3.33 \tag{22}
\end{equation*}
$$

for $j=0,1, \cdots, N-1$. Similarly, we can show that $\operatorname{PMEPR}\left(X_{2, j}^{T}\right) \leq 3.33, \operatorname{PMEPR}\left(X_{3, N-1-j}^{T}\right) \leq 3.33$ and $\operatorname{PMEPR}\left(X_{4, N-1-j}^{T}\right) \leq 3.33$. This completes the proof.

Example 2: Let $n=3$ with $L=24, Z=16$ and $N=32$. Let us consider a (24,16)-ZCP $(\boldsymbol{a}, \boldsymbol{b})$ constructed from [9] with $\boldsymbol{a}=[+-+-++-------++-+-+-++--]$ and $\boldsymbol{b}=[-++-----++--+-+--++-----]$. We take a GCP $(\boldsymbol{x}, \boldsymbol{y})$ of length 32 with $\boldsymbol{x}=[+++++-$ $+---+++--+++--+--+----+-+-]$ and $\boldsymbol{y}=[-----+-+++--++-++--$ $+--+----+-+-]$. Based on Theorem 1, we construct four arrays $\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}$ and $\boldsymbol{X}_{4}$ of equal size $24 \times 32$. We show the absolute values of 2D-AACF sums for four arrays $\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}$ and $\boldsymbol{X}_{4}$ in Fig. 2 . We can see that the set


Fig. 2: The sum of 2D-AACFs from Example 2


Fig. 3: The IEPRs of column sequences for $\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}$ and $\boldsymbol{X}_{4}$ given in Example 2
$\left\{\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}, \boldsymbol{X}_{4}\right\}$ forms a binary $2 \mathrm{D}-\mathrm{ZCAQ}$ with array size $24 \times 32$ and $2 \mathrm{D}-\mathrm{ZCZ}$ width $16 \times 32$. The maximum column sequence PMEPR of each column of $\boldsymbol{X}_{1}$ and $\boldsymbol{X}_{3}$ is 3.197 . The maximum column sequence PMEPR of each column of $\boldsymbol{X}_{2}$ and $\boldsymbol{X}_{4}$ is 2.851 . For example, we consider the column sequence IEPR of the first column of $\boldsymbol{X}_{2}$ as illustrated in Fig. 3. Note that the sub-carrier spacing is normalized to 1 for Fig. 3. According to Fig. 3, we have

$$
\begin{equation*}
\operatorname{PMEPR}\left(X_{2,0}^{T}\right)=\sup _{0 \leq t \leq 1} \operatorname{IEPR}_{X_{2,0}^{T}}(t)=2.851 \tag{23}
\end{equation*}
$$

whereas the theoretical PMEPR upper bound is 3.33 (given by (22)) from Proposition 1. Similarly, the maximum column sequence PMEPR of each column of $\boldsymbol{X}_{1}$ is 3.197 as observed in Fig. 3.

Proposition 2: The column sequence PMEPR of the constructed 2D-ZCAQs by Theorem 1 is upper bounded by 4 when a seed $(L, Z)$-ZCP from [14] is used with $L=2 N+2$ and $Z=3 N / 2+1$.

Proof: By using [14, (16), (17) and (18)], the constructed $(L, Z)$-ZCP $(\boldsymbol{a}, \boldsymbol{b})$ from [14] has the following autocorrelation property given by

$$
\rho_{a}(\tau)+\rho_{b}(\tau)=\left\{\begin{array}{l}
0, \text { if } 0<\tau \leq 3 N / 2  \tag{24}\\
\pm 4, \text { if } 3 N / 2<\tau \leq 2 N, \\
0, \text { if } \tau=2 N+1,
\end{array}\right.
$$

Based on (19), the $j$-th column sequence $X_{1, j}^{T}$ of the array $\boldsymbol{X}_{1}$ is given by

$$
\begin{equation*}
\operatorname{PMEPR}\left(X_{1, j}^{T}\right) \leq 2+\frac{2}{2 N+2} \sum_{\tau=3 N / 2+1}^{2 N} 4<4 \tag{25}
\end{equation*}
$$

for $j=0,1, \cdots, N-1$. Similarly, we can show that $\operatorname{PMEPR}\left(X_{2, j}^{T}\right) \leq 4, \operatorname{PMEPR}\left(X_{3, N-1-j}^{T}\right) \leq 4$ and $\operatorname{PMEPR}\left(X_{4, N-1-j}^{T}\right) \leq 4$. This completes the proof.

Example 3: Let $L=18, Z=13$ and $N=26$. Let us take a binary $(18,13)-\mathrm{ZCP}(\boldsymbol{a}, \boldsymbol{b})$ constructed from [14] with $\boldsymbol{a}=[-+++-++-+-+---+++-]$ and $\boldsymbol{b}=$ $[++++---+--+--+----]$. We consider a binary GCP $(\boldsymbol{x}, \boldsymbol{y})$ of length 26 with $\boldsymbol{x}=[++++-++-$ $-+-+-+--+-+++--+++]$ and $\boldsymbol{y}=[++++-+$ $+--+-+++++-+---++---]$. According to Theorem 1, we construct four binary arrays $\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}$ and $\boldsymbol{X}_{4}$ of equal size $18 \times 26$. The absolute values of 2 D AACF sums for four arrays $\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}$ and $\boldsymbol{X}_{4}$ are shown in Fig. 4. We can observe that the set $\left\{\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}, \boldsymbol{X}_{4}\right\}$ forms a binary $2 \mathrm{D}-\mathrm{ZCAQ}$ with array size $18 \times 26$ and $2 \mathrm{D}-\mathrm{ZCZ}$ width $13 \times 26$. The maximum column sequence PMEPR of each column of $\boldsymbol{X}_{1}$ and $\boldsymbol{X}_{3}$ is 2.797. The maximum column sequence PMEPR of each column of $\boldsymbol{X}_{2}$ and $\boldsymbol{X}_{4}$ is 2.706. For example, we consider the column sequence IEPR of the first column of $X_{1}$ as illustrated in Fig. 5. Based on Fig. 5, we have

$$
\begin{equation*}
\operatorname{PMEPR}\left(X_{1,0}^{T}\right)=\sup _{0 \leq t \leq 1} \operatorname{IEPR}_{X_{1,0}^{T}}(t)=2.797, \tag{26}
\end{equation*}
$$

whereas the theoretical PMEPR upper bound is 4 from Proposition 2. Similarly, the maximum column sequence PMEPR of each column of $\boldsymbol{X}_{2}$ is 2.706 as depicted in Fig. 5.

Proposition 3: The column sequence PMEPR of the constructed 2D-ZCAQs by Theorem 1 is upper bounded by 3.714 when a seed $(L, Z)$-ZCP from [13, Th. 1] is used with $L=2^{n+3}+2^{n+2}+2^{n+1}$ and $Z=2^{n+3}$.

Proof: The constructed $(L, Z)$-ZCP $(\boldsymbol{a}, \boldsymbol{b})$ from [13, Th. 1] has the following auto-correlation property [17] given by

$$
\begin{align*}
& \rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau) \\
& =\left\{\begin{array}{l}
2^{n+4}+2^{n+3}+2^{n+2}, \text { when } \tau=0, \\
\pm 2^{n+2}, \text { when } \tau=2^{n+3}+l \cdot 2^{n+1} \text { for } l=0,1,2, \\
0, \text { otherwise. }
\end{array}\right. \tag{27}
\end{align*}
$$



Fig. 4: The sum of 2D-AACFs from Example 3


Fig. 5: The IEPRs of column sequences for $\boldsymbol{X}_{1}, \boldsymbol{X}_{2}, \boldsymbol{X}_{3}$ and $\boldsymbol{X}_{4}$ given in Example 3

According to (19), the $j$-th column sequence $X_{1, j}^{T}$ of the array $X_{1}$ is given by

$$
\begin{align*}
\operatorname{PMEPR}\left(X_{1, j}^{T}\right) & \leq 2+\frac{2}{2^{n+3}+2^{n+2}+2^{n+1}} \cdot 3 \cdot 2^{n+2} \\
& =2+\frac{12}{7} \approx 3.714 \tag{28}
\end{align*}
$$

for $j=0,1, \cdots, N-1$. Similarly, we can show that $\operatorname{PMEPR}\left(X_{2, j}^{T}\right) \leq 3.714, \operatorname{PMEPR}\left(X_{3, N-1-j}^{T}\right) \leq 3.714$ and $\operatorname{PMEPR}\left(X_{4, N-1-j}^{T}\right) \leq 3.714$.

## V. Conclusion

In this paper, we have investigated a new construction for 2D-ZCAQs with various array sizes $L \times N$, where $N=2^{\alpha} 10^{\beta} 26^{\gamma}$ or $2^{a+u} 3^{b} 5^{c} 11^{d} 13^{e}$ and $L$ is any positive integer $\geq 2$. The admissible lengths of 1D-GCPs and 1DZCPs allow us to offer more feasible array sizes of 2DZCAQs. Furthermore, we have shown that the column sequence PMEPR of the proposed 2D-ZCAQ is upper bounded by 4 when we use seed 1D-ZCPs from [9], [14] and [13, Th. 1]. The proposed 2D-ZCAQs with low column sequence PMEPRs can be utilized as alternatives to 2D-GCASs in UWB MC-CDMA systems.

## REFERENCES

[1] M. J. E. Golay, "Complementary Series," IRE Trans. Inf. Theory, vol. IT-7, no. 2, pp. 82-87, Apr. 1961.
[2] J. A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Muller codes," IEEE Trans. Inf. Theory, vol. 45, no. 7, pp. 2397-2417, Nov. 1999.
[3] K. G. Paterson, "Generalized Reed-Muller codes and power control in OFDM modulation," IEEE Trans. Inf. Theory, vol. 46, no. 1, pp. 104-120, 2000.
[4] P. Spasojevic and C. N. Georghiades, "Complementary sequences for ISI channel estimation," IEEE Trans. Inf. Theory, vol. 47, no. 3, pp. 1145-1152, 2001.
[5] S. Z. Budišin, "New complementary pairs of sequences," Electron. Lett., vol. 26, pp. 881-883(2), June 1990.
[6] P. Fan, W. Yuan, and Y. Tu, "Z-complementary binary sequences," IEEE Signal Process. Lett., vol. 14, no. 8, pp. 509-512, Aug. 2007.
[7] X. Li, P. Fan, X. Tang, and Y. Tu, "Existence of binary Zcomplementary pairs," IEEE Signal Process. Lett., vol. 18, no. 1, pp. 63-66, 2011.
[8] Z. Liu, U. Parampalli, and Y. L. Guan, "Optimal odd-length binary Z-complementary pairs," IEEE Trans. Inf. Theory, vol. 60, no. 9, pp. 5768-5781, Sep. 2014.
[9] _-, "On even-period binary Z-complementary pairs with large ZCZs," IEEE Signal Process. Lett., vol. 21, no. 3, pp. 284-287, Mar. 2014.
[10] C. Chen, "A novel construction of Z-complementary pairs based on generalized Boolean functions," IEEE Signal Process. Lett., vol. 24, no. 7, pp. 987-990, Jul. 2017.
[11] S. Das, S. Majhi, S. Budišin, Z. Liu, and Y. L. Guan, "A novel multiplier-free generator for complete complementary codes," in Proc. 23rd Asia-Pacific Conference on Communications (APCC), Dec. 2017.
[12] S. Das, S. Budišin, S. Majhi, Z. Liu, and Y. L. Guan, "A multiplierfree generator for polyphase complete complementary codes," IEEE Trans. Signal Process., vol. 66, no. 5, pp. 1184-1196, Nov. 2017.
[13] C. Xie and Y. Sun, "Constructions of even-period binary Zcomplementary pairs with large ZCZs," IEEE Signal Process. Lett., vol. 25, no. 8, pp. 1141-1145, Aug. 2018.
[14] A. R. Adhikary, S. Majhi, Z. Liu, and Y. L. Guan, "New sets of evenlength binary Z-complementary pairs with asymptotic ZCZ ratio of 3/4," IEEE Signal Process. Lett., vol. 25, no. 7, pp. 970-973, Jul. 2018.
[15] S. Das, S. Majhi, and Z. Liu, "A novel class of complete complementary codes and their applications for APU matrices," IEEE Signal Process. Lett., vol. 25, no. 9, pp. 1300-1304, Jul. 2018.
[16] S. Das, S. Majhi, S. Budišin, and Z. Liu, "A new construction framework for polyphase complete complementary codes with various lengths," IEEE Trans. Signal Process., vol. 67, no. 10, pp. 2639-2648, Mar. 2019.
[17] C. Chen and C. Pai, "Binary Z-complementary pairs with bounded peak-to-mean envelope power ratios," IEEE Commun. Lett., vol. 23, no. 11, pp. 1899-1903, Nov. 2019.
[18] S. Das, U. Parampalli, S. Majhi, and Z. Liu, "Construction of new optimal Z-complementary code sets from Z-paraunitary matrices," in IEEE International Workshop on Signal Design and its Applications in Communications (IWSDA), 2019, pp. 1-5.
[19] S. Das, U. Parampalli, S. Majhi, Z. Liu, and S. Budišin, "New optimal $Z$-complementary code sets based on generalized paraunitary matrices," IEEE Trans. Signal Process., vol. 68, pp. 5546-5558, 2020.
[20] S. Das, A. Banerjee, and Z. Liu, "New family of cross Zcomplementary sequences with large ZCZ width," in IEEE International Symposium on Information Theory (ISIT), 2022, pp. 522-527.
[21] C. Pai, S. Wu, and C. Chen, "Z-complementary pairs with flexible lengths from generalized Boolean functions," IEEE Commun. Lett., vol. 24, no. 6, pp. 1183-1187, 2020.
[22] F. Li, Y. Jiang, C. Du, and X. Wang, "Construction of Golay complementary matrices and its applications to MIMO omnidirectional transmission," IEEE Trans. Signal Process., vol. 69, pp. 2100-2113, 2021.
[23] Z. Wang, D. Ma, G. Gong, and E. Xue, "New construction of complementary sequence (or array) sets and complete complementary codes," IEEE Trans. Inf. Theory, vol. 67, no. 7, pp. 4902-4928, 2021.
[24] C. Y. Pai and C. Y. Chen, "Two-dimensional Golay complementary array pairs/sets with bounded row and column sequence PAPRs," IEEE Trans. Commun., vol. 70, no. 6, pp. 3695-3707, 2022.
[25] R. Frank, "Polyphase complementary codes," IEEE Trans. Inf. Theory, vol. 26, no. 6, pp. 641-647, 1980.
[26] C. Liu, S. Liu, X. Lei, A. R. Adhikary, and Z. Zhou, "Threephase Z-complementary triads and almost complementary triads," Cryptography and Communications, vol. 13, no. 5, pp. 763-773, 2021.
[27] R. Craigen, W. Holzmann, and H. Kharaghani, "Complex Golay sequences: structure and applications," Discrete Math., vol. 252, no. 1, pp. 73-89, 2002.
[28] C. Zhang, X. Lin, and M. Hatori, "Novel two dimensional complementary sequences in ultra wideband wireless communications," in IEEE Conference on Ultra Wideband Systems and Technologies, 2003, 2003, pp. 398-402.
[29] M. Turcsány and P. Farkaš, "New 2D-MC-DS-SS-CDMA techniques based on two-dimensional orthogonal complete complementary codes," in Multi-Carrier Spread-Spectrum. Springer, 2004, pp. 49-56.
[30] P. Farkas and M. Turcsany, "Two-dimensional orthogonal complete complementary codes," in Proc. Joint 1st Workshop on Mobile Future and Symposium on Trends in Communications, Oct. 2003, pp. 21-24.
[31] M. Turcsany, P. Farkas, P. Duda, and J. Kralovic, "Performance evaluation of two-dimensional quasi orthogonal complete complementary codes in fading channels," in Joint IST Workshop on Mobile Future, 2006 and the Symposium on Trends in Communications. SympoTIC, 2006, pp. 84-87.
[32] Y. Li and C. Xu, "Construction of two-dimensional periodic complementary array set with zero-correlation zone," in Proc. Fifth International Workshop on Signal Design and Its Applications in Communications, 2011, pp. 104-107.
[33] C. Y. Pai, Y. T. Ni, and C. Y. Chen, "Two-dimensional binary Zcomplementary array pairs," IEEE Trans. Inf. Theory, vol. 67, no. 6, pp. 3892-3904, 2021.
[34] A. Roy, P. Sarkar, and S. Majhi, "A direct construction of q-ary 2-D Z-complementary array pair based on generalized Boolean functions," IEEE Commun. Lett., vol. 25, no. 3, pp. 706-710, 2021.
[35] C. Y. Pai and C. Y. Chen, "A novel construction of two-dimensional Z-complementary array pairs with large zero correlation zone," IEEE Signal Process. Lett., vol. 28, pp. 1245-1249, 2021.
[36] S. Das and S. Majhi, "Two-dimensional Z-complementary array code sets based on matrices of generating polynomials," IEEE Trans. Signal Process., vol. 68, pp. 5519-5532, 2020.
[37] H. Zhang, C. Fan, and S. Mesnager, "New constructions of $q$-ary 2-D Z-complementary array pairs," 2021. [Online]. Available: https://arxiv.org/abs/2107.11599
[38] Z. Liu, Y. L. Guan, and U. Parampalli, "New complete complementary codes for peak-to-mean power control in multi-carrier CDMA," IEEE Trans. Commun., vol. 62, no. 3, pp. 1105-1113, Mar. 2014.
[39] K. Feng, P. J. S. Shiue, and Q. Xiang, "On aperiodic and periodic complementary binary sequences," IEEE Trans. Inf. Theory, vol. 45, no. 1, pp. 296-303, 1999.

