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Abstract

We consider the problem of approximating a general Gaussian location mixture by finite mixtures. The minimum order of finite mixtures that achieve a prescribed accuracy (measured by various f-divergences) is determined within constant factors for the family of mixing distributions with compactly support or appropriate assumptions on the tail probability including subgaussian and subexponential. While the upper bound is achieved using the technique of local moment matching, the lower bound is established by relating the best approximation error to the low-rank approximation of certain trigonometric moment matrices, followed by a refined spectral analysis of their minimum eigenvalue. In the case of Gaussian mixing distributions, this result corrects a previous lower bound in [WV10b].

Keywords— Gaussian mixture, approximation error, channel capacity, moment matrix, orthogonal polynomials.

Contents

Intr	Introduction 2				
1.1	Main results	3			
1.2	Comparison with previous results	5			
1.3	Related work	6			
1.4	Organization	7			
1.5	Notations	7			
Pre	liminaries	7			
2.1	Trigonometric moment matrices	7			
2.2	Orthogonal polynomials	8			
Achievability via moment matching					
3.1	Compactly supported distributions	9			
3.2	Distribution families under tail conditions	10			
Cor		12			
4.1	General framework	12			
	4.1.1 Lower bound via trigonometric moments	12			
	4.1.2 Spectral analysis of trigonometric moment matrices	13			
4.2	Distribution families under tail conditions	14			
4.3	Compactly supported distributions	15			
4.4	Inconsistency regime	17			
	 1.1 1.2 1.3 1.4 1.5 Pree 2.1 2.2 Ach 3.1 3.2 Cor 4.1 4.2 4.3 	1.1 Main results			

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5	Applications and extensions		
	5.1 Convergence rates of nonparametric maximum likelihood estimator	19	
	5.2 Mixing distributions under moment constraints		
	5.3 Generalization to other mixture models		
Α	Preliminaries	22	
	A.1 Distances between probability measures	22	
	A.2 Gauss quadrature		
	A.3 Orthogonal polynomials		
в	Alternative proofs of lower bounds via orthogonal expansion	23	
	B.1 Lower bound for Gaussian distributions	23	
	B.2 Lower bound for uniform distributions		
С	Proof of the approximation rates	27	
	C.1 Proofs of main theorems	27	
	C.2 Proofs for Section 5.2.		
	C.3 Proofs for Section 5.3		
D	Proof of the convergence rates of NPMLE	33	

1 Introduction

Gaussian mixtures are widely applied in statistical modeling of heterogeneous populations. Let ϕ denote the standard Gaussian density. For a probability distribution P on the real line, denote by f_P the marginal density of the Gaussian convolution $P * \phi$, that is

$$f_P(x) = \int \phi(x-\theta) dP(\theta).$$

We refer to P and $P * \phi$ as the *mixing distribution* and the *mixture*, respectively. Given a general mixture $P * \phi$, the problem of interest is how to best approximate it by a finite mixture $P_m * \phi$, where the support size of P_m is at most m (i.e., m-atomic).

Let d(f,g) denote a loss function that measures the approximation error of g by f. Concrete examples include L_p distances or f-divergences [Csi75], the latter of which, including the total variation $\mathsf{TV}(f,g)$, squared Hellinger distance $H^2(f,g)$, the Kullback-Leibler divergence $\mathsf{KL}(f||g)$, and the χ^2 -divergence $\chi^2(f||g)$, are the focus of the present paper¹ (see Appendix A.1 for a quick review). The best approximation error of f_P by an m-component mixture is defined as

$$\mathcal{E}^{\star}(m, P, d) \triangleq \inf_{P_m \in \mathcal{P}_m} d\left(f_{P_m}, f_P\right), \tag{1}$$

where \mathcal{P}_m denotes the collection of all distributions that are at most *m*-atomic. Considering the worst instance of this pointwise quantity, we define

$$\mathcal{E}^{\star}(m, \mathcal{P}, d) \triangleq \sup_{P \in \mathcal{P}} \mathcal{E}^{\star}(m, P, d)$$
(2)

as the worst-case approximation error over a family \mathcal{P} of mixing distributions by *m*-component mixtures. It is well-known that the optimization problem (1) is *nonconvex* (in the location parameters) and is generally hard to solve. This shares the essential difficulty of approximation by neural nets with one hidden layer [Bar93].

¹If f and g are densities for distribution P and Q, respectively, we also write d(P,Q) = d(f,g).

In information theory, the Gaussian convolution structure arises in the context of Gaussian channels [CT06], where the input and output distributions correspond to P and $P * \phi$, respectively. The channel capacity determines the maximal rate at which information can be reliably transmitted, which, under the second moment constraint, is achieved by a Gaussian input distribution. In practice, the input may be constrained to be finitely valued due to modulation. To address this issue, [WV10b] studied the Gaussian channel capacity under input cardinality constraints (finite-constellation capacity), in particular, the rate of convergence to the Gaussian channel capacity when the cardinality grows. It turns out that this capacity gap is precisely characterized by \mathcal{E}^* under the KL divergence for the unconstrained capacity-achieving input distribution (Gaussian) – see (8).

The problem of approximation by finite mixtures also naturally arises in nonparametric statistics and empirical process theory. Classical results show that the complexity of a class of distributions, as manifested by their metric entropy, plays a crucial role in determining the rate of convergence of nonparametric density estimation [YB99,vdVW96]. If the distribution family is parametric, its entropy is often determined by the dimension of the parameter space. However, nonparametric families are infinite-dimensional and determining its entropy entails more delicate analysis including finite-dimensional approximation, which, for the Gaussian mixture family, requires approximation by finite mixtures. To describe the most economical approximation by finite mixtures, let us define

$$m^{\star}(\epsilon, P, d) \triangleq \min\{m \in \mathbb{N} : \exists P_m \in \mathcal{P}_m, d(f_{P_m}, f_P) \leq \epsilon\},\$$

i.e., the smallest order of a finite mixture that approximates a given mixture f_P within a prescribed accuracy ϵ . For uniform approximation over \mathcal{P} , define

$$m^{\star}(\epsilon, \mathcal{P}, d) \triangleq \sup_{P \in \mathcal{P}} m^{\star}(\epsilon, P, d)$$

which offers a meaningful complexity measure for the class $\{f_P : P \in \mathcal{P}\}\$ and is closely related to more classical complexity notions such as the metric entropy. In fact, most of the existing constructive bounds on the metric or bracketing entropy for general Gaussian mixtures are obtained by first finding a discrete approximation then quantizing the weights and atoms, and the resulting upper bounds increase with m^* [GvdV07a, GvdV01, Zha09, SG20]. Hence, tightened upper bounds for metric entropy immediately follow. See Section 5.1 for a detailed discussion.

Clearly, determining m^* and that of \mathcal{E}^* are equivalent by the formula

$$m^{\star}(\epsilon, \mathcal{P}, d) = \inf \left\{ m : \mathcal{E}^{\star}(m, \mathcal{P}, d) \le \epsilon \right\}.$$

Next, we state our main results in terms of m^* , which follow from \mathcal{E}^* in Sections 3 and 4.

1.1 Main results

Our main results give non-asymptotic rates of m^* . First, we explore the family of compactly supported distributions and provide a tight convergence rate in Theorem 1, which improves upon previous results [GvdV01, Zha09]. Next, we turn our attention to distribution families with exponential tail decay (e.g., subgaussian and subexponential families) in Theorem 2. In both regimes, we give tight lower and upper bounds. An overview of the proof strategy is shown at the end of this subsection.

To begin with, consider the distributions supported on [-M, M] for M > 0:

$$\mathcal{P}_M^{\mathsf{Bdd}} \triangleq \{P : P[-M, M] = 1\}.$$

The following optimal rate of complexity level is established:

Theorem 1. Suppose $M \leq \epsilon^{-c_1}$ for some universal constant $0 < c_1 < 1$. Then, for $\epsilon \in (0,1)$ and $d \in \{\mathsf{TV}, H, \mathsf{KL}, \chi^2\}$,

$$m^{\star}(\epsilon, \mathcal{P}_{M}^{Bdd}, d) \asymp \frac{\log \frac{1}{\epsilon}}{\log \left(1 + \frac{1}{M}\sqrt{\log \frac{1}{\epsilon}}\right)} \vee 1.^{2}$$
 (3)

We provide some interpretations of Theorem 1. By definition, m^* increases with M and decreases with ϵ . In fact, (3) captures an "elbow-effect" depending on the relationship between M and ϵ . If the support of mixing distributions is not too wide, e.g., $M \leq \left(\log \frac{1}{\epsilon}\right)^{\frac{1}{2}-\delta}$ for a constant $\delta > 0$, the complexity m^* has a slower growth with respect to ϵ as $\frac{\log \epsilon^{-1}}{\log \log \epsilon^{-1}}$. When $\sqrt{\log \frac{1}{\epsilon}} \leq M \leq \epsilon^{-c_1}$, the finite mixture needs to cover a wider range, and consequently, m^* grows faster as $M\sqrt{\log \frac{1}{\epsilon}}$. Finally, when $M \gtrsim \epsilon^{-c_1}$ for any $c_1 \in (0, 1)$, our Theorem 7 shows that we need at least $\Omega(M)$ components to encompass an extensive range.

Next, we turn to the families of distributions supported on the whole real line. The following notions are introduced to characterize the tail conditions. Given a function $\psi : [0, \infty) \mapsto [0, \infty)$ that is non-decreasing and satisfies that $\psi(0) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$, define

$$||X||_{\psi} \triangleq \inf \left\{ t > 0 : \mathbb{E}\psi\left(\frac{|X|}{t}\right) \le 1 \right\}.$$

This concept is commonly referred to as the ψ -Orlicz norm, originally introduced in [KR61]. It also implies the following tail bound [KC22]:

$$\mathbb{P}[|X| \ge t] \le \frac{2}{\psi(t/\|X\|_{\psi}) + 1}, \quad \forall t \ge 0.$$
(4)

We focus on $\psi(x) = e^{x^{\alpha}} - 1$ for $\alpha > 0$. Define

$$\mathcal{P}_{\alpha}(\beta) \triangleq \left\{ P : X \sim P, \|X\|_{\psi} \le \beta, \psi(x) = e^{x^{\alpha}} - 1 \right\},$$
(5)

which is also referred to as sub-Weibull family (see [VGNA20,ZW22,KC22] for various equivalent definitions). For instance, the special cases of $\alpha = 2$ and $\alpha = 1$ correspond to the β^2 -subgaussian and β -subexponential families, respectively [Ver18]. The tail probability bound (4) becomes [KC22, Eq. (2.3)]

$$P[|X| \ge t] \le 2 \exp\left[-\left(\frac{t}{\beta}\right)^{\alpha}\right], \quad P \in \mathcal{P}_{\alpha}(\beta).$$
(6)

We assume that ψ (and thus α therein) is fixed, while β could vary with m.

Theorem 2. Suppose that $\beta \leq \epsilon^{-c_1}$ for some universal constant $0 < c_1 < 1$. Then, for $\epsilon \in (0, 1)$ and $d \in \{\mathsf{TV}, H, \mathsf{KL}, \chi^2\}$,

$$\beta \left(\log \frac{1}{\epsilon} \right)^{\frac{2+\alpha}{2\alpha}} \lesssim_{\alpha} m^{\star}(\epsilon, \mathcal{P}_{\alpha}(\beta), d) \lesssim_{\alpha} \frac{\log \frac{1}{\epsilon}}{\log \left(1 + \frac{1}{\beta} \left(\log \frac{1}{\epsilon} \right)^{\frac{\alpha-2}{2\alpha}} \right)} \vee 1$$

In particular, when $\left(\log \frac{1}{\epsilon}\right)^{\frac{\alpha-2}{2\alpha}} \leq \beta \leq \epsilon^{-c_1}$, we have

$$\underline{m^{\star}(\epsilon, \mathcal{P}_{\alpha}(\beta), d)} \asymp_{\alpha} \beta \left(\log \frac{1}{\epsilon} \right)^{\frac{2+\alpha}{2\alpha}}.$$

²For $x, y \in \mathbb{R}$, $x \vee y \triangleq \max\{x, y\}$ and $x \wedge y \triangleq \min\{x, y\}$. For two positive sequences a_n and b_n , write $a_n \leq b_n$ or $a_n = O(b_n)$ when $a_n \leq Cb_n$ for some absolute constant C > 0, $a_n \geq b_n$ or $a_n = \Omega(b_n)$ if $b_n \leq a_n$, and $a_n \approx b_n$ or $a_n = \Theta(b_n)$ if both $b_n \geq a_n$ and $a_n \geq b_n$ hold. We write $a_n = O_\alpha(b_n)$ and $a_n \leq_\alpha b_n$ if C may depend on parameter α .

Theorem 2 shows that the complexity level grows in a logarithm rate, which equivalently implies an exponential convergence rate of the approximation error \mathcal{E}^{\star} . The tight result is applicable when $\alpha \in (0, 2]$ and β is a fixed constant, such as in the case of the subgaussian and subexponential family. Specifically, for the σ^2 -subgaussian family $\mathcal{P}(2, \sigma)$ with $c_0 \leq \sigma \leq \epsilon^{-c_1}$, we have

$$m^*(\epsilon, \mathcal{P}(2, \sigma), d) \asymp \sigma \log \frac{1}{\epsilon}.$$
 (7)

One way to reconcile Theorems 1 and 2 is to notice that each distribution $P \in \mathcal{P}_{\alpha}(\beta)$ is effectively supported (except for a total mass that is polynomially small in ϵ) on the interval $\left[-C\beta\left(\log\frac{1}{\epsilon}\right)^{\frac{1}{\alpha}}, C\beta\left(\log\frac{1}{\epsilon}\right)^{\frac{1}{\alpha}}\right]$ for some large constant C, so that the approximation complexity of the class $\mathcal{P}_{\alpha}(\beta)$ coincides with that of $\mathcal{P}_{M}^{\mathsf{Bdd}}$ with $M \asymp \beta\left(\log\frac{1}{\epsilon}\right)^{\frac{1}{\alpha}}$. As will be shown next, our upper bound essentially pursues this idea, while the lower bound is based on a different approach.

We now briefly discuss the proof strategies for the main results. We prove the upper bound under the χ^2 -divergence and the lower bound under the TV distance. The results for $d \in \{\text{TV}, H, \text{KL}, \chi^2\}$ quickly follow by the inequalities between *f*-divergences in Appendix A.1. For the upper bound, we extend the local moment matching argument in previous work [Zha09], which constructs a discrete approximation by matching the moments for the mixing distribution conditioned on each subinterval in a partition of the effective support of the mixture. This approach can be further generalized to distribution families characterized by various tail conditions. The matching lower bound is the major contribution of this paper, where we relate the best approximation error to the *low-rank approximation* of *trigonometric moment matrices*, and then conduct a refined analysis based on the classical spectrum theory. The application of orthogonal polynomials also plays a crucial role in our analysis.

Remark 1 (Gaussian location mixture with non-unitary variance). Consider the problem of approximating $P * \phi_{\sigma}$ by an *m*-component mixture $P_m * \phi_{\sigma}$, where ϕ_{σ} denotes the density of $N(0, \sigma^2)$. This problem appears in heteroscedastic settings of nonparametric density estimation [Jia20, SGS21]. By the scale invariance of the *f*-divergences, we have $d(P_{\sigma} * \phi, Q_{\sigma} * \phi) = d(P * \phi_{\sigma}, Q * \phi_{\sigma})$ for $d \in \{\mathsf{TV}, H, \mathsf{KL}, \chi^2\}$, where P_{σ} denotes the distribution of X/σ for $X \sim P$. Consequently, the minimum number of components is equal to $m^*(\epsilon, P_{\sigma}, d)$.

1.2 Comparison with previous results

Below we give an overview of previous results. The upper bound for the compact support case is discussed in [GvdV01, GvdV07a, Zha09]. Among them, the strongest result [Zha09, Lemma 1] gives an upper bound of $m^*(\epsilon, \mathcal{P}_M^{\mathsf{Bdd}}, \mathsf{TV}) \lesssim M\sqrt{\log \epsilon^{-1}} \lor \log \epsilon^{-1}$. Theorem 1 strengthens this result by bounding the χ^2 -divergence and establishing the optimal rate. Correspondingly, for general σ , it follows from Remark 1 that the $\frac{M}{\sigma}\sqrt{\log \epsilon^{-1}} \lor \log \epsilon^{-1}$ upper bound from [Jia20, Lemma 3] can be improved to $\frac{\log \epsilon^{-1}}{\log (1+\sigma\sqrt{\log \epsilon^{-1}/M})} \lor 1$. In addition, for Hellinger distance, [PW20]

conjectures that $m^{\star}(\epsilon, \mathcal{P}_1^{\mathsf{Bdd}}, H) \asymp \frac{\log \frac{1}{\epsilon}}{\log \log \frac{1}{\epsilon}}$, which is proved by our Theorem 1.

For the subgaussian case, [PW20, Lemma 7] gives a log ϵ^{-1} upper bound for 1–subgaussian family, which is further extended to sub-Weibull families by Theorem 2. The specific problem of approximating $P = N(0, \sigma^2)$ is studied in [WV10a, WV10b] in the context of the finite-constellation capacity. In fact,

$$\mathcal{E}^{\star}(m, N(0, \sigma^2), \mathsf{KL}) = C - C_m \tag{8}$$

where

$$C = \max_{P_X: \mathbb{E}[X^2] \le \sigma^2} I(X; X + Z) = \frac{1}{2} \log(1 + \sigma^2)$$

is the Gaussian channel capacity with input X and additive noise $Z \sim N(0, 1)$, and

$$C_m = \max_{P_X \in \mathcal{P}_m: \mathbb{E}[X^2] \le \sigma^2} I(X; X + Z)$$

is the capacity under the additional input cardinality constraint. While quantized Gaussian only achieves an error that is polynomial in m, an exponential upper bound is $\mathcal{E}^{\star}(m, N(0, \sigma^2), \mathsf{KL}) \leq \sigma^2 (\frac{\sigma^2}{1+\sigma^2})^{2m}$ is shown in [WV10b, Theorem 8] using the Gauss quadrature. As a corollary of our main result, (7) generalizes this result to subgaussian family with an improved exponent for large σ .

Compared with these constructive upper bounds, the lower bound is far less understood. For the Gaussian distributions, [WV10b, Eq. 66] claims that $\mathcal{E}^*(m, N(0, \sigma^2), \mathsf{KL}) \geq (\frac{\sigma^2}{2+\sigma^2} + o(1))^{2m}$; however, the sketched proof turns out to be flawed, which results in a wrong dependency of the exponent on large σ . This is now corrected in Theorem 2 (see also Theorems 4 and 5), which shows that the exponential convergence is indeed tight but the optimal exponent (for fixed σ) is $\Theta(\frac{1}{\sigma})$ as opposed to $\Theta(\frac{1}{\sigma^2})$. The exact optimal exponent, as a function of σ , however, remains open.

1.3 Related work

The problem of approximation by location mixtures is first addressed by the celebrated Tauberian theorem of Wiener [Wie32], which gives a general characterization of whether the translation family of a given function is dense in $L_1(\mathbb{R}^d)$ or $L_2(\mathbb{R}^d)$ in terms of its Fourier transform. Convergence rates have been studied over the past few decades, with a wide range of applications in approximation theory, machine learning, and information theory [Bar93, Fer97, CABR02, WV10b, WV10a]. For example, for the location and scale *m*-mixture class of sigmoidal functions, Barron [Bar93] obtained dimension-free convergence rate for approximating functions whose gradient has integrable Fourier transform, a fundamental result in the theory of neural networks. For Gaussian models, Wu and Verdú [WV10b] linked this problem to the Gaussian channel capacity under input cardinality constraint (cf. (8)). More recently, [NM19, NNCM20, NCNM22] showed the consistent approximation over various families with general location-scale mixtures. Another related problem is the Gaussian mixture reduction, which requires approximating a high-order Gaussian mixture with a low-order one. This problem broadly arises in applications including belief propagation and Bayesian filtering. Although there are many numerical algorithms by means of clustering, optimization, or the greedy algorithm, convergence rates and optimal approximators are still left to be discovered. See [ZZC23, SLP23, DMIO23] for some recent works and **[CWPS11]** for a review.

In the statistics literature, understanding the complexity of a distribution class plays an important role in nonparametric density and functional estimation [vdVW96]. Information-theoretic risk bounds are obtained on the basis of metric entropy for a variety of loss functions (see [PW24, Chapter 32] for an overview). In addition, metric entropy of the Gaussian mixture family is crucial in the statistical analysis of the sieve and nonparametric maximum likelihood estimator (MLE) in mixture models as well as posterior concentration [SW94, WS95, GW00, GvdV01, GvdV07a, Zha09, SG20, Jia20, SGS21]. These results all rely on metric entropy of the Gaussian mixture class obtained via approximation by finite mixtures. In this vein, the quantity $m^*(n^{-1/2}, \mathcal{P}, d)$ is referred to as the *statistical degree* of \mathcal{P} [PW20], which represents the smallest m so that any density in \mathcal{P} can be made statistically indistinguishable (on the basis of n observations) from some density in \mathcal{P}_m .

A related quantity to finite mixture approximation, known as smoothed empirical empirical distribution, has been studied in recent literature [GGNWP20,CNW22,BJPR22]. Given a sample $(X_1, \ldots, X_m) \stackrel{\text{i.i.d.}}{\sim} P$, we approximate $P * \phi$ by $P_m * \phi$, where $P_m = \frac{1}{m} \sum_{i=1}^m \delta_{X_i}$ is the empirical measure. For mixing distributions with bounded support or subgaussian tails, the convergence rate of this approximation is shown to be dimension-free and polynomial in m, measured by

different Wasserstein distances and f-divergences (e.g., $O(m^{-1/2})$ under TV and $O(m^{-1})$ under χ^2). Note that P_m is a special m-atomic distribution. As noted in [WV10b, Sec. VII], the achieved approximation error is consistent with CLT-type of rates which are markedly slower than the optimal rates that are typically exponentially.

1.4 Organization

The rest of this paper is organized as follows. In Section 2, we provide the background on trigonometric moment matrices and orthogonal polynomials on the real line and on the unit circle. Section 3 and 4 contain upper and lower bounds on the best approximation error \mathcal{E}^* and present the main proof ideas. Section 5 discusses applications to the convergence rate of nonparametric maximum likelihood estimator and the extensions to other classes of mixing distributions and mixture models. Additional backgrounds and proof details are presented in the Appendices.

1.5 Notations

Let $[k] = \{1, \ldots, k\}$ for $k \in \mathbb{N}$. Denote by $\delta_{jk} = \mathbf{1}\{j = k\}$ the Kronecker's delta notation. We use bold symbols to represent vectors and matrices. For a vector \mathbf{x} , denote \mathbf{x}^{\top} and \mathbf{x}^{\star} as the transpose and the Hermitian transpose, respectively, and $\text{Diag}(\mathbf{x})$ as the corresponding diagonal matrix. Denote $\|\cdot\|$ as the Euclidean norm for vectors or spectral norm for matrices, and let $\|\cdot\|_F$ be the Frobenius norm. Write $\lambda_{\min}(\mathbf{A})$ as the smallest eigenvalue of a Hermitian matrix \mathbf{A} .

2 Preliminaries

2.1 Trigonometric moment matrices

The theory of moments is fundamental in many areas of probability, statistics, and approximation theory [Usp37]. Given a distribution P and $X \sim P$, denote its k^{th} moment by $m_k = m_k(P) = m_k(X) = \mathbb{E}_P[X^k]$. The moment matrix associated with P of order n + 1is the following *Hankel* matrix:

$$\mathbf{M}_{n}(X) = \begin{bmatrix} m_{0} & m_{1} & \cdots & m_{n} \\ m_{1} & m_{2} & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n} & m_{n+1} & \cdots & m_{2n} \end{bmatrix}_{(n+1)\times(n+1)} .$$
(9)

Denote the vector of monomials as $\mathbf{X}_n = (1, X, \dots, X^n)^{\top}$. The moment matrix of P can be equivalently represented as $\mathbf{M}_n(X) = \mathbb{E}_P[\mathbf{X}_n \mathbf{X}_n^{\top}]$. Consequently, if P is discrete with no more than m atoms, the moment matrix of any order is of rank at most m, and P can be uniquely determined by its first 2m - 1 moments [Usp37].

The above formulation can also be adapted to the trigonometric moments. For $k \in \mathbb{Z}$, denote $t_k = t_k(P) = t_k(X) = \mathbb{E}_P[e^{ikX}]$ as the k^{th} order Fourier coefficients (or characteristic functions) of P. Define the *Toeplitz* matrix

$$\mathbf{T}_{n}(X) = \begin{bmatrix} t_{0} & t_{1} & \cdots & t_{n} \\ t_{-1} & t_{0} & \cdots & t_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{-n} & t_{-(n-1)} & \cdots & t_{0} \end{bmatrix}_{(n+1)\times(n+1)}$$
(10)

as the trigonometric moment matrix associated with P of order n + 1. $\mathbf{T}_n(X)$ is equivalently the ordinary moment matrix of $Z = e^{iX}$ in the sense that $\mathbf{T}_n(X) = \mathbb{E}_P[\mathbf{Z}_n \mathbf{Z}_n^*]$ for $\mathbf{Z}_n = (1, Z, \dots, Z^n)^{\top}$. Note that both \mathbf{M}_n and \mathbf{T}_n are positive semidefinite matrices.

Our proof of the converse results relies on classical theory of moment matrices. For Hankel moment matrices, the seminal work [Sze36] studied the asymptotic behavior of the eigenvalues for Gaussian and exponential weights. Systematical treatments for general distribution classes are given by a series of work [WW66, CL99, BCI99, CL04]. [ŠŠ19] proposes a generalized result for certain forms of weighted Hankel matrices. [KMS53] gives characterization for eigenvalues of Toeplitz forms, which applies in particular to the trigonometric moment matrices.

The moment matrices are Hermitian and positive definite, provided that the corresponding distribution has infinite support [Sze36]. It turns out that the smallest eigenvalue of the moment matrix plays an important role in the derivation of the lower bound. [PD92] gives bounds on eigenvalues for Gaussian Toeplitz matrices, but the lower bound is suboptimal. [CL99] introduces a framework of bounding the smallest eigenvalue, extending the method of [Sze36]. Specifically, the ordinary moment matrix (9) is related to the orthogonal polynomials on the real line, and the trigonometric moment matrix (10) is related to the orthogonal polynomials on the unit circle (see Section 4 for further discussions).

2.2 Orthogonal polynomials

Orthogonal polynomials on the real line and on the unit circle serve as useful tools in our proofs. Given a weight function $w : \mathbb{R} \mapsto [0, \infty)$, denote by $\{p_n(x)\}$ the set of monic (leading coefficient equal to one) orthogonal polynomials on the real line associated with the weight w(x), such that the degree of $p_n(x)$ is n and

$$\int p_j(x)p_k(x)w(x)\mathrm{d}x = h_j\delta_{jk}, \quad \forall j,k \ge 0.$$
(11)

When w(x) is the density of a probability distribution P, $\{p_n(x)\}$ is said to be the set of orthogonal polynomials associated with P.

For a density function $f(\theta)$ supported on a subset of \mathbb{R} , let $\{\Phi_n(z)\}$ be the associated set of monic orthogonal polynomials on the unit circle, that is, Φ_n is of degree n, and

$$\int \Phi_j(e^{i\theta}) \overline{\Phi_k(e^{i\theta})} f(\theta) d\theta = \kappa_j^{-2} \delta_{jk}, \quad \forall j,k \ge 0.$$
(12)

The formula of Szegö [Sze75, Theorem 11.5] shows the relation between orthogonal polynomials on the unit circle and on the real line, provided that the weight functions of the two orthogonal systems are related. This result is further developed in the follow-up work [Zhe98, Kra03]. See Section 4.3 for a detailed exposition and applications.

In Appendix A.3, we describe several concrete examples of orthogonal polynomials associated with certain distributions that will be used in our proof. We refer the readers to [Sze75] for a comprehensive review of orthogonal polynomials and [Sim05] for a special treatment of orthogonal polynomials on the unit circle.

3 Achievability via moment matching

In this section, we give upper bounds of finite mixture approximation of the distribution family \mathcal{P} . For any $P \in \mathcal{P}$, we need to construct an *m*-atomic distribution P_m achieving a small approximation error measured by $d(f_{P_m}, f_P)$. Previous results have shown that comparing moments is useful in determining the approximation accuracy. The Gauss quadrature, which is briefly explained in Appendix A.2, serves as a classical and effective approach to the global discrete approximation in the sense of matching moments. For example, [WV10b, Theorem 8] considers

the *m*-point Gaussian quadrature with a scale parameter σ that matches the first 2m - 1 moments of $N(0, \sigma^2)$. For compactly supported distributions, [Zha09, Lemma 1] provides another moment matching approximation.

In our non-asymptotic analysis framework, when the relative scale of the parameter of the distribution family compared with m changes, different treatments are needed. Consider the family of compactly supported distributions as an example. The previous result [Zha09] becomes suboptimal unless M grows with m at a certain rate, as shown in (13). This problem also arises for distribution families with tail conditions.

Aiming at tight upper bounds, we carry out a refined analysis based on the technique of moment matching. The following is a road map for this section. We start from the compactly supported case in Section 3.1, where we bound the χ^2 -divergence by moment differences and construct moment matching approximations both globally and locally depending on the relationship between parameters. Then, in Section 3.2, convergence rates for distribution families with tail conditions are obtained via an extra truncation argument. We derive rate upper bounds for the $\mathcal{P}_{\alpha}(\beta)$ family, and the proof idea also works for general choices of ψ by plugging in the tail conditions (see Section 5.2 for an extension).

3.1 Compactly supported distributions

The following lemma is useful for upper bounding the approximation error in χ^2 -divergence by comparing moments.

Lemma 1. Suppose that
$$P, Q \in \mathcal{P}_M^{\mathsf{Bdd}}$$
. For any $J > 4M^2$, if $m_j(P) = m_j(Q)$ for $j \in [J-1]$,
 $\chi^2(f_P || f_Q) \le 4 \exp\left(\frac{M^2}{2}\right) \left(\frac{4eM^2}{J}\right)^J$.

Proof. Denote $\mu = \mathbb{E}[Q]$ and $\sigma^2 = \operatorname{var}[Q]$. Let $U \sim P$, $V \sim Q$, $U - \mu \sim P'_{\mu}$, and $V - \mu \sim Q'_{\mu}$. Since $m_j(P) = m_j(Q)$ for $j \in [J-1]$, it holds that $m_j(P'_{\mu}) = m_j(Q'_{\mu})$ for $j \in [J-1]$. Applying [WY20, Lemma 9] with $\mathbb{E}_{Q'_{\mu}}[X] = 0$, we have

$$\chi^{2}(f_{P}||f_{Q}) = \chi^{2}\left(f_{P'_{\mu}}||f_{Q'_{\mu}}\right) \le e^{\frac{\sigma^{2}}{2}} \sum_{j\ge J} \frac{\left(m_{j}(P'_{\mu}) - m_{j}(Q'_{\mu})\right)^{2}}{j!}$$

Since $P'_{\mu}, Q'_{\mu} \in \mathcal{P}_{2M}^{\mathsf{Bdd}}$, we have $|m_j(P'_{\mu})|, |m_j(Q'_{\mu})| \leq (2M)^j$. Since $\sigma^2 \leq M^2$, it follows that

$$\chi^2(f_P || f_Q) \le \exp\left(\frac{M^2}{2}\right) \sum_{j\ge J} \frac{[2(2M)^j]^2}{j!} \le 4 \exp\left(\frac{M^2}{2}\right) \left(\frac{4eM^2}{J}\right)^J$$

where the last inequality follows from the Poisson tail bound $\mathbb{P}[X \ge J] \le e^{-4M^2} \left(\frac{4eM^2}{J}\right)^J$ for $X \sim \text{Poisson}(4M^2)$ and $J > 4M^2$ [Ver18, Exercise 2.3.3].

Now we are ready to show the χ^2 approximation error bound for compactly supported mixing distributions. When M is small, we construct a discrete distribution that matches the first few moments. However, this approach is loose for large M and can be remedied using the idea of local approximation. Specifically, we partition the support into subintervals and approximate each conditional distribution via moment matching; similar construction of local moment matching has appeared in the statistics literature [Zha09, SW23] by applying Caratheódory's theorem to the truncated moments. The overall approximation error can be bounded combining that in each subinterval.

Theorem 3. There exists a universal constant $\kappa > 0$ such that for any $m \in \mathbb{N}$ and M > 0:

$$\mathcal{E}^{\star}(m, \mathcal{P}_{M}^{\mathsf{Bdd}}, \chi^{2}) \leq \begin{cases} \exp\left(-m\log\frac{m}{M^{2}}\right), & m \geq \kappa M^{2};\\ \exp\left(-\frac{\log\kappa}{4\kappa}\frac{m^{2}}{M^{2}}\right), & 3\sqrt{\kappa}M \leq m \leq \kappa M^{2}. \end{cases}$$
(13)

Proof. Suppose that $m \geq \kappa M^2$ for some universal constant $\kappa \geq 16e^3$. For $P \in \mathcal{P}_M^{\mathsf{Bdd}}$, by the Gauss quadrature rule (see Appendix A.2), there exists $P_m \in \mathcal{P}_M^{\mathsf{Bdd}} \cap \mathcal{P}_m$ that matches the first 2m-1 moments of P. By Lemma 1, we have

$$\chi^{2}\left(f_{P_{m}}\|f_{P}\right) \leq 4\exp\left(-2m\log\frac{m}{M^{2}} + \frac{M^{2}}{2} + \log(4e^{2})m\right)$$
$$\leq \exp\left(-m\log\frac{m}{M^{2}}\right).$$
(14)

Suppose that $3\sqrt{\kappa}M \leq m \leq \kappa M^2$ holds. Set $K = \left\lfloor \frac{3\kappa M^2}{m} \right\rfloor \geq 3$. We partition the interval [-M, M] into K subintervals $I_j = [-M + (j-1)\frac{2M}{K}, -M + j\frac{2M}{K}]$ for $j \in [K]$. Let $P_{(j)}$ be the conditional distribution of P on I_j , that is, for any Borel set A, $P_{(j)}(A) = P(A|I_j)$. Denote $\tilde{m} = \lfloor m/K \rfloor$. Note that the condition $\tilde{m} \geq \kappa \left(\frac{M}{K}\right)^2$ holds by

$$K^{2}\left\lfloor\frac{m}{K}\right\rfloor \geq \left\lfloor\frac{3\kappa M^{2}}{m}\right\rfloor^{2}\left\lfloor\frac{m^{2}}{3\kappa M^{2}}\right\rfloor \geq \left(\frac{3}{4}\frac{3\kappa M^{2}}{m}\right)^{2}\left(\frac{3}{4}\frac{m^{2}}{3\kappa M^{2}}\right) \geq \kappa M^{2},$$

where the second inequality follows from the facts that $\lfloor x \rfloor \geq \frac{c}{c+1}x$ for all $x \geq c, c \in \mathbb{N}$, and $\min\left\{\frac{m^2}{3\kappa M^2}, \frac{3\kappa M^2}{m}\right\} \geq 3$. Applying (14) to each $P_{(j)}$ with a translation, there exists \tilde{P}_j supported on at most \tilde{m} atoms such that

$$\chi^2(f_{\tilde{P}_j} \| f_{P_{(j)}}) \le \exp\left(-\left\lfloor \frac{m}{K} \right\rfloor \log \frac{\left\lfloor \frac{m}{K} \right\rfloor}{\left(\frac{M}{K}\right)^2}\right) \le \exp\left(-\frac{3}{4} \frac{m^2}{3\kappa M^2} \log \kappa\right) = \exp\left(-\frac{\log \kappa}{4\kappa} \frac{m^2}{M^2}\right).$$

Define $P_m = \sum_{j=1}^{K} P(I_j) \tilde{P}_j$ supported on at most *m* atoms. Since $P = \sum_{j=1}^{K} P(I_j) P_{(j)}$, by Jensen's inequality and the convexity of *f*-divergences,

$$\chi^{2}(f_{P_{m}} \| f_{P}) \leq \sum_{j=1}^{K} P(I_{j}) \chi^{2}(f_{\tilde{P}_{j}} \| f_{P_{(j)}}) \leq \exp\left(-\frac{\log \kappa}{4\kappa} \frac{m^{2}}{M^{2}}\right).$$

3.2 Distribution families under tail conditions

In this subsection, we extend our analysis to the sub-Weibull family $\mathcal{P}_{\alpha}(\beta)$ using a truncation argument. For each $P \in \mathcal{P}_{\alpha}(\beta)$, we reduce the problem to approximating the conditional distribution P_t of P on [-t, t], where the approximate error is bounded by Theorem 3 below. The final result follows by optimizing t. Moreover, this approach is applicable for general distribution families by incorporating the corresponding tail probability bounds.

The following lemma upper bounds the χ^2 -divergence between two (general) mixtures by truncating one of the mixing distributions:

Lemma 2. Let A be a Borel set such that $P(A), P(A^c) > 0$. Denote by P_A the conditional distribution of P on A, i.e., $P_A(\cdot) = P(\cdot \cap A)/P(A)$. For any distribution Q, we have

$$\chi^2(f_Q \| f_P) \le \frac{2}{P(A)} \left(\chi^2(f_Q \| f_{P_A}) + P(A^c) \right).$$

Proof. Define P_{A^c} as P conditioned on the complement A^c . By linearity, the mixture can be decomposed as $f_P = P(A)f_{P_A} + P(A^c)f_{P_{A^c}}$. Then

$$\chi^{2}(f_{Q}||f_{P}) = \int \frac{(f_{Q} - P(A)f_{P_{A}} - P(A^{c})f_{P_{Ac}})^{2}}{f_{P}}$$

$$\leq 2\int \frac{(f_{Q} - f_{P_{A}})^{2} + [P(A^{c})(f_{P_{A}} - f_{P_{Ac}})]^{2}}{f_{P}}$$

$$\leq \frac{2\chi^{2}(f_{Q}||f_{P_{A}})}{P(A)} + 2P(A^{c})^{2}\int \frac{(f_{P_{A}} - f_{P_{Ac}})^{2}}{f_{P}}$$

Since $f_P \ge P(A)f_{P_A}$ and $f_P \ge P(A^c)f_{P_{A^c}}$, it follows that

$$\int \frac{(f_{P_A} - f_{P_{A^c}})^2}{f_P} \le \int \frac{f_{P_A}^2 + f_{P_{A^c}}^2}{f_P} \le \frac{1}{P(A)} + \frac{1}{P(A^c)} = \frac{1}{P(A)P(A^c)}.$$

Consequently,

$$\chi^{2}(f_{Q}||f_{P}) \leq \frac{2}{P(A)} \left(\chi^{2}(f_{Q}||f_{P_{A}}) + P(A^{c}) \right).$$

The next proposition directly follows from Lemma 2 with $A = I_t \triangleq [-t, t]$ and the definition of \mathcal{E}^* .

Proposition 1. For any distribution family \mathcal{P} , with the same notation as in Lemma 2, we have

$$\mathcal{E}^{\star}(m,\mathcal{P},\chi^2) \leq \inf_{t>0} \sup_{P\in\mathcal{P}} \frac{2}{P(I_t)} \left(\mathcal{E}^{\star}(m,\mathcal{P}_t^{Bdd},\chi^2) + P(I_t^c) \right).$$

Now we can derive upper bounds for distribution classes under tail conditions. In the following, we state the universal approximation rate for the family $\mathcal{P}_{\alpha}(\beta)$. Extensions to other families are discussed in Section 5.2.

Theorem 4. There are constants c_{α}, C_{α} , such that for all $m \geq C_{\alpha}\beta$,

$$\mathcal{E}^{\star}(m, \mathcal{P}_{\alpha}(\beta), \chi^2) \le \exp\left\{-c_{\alpha}m\log\left(1 + \frac{m^{\frac{\alpha-2}{\alpha+2}}}{\beta^{\frac{2\alpha}{\alpha+2}}}\right)\right\}.$$
(15)

Proof. Denote $\mathcal{P} = \mathcal{P}_{\alpha}(\beta)$. By Theorem 3, when $m \geq 3\sqrt{\kappa}t$, we have that

$$\mathcal{E}^{\star}(m, \mathcal{P}_t^{\mathsf{Bdd}}, \chi^2) \le \exp\left[-\Theta_{\alpha}\left(m\log\left(1 + \frac{m}{t^2}\right)\right)\right].$$
(16)

Set $t = c_{\alpha}\beta \left[m \log \left(1 + \frac{m^{\frac{\alpha-2}{\alpha+2}}}{\beta^{\frac{\alpha}{\alpha+2}}}\right)\right]^{\frac{1}{\alpha}}$ for some c_{α} only depending on α . From the assumption $m \ge C_{\alpha}\beta$ for sufficiently large C_{α} , it can be verified that there exists c_{α} such that $m \ge 3\sqrt{\kappa}t$ and $t \ge \Omega_{\alpha}(\beta)$ hold. By the tail condition (6), we have $\sup_{P \in \mathcal{P}} P(I_t^c) \le 2 \exp(-(\frac{t}{\beta})^{\alpha})$, and hence $P(I_t) = \Omega_{\alpha}(1)$. The desired result follows from applying Proposition 1 with our choice of t and the upper bounds (16) and (6).

As a special case, consider the family $\mathcal{P}_2(\sigma)$ of all σ^2 -subgaussian distributions. Then for some universal constant C,

$$\mathcal{E}^{\star}(m, \mathcal{P}_2(\sigma), \chi^2) \le \exp\left(-Cm\log\left(1+\frac{1}{\sigma}\right)\right).$$

In Theorems 3 and 4, we assume that the order of m is at least proportional to the scaling parameter (i.e., M and β , respectively). Otherwise, if the scaling parameter of the mixing distribution gets larger, the distribution becomes too heavy-tailed to be well approximated by any m-order finite mixture. This intuition is made precise by the lower bound in Theorem 7, which implies that consistent approximation is impossible unless m grows proportionally to the scale parameters.

4 Converse via spectrum of moment matrices

In this section, we give lower bounds of the best approximation error by finite mixtures. In Section 4.1, we first propose a general framework applicable to any finite mixture in terms of trigonometric moment matrices. This strategy is then applied to the class of sub-Weibull distributions $\mathcal{P}_{\alpha}(\beta)$ and compactly supported distributions $\mathcal{P}_{M}^{\mathsf{Bdd}}$ in Sections 4.2 and 4.3, respectively. For each distribution family \mathcal{P} of interest, we exhibit a *test distribution* $P \in \mathcal{P}$ and obtain lower bound of $\mathcal{E}^{*}(\epsilon, P, d)$ and hence also $\mathcal{E}^{*}(m, \mathcal{P}, d)$. These test distributions are essentially the hardest to approximate in their respective distribution classes, and are amenable to analysis. Finally, in Section 4.4 we focus on the regime of large β and M and prove lower bound on msuch that consistent approximation is possible.

4.1 General framework

We state a general procedure to lower bound the approximation error $\mathcal{E}^{\star}(m, P, d)$, where P is not finitely supported. At the high level, we relate the approximation error in TV to that of the trigonometric moments. As mentioned in Section 2.1, the trigonometric moment matrix of P of any degree is full-rank, while that of any *m*-atomic distribution has rank at most *m*. Thus, the question boils down to the smallest eigenvalue of the trigonometric moment matrix of P, which can be lower bounded using classical results in spectral analysis. For this step, we provide two different methods, both of which are needed for proving the main results.

4.1.1 Lower bound via trigonometric moments

The following lemma states a classical result of the best low-rank approximation, known as the Eckart-Young-Mirsky theorem (see, e.g., [Mir60]).

Lemma 3. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix with eigenvalues $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$. Then for any $k \in [n]$,

$$\min_{B \in \mathbb{C}^{n \times n}, \text{ rank}(B) = k} \|A - B\|_F = \sqrt{\sigma_{k+1}^2 + \dots + \sigma_n^2}.$$

Recall that $\mathbf{T}_m(X) = (t_{k-j}(X))_{j,k=0}^m$ denotes the trigonometric moment matrix for a random variable X of order m + 1 as in (10). The following proposition relates the approximation error in TV to the smallest eigenvalue of trigonometric moment matrices.

Proposition 2. For any random variable $X \in \mathbb{R}$ with distribution P,

$$\mathcal{E}^{\star}(m, P, \mathsf{TV}) \ge \sup_{\delta > 0} \frac{\lambda_{\min}(\mathbf{T}_m(\delta X))}{2(m+1)\exp(m^2\delta^2/2)}$$

Proof. Denote $Z \sim N(0,1)$ and $X_m \sim P_m$, where $P_m \in \mathcal{P}_m$ is an arbitrary *m*-atomic distribution. Set the test function as $f_{\omega}(x) = \exp(i\omega x)$. By the variational representation formula in Lemma 7, we obtain that

$$\mathsf{TV}(f_{P_m}, f_P) \ge \frac{1}{2} \sup_{\omega} |\mathbb{E}[f_{\omega}(X+Z)] - \mathbb{E}[f_{\omega}(X_m+Z)]|$$
$$= \frac{1}{2} \sup_{\omega} e^{-\frac{\omega^2}{2}} |\mathbb{E}[f_{\omega}(X)] - \mathbb{E}[f_{\omega}(X_m)]|.$$

Set $\omega \in \delta \mathbb{Z} \triangleq \{k\delta : k \in \mathbb{Z}\}$, where $\delta > 0$ is a frequency parameter to be determined. Then

$$\mathsf{TV}(f_{P_m}, f_P) \ge \max_{k \in \mathbb{Z}, |k| \le m} \frac{|\mathbb{E}[\exp(ik\delta X_m)] - \mathbb{E}[\exp(ik\delta X)]|}{2\exp(k^2\delta^2/2)} = \max_{|k| \le m} \frac{|t_k(\delta X_m) - t_k(\delta X)|}{2\exp(k^2\delta^2/2)}.$$

Note that $t_k(\delta X_m)$ and $t_k(\delta X)$ are entries of $(m+1) \times (m+1)$ Toeplitz matrices $\mathbf{T}_m(\delta X_m)$ and $\mathbf{T}_m(\delta X)$, respectively. Since rank $(\mathbf{T}_m(\delta X_m)) \leq m$, applying Lemma 3 yields

$$\mathsf{TV}\left(f_{P_m}, f_P\right) \ge \frac{\|\mathbf{T}_m(\delta X_m) - \mathbf{T}_m(\delta X)\|_F}{2(m+1)\exp(m^2\delta^2/2)} \ge \frac{\lambda_{\min}(\mathbf{T}_m(\delta X))}{2(m+1)\exp(m^2\delta^2/2)}.$$

4.1.2 Spectral analysis of trigonometric moment matrices

To lower bound the smallest eigenvalue of $\mathbf{T}_m(\delta X)$, we propose two parallel approaches, by applying the techniques of *wrapped density* and *orthogonal expansion*, respectively.

Wrapped density. Given a density function g on \mathbb{R} , define the corresponding wrapped density g^{wrap} on $[0, 2\pi]$ as

$$g^{\mathsf{wrap}}(\theta) \triangleq \begin{cases} \sum_{j=-\infty}^{\infty} g(\theta - 2\pi j), & \theta \in [0, 2\pi]; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if Y has density g, then $Y^{\mathsf{wrap}} \triangleq Y \mod 2\pi\mathbb{Z}$ has density g^{wrap} . By definition, we have that $t_k(Y) = t_k(Y^{\mathsf{wrap}})$ for any $k \in \mathbb{N}$, and then $\mathbf{T}_m(Y^{\mathsf{wrap}}) = \mathbf{T}_m(Y)$. Furthermore, the classical theory for the spectrum of Toeplitz matrices shows that the eigenvalues of $\mathbf{T}_m(Y^{\mathsf{wrap}})$ are bounded by the minimum and maximum of the wrapped density (cf. e.g. [KMS53, Eq. (1.11)]), that is,

$$\lambda_{\min}(\mathbf{T}_m(Y)) = \lambda_{\min}(\mathbf{T}_m(Y^{\mathsf{wrap}})) \ge \inf\{2\pi g^{\mathsf{wrap}}(\theta) : \theta \in [0, 2\pi]\}.$$
 (17)

To see this, writing $\mathbf{T} \equiv \mathbf{T}_m(Y^{\text{wrap}})$, the smallest eigenvalue is expressed as the minimum Rayleigh quotient

$$\lambda_{\min}(\mathbf{T}) = \min_{\mathbf{x} \in \mathbb{C}^{m+1} \setminus \{\mathbf{0}\}} \frac{\mathbf{x}^* \mathbf{T} \mathbf{x}}{\|\mathbf{x}\|^2},\tag{18}$$

For any $\mathbf{x} = (x_0, \dots, x_m)^\top \in \mathbb{C}^{m+1}$, let $\pi_m(w) = \sum_{j=0}^m x_j w^j$. Let $\kappa \triangleq \inf_{\theta \in [0,2\pi]} g^{\mathsf{wrap}}(\theta)$. Then

$$\mathbf{x}^{\star}\mathbf{T}\mathbf{x} = \mathbb{E}[|\pi_m(e^{iY^{\mathsf{wrap}}})|^2] = \int_0^{2\pi} g^{\mathsf{wrap}}(\theta) |\pi_m(e^{i\theta})|^2 \mathrm{d}\theta \ge \kappa \int_0^{2\pi} |\pi_m(e^{i\theta})|^2 \mathrm{d}\theta = 2\pi\kappa \cdot \mathbf{x}^{\star}\mathbf{x}, \quad (19)$$

where the last step applies the orthogonality of the monomials on the unit circle $\int_0^{2\pi} d\theta e^{i(j-k)\theta} = 2\pi \delta_{jk}$.

Orthogonal expansion. Orthogonal expansion serves as an alternative approach to lower bound the smallest eigenvalue of $\mathbf{T}_m(\delta X)$. Let $\{\varphi_n\}$ be the orthonormal polynomials on the unit circle associated with $Y = \delta X$, that is,

$$\mathbb{E}\left[\varphi_j(e^{iY})\overline{\varphi_k(e^{iY})}\right] = \delta_{jk}, \quad j,k \ge 0.$$

Let $\mathbf{R}_{m,\delta} = (R_{nj})_{n,j=0}^m$ be the $(m+1) \times (m+1)$ lower triangular matrix that encodes the coefficients of $\{\varphi_k\}_{k=0}^m$, namely, $\varphi_n(t) = \sum_{j=0}^n R_{nj}t^j$ and $R_{nj} = 0$ if j > n. We call $\mathbf{R}_{m,\delta}$ the associated coefficient matrix of $\{\varphi_k\}$ of order m+1.

The next proposition gives a lower bound of the minimum eigenvalue based on the method in [CL99]. In a nutshell, we expand the quadratic form in the Rayleigh quotient under the basis of orthogonal polynomials, and bound the norm of the coefficient matrix.

Proposition 3. For any $X \sim P$ and $\delta > 0$, with the above notations,

$$\lambda_{\min}(\mathbf{T}_m(\delta X)) \ge \frac{1}{\|\mathbf{R}_{m,\delta}\|_F^2}.$$

Proof. For any $\mathbf{x} = (x_0, \ldots, x_m)^\top \in \mathbb{C}^{m+1} \setminus \{0\}$, let $\pi_m(w) = \sum_{j=0}^m x_j w^j$ with $w = w(z) = e^{i\delta z}$. Expand π_m under the orthogonal basis $\{\varphi_n\}$ as $\pi_m(w) = \sum_{k=0}^m c_k \varphi_k(w)$. Letting $\mathbf{c} = (c_0, \ldots, c_m)^\top$ and $\mathbf{T} \equiv \mathbf{T}_m(\delta X)$, the orthogonality of $\{\varphi_k\}$ implies that

$$\|\mathbf{c}\|^2 = \mathbb{E}\left[|\pi_m(e^{i\delta X})|^2\right] = \mathbf{x}^* \mathbf{T} \mathbf{x}.$$

Denote $\mathbf{R} \equiv \mathbf{R}_{m,\delta}$ as the coefficient matrix of $\{\varphi_k\}$ of order m+1. By comparing the coefficients, we have that $\mathbf{x}^{\top} = \mathbf{c}^{\top} \mathbf{R}$, which implies that $\|\mathbf{x}\| \leq \|\mathbf{c}\| \|\mathbf{R}\| \leq \|\mathbf{c}\| \|\mathbf{R}\|_F$. Finally, applying (18), we have

$$\lambda_{\min}(\mathbf{T}) \ge \frac{\|\mathbf{c}\|^2}{\|\mathbf{c}\|^2 \|\mathbf{R}\|_F^2} = \frac{1}{\|\mathbf{R}\|_F^2}.$$

Proposition 3 reduces the task of lower bounding the smallest eigenvalue to upper bounding $\|\mathbf{R}\|_{F}^{2}$, which is feasible if the orthogonal polynomial under the test distribution P has an explicit representation. Alternatively, as will be shown in Section 4.3, we can control $\|\mathbf{R}\|_{F}^{2}$ by the Szegö recurrence formula [Sim05, Sec.1.5]. Finally, the error lower bound is obtained by optimizing the frequency parameter δ .

In the next two subsections, we sketch the application of this approach to uniform and Gaussian distributions, the associated orthogonal polynomials of which are the standard monomials and the Rogers-Szegö polynomials (see Appendix A.3), respectively. The wrapped density approach only uses the density lower bound, while the orthogonal expansion approach requires understanding the coefficients of the orthogonal polynomials; nevertheless, as we will see in the case of compactly supported distributions, the latter approach yields optimal results while the former yields a trivial lower bound.

4.2 Distribution families under tail conditions

In this subsection, we establish the lower bound for the class $\mathcal{P}_{\alpha}(\beta)$ of sub-Weibull distributions (recall (5)) by applying the wrapped density approach in Section 4.1.2. We focus on the case where α is fixed and the scale parameter β may vary with m. Let $X_{\alpha,\beta} \sim P_{\alpha,\beta}$ with the density function

$$f_{\alpha,\beta}(x) = \frac{C_{\alpha}}{\beta} \exp\left[-\left|\frac{x}{\beta}\right|^{\alpha}\right],\tag{20}$$

where $C_{\alpha} = \left(\int_{-\infty}^{\infty} \exp\left(-|x|^{\alpha}\right) dx\right)^{-1}$ is the normalization constant. We establish the following lower bound of the approximation error measured by total variation distance.

Theorem 5. For any $m \in \mathbb{N}$ and $\alpha, \beta > 0$,

$$\mathcal{E}^{\star}(m, P_{\alpha, \beta}, \mathsf{TV}) \ge \beta^{-\frac{2}{2+\alpha}} m^{-\frac{\alpha}{2+\alpha}} \exp\left[-\Theta_{\alpha}\left(\left(\frac{m}{\beta}\right)^{\frac{2\alpha}{2+\alpha}}\right)\right].$$

Proof. Denote $X \sim P_{\alpha,\beta}$ and $Y = \delta X \sim g$, where $g(\theta) = \frac{1}{\delta} f_{\alpha,\beta}(\frac{\theta}{\delta})$. By Proposition 2 and (17), we have

$$\mathcal{E}^{\star}(m, P_{\alpha, \beta}, \mathsf{TV}) \ge \sup_{\delta > 0} \frac{\lambda_{\min}(\mathbf{T}_m(Y))}{2(m+1)\exp(m^2\delta^2/2)} \ge \sup_{\delta > 0} \frac{\pi \min_{0 \le \theta \le 2\pi} g^{\mathsf{wrap}}(\theta)}{(m+1)\exp(m^2\delta^2/2)}.$$
 (21)

Note that $g^{\mathsf{wrap}}(\theta) \ge \inf_{\theta \in [0,2\pi)} g(\theta) = \frac{1}{\delta} f_{\alpha,\beta}\left(\frac{2\pi}{\delta}\right)$. Choosing $\delta = \left(\frac{1}{m}\right)^{\frac{2}{2+\alpha}} \left(\frac{2\pi}{\beta}\right)^{\frac{\alpha}{2+\alpha}}$, we have

$$\mathcal{E}^{\star}(m, P_{\alpha, \beta}, \mathsf{TV}) \geq \frac{\pi f_{\alpha, \beta}\left(\frac{2\pi}{\delta}\right)}{2m\delta \exp(m^2 \delta^2/2)} \geq C_1 \beta^{-\frac{2}{2+\alpha}} m^{-\frac{\alpha}{2+\alpha}} \exp\left[-C_2\left(\frac{m}{\beta}\right)^{\frac{2\alpha}{2+\alpha}}\right],$$

for some constants C_1, C_2 depending only on α .

As concrete examples, we consider the case where P is the Laplace or Gaussian distribution. Consider the Laplace(λ) distribution with density $f(x) = (2\lambda)^{-1}e^{-|x|/\lambda}$. Applying Theorem 5 with $(\alpha, \beta) = (1, \lambda)$, it follows that

$$\mathcal{E}^{\star}(m, \operatorname{Laplace}(\lambda), \mathsf{TV}) \geq \frac{\pi}{4\sqrt[3]{2\pi m\lambda^2}} \exp\left[-\left(\frac{2\pi m}{\lambda}\right)^{\frac{2}{3}}\right],$$

where the explicit constants are obtained from (21) with $g(\theta) = \frac{1}{\delta} f(\frac{\theta}{\delta})^3$. Similarly, for the $N(0, \sigma^2)$ distribution with $(\alpha, \beta) = (2, \sqrt{2}\sigma)$,

$$\mathcal{E}^{\star}(m, N(0, \sigma^2), \mathsf{TV}) \ge \frac{1}{2\sqrt{2m\sigma}} \exp\left(-\frac{\pi m}{\sigma}\right).$$
 (22)

We finally remark that Theorem 5 also applies to lower bound $\mathcal{E}^{\star}(m, \mathcal{P}_{\alpha}(\beta), \mathsf{TV})$ by noting that $P_{\alpha,\beta} \in \mathcal{P}_{\alpha}(k_{\alpha}\beta)$ for $k_{\alpha} = (1 - 2^{-\alpha})^{-\frac{1}{\alpha}}, \alpha > 0.$

Remark 2. Alternatively, (22) can be shown using Proposition 3 and orthogonal polynomials for the wrapped Gaussian distribution. This second proof is given in Appendix B.1.

4.3 Compactly supported distributions

We focus on the class $\mathcal{P}_M^{\mathsf{Bdd}}$ of all distributions supported on [-M, M]. For $M \leq m \leq M^2$, the desired lower bound can be directly obtained by applying Proposition 2 to the trigonometric moment matrix of the uniform distribution. However, the case of $m \geq M^2$ is more challenging, which requires a small frequency parameter δ . For X supported on [-M, M], $\delta X \mod 2\pi \mathbb{Z}$ is the same as as δX for small δ . As such, the minimum value of the wrapped density function is zero, which only yields a trivial lower bound. Instead, we apply the second approach based on orthogonal expansion to distributions supported on an arc of the unit circle.

Given a distribution P with density f, let $\{\Phi_n\}$ be the associated set of *monic* orthogonal polynomials on the unit circle introduced in (12). Then the orthonormal version is given by $\varphi_n(z) = \kappa_n \Phi_n(z) \triangleq \sum_{j=0}^n R_{nj} z^j$ with $R_{nn} = \kappa_n$. Then, $\mathbf{R}_m \triangleq (R_{nj})_{n,j=0}^m$ is the coefficient matrix of $\{\varphi_n\}$ of order m+1. The Frobenius norm of \mathbf{R}_m is upper bounded by the following lemma:

Lemma 4. For any distribution P on \mathbb{R} , under the notations above,

$$\|\mathbf{R}_m\|_F^2 \le \sum_{n=0}^m 2^{2n} \kappa_n^2.$$

Proof. Let $a_n = -\overline{\Phi_{n+1}(0)}, n \ge 0$. [Sim05, Eq. (1.5.22)] shows that $\kappa_n = \prod_{j=0}^{n-1} (1 - |a_j|^2)^{-\frac{1}{2}}$. Then, by [Sim05, Eq. (1.5.30)], it holds for all |z| = 1 that

$$|\varphi_n(z)| \le \prod_{j=0}^{n-1} \frac{2}{\sqrt{1-|a_j|^2}} |\varphi_0(z)| = 2^n \kappa_n, \quad n \in \mathbb{N}.$$

It follows that

$$\sum_{j=0}^{n} |R_{nj}|^2 = \frac{1}{2\pi} \oint_{|z|=1} |\varphi_n(z)|^2 dz \le 2^{2n} \kappa_n^2$$

Finally, the result follows from $\|\mathbf{R}_m\|_F^2 = \sum_{n=0}^m \sum_{j=0}^n |R_{nj}|^2$.

³Slightly different from the proof of Theorem 5, here we lower bound the wrapped density with $g^{\text{wrap}}(\theta) \ge \inf_{\theta \in [0,\pi]} g(\theta)$ for symmetric distributions, which improves the multiplicative constants by a factor of 2.

It remains to construct a distribution supported on an arc of the unit circle and control the coefficients κ_n in Lemma 4. To this end, we apply the following lemma [Kra03, Lemma 2.1] connecting the orthogonal system on an arc of the unit circle to the orthogonal system on an interval of the real line. Let w be a positive, *even* weight function on [-1, 1] normalized to $\int w(x)dx = 1$, and $\{p_n\}$ be the set of monic orthogonal polynomials associated with w with

$$\int p_j(x)p_k(x)w(x)\mathrm{d}x = h_j\mathbf{1}\{j=k\}.$$

Let $f(\theta) = \frac{1}{2\gamma}w\left(\gamma^{-1}\cos\frac{\theta}{2}\right)\sin\frac{\theta}{2}$ if $\alpha \leq \theta \leq 2\pi - \alpha$, and $f(\theta) = 0$ otherwise, where $\gamma = \cos\frac{\alpha}{2}$ and $\alpha \in [0, \pi]$. Since $f \geq 0$ and $\int f(x)dx = \int w(x)dx = 1$, f is a density. Let $\{\Phi_n\}$ be the set of monic orthogonal polynomials on the unit circle associated with f. The following statement holds:

Lemma 5. Set $x = \gamma^{-1} \cos \frac{\theta}{2} = (2\gamma)^{-1}(z^{1/2} + z^{-1/2})$ with $z = e^{i\theta}$. Then,

$$\begin{split} \Phi_n(z) &= \frac{(2\gamma)^{n+1} z^{n/2}}{z-1} \left(z^{1/2} p_{n+1}(x) - \frac{p_{n+1}\left(\gamma^{-1}\right)}{p_n\left(\gamma^{-1}\right)} p_n(x) \right), \\ \kappa_n^{-2} &= 2^{2n} \gamma^{2n+1} \frac{p_{n+1}\left(\gamma^{-1}\right)}{p_n\left(\gamma^{-1}\right)} h_n. \end{split}$$

Now, with Lemmas 4 and 5, we are ready to show the lower bound for $\mathcal{P}_M^{\mathsf{Bdd}}$:

Theorem 6. For any M > 0 and $m \in \mathbb{N}$,

$$\mathcal{E}^{\star}(m, \operatorname{Unif}[-M, M], \mathsf{TV}) \ge \frac{1}{4m} \exp\left(-\frac{\pi^2 m^2}{2M^2}\right).$$
(23)

Furthermore, there exists a universal constant C_1 and C' such that when $m \ge C'M^2$,

$$\mathcal{E}^{\star}(m, \mathcal{P}_{M}^{Bdd}, \mathsf{TV}) \ge \exp\left(-C_{1}m\log\left(\frac{m}{M^{2}}\right)\right).$$
 (24)

Proof. First, we prove (23). Let $X \sim P = \text{Unif}[-M, M]$ and $\delta = \frac{\pi}{M}$. Then $\delta X \sim \text{Unif}[-\pi, \pi]$, and $t_k(\delta X) = \mathbb{E}[e^{ik\delta X}] = \mathbf{1}\{k = 0\}$ for $k \in \mathbb{Z}$. Consequently, $\mathbf{T} = \mathbf{T}_m(\delta X) = I_{m+1}$ is the identity matrix and so $\lambda_{\min}(\mathbf{T}) = 1$. By Proposition 2,

$$\mathcal{E}^{\star}(m, P, \mathsf{TV}) \ge \frac{1}{2(m+1)\exp(m^2\delta^2/2)} \ge \frac{1}{4m}\exp\left(-\frac{\pi^2 m^2}{2M^2}\right).$$

Next, we prove (24) by constructing a distribution supported on [-M, M]. Let $b \in (0, \pi)$ and $\gamma = \sin \frac{b}{2}$. Let \tilde{P} be the distribution supported on [-b, b] with a density function

$$\tilde{f}(\theta) = \begin{cases} \frac{\sqrt{\gamma^2 - \sin^2(\theta/2)}\cos(\theta/2)}{\pi\gamma^2}, & \theta \in [-b, b], \\ 0, & \text{otherwise.} \end{cases}$$

Let $\delta = b/M$ and $P \in \mathcal{P}_M^{\mathsf{Bdd}}$ be the distribution of X/δ for $X \sim \tilde{P}$. To apply Proposition 3, it remains to evaluate the coefficients of the orthonormal polynomials associated with \tilde{P} , which can be related to Chebyshev polynomials using Lemma 5. To this end, let $w(x) = \frac{2}{\pi}\sqrt{1-x^2}$ for $x \in [-1,1]$, and define the translated distribution $Q(\theta) \triangleq \tilde{P}(\theta - \pi)$ for $\theta \in [\pi - b, \pi + b]$, with density $\tilde{f}(\theta - \pi) = \frac{1}{2\gamma}w\left(\gamma^{-1}\cos\frac{\theta}{2}\right)\sin\frac{\theta}{2}$. Let U_n denote the degree-*n* Chebyshev polynomial of the second kind [Sze75, Eq. (1.12.3)] and $p_n(x) = \frac{1}{2^n}U_n(x)$. Then, p_n is monic with

$$\int_{-1}^{1} p_n(x) p_m(x) w(x) \mathrm{d}x = h_n \delta_{mn}, \quad h_n = \frac{1}{2^{2n}}.$$

Let $\{\Psi_n\}$ and $\{\tilde{\Phi}_n\}$ be the sets of monic orthogonal polynomials on the unit circle associated with Q and \tilde{P} as defined in (12), and $\psi_n = \kappa_n \Psi_n$, $\tilde{\varphi}_n = \tilde{\kappa}_n \tilde{\Phi}_n$ be the orthonormal polynomials, respectively. By definition, $\tilde{\Phi}_n(z) = (-1)^n \Psi_n(-z)$ and $\tilde{\kappa}_n = \kappa_n$. Let $\tilde{\mathbf{R}}_m$ be the coefficient matrix of $\{\tilde{\Phi}_n\}$ of order m + 1. By Lemmas 4 and 5, we obtain that

$$\|\tilde{\mathbf{R}}_m\|_F^2 \le \sum_{n=0}^m 2^{2n} \tilde{\kappa}_n^2 = \sum_{n=0}^m 2^{2n} \kappa_n^2 = \sum_{n=0}^m \frac{1}{\gamma^{2n+1} h_n} \frac{p_n\left(\gamma^{-1}\right)}{p_{n+1}\left(\gamma^{-1}\right)}.$$

Consider $r_n(x) \triangleq p_{n+1}(x)/p_n(x)$. By definition, $r_0(x) = x$. It follows from the recurrence relation $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ [Sze75, Sec.4.5.] that

$$r_n(x) = x - \frac{1}{4r_{n-1}(x)}, \quad n \in \mathbb{N}$$

Note that $r_0(x) \ge x/2$ for $x \ge 0$, and if $r_{n-1}(x) \ge x/2$, then $r_n(x) \ge x - \frac{1}{2x} \ge \frac{x}{2}$ for $x \ge 1$. By induction, $r_n(x) \ge \frac{x}{2}$ for all $x \ge 1$ and $n \in \mathbb{N}$. Since $\gamma^{-1} > 1$, we have

$$\|\tilde{\mathbf{R}}_m\|_F^2 \le \sum_{n=0}^m 2^{2n+1} \gamma^{-2n} = 2 \frac{(2/\gamma)^{2m+2} - 1}{(2/\gamma)^2 - 1}.$$

Assuming $m \ge \frac{eM^2}{2}$, choose $b = \sqrt{\frac{M^2}{m} \log \frac{2m}{eM^2}}$, which satisfies $b \le \frac{\sqrt{2}}{e}$. Then $\gamma = \sin \frac{b}{2} \in \left[\frac{b}{4}, \frac{b}{2}\right]$. Applying Propositions 2 and 3, we obtain that

$$\begin{split} \mathcal{E}^{\star}(m, P, \mathsf{TV}) &\geq \frac{1}{2(m+1)\exp(m^2\delta^2/2) \|\tilde{\mathbf{R}}_m\|_F^2} \geq \frac{(2/\gamma)^2 - 1}{8m[(2/\gamma)^2 - (2/\gamma)^{-2m}]} \exp\left(-\frac{m^2b^2}{2M^2} - 2m\log\frac{2}{\gamma}\right) \\ &\geq \frac{1 - (b/4)^2}{8m} \exp\left(-\frac{m^2b^2}{2M^2} - 2m\log\frac{8}{b}\right). \end{split}$$

We conclude the proof of (24) by plugging in the choice of b.

Remark 3. Although (24) is proved by choosing a non-uniform distribution on [-M, M], we can show that this lower bound also holds for Unif[-M, M]. For this result, we first lower bound the approximation error in χ^2 , which can be related to, via the variational representation of the χ^2 -divergence, to the smallest eigenvalue of a certain *weighted moment matrix*. Then, we convert the χ^2 -divergence lower bound to a TV lower bound using a truncation argument for Gaussian mixtures. This result in presented in Appendix B.2 – see Theorem 11 therein.

4.4 Inconsistency regime

In this subsection, we consider the regime when the scale parameter M or β diverges. In this case, more components are needed to achieve a prescribed approximation accuracy. Hence, it is natural to conjecture that consistent approximation is impossible unless m is at least proportional to the scale parameter. This however is not captured by Theorems 5 and 6 where the lower bound still converges to zero when $m \approx M$ (or $m \approx \beta$) due to the polynomial term in m.

The next result makes the intuition precise in a strong sense for the test distributions used in the previous subsections. Instead of applying Proposition 2 and bounding the minimum eigenvalue, we execute a different but more straightforward proof strategy: If the mixing distribution P is smooth enough with a large M or β , then f_P can be flat over a large region. In comparison, when m is too small, f_{P_m} inevitably has many modes and thus cannot approximate f_P consistently.

Theorem 7. The following statements hold:

1. Let $P_{\alpha,\beta}$ and C_{α} be defined in (20). Denote $\tilde{C}_{\alpha} = \sqrt{2\pi}C_{\alpha}$. For any $\beta \geq 2\tilde{C}_{\alpha}m$,

$$\mathcal{E}^{\star}(m, P_{\alpha, \beta}, \mathsf{TV}) \ge 1 - 5 \frac{\tilde{C}_{\alpha} m}{\beta} \sqrt{\log \frac{\beta}{\tilde{C}_{\alpha} m}}.$$
 (25)

2. For $C = \sqrt{\frac{\pi}{2}}$ and any $M \ge \sqrt{2\pi}m$,

$$\mathcal{E}^{\star}(m, \operatorname{Unif}[-M, M], \mathsf{TV}) \ge 1 - 5\frac{Cm}{M}\sqrt{\log\frac{M}{Cm}}.$$
 (26)

Proof. For $P = P_{\alpha,\beta}$, note that

$$f_P(x) = \int \phi(x-\theta) f_{\alpha,\beta}(\theta) d\theta \le \frac{C_\alpha}{\beta} \int \phi(x-\theta) d\theta = \frac{C_\alpha}{\beta}$$

Fix any *m*-atomic distribution $P_m = \sum_{i=1}^m w_i \delta_{\theta_i}$. Define

$$R_i \triangleq \left\{ x : w_i \phi(x - \theta_i) \ge \frac{C_\alpha}{\beta} \right\}$$

as the peak region of the *i*th mixture component. Denote $\eta_i = \eta(w_i) \triangleq \sqrt{2 \log \frac{\beta w_i}{\tilde{C}_{\alpha}}}$ if $w_i \ge \frac{\tilde{C}_{\alpha}}{\beta}$. Then, $R_i = [\theta_i - \eta_i, \theta_i + \eta_i]$ if $w_i \ge \frac{\tilde{C}_{\alpha}}{\beta}$, and $R_i = \emptyset$ otherwise. Let F_m , F, and G_{θ} be the probability measures corresponding to densities f_{P_m} , f_P , and $\phi(\cdot - \theta)$, respectively. Define $R \triangleq \bigcup_{i=1}^m R_i$. By the union bound,

$$\mathsf{TV}(f_{P_m}, f_P) \ge F_m(R) - F(R) \ge \sum_{i=1}^m w_i G_{\theta_i}(R_i) - \sum_{i=1}^m F(R_i).$$

Since $f_P \leq \frac{C_{\alpha}}{\beta}$, we have $R_i \subseteq \{x : w_i \phi(x - \theta_i) \geq f_P(x)\}$. Consequently, $w_i G_{\theta_i}(R_i) - F(R_i) \geq 0$ for all $i \in [m]$. Consider the following set of indices of mixture components with large weights

$$\mathcal{I} \triangleq \left\{ i \in [m] : w_i \ge \frac{2\tilde{C}_{\alpha}}{\beta} \right\}.$$

Note that our assumption $\beta \geq 2\tilde{C}_{\alpha}m$ implies that $\{i \in [m] : mw_i \geq 1\} \subseteq \mathcal{I}$. Hence, by the pigeonhole principle, \mathcal{I} is nonempty. For each $i \in \mathcal{I}$, we have $R_i = [\theta_i - \eta_i, \theta_i + \eta_i]$ and $\eta_i \geq \sqrt{2\log 2}$. It follows that

$$w_i G_{\theta_i}(R_i) - F(R_i) \ge w_i \left(1 - 2\Phi_c(\eta_i)\right) - \frac{2\eta_i C_\alpha}{\beta} \ge w_i \left[1 - \frac{2C_\alpha}{\beta w_i} \left(\frac{1}{\eta_i} + \eta_i\right)\right],$$
(27)

where the second inequality used $\Phi_c(\eta_i) \triangleq \int_{\eta_i}^{\infty} \phi(u) du \leq \frac{1}{\eta_i} \phi(\eta_i)$. Then,

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$$(f_{P_m}, f_P) \ge \sum_{i \in \mathcal{I}} w_i G_{\theta_i}(R_i) - F(R_i)$$

$$\stackrel{(a)}{\ge} 1 - \sum_{i \notin \mathcal{I}} w_i - \sum_{i \in \mathcal{I}} \frac{4C_{\alpha}}{\beta} \eta(w_i)$$

$$\stackrel{(b)}{\ge} 1 - m \frac{2\tilde{C}_{\alpha}}{\beta} - \frac{4C_{\alpha}}{\beta} |\mathcal{I}| \eta(|\mathcal{I}|^{-1})$$

$$= 1 - \frac{2\tilde{C}_{\alpha}m}{\beta} - \frac{4}{\sqrt{\pi}} \frac{\tilde{C}_{\alpha}|\mathcal{I}|}{\beta} \sqrt{\log \frac{\beta}{\tilde{C}_{\alpha}|\mathcal{I}|}}$$

$$\stackrel{(c)}{\ge} 1 - \frac{2\tilde{C}_{\alpha}m}{\beta} - \frac{4}{\sqrt{\pi}} \frac{\tilde{C}_{\alpha}m}{\beta} \sqrt{\log \frac{\beta}{\tilde{C}_{\alpha}m}},$$

where (a) follows from (27), (b) uses the definition of \mathcal{I} and the concavity of $x \mapsto \sqrt{\log x}$ on $x \ge 1$, and (c) holds since $\frac{\beta}{\tilde{C}_{\alpha}|\mathcal{I}|} \ge \frac{\beta}{\tilde{C}_{\alpha}m} \ge 2$ and $\frac{\sqrt{\log x}}{x}$ is decreasing in $x \ge \sqrt{e}$. Consequently,

$$\mathsf{TV}\left(f_{P_m}, f_P\right) \ge 1 - \left(\frac{2}{\sqrt{\log 2}} + \frac{4}{\sqrt{\pi}}\right) \frac{\tilde{C}_{\alpha}m}{\beta} \sqrt{\log \frac{\beta}{\tilde{C}_{\alpha}m}} \ge 1 - 5\frac{\tilde{C}_{\alpha}m}{\beta} \sqrt{\log \frac{\beta}{\tilde{C}_{\alpha}m}}$$

The lower bound holds uniformly for all $P_m \in \mathcal{P}_m$, and hence the result (25) follows.

For P = Unif[-M, M], the derivation is essentially the same as above. Note that $f_P(x) = \frac{1}{2M} \int_{-M}^{M} \phi(x-\theta) d\theta \leq \frac{1}{2M}$. For any $P_m = \sum_{i=1}^{m} w_i \delta_{\theta_i} \in \mathcal{P}_m$, define $R_i = \{x : w_i \phi(x-\theta_i) \geq (2M)^{-1}\}$ and $\mathcal{I} \triangleq \{i \in [m] : w_i \geq \sqrt{2\pi}M^{-1}\}$ correspondingly. Finally, (26) is proved via a similar argument.

5 Applications and extensions

In this section, we discuss a few applications and extensions of the paper.

5.1 Convergence rates of nonparametric maximum likelihood estimator

Given a sample of *n* independent observations $X = (X_1, \ldots, X_n)$, where $X_i \stackrel{\text{i.i.d.}}{\sim} f_{P^*}$, the nonparametric maximum likelihood estimator (NPMLE) is defined as [KW56]

$$\hat{P} \in \operatorname*{arg\,max}_{Q \in \mathcal{Q}} \frac{1}{n} \sum_{i=1}^{n} \log f_Q(X_i), \tag{28}$$

where the optimization is over a class Q of priors. The finite-order approximation of a family of distributions naturally yields upper bounds of the metric entropy of that family, which further determines the convergence rate of the NPMLE [WS95, vdVW96, GvdV01, JZ09]. Following the analysis in [Zha09], the Hellinger distance $H(f_{\hat{P}}, f_{P^*})$ can be upper bounded in terms of $m^*(\epsilon, Q, L_{\infty}[-M, M])$, where M is a parameter depending on the tail of P^* . Hence, our analysis can recover the convergence rates derived in [Zha09].

Moreover, suppose P^* belongs to a given family of distributions \mathcal{P} , Theorem 8 below in fact establishes a faster rate for the constrained NPMLE with $\mathcal{Q} = \mathcal{P}$, which is characterized by $m^*(H, \mathcal{P}, n^{-1/2})\frac{\log n}{n}$ with respect to the Hellinger distance.⁴ The improvement arises from our tight approximation rates. In contrast to [Zha09, Theorem 1] which yield the same convergence rates for the families $\mathcal{P}_M^{\text{Bdd}}$ and $\mathcal{P}_{\alpha}(\beta)$, our result gives a strictly faster rate for the former. In addition, another technical contribution is a a simplified proof: Compared with that of [Zha09, Theorem 1], we simplify the argument by first proving a L_{∞} -entropy estimate then directly bounding the square root of the mixture density. The proofs are deferred to Appendix D.

Theorem 8. 1. Let $P^* \in \mathcal{P}_M^{\mathsf{Bdd}}$, where $M \leq n^c$ for some constant c > 0. Let \hat{P} be a solution of (28) with $\mathcal{Q} = \mathcal{P}_M^{\mathsf{Bdd}}$. There exist constants $s^*, c' > 0$ such that for any $s \geq s^*$,

$$\mathbb{P}[H(f_{\hat{P}}, f_{P^{\star}}) \ge s\epsilon_n] \le n^{-c's^2},\tag{29}$$

where

$$\epsilon_n = \frac{\log n}{\sqrt{n\log\left(1 + \sqrt{\log n}/M\right)}}.$$
(30)

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⁴We note that existing minimax lower bounds in [PW21, Theorem 20] agree with $m^*(H, \mathcal{P}, n^{-1/2})/n$. However, bridging this gap remains an open problem.

2. Let $P^* \in \mathcal{P}_{\alpha}(\beta)$, where $\alpha > 0$ is fixed and $\beta \leq n^c$ for some constant c > 0. Let \hat{P} be a solution of (28) with $\mathcal{Q} = \mathcal{P}_{\alpha}(\beta)$. There exist $s^*, c' > 0$ which depend on α such that for any $s \geq s^*$, (29) holds with

$$\epsilon_n = \frac{\log n}{\sqrt{n \log \left(1 + (\log n)^{\frac{\alpha - 2}{2\alpha}} / \beta\right)}}.$$
(31)

5.2 Mixing distributions under moment constraints

So far, our results deal with mixing distributions that exhibit exponential decay in their tails. In fact, our analysis framework also allows for extension to distributions with heavier tails. Let us consider mixing distributions under moment conditions, which were well-studied in mixture models [GvdV07b, Zha09, SG20, SW23] Let $\psi(x) = x^{\alpha}$ for $\alpha > 0$. Then the corresponding ψ -Orlicz norm is simply the usual L_{α} -norm $||X||_{\psi} = ||X||_{\alpha} \triangleq \mathbb{E}[|X|^{\alpha}]^{\frac{1}{\alpha}}$. Define

$$\mathcal{M}_{\alpha}(\beta) \triangleq \{P : X \sim P, \|X\|_{\alpha} \leq \beta\},\$$

that is, the family of distributions whose α^{th} moment is no larger than β^{α} . For $P \in \mathcal{M}_{\alpha}(\beta)$, (4) implies the polynomial-type tail bounds $P[|X| \ge t] \le 2\left(\frac{\beta}{t}\right)^{\alpha}$.

Theorem 9. There exists c_1, c_2 (which may depend on α) such that the following holds. When $m \ge c_1\beta$ and $\beta \ge c_2$,

$$\frac{1}{m\log\frac{m}{\beta}} \left(\frac{\beta}{m}\right)^{\alpha} \lesssim_{\alpha} \mathcal{E}^{\star}(m, \mathcal{M}_{\alpha}(\beta), \mathsf{TV}) \lesssim_{\alpha} \left(\frac{\beta}{m} \sqrt{\log\frac{m}{\beta}}\right)^{\alpha}.$$

In contrast to the sub-Weibull family, distributions in $\mathcal{M}_{\alpha}(\beta)$ have much heavier tails and thus require more components to approximate. As such, the best approximation error is no longer exponential but polynomial in m for light-tailed, as shown in Theorem 9. The derivation of the upper and lower bounds essentially follows the same steps for Theorems 4 and 5, respectively, and the proofs are deferred to Appendix C.2.

There is a gap between the lower and upper bound polynomial in m. Following our derivation, the gap appears due to the extra m^{-1} term in the Proposition 2. In Theorems 1 and 2, the extra polynomial term does not affect the tightness of the results since the exponential term dominates the rate. On the other hand, Theorem 9 is limited to the TV distance, since other divergences (e.g., the KL and χ^2 divergence) may vary within an exponent from the TV distance and thus widen the gap of the polynomial rate. A potential remedy is to characterize the information geometry of the Gaussian mixture families, as carried out in [JPW23] for compactly supported and subgaussian mixing distributions. The tight rate is still left to be discovered.

5.3 Generalization to other mixture models

So far we have focused on approximating general one-dimensional Gaussian location mixture by finite mixtures. Our techniques of finite-order approximation can also adapt to other models, including the Gaussian location-scale mixtures and multi-dimensional mixtures. In these contexts, most of the existing approximation results [SG20, Jia20, SGS21, SW23] in the empirical Bayes literature rely on Taylor expansion and local moment matching.

Location-scale Gaussian mixtures Consider the location-scale Gaussian mixture with density

$$f_H(x) = \int \frac{1}{\sigma} \phi\left(\frac{x-\theta}{\sigma}\right) dH(\theta,\sigma),$$

where H is the joint mixing distribution of location and scale parameters. In other words, if $(X, S) \sim H$ and $Z \sim N(0, 1)$ are independent, then $X + SZ \sim f_H$. Let H_m denote an *m*-atomic approximation with $X_m + S_m Z \sim f_{H_m}$.

Previous works applied moment-based approach to construct approximators. For instance, when H is supported in $[-1,1]^2$, by applying Taylor expansion and Carathéodory's theorem, [GvdV01, Lemma 3.4] constructs an H_m with $m \leq O(\log^2 \frac{1}{\epsilon})$ such that $||f_H - f_{H_m}||_{\infty} \leq \epsilon$. In addition, if X and S are independent, both X_m and S_m are supported on $O(\log \frac{1}{\epsilon})$ points.

For the lower bound, applying Lemma 7 yields that

$$\mathsf{TV}\left(f_{H_m}, f_H\right) \ge \frac{1}{2} \sup_{\omega} \left| \mathbb{E}\left[\exp\left(i\omega X - \frac{1}{2}\omega^2 S^2\right) \right] - \mathbb{E}\left[\exp\left(i\omega X_m - \frac{1}{2}\omega^2 S_m^2\right) \right] \right|.$$
(32)

It is unclear how to lower bound this uniformly over (X_m, H_m) that is *m*-atomic. However, for the special case of scale mixtures with $X \equiv 0$ and *S* bounded away from zero and infinity, we can further lower bound (32) as follows. First, by choosing $\omega \in \{\sqrt{k} : k \in \{0, \ldots, 2m\}\}$, using the same argument as in Proposition 2, (32) can be related to the minimum eigenvalue λ_{\min} of the Hankel matrix $\left(\mathbb{E}\exp\left(-\frac{(j+k)S^2}{2}\right)\right)_{j,k=0}^m$. Applying the method based on orthogonal expansion in Proposition 3 (cf. Appendix B.2), one can show that $\lambda_{\min} \geq e^{-\Omega(m)}$, leading to the lower bound that at least $\Omega(\log \frac{1}{\epsilon})$ components are needed to achieve an approximation error ϵ in the scale mixture model. Establishing lower bounds for general location-scale mixtures remains an outstanding challenging.

Multiple dimensions Consider the Gaussian location mixture on \mathbb{R}^d :

$$f_P(\mathbf{x}) = \int \phi_{(d)}(\mathbf{x} - \boldsymbol{\theta}) \mathrm{d}P(\boldsymbol{\theta}),$$

where $\phi_{(d)}(z) \triangleq (2\pi)^{-d/2} \exp(-||z||^2/2)$ denotes the *d*-dimensional standard Gaussian density.

Let $\mathcal{P}_{M,d}^{\mathsf{Bdd}}$ denote the family of distributions supported on the *d*-dimensional Euclidean ball $B_d(M) \triangleq \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leq M\}$. When $P \in \mathcal{P}_{M,d}^{\mathsf{Bdd}}$ is a product measure, our one-dimensional results can be applied to approximate each coordinate separately. For general $P \in \mathcal{P}_{M,d}^{\mathsf{Bdd}}$, we need to extend Lemma 1 to the multidimensional case by comparing moment tensors [DWYZ23]. Then, we establish the following upper bound, which improves the previous result [SG20, Lemma D.3] by bounding the χ^2 -divergence and deriving a tighter bound than the $\exp(-\Theta_d(m^{\frac{1}{d}}))$ result in [SG20, Lemma D.3] under the case $m^{\frac{1}{d}} \gtrsim_d M^2$.

Proposition 4. There exists $\kappa_d > 0$ which only depends on d such that for any $m \in \mathbb{N}$ and M > 0:

$$\mathcal{E}^{\star}(m, \mathcal{P}_{M,d}^{\mathsf{Bdd}}, \chi^2) \leq \begin{cases} \exp\left(-\frac{m^{\frac{1}{d}}}{2}\log\frac{m^{\frac{1}{d}}}{M^2}\right), & m^{\frac{1}{d}} \geq \kappa_d M^2; \\ \exp\left(-\frac{\log\kappa_d}{42\kappa_d}\frac{m^{\frac{2}{d}}}{M^2}\right), & 6\sqrt{3\kappa_d}M \leq m^{\frac{1}{d}} \leq \kappa_d M^2. \end{cases}$$
(33)

The lower bound is derived following the roadmap in Section 4.1. For the multidimensional regime, the trigonometric moment matrices are replaced by multilevel Toeplitz matrices [TZ98, Pes19], and the wrapped density approach is then applied to yield the following result.

Proposition 5. For any $m \in \mathbb{N}$,

$$\mathcal{E}^{\star}(m, \operatorname{Unif}([-M/\sqrt{d}, M/\sqrt{d}]^d), \mathsf{TV}) \ge \frac{1}{2^{d+1}m} \exp\left(-\frac{d^2 \pi^2 m^{\frac{2}{d}}}{2M^2}\right),$$

The approximation rates for the distribution classes under tail conditions can be similarly extended. The additional results and proofs are presented in Appendix C.3.

A Preliminaries

A.1 Distances between probability measures

The loss function $d(\cdot, \cdot)$ for measuring the approximation error is chosen among f-divergences, including

- χ^2 -divergence $\chi^2(P||Q) = \int (\frac{(p-q)^2}{q} d\mu;$
- Total variation distance $\mathsf{TV}(P,Q) = \frac{1}{2} \int |p-q| \mathrm{d}\mu;$
- Kullback-Leibler (KL) divergence $\mathsf{KL}(P,Q) = \int p \log \frac{p}{q} d\mu;$
- Squared Hellinger distance $H^2(P,Q) = \int (\sqrt{p} \sqrt{q})^2 d\mu$,

where $p = \frac{dP}{d\mu}$ and $q = \frac{dQ}{d\mu}$ for some measure μ dominating P and Q. We refer the reader to [PW24, Chapter 7] for a comprehensive over of f-divergences, including the following inequalities [PW24, Sec. 7.6]:

Lemma 6.

$$\begin{split} \frac{1}{2}H^2(P,Q) &\leq \mathsf{TV}(P,Q) \leq \sqrt{\frac{1}{2}D(P,Q)} \leq \sqrt{\frac{1}{2}\chi^2(P,Q)},\\ \mathsf{TV}(P,Q) &\leq H(P,Q) \leq \sqrt{D(P,Q)}. \end{split}$$

We also need the following variational representation of f-divergences (see, e.g, [PW24, Sec. 7.13]):

Lemma 7. For two probability measures P and Q supported on a space \mathcal{X} ,

$$\chi^{2}(P||Q) = \sup_{g:\mathcal{X}\to\mathbb{R}} \frac{\left(\mathbb{E}_{P}g - \mathbb{E}_{Q}g\right)^{2}}{\mathsf{var}_{Q}[g]},\tag{34}$$

$$\mathsf{TV}(P,Q) = \frac{1}{2} \sup_{\|g\|_{\infty} \le 1} |\mathbb{E}_P g - \mathbb{E}_Q g|, \qquad (35)$$

where $g: \mathcal{X} \to \mathbb{C}$ in (35) and $||g||_{\infty} \triangleq \sup_{x \in \mathcal{X}} |g(x)|$ denotes the L_{∞} -norm.

A.2 Gauss quadrature

Gauss quadrature is the best discrete approximation of a given distribution in the sense of moments; cf. [SB02, Section 3.6]. Given a distribution P supported on an interval $[a, b] \in \mathbb{R}$, an m-point Gauss quadrature is an m-atomic distribution $P_m \in \mathcal{P}_m$, also supported on [a, b], such that, for any polynomial g of degree no more than 2m - 1,

$$\mathbb{E}_{P}\left[g(X)\right] = \mathbb{E}_{P_{m}}\left[g(X)\right].$$
(36)

For a basic algorithm to compute the Gauss quadrature, see [GW69]. Note that the generic result of Caratheódory's theorem [GvdV01, Lemma 1] provides a distribution that match the first m orders of moments, which is improved by the Gauss quadrature by a factor of two.

A.3 Orthogonal polynomials

In this section, we give a brief introduction to certain classes of orthogonal polynomials that will be applied to prove the results in Appendix B.

• (Probabilists') Hermite polynomials

$$H_n(x) = r! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1/2)^j}{j!(n-2j)!} x^{n-2j}.$$
(37)

For $Z \sim N(0,1)$, it holds that $\mathbb{E}[H_j(Z)H_k(Z)] = k!\delta_{jk}$ and $\mathbb{E}[H_k(x+Z)] = x^k$.

• Scaled Legendre polynomials

$$L_n(x) = \frac{\sqrt{2n+1}}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}$$
(38)

with $\mathbb{E}[L_j(U)L_k(U)] = \delta_{jk}$ for $U \sim \text{Unif}[-1, 1]$.

• Rogers-Szegö polynomials [Sim05, Example 1.6.5 and Eq. (1.6.51)]

$$R_n(x) = \frac{1}{\sqrt{(q)_n}} \sum_{j=0}^n (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{(n-j)/2} x^j$$
(39)

which are orthogonal with respect to the "wrapped Gaussian" weight on the unit circle, namely $\mathbb{E}\left[R_k(e^{iaZ})\overline{R_j(e^{iaZ})}\right] = \delta_{jk}$, where $a^2 = -\log q$ and $Z \sim N(0,1)$. Here in (39), $(q)_n \triangleq (1-q)(1-q^2)\dots(1-q^n), (0)_n \triangleq 1$, and $\begin{bmatrix} n\\ j \end{bmatrix}_q \triangleq \frac{(q)_n}{(q)_j(q)_{n-j}}$ is known as the qbinomial coefficient. By definition, $(q)_n$ is decreasing and converges to $(q)_\infty \triangleq \prod_{n=1}^{\infty} (1-q^n)$ as $n \to \infty$. $(q)_\infty$ is referred to as the *Euler function*.

B Alternative proofs of lower bounds via orthogonal expansion

B.1 Lower bound for Gaussian distributions

As mentioned in Remark 2, the orthogonal expansion approach is applicable if the explicit expressions of the orthogonal polynomials are available. Specially, for the case where the mixing distribution is Gaussian, we state the result in Theorem 10. Compared with the lower bound (22), it recovers the order $\Theta(m/\sigma)$ in the exponent with a larger multiplicative constant.

The following lemma upper bounds the Euler function $(q)_{\infty}$, which is helpful to control the norm of the polynomial coefficient matrix.

Lemma 8. Let $q \in (0,1)$ and $\exp(-t) = q$. For any $t_0 > 0$ and $t \in (0,t_0]$, there exists a constant C which depends on t_0 such that

$$\frac{1}{(q)_{\infty}} \le C \exp\left(\frac{\pi^2}{6t}\right).$$

Proof. Applying the asymptotic result in [Wat36, p. 57], as $q \to 1_{-}$,

$$\frac{1}{(q)_{\infty}} = \sqrt{\frac{t}{2\pi}} \exp\left(\frac{\pi^2}{6t} - \frac{t}{24}\right) + o(1).$$

Then, there exists a small $\delta_0 > 0$ such that $\frac{1}{(q)_{\infty}} \leq \exp\left(\frac{\pi^2}{6t}\right)$ for $t \in (0, \delta_0)$, and the ratio $\frac{1}{(q)_{\infty} \exp\left(\frac{\pi^2}{6t}\right)}$ is uniformly bounded in $[\delta_0, t_0]$. Hence, the desired result follows.

Theorem 10. Suppose that $m \ge 2\sigma$. There exists some universal constant c such that

$$\mathcal{E}^{\star}(m, N(0, \sigma^2), \mathsf{TV}) \ge cm^{-3} \exp\left(-\left(4 + \frac{\pi^2}{24}\right) \frac{m}{\sigma}\right).$$

Proof. Let $\{R_n\}$ be the orthonormal Rogers-Szegö polynomial (39) with $a = \delta \sigma$ satisfying that $\mathbb{E}\left[R_k(e^{i\delta\sigma Z})\overline{R_j(e^{i\delta\sigma Z})}\right] = \delta_{jk}$. Let $\mathbf{R} = (R_{nj})_{n,j=0}^m$ be the associated coefficient matrix of $\{\varphi_n\}$ as defined in Section 4.1. To upper bound $\|\mathbf{R}\|_F$, we apply the explicit form of R_n in (39), i.e.,

$$R_n(w) = \frac{1}{\sqrt{(q)_n}} \sum_{j=0}^n (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{(n-j)/2} w^j$$
(40)

where $q = \exp(-\delta^2 \sigma^2)$.

Set $\delta = \sqrt{\frac{4}{m\sigma}}$, with $q = \exp(-\frac{4\sigma}{m})$ and $t \triangleq -\log(q) = \frac{4\sigma}{m}$. By our assumption, $t \in (0, 2]$, $q \in [e^{-2}, 1)$. In this region, applying Lemma 8, there exists a universal constant C such that

$$\frac{1}{(q)_{\infty}} \le C \exp\left(\frac{\pi^2}{6t}\right) = C \exp\left(\frac{\pi^2 m}{24\sigma}\right).$$

By the q-binomial theorem [DLM, 17.2.35, 17.2.39], for any $q \in [0, 1]$, we have

$$\sum_{j=0}^{n} q^{j} \begin{bmatrix} n \\ j \end{bmatrix}_{q^{2}} = \prod_{k=1}^{n} (1+q^{k}) \le \exp\left(\sum_{k=0}^{\infty} q^{k}\right) = \exp\left(\frac{1}{1-q}\right).$$

It follows from (40) that

$$\|\mathbf{R}\|_F \le \sum_{n=0}^m \sum_{j=0}^n |R_{nj}| \le \sum_{n=0}^m \frac{1}{\sqrt{(q)_n}} \sum_{j=0}^n [j]_q q^{j/2} \le \sum_{n=0}^m \frac{1}{\sqrt{(q)_n}} \exp\left(\frac{1}{1-\sqrt{q}}\right).$$

Set $s = \sigma/m$ in $1 - s - \exp(-2s) > 0$ for $s \in [0, 1/2]$, we have $\frac{1}{1 - \sqrt{q}} = \frac{1}{1 - \exp(-2s)} \le \frac{1}{s} = \frac{m}{\sigma}$. Then

$$\|\mathbf{R}\|_F^2 \le m^2 \frac{\exp(2m/\sigma)}{(q)_{\infty}} \le Cm^2 \exp\left(\left(2 + \frac{\pi^2}{24}\right)\frac{m}{\sigma}\right).$$
(41)

Consequently, applying Propositions 2 and 3 with (41), we obtain

$$\mathcal{E}^{\star}(m, N(0, \sigma^2), \mathsf{TV}) \ge cm^{-3} \exp\left(-\left(4 + \frac{\pi^2}{24}\right) \frac{m}{\sigma}\right)$$

for some universal constant c.

B.2 Lower bound for uniform distributions

We give an alternative proof of the lower bound (24) for approximating uniform mixing distribution. The framework resembles the approach in Section 4.1, while we consider the χ^2 divergence instead of working with the TV distance directly. First, we show that it suffices to derive the same lower bound (within a constant in the exponent) under the χ^2 -divergence. Next, we relate the χ^2 variational lower bound to the spectrum of some weighted moment matrix, and then apply orthogonal expansion to bound the corresponding coefficient matrix as carried out in Proposition 3.

The following lemma shows a converse f-divergence inequality between two Gaussian mixtures, with respect to Lemma 6.

Lemma 9 ([JPW23, Theorem 21]). For any $P, Q \in \mathcal{P}_a^{Bdd}$ with $a \geq 2$,

 $\chi^2(f_P || f_Q) \le 2 \exp(50a^2) H^2(f_P, f_Q).$

Proposition 6. There exist universal positive constants ζ, η , and C' such that the following holds. If $m \geq C'(M \vee 1)^2$, then for any $P \in \mathcal{P}_M^{\mathsf{Bdd}}$,

$$\mathcal{E}^{\star}(m, P, \mathsf{TV}) \ge \eta \mathcal{E}^{\star}(m, P, \chi^2)^{\zeta}.$$
(42)

Proof. Suppose that $Q \in \mathcal{P}_m$ satisfying $\mathsf{TV}(f_P, f_Q) = \mathcal{E}^*(m, P, \mathsf{TV}) \triangleq \epsilon$. Let Q_a be the conditional version of Q on $I_a = [-a, a]$. Suppose that $a \ge 2(M \lor 1)$, and denote $U \sim Q, V \sim P$. Then

$$\mathsf{TV}(f_{Q_a}, f_P) \le \mathsf{TV}(f_P, f_Q) + \mathsf{TV}(f_Q, f_{Q_a}) \le \epsilon + \mathbb{P}[|U| \ge a].$$
(43)

Since $P, Q_a \in \mathcal{P}_a^{\mathsf{Bdd}}$, by Lemmas 6 and 9,

$$\mathsf{TV}(f_{Q_a}, f_P) \ge \frac{1}{2} H^2(f_{Q_a}, f_P) \ge \frac{e^{-50a^2}}{4} \chi^2(f_{Q_a} \| f_P).$$
(44)

Also, we have

$$\epsilon \geq \mathsf{TV}(f_P, f_Q) \geq \sup_t \mathbb{P}\left[U + Z \geq t\right] - \mathbb{P}\left[V + Z \geq t\right]$$
$$= \sup_t \mathbb{E}\left[\Phi_c(t - U)\right] - \mathbb{E}\left[\Phi_c(t - V)\right]$$
$$\geq \sup_{t \geq a} \mathbb{P}\left[U \geq a\right] \Phi_c(t - a) - \Phi_c(t - M),$$

where the last inequality follows from Markov inequality and $|V| \leq M$. Then,

$$\mathbb{P}\left[U \ge a\right] \le \inf_{t \ge a} \frac{\epsilon + \Phi_c(t - M)}{\Phi_c(t - a)}.$$
(45)

Likewise, the same upper bound holds for $\mathbb{P}[U \leq -a]$. It follows from (43), (44) and (45) that, for $c_0 = 50$ and any $a \geq 2(M \vee 1)$,

$$\begin{split} \chi^2 \left(f_{Q_a} \| f_P \right) &\leq 4 e^{c_0 a^2} \operatorname{\mathsf{TV}} \left(f_{Q_a}, f_P \right) \\ &\leq 4 e^{c_0 a^2} \left(\epsilon + \mathbb{P} \left[|U| \geq a \right] \right) \\ &\leq 4 e^{c_0 a^2} \left(\epsilon + 2 \inf_{t \geq a} \frac{\epsilon + \Phi_c(t - M)}{\Phi_c(t - a)} \right). \end{split}$$

By applying the Gaussian tail bound $\frac{t\phi(t)}{1+t^2} \leq \Phi_c(t) \leq \frac{\phi(t)}{t}$ and setting $t = (2c_0 + 1)a$ for $a \geq 2(M \vee 1)$, we have

$$\frac{\epsilon + \Phi_c(t - M)}{\Phi_c(t - a)} \le \frac{1 + (t - a)^2}{t - a} \frac{\epsilon}{\phi(t - a)} + \frac{1 + (t - a)^2}{(t - a)(t - M)} \frac{\phi(t - M)}{\phi(t - a)}$$
$$\le 4c_0 a \frac{\epsilon}{\phi(2c_0 a)} + 2 \frac{\phi((2c_0 + 0.5)a)}{\phi(2c_0 a)}$$
$$\le 4\sqrt{2\pi}c_0 \epsilon e^{(2c_0^2 + 1)a^2} + 2e^{-(c_0 + 1/8)a^2}.$$

Then, with constants $c_1 = 4 + 32\sqrt{2\pi}c_0$, $c_2 = 2c_0^2 + c_0 + 1$, and $c_3 = 1/8$,

$$\chi^2(f_{Q_a} \| f_P) \le \inf_{a \ge 2(M \lor 1)} c_1(\epsilon e^{c_2 a^2} + e^{-c_3 a^2}).$$
(46)

By Theorem 3, there exists a universal constant C' such that, for any $m \ge C'(M \lor 1)^2$,

$$\epsilon \le \exp(-(c_2 + c_3)[2(M \lor 1)]^2).$$

Choose a such that $\epsilon = e^{-(c_2+c_3)a^2}$ hold, and it follows that $a \ge 2(M \lor 1)$. Then, (46) implies

$$\chi^2(f_{Q_a} \| f_P) \le 2c_1 \epsilon^{\frac{c_3}{c_2 + c_3}} = 2c_1 \mathcal{E}^*(m, P, \mathsf{TV})^{\frac{c_3}{c_2 + c_3}}$$

Since $Q_a \in \mathcal{P}_m$ by definition, we obtain the desired result (42) with $\zeta = \frac{c_2 + c_3}{c_3}$ and $\eta = \frac{1}{(2c_1)^{\zeta}}$.

Theorem 11. There exists universal constants C_1 and C' such that the following holds. When $m \ge C'(M \lor 1)^2$,

$$\mathcal{E}^{\star}(m, \operatorname{Unif}[-M, M], \mathsf{TV}) \ge \exp\left(-C_1 m \log\left(\frac{m}{M^2}\right)\right)$$

Proof. We first prove a χ^2 lower bound by applying the variational representation (34), and finally upgrade it to the total variation via Proposition 6. We choose the Hermite polynomials H_k in (37) as the test functions. Denote $X \sim P = \text{Unif}[-M, M], Y = X + Z$, and $Y_m = X_m + Z$ for some $X_m \sim P_m \in \mathcal{P}_m$. Then,

$$\mathbb{E}\left[H_k(Y)\right] = \mathbb{E}\left[\mathbb{E}\left[H_k(X+Z)|X\right]\right] = \mathbb{E}X^k$$

Likewise, $\mathbb{E}[H_k(Y_m)] = \mathbb{E}X_m^k$. For $m \ge M^2$ and $k \in [2m]$, by [GR07, 7.374.7], we have

$$\operatorname{var}[H_k(Y)] \le \mathbb{E}[H_k^2(Y)] = k! \sum_{l=0}^k \binom{k}{k-l} \frac{\mathbb{E}[X^{2l}]}{l!} \le k! \sum_{l=0}^k 2^k \frac{M^{2l}}{l!} \le k! (4e)^m.$$

Applying (34) yields

$$\chi^{2}(f_{P_{m}}||f_{P}) \ge \sup_{k \in \mathbb{N}} \frac{\left(\mathbb{E}[X^{k}] - \mathbb{E}[X_{m}^{k}]\right)^{2}}{\mathsf{var}[H_{k}(Y)]}$$

$$\tag{47}$$

$$\geq \frac{1}{2m} \sum_{k=1}^{2m} \frac{\left(\mathbb{E}[X^k] - \mathbb{E}[X_m^k]\right)^2}{k! (4e)^m}$$
(48)

$$= \frac{1}{2m(4e)^m} \sum_{k=1}^{2m} M^{2k} \frac{\left(\mathbb{E}[U^k] - \mathbb{E}[U_m^k]\right)^2}{k!}$$
(49)

where $U \sim \text{Unif}[-1, 1]$ and $U_m = \frac{X_m}{M}$.

We lower bound (49) by analyzing weighted moment matrix. Specifically, for $i, j \in \mathbb{N}$ satisfying i + j = k,

$$M^{2k} \frac{\left(\mathbb{E}[U^{k}] - \mathbb{E}[U_{m}^{k}]\right)^{2}}{k!} \geq \frac{M^{2(i+j)}}{2^{i+j}i!j!} \left(\mathbb{E}[U^{k}] - \mathbb{E}[U_{m}^{k}]\right)^{2} \\ = \left(V_{ij}(U) - V_{ij}(U_{m})\right)^{2},$$

where $C_i \triangleq \frac{M^i}{\sqrt{2^i i!}}$ and $V_{ij}(U) \triangleq \mathbb{E}[(C_i U^i)(C_j U^j)]$. The weighted moment matrix $\mathbf{V}_m(U)$ of order m+1 consists of $V_{ij}(U)$ in its $(i, j)^{\text{th}}$ entry for $i, j = 0, \ldots, m$. Let $\mathbf{C}_m = \text{Diag}(C_0, C_1, \ldots, C_m)$. Then $\mathbf{V}_m(U) = \mathbf{C}_m \mathbf{M}_m(U) \mathbf{C}_m$. Notably, since U_m is *m*-atomic, we also have rank $(\mathbf{V}_m(U_m)) \leq m$. By Lemma 3, we have

$$\sum_{k=1}^{2m} M^{2k} \frac{\left(\mathbb{E}[U^k] - \mathbb{E}[U_m^k]\right)^2}{k!} \ge \frac{\|\mathbf{V}_m(U) - \mathbf{V}_m(U_m)\|_F^2}{m+1} \ge \frac{\lambda_{\min}^2(\mathbf{V}_m(U))}{m+1}.$$
 (50)

It remains to bound the smallest eigenvalue of $\mathbf{V} = \mathbf{V}_m(U)$, given by

$$\lambda_{\min}(\mathbf{V}) = \min_{\mathbf{x} \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}} \frac{\mathbf{x}^\top \mathbf{V} \mathbf{x}}{\|\mathbf{x}\|^2}.$$
(51)

Let L_k be the orthonormal Legendre polynomial defined in (38), and $\pi_m(t) = \sum_{j=0}^m C_j x_j t^j$, with an expansion $\pi_m(t) = \sum_{k=0}^m c_k L_k(t)$. Denote $\mathbf{c} = (c_0, \ldots, c_m)^\top$ and $\mathbf{t} = (1, t, \ldots, t^m)^\top$. Then, by a similar derivation of Proposition 3, we have

$$\|\mathbf{c}\|_2^2 = \mathbb{E}\left[\pi_m^2(U)\right] = \mathbf{x}^\top \mathbf{V} \mathbf{x}.$$

Let $\mathbf{L}_m = (L_{nj})_{n,j=0}^m$ be the associated coefficient matrix of $\{L_k\}$ of order m+1. By definition,

$$\mathbf{x}^{\top} \mathbf{C}_m^{\top} \mathbf{t} = \pi_m(t) = \sum_{k=0}^m c_k L_k(t) = \mathbf{c}^{\top} \mathbf{L}_m \mathbf{t}, \quad \forall t,$$

which implies that $\mathbf{x} = \mathbf{C}_m^{-1} \mathbf{L}_m^\top \mathbf{c}$. Hence

$$\lambda_{\min}(\mathbf{V}) = \min_{\mathbf{c}\neq 0} \frac{\|\mathbf{c}\|_2^2}{\mathbf{c}^\top \mathbf{L}_m \mathbf{C}_m^{-2} \mathbf{L}_m^\top \mathbf{c}} \ge \frac{1}{\|\mathbf{L}_m \mathbf{C}_m^{-1}\|_F^2}.$$
(52)

Applying the explicit formula of L_m in (38), we have

$$\|\mathbf{L}_{m}\mathbf{C}_{m}^{-1}\|_{F}^{2} \leq \sum_{n=0}^{m} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2n+1}{2^{2n}} \frac{[(2n-2k)!]^{2}}{(k!)^{2}(n-2k)![(n-k)!]^{2}} \left(\frac{2}{M^{2}}\right)^{n-2k}$$

$$\leq (2m+1) \sum_{n=0}^{m} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^{2n}} 2^{4(n-k)} \frac{n!}{(k!)^{2}} \left(\frac{2}{M^{2}}\right)^{n-2k}$$

$$= (2m+1) \sum_{n=0}^{m} n! \left(\frac{8}{M^{2}}\right)^{n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{(M^{2}/8)^{k}}{k!}\right)^{2}$$

$$\leq 3m^{2}m! \left(\frac{8}{M^{2}}\right)^{m} \left(\sum_{k=0}^{\infty} \frac{(m/8)^{k}}{k!}\right)^{2}$$

$$\leq Cm^{2.5} \left(\frac{8m}{eM^{2}}\right)^{m} \exp\left(\frac{m}{4}\right)$$
(53)

for some constant C when $m \ge M^2$. Combining (49), (50), (52), and (53), we obtain that

$$\mathcal{E}^{\star}(m, P, \chi^2) \geq \frac{1}{2m(4e)^m} \sup_{P_m \in \mathcal{P}_m} \sum_{k=1}^{2m} M^{2k} \frac{\left(\mathbb{E}[U^k] - \mathbb{E}[U_m^k]\right)^2}{k!}$$
$$\geq \frac{1}{Cm^7} \left(\frac{e}{256}\right)^m \exp\left[-2m\log\left(\frac{m}{M^2}\right) - \frac{m}{2}\right]$$
$$\geq c \exp\left[-2m\log\left(\frac{m}{M^2}\right) - 6m\right]$$

for some small constant c.

Finally, by Proposition 6, there exists universal constants C_1 and C' such that

$$\mathcal{E}^{\star}(m, P, \mathsf{TV}) \ge \exp\left(-C_1 m \log\left(\frac{m}{M^2}\right)\right), \quad m \ge C' M^2.$$

C Proof of the approximation rates

C.1 Proofs of main theorems

For convenience, we restate the dual formula of m^* and \mathcal{E}^* in the following:

$$m^{\star}(\epsilon, \mathcal{P}, d) = \inf \left\{ m : \mathcal{E}^{\star}(m, \mathcal{P}, d) \le \epsilon \right\},$$
(54)

$$\mathcal{E}^{\star}(m,\mathcal{P},d) = \inf\left\{\epsilon : m^{\star}(\epsilon,\mathcal{P},d) \le m\right\}.$$
(55)

Proof of Theorem 1. Pick any $d \in \{\mathsf{TV}, H, \mathsf{KL}, \chi^2\}$ and denote $m = m^*(\epsilon, \mathcal{P}_M^{\mathsf{Bdd}}, d)$. Combining the result in Theorems 3 and 6, we have that $\log \frac{1}{\epsilon} \asymp m \log \frac{m}{M^2}$ when $m \ge CM^2$ for some constant C, and $\log \frac{1}{M\epsilon} \lesssim \frac{m^2}{M^2} \lesssim \log \frac{1}{\epsilon}$ when $3\sqrt{\kappa}M \le m \le CM^2$. Also, the condition $M \le \epsilon^{-c_1}, c_1 \in (0, 1)$ implies that $\log \frac{1}{M\epsilon} \asymp \log \frac{1}{\epsilon}$. Following (54), we can rewrite the result as

$$m \asymp \begin{cases} \frac{\log \epsilon^{-1}}{\log \log \epsilon^{-1} - \log M^2}, & M \le c\sqrt{\log \epsilon^{-1}}; \\ M\sqrt{\log \epsilon^{-1}}, & c\sqrt{\log \epsilon^{-1}} < M \le \epsilon^{-c_1}, \end{cases}$$

for some small constant c, which is equivalent to the desired result (3).

Proof of Theorem 2. Consider any $d \in \{\mathsf{TV}, H, \mathsf{KL}, \chi^2\}$. Denote $m = m^*(\epsilon, \mathcal{P}_{\alpha}(\beta), d)$.

First we prove the lower bound. By Theorem 5,

$$\frac{1}{\epsilon^2 \beta^{\frac{4}{2+\alpha}} m^{\frac{2\alpha}{2+\alpha}}} \log \frac{1}{\epsilon^2 \beta^{\frac{4}{2+\alpha}} m^{\frac{2\alpha}{2+\alpha}}} = O_\alpha \left(\frac{1}{\beta^2 \epsilon^2}\right)$$

which implies that $m = \Omega_{\alpha} \left[\beta \left(\log \frac{1}{\beta^2 \epsilon^2} \right)^{\frac{2+\alpha}{2\alpha}} \right]$. By the relation $\beta \leq \epsilon^{-c_1}$ for $c_1 < 1$, we have $m = \Omega_{\alpha} \left[\beta \left(\log \frac{1}{\epsilon^2} \right)^{\frac{2+\alpha}{2\alpha}} \right]$.

Next, we show the upper bound by applying Theorem 4, where the condition $m \ge \Omega_{\alpha}(\beta)$ is satisfied by our lower bound. We have that

$$\log \frac{1}{\epsilon} \gtrsim_{\alpha} \begin{cases} m \log \frac{m^{\alpha-2}}{\beta^{2\alpha}}, & m^{\alpha-2} \gtrsim_{\alpha} \beta^{2\alpha}; \\ \left(\frac{m}{\beta}\right)^{\frac{2\alpha}{2+\alpha}}, & m^{\alpha-2} \lesssim_{\alpha} \beta^{2\alpha}, \end{cases}$$

which by (54) implies that

$$m \lesssim_{\alpha} \frac{\log \frac{1}{\epsilon}}{\log \left(1 + \frac{1}{\beta} \left(\log \frac{1}{\epsilon}\right)^{\frac{\alpha - 2}{2\alpha}}\right)}.$$

Particularly, if $m^{\alpha-2}\beta^{-2\alpha} = O_{\alpha}(1)$ holds, we have $m = \Theta_{\alpha}\left(\beta\left(\log \frac{1}{\epsilon}\right)^{\frac{2+\alpha}{2\alpha}}\right)$.

C.2 Proofs for Section 5.2

The following two lemmas give upper and lower bounds for the approximation error, respectively:

Proposition 7. Suppose that $m \ge (16e^3)^{\frac{9}{8\alpha}}\beta$ and $\beta \ge \frac{8\alpha}{e\log 16e^3}$ hold. Then, it holds for $d \in \{\mathsf{TV}, H^2, \mathsf{KL}, \chi^2\}$ that

$$\mathcal{E}^{\star}(m, \mathcal{M}_{\alpha}(\beta), d) \le O_{\alpha} \left[\left(\frac{\beta}{m} \sqrt{\log \frac{m}{\beta}} \right)^{\alpha} \right].$$
(56)

Proof. Let $\kappa = 16e^3$ and $t = m\sqrt{\frac{\log \kappa}{4\kappa}\frac{1}{2\alpha \log(m/\beta)}}$, that satisfies

$$3\sqrt{\kappa}t = m\sqrt{\frac{9\log\kappa}{8\alpha}\frac{1}{\log(m/\beta)}} \le m;$$

$$\kappa t^2 = \kappa m\beta \frac{\log\kappa}{8\alpha\kappa} \frac{m/\beta}{\log(m/\beta)} \ge m\beta \frac{e\log\kappa}{8\alpha} \ge m$$

By Theorem 3, we have

$$\mathcal{E}^{\star}(m, \mathcal{P}_t^{\mathsf{Bdd}}, \chi^2) \le \exp\left(-\frac{\log \kappa}{4\kappa} \frac{m^2}{t^2}\right) = \left(\frac{\beta}{m}\right)^{2\alpha}$$

On the other hand, It follows from the tail bound (4) that

$$\sup_{P \in \mathcal{M}_{\alpha}(\beta)} P(I_t^c) \le 2\left(\frac{\beta}{t}\right)^{\alpha} = 2\left(\frac{\beta}{m}\sqrt{\frac{4\kappa}{\log\kappa}2\alpha\log\frac{m}{\beta}}\right)^{\alpha} \asymp_{\alpha}\left(\frac{\beta}{m}\sqrt{\log\frac{m}{\beta}}\right)^{\alpha}.$$

By Proposition 1, (56) holds for $d = \chi^2$. For the total variation, we have

$$\mathsf{TV}(f_Q, f_P) \stackrel{\text{(a)}}{\leq} P(I_t) \mathsf{TV}(f_Q, f_{P_t}) + P(I_t^c) \mathsf{TV}(f_Q, f_{P_t^c})$$

$$\stackrel{\text{(b)}}{\leq} \sqrt{\frac{1}{2}\chi^2(f_Q || f_{P_t})} + P(I_t^c),$$

where (a) applies Jensen's inequality, and (b) follows from Lemma 6. Hence,

$$\mathcal{E}^{\star}(m, \mathcal{P}, \mathsf{TV}) \leq \inf_{t>0} \left(\sqrt{\frac{1}{2} \mathcal{E}^{\star}(m, \mathcal{P}_t^{\mathsf{Bdd}}, \chi^2)} + \sup_{P \in \mathcal{P}} P(I_t^c) \right),$$

and thus (56) holds for $d = \mathsf{TV}$. Finally, (56) holds for $d \in \{\mathsf{TV}, H^2, \mathsf{KL}, \chi^2\}$ by Lemma 6.

Proposition 8. There exist C_{α} and D_{α} that only depend on α such that the following holds. If $m \geq D_{\alpha}\beta$, then, there exists $P \in \mathcal{M}_{\alpha}(\beta)$ such that

$$\mathcal{E}^{\star}(m, P, \mathsf{TV}) \ge \frac{C_{\alpha}}{m \log \frac{m}{\beta}} \left(\frac{\beta}{m}\right)^{\alpha}$$

Proof. Set $D_{\alpha} = \min\left\{x \ge e : \frac{1}{x}\left(\frac{1}{\alpha \log x}\right)^{\frac{1}{\alpha}} \le \frac{1}{3}\left(\frac{1-\log^{-1}x}{1+\alpha}\right)^{\frac{1}{\alpha}}\right\}$, which is a finite value since $\frac{1}{x}\left(\frac{1}{\alpha \log x}\right)^{\frac{1}{\alpha}} \to 0$ and $\left(\frac{1-\log^{-1}x}{1+\alpha}\right)^{\frac{1}{\alpha}} \to (1+\alpha)^{-\frac{1}{\alpha}} > 0$ as $x \to \infty$. Suppose that $r = \frac{m}{\beta} \ge D_{\alpha}$. Consider the truncated Pareto random variable $X \sim P$ with the density function

$$h_{\alpha}(x) = \frac{a}{x^{\alpha+1}} \mathbf{1}\{b \le x \le kb\},\tag{57}$$

where the parameters are set as $k = r(\log r)^{\frac{1}{\alpha}}$, $a = \frac{\beta^{\alpha}}{\log k}$, and $b = [\alpha^{-1}a(1-k^{-\alpha})]^{\frac{1}{\alpha}}$. Direct calculation shows that h_{α} is a well-defined density with $||X||_{\alpha} = \beta$, i.e., $P \in \mathcal{M}_{\alpha}(\beta)$. Note that

$$a \in \left[\frac{\alpha}{1+\alpha}\frac{\beta^{\alpha}}{\log r}, \frac{\beta^{\alpha}}{\log r}\right]; \quad b \in \left[\left(\frac{1-\log^{-1}D_{\alpha}}{1+\alpha}\right)^{\frac{1}{\alpha}}\frac{\beta}{(\log r)^{1/\alpha}}, \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha}}\frac{\beta}{(\log r)^{1/\alpha}}\right]$$

Choose $\delta = \frac{\pi D_{\alpha} (\alpha \log D_{\alpha})^{1/\alpha}}{m}$. Then, we have

$$b \le \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha}} \frac{\beta}{(\log r)^{1/\alpha}} \le \frac{m}{D_{\alpha}} \left(\frac{1}{\alpha \log D_{\alpha}}\right)^{1/\alpha} = \frac{\pi}{\delta};$$
$$kb \ge \left(\frac{1 - \log^{-1} D_{\alpha}}{1 + \alpha}\right)^{\frac{1}{\alpha}} \frac{\beta r (\log r)^{\frac{1}{\alpha}}}{(\log r)^{1/\alpha}} = m \left(\frac{1 - \log^{-1} D_{\alpha}}{1 + \alpha}\right)^{\frac{1}{\alpha}} \ge \frac{3m}{D_{\alpha}} \left(\frac{1}{\alpha \log D_{\alpha}}\right)^{1/\alpha} = \frac{3\pi}{\delta}.$$

With $Y \triangleq \delta X \sim g$, we have

$$\mathcal{E}^{\star}(m, P, \mathsf{TV}) \stackrel{(a)}{\geq} \sup_{\delta > 0} \frac{\pi \min_{0 \le \theta \le 2\pi} g^{\mathsf{wrap}}(\theta)}{(m+1) \exp(m^2 \delta^2 / 2)} \stackrel{(b)}{\geq} \frac{\pi h_{\alpha}\left(\frac{3\pi}{\delta}\right)}{2m\delta \exp(m^2 \delta^2 / 2)}$$

where (a) follows from Proposition 2 and (17), and (b) holds by the inequality

$$g^{\mathsf{wrap}}(\theta) \ge \inf_{\theta \in [\delta b, \delta b + 2\pi)} g(\theta) = \frac{1}{\delta} h_{\alpha} \left(b + \frac{2\pi}{\delta} \right) \ge \frac{1}{\delta} h_{\alpha} \left(\frac{3\pi}{\delta} \right), \quad \theta \in [0, 2\pi]$$

Since $\frac{3\pi}{\delta} \in [b, kb]$, we have $h_{\alpha}\left(\frac{3\pi}{\delta}\right) \gtrsim_{\alpha} \frac{1}{\log r} \frac{\beta^{\alpha}}{m^{\alpha+1}}$, and the desired result follows.

C.3 Proofs for Section 5.3

To begin with, we introduce some basic concepts of moment tensors. For a random variable $U \in \mathbb{R}^d$, define its order- ℓ moment tensor in $(\mathbb{R}^d)^{\otimes \ell}$ as

$$M_{\ell}(U) \triangleq \mathbb{E}[\underbrace{U \otimes \cdots \otimes U}_{\ell \text{ times}}].$$

For example, $M_1(U) = \mathbb{E}[U]$, and $M_2(U - \mathbb{E}[U])$ is the covariance matrix of U. Then, for a multi-index $\mathbf{j} = (j_1, \ldots, j_\ell) \in [d]^\ell$, the \mathbf{j}^{th} entry of $M_\ell(U)$ is $m_{\mathbf{j}}(U) \triangleq \mathbb{E}[U_{j_1} \cdots U_{j_\ell}]$. Also, with $U \sim P$, we write $M_\ell(P) = M_\ell(U)$. The Frobenius norm of a tensor $T \in (\mathbb{R}^d)^{\otimes \ell}$ is defined as $||T||_F \triangleq \sqrt{\langle T, T \rangle}$, where the tensor inner product is $\langle S, T \rangle = \sum_{\mathbf{j} \in [d]^\ell} S_{j_1, \ldots, j_\ell} T_{j_1, \ldots, j_\ell}$. The following result is an extension of Lemma 1 to multivariate distributions.

Lemma 10. Suppose that P, Q are supported on the d-dimensional ℓ_2 -ball of radius R. If $L > 4(dR)^2$ and $M_\ell(P) = M_\ell(Q)$ for $\ell \in [L-1]$, then

$$\chi^2(f_P || f_Q) \le 4 \exp\left(\frac{R^2}{2}\right) \left(\frac{4ed^2R^2}{L}\right)^L.$$

Proof. Denote $\mu = \mathbb{E}[Q]$ and let $U \sim P$, $V \sim Q$, $U - \mu \sim P'_{\mu}$, and $V - \mu \sim Q'_{\mu}$. Following the proof argument of [DWYZ23, Theorem 4.2], we have that

$$\chi^{2}(f_{P}||f_{Q}) = \chi^{2}\left(f_{P'_{\mu}}||f_{Q'_{\mu}}\right) \le e^{\frac{R^{2}}{2}} \sum_{\ell \ge 1} \frac{\left\|M_{\ell}(P'_{\mu}) - M_{\ell}\left(Q'_{\mu}\right)\right\|_{\mathrm{F}}^{2}}{\ell!} d^{\ell}.$$
(58)

Note that $M_{\ell}(P) = M_{\ell}(Q)$ for all $\ell \in [L-1]$ implies that $M_{\ell}(P'_{\mu}) = M_{\ell}(Q'_{\mu})$ for all $\ell \in [L-1]$. Since $P'_{\mu}, Q'_{\mu} \in \mathcal{P}_{2R,d}^{\mathsf{Bdd}}$, we have $\|M_{\ell}(P'_{\mu})\|_{F}^{2}, \|M_{\ell}(Q'_{\mu})\|_{F}^{2} \leq d^{\ell}(2R)^{2\ell}$. It follows from (58) that

$$\chi^2(f_P \| f_Q) \le e^{\frac{R^2}{2}} \sum_{\ell \ge L} \frac{d^{2\ell} [2(2R)^{\ell}]^2}{\ell!} \le 4e^{\frac{R^2}{2}} \left(\frac{4ed^2 R^2}{L}\right)^L,$$

where the last inequality follows from the Poisson tail bound $\mathbb{P}[X \ge L] \le e^{-4d^2R^2} \left(\frac{4ed^2R^2}{L}\right)^L$ for $X \sim \text{Poisson}(4d^2R^2)$ and $L > 4d^2R^2$ [Ver18, Exercise 2.3.3].

Proof of Proposition 4. Note that there are $\binom{d+\ell-1}{\ell}$ distinct entries in $M_{\ell}(P)$. Hence, the total number of distinct entries in $M_1(P), \ldots, M_{L-1}(P)$ is $C_{L,d} \triangleq \sum_{i=1}^{L-1} \binom{d+\ell-1}{\ell} = \binom{d+L-1}{L-1}$. Denote $T_L(P) \in \mathbb{R}^{C_{L,d}}$ as the tuple consisting of these distinct entries. Consider the following convex set of tuples

$$\left\{T_L(P): P \in \mathcal{P}_{M,d}^{\mathsf{Bdd}}\right\} \subseteq \mathbb{R}^{C_{L,d}}.$$

By Carathéodory's theorem, there exists a distribution P' supported on no more than $C_{L,d} \leq L^d$ atoms such that $m_{\mathbf{j}}(P) = m_{\mathbf{j}}(P')$ for all $\mathbf{j} \in [d]^{\ell}, \ell \in [L-1]$. By Lemma 10, we have

$$\chi^2(f_{P'}||f_P) \le 4 \exp\left(-L\log\frac{L}{M^2} + \frac{M^2}{2} + \log(4ed^2)L\right).$$

Consequently, when $L \ge \kappa_d M^2$ for $\kappa_d = 64e^3d^4$,

$$\mathcal{E}^{\star}(L^{d}, \mathcal{P}_{M,d}^{\mathsf{Bdd}}, \chi^{2}) \le \exp\left(-\frac{L}{2}\log\frac{L}{M^{2}}\right).$$
(59)

Suppose that $6\sqrt{3\kappa_d}M \leq L \leq \kappa_d M^2$ holds. Set $K = \left\lfloor \frac{6\kappa_d M^2}{L} \right\rfloor \geq 6$. Denote $r = \frac{M}{K}$. Let $\mathcal{B} = \{B_i = B(u_i, r) : i \in [N]\}$ be a *r*-covering of $B_d(M)$ under the Euclidean distance,

where $N \leq (3K)^d$ by [Ver18, Corollary 4.2.13]. Define $\tilde{B}_i = \{u \in B_d(M) : i = \min\{j : j \in \arg\min_{k \in [N]} ||u_k - u||\}\}$. Since $||u_i - u|| = \min_{j \in [N]} ||u_j - u|| \leq r$ for any $u_i \in \tilde{B}_i$, we have $\tilde{B}_i \subset B(u_i, r)$. By definition, $\{\tilde{B}_i\}_{i=1}^N$ consists of a partition of $B_d(M)$ (that is, each \tilde{B}_i is disjoint and $B_d(M) = \bigcup_{i=1}^N \tilde{B}_i$). Let P_k denote the conditional distribution of P on \tilde{B}_k . Let $m = L^d$, $\tilde{m} = \left\lfloor \frac{m}{(3K)^d} \right\rfloor = \left\lfloor \left(\frac{L}{3K}\right)^d \right\rfloor$, and $\tilde{L} = \lfloor \tilde{m}^{1/d} \rfloor$. Note that the condition $\tilde{L} \geq \kappa_d \left(\frac{M}{K}\right)^2$ holds by

$$K^{2}\tilde{L} \stackrel{(a)}{\geq} K^{2} \left\lfloor \frac{L}{3K} \right\rfloor \geq \left\lfloor \frac{6\kappa_{d}M^{2}}{L} \right\rfloor^{2} \left\lfloor \frac{L^{2}}{18\kappa_{d}M^{2}} \right\rfloor \stackrel{(b)}{\geq} \left(\frac{6}{7} \frac{6\kappa_{d}M^{2}}{L} \right)^{2} \left(\frac{6}{7} \frac{L^{2}}{18\kappa_{d}M^{2}} \right) \geq \kappa_{d}M^{2},$$

where (a) follows from $\lfloor x^d \rfloor \geq \lfloor x \rfloor^d$ for $d \in \mathbb{N}$ and $x \geq 0$, and (b) applies $\lfloor x \rfloor \geq \frac{c}{c+1}x$ for all $x \geq c \in \mathbb{N}$ and $\min\left\{\frac{L^2}{18\kappa_d M^2}, \frac{6\kappa_d M^2}{L}\right\} \geq 6$. Then (59) implies that, for each P_k , there exists \tilde{P}_k supported on at most \tilde{m} atoms such that

$$\chi^2(f_{\tilde{P}_k} \| f_{P_k}) \le \exp\left(-\frac{\tilde{L}}{2}\log\frac{\tilde{L}}{\left(\frac{M}{K}\right)^2}\right) \le \exp\left(-\frac{L^2}{42\kappa_d M^2}\log\kappa_d\right).$$

Finally, define $P_m \triangleq \sum_{k=1}^{N} P(\tilde{B}_k) \tilde{P}_k$ supported on at most $\tilde{m}N \leq m$ atoms. Note that $P = \sum_{k=1}^{N} P(\tilde{B}_k) P_k$ since $\{\tilde{B}_k\}_{k=1}^{N}$ is a partition of $B_d(M)$. By Jensen's inequality,

$$\chi^{2}(f_{P_{m}}||f_{P}) \leq \sum_{k=1}^{N} P(\tilde{B}_{k})\chi^{2}(f_{\tilde{P}_{k}}||f_{P_{k}}) \leq \exp\left(-\frac{\log \kappa_{d}}{42\kappa_{d}}\frac{L^{2}}{M^{2}}\right).$$

To prove the lower bound, we first introduce the multilevel Toeplitz matrices, which are generalizations of Toeplitz matrices that naturally arise in multidimensional Fourier analysis [Pes19, TZ98]. Let $\mathbf{u}_L : \mathbb{R}^d \mapsto \mathbb{C}^{L^d}$ denote the multivariate trigonometric functions ordered by the *L*-base representation

$$(\mathbf{u}_L(\theta))_k = \exp\left(i\sum_{n=0}^{d-1} j_n \theta_n\right), \text{ if } k = \sum_{n=0}^{d-1} j_n L^n, j_n \in \{0, 1, \dots, L-1\}.$$

For a random variable X (or the probability distribution P thereof) supported on \mathbb{R}^d , define

$$\mathbf{A}_L(X) = \mathbf{A}_L(P) = \mathbb{E}[\mathbf{u}_L(X)\mathbf{u}_L^{\star}(X)].$$

By definition, $\mathbf{A}_L(P)$ has a nested block structure with $L \times L$ blocks, and each block contains $L \times L$ smaller blocks. For example, when d = 2 and L = 2,

$$\mathbf{A}_{L}(P) = \begin{pmatrix} a_{(0,0)} & a_{(0,-1)} & a_{(-1,0)} & a_{(-1,-1)} \\ a_{(0,1)} & a_{(0,0)} & a_{(-1,1)} & a_{(-1,0)} \\ a_{(1,0)} & a_{(1,-1)} & a_{(0,0)} & a_{(0,-1)} \\ a_{(1,1)} & a_{(1,0)} & a_{(0,1)} & a_{(0,0)} \end{pmatrix},$$

where $a_{\mathbf{j}} = \mathbb{E} \exp(i X^{\top} \mathbf{j})$ denotes the trigonometric moment.

Let $X \sim P$ and $X_m \sim P_m$ for any $P_m \in \mathcal{P}_m$. Note that the rank of $\mathbf{A}_L(P_m)$ is at most m. Applying (35) yields that, for any $\delta > 0$,

$$\begin{aligned} \mathsf{TV}\left(f_{P_{m}}, f_{P}\right) &\geq \frac{1}{2} \sup_{\omega \in \mathbb{R}^{d}} \exp\left(-\frac{\|\omega\|^{2}}{2}\right) |\mathbb{E}\left[\exp\left(i\langle\omega, X\rangle\right)\right] - \mathbb{E}\left[\exp\left(i\langle\omega, X_{m}\rangle\right)\right]| \\ &\geq \frac{1}{2} \exp\left(-\frac{d\delta^{2}(L-1)^{2}}{2}\right) \max_{\mathbf{j} \in \{-L+1,\dots,L-1\}^{d}} |\mathbb{E}\left[\exp\left(i\langle\mathbf{j},\delta X\rangle\right)\right] - \mathbb{E}\left[\exp\left(i\langle\mathbf{j},\delta X_{m}\rangle\right)\right]| \\ &\geq \frac{1}{2L^{d}} \exp\left(-\frac{d\delta^{2}(L-1)^{2}}{2}\right) \|\mathbf{A}_{L}(\delta X) - \mathbf{A}_{L}(\delta X_{m})\|_{F}. \end{aligned}$$

By Lemma 3, if $L^d \ge m+1$,

$$\mathcal{E}^{\star}(m, P, \mathsf{TV}) \ge \sup_{\delta > 0} \frac{1}{2L^d} \exp\left(-\frac{d\delta^2(L-1)^2}{2}\right) \lambda_{\min}(\mathbf{A}_L(\delta X)).$$
(60)

Then, we extend the wrapped density approach to multi-dimensional case. Given a density function g supported on a subset of \mathbb{R}^d , define the corresponding wrapped density g^{wrap} on $[-\pi,\pi]^d$ as

$$g^{\mathsf{wrap}}(\theta) = \begin{cases} \sum_{j \in \mathbb{Z}^d} g(\theta - 2\pi j), & \theta \in [-\pi, \pi]^d; \\ 0, & \text{otherwise.} \end{cases}$$

Denote $Y = \delta X \sim g$ and $Y^{\text{wrap}} \sim g^{\text{wrap}}$. Be definition, $\mathbf{A}_L(Y) = \mathbf{A}_L(Y^{\text{wrap}})$. The following result is a straightforward corollary of [Pes19, Lemma 4.1], which can be viewed as a multidimensional version of (17) and follows from the same argument as (19).

Lemma 11. Let g be a density function on \mathbb{R}^d that is symmetric about 0. Then, for $Y \sim g$,

$$\lambda_{\min}(\mathbf{A}_L(Y)) = \lambda_{\min}(\mathbf{A}_L(Y^{\mathsf{wrap}})) \ge \min\{(2\pi)^d g^{\mathsf{wrap}}(\theta) : \theta \in [-\pi, \pi]^d\}.$$

Proof of Proposition 5. Let $L = \lceil (m+1)^{1/d} \rceil$ satisfying $m+1 \leq L^d \leq 2^d (L-1)^d \leq 2^d m$. Let $X \sim P = \text{Unif}([-M/\sqrt{d}, M/\sqrt{d}]^d)$ and $\delta = \frac{\pi\sqrt{d}}{M}$. Then, $\mathbf{A}_L(\delta X) = I_{L^d}$ and thus the conclusion follows from (60).

Next, by applying similar analyses for the upper and lower bounds, we present the extension for approximating multivariate Gaussian distributions.

Proposition 9. Suppose that $\sigma \geq \sigma_0 > 0$. There exist C_0, C_1 which depend on d and σ_0 such that, when $m^{1/d} \geq C_0 \sigma$,

$$\mathcal{E}^{\star}(m, N(0, \sigma^2 I_d), \chi^2) \le \exp\left(-C_1 \frac{m^{\frac{1}{d}}}{\sigma}\right)$$

Furthermore, for any $m \in \mathbb{N}$ and $\sigma > 0$,

$$\mathcal{E}^{\star}(m, N(0, \sigma^2 I_d), \mathsf{TV}) \geq \frac{1}{2^{\frac{d}{2}+1}m^{\frac{1}{2}}\sigma^{\frac{d}{2}}} \exp\left(-\frac{d\pi m^{\frac{1}{d}}}{\sigma}\right).$$

Proof. Let $X \sim P = N(0, \sigma^2 I_d)$. There exist sufficiently large C_0 and C' depending on d and σ_0 such that, if $\tilde{m} \triangleq m^{1/d} \ge C_0 \sigma$ and $t = C' \sqrt{\tilde{m}\sigma}$, then $6\sqrt{3\kappa_d}t \le \tilde{m} \le \kappa_d t^2$ and $t \ge 2\sigma\sqrt{d}$ hold. Let $A = B_d(t)$ and P_A denote the conditional distribution of P on A. Applying Lemma 2, we have

$$\begin{aligned} \mathcal{E}^{\star}(m, P, \chi^2) &\leq \frac{2}{P(A)} \left(\mathcal{E}^{\star}(m, P_A, \chi^2) + P(A^c) \right) \\ &\stackrel{(a)}{\leq} \frac{2}{P(A)} \left[\exp\left(-\frac{\log \kappa_d}{42\kappa_d} \frac{\tilde{m}^2}{t^2} \right) + \exp\left(-\frac{1}{2} \left(\frac{t}{\sigma} - \sqrt{d} \right)^2 \right) \right] \\ &\leq \frac{2}{P(A)} \left[\exp\left(-\frac{\log \kappa_d}{42\kappa_d} \frac{\tilde{m}^2}{t^2} \right) + \exp\left(-\frac{t^2}{8\sigma^2} \right) \right] \\ &\leq \exp\left(-C_1 \frac{\tilde{m}}{\sigma} \right), \end{aligned}$$

where (a) applies the tail bound of the χ^2 distribution $P[||X/\sigma||_2 \ge \sqrt{d} + \sqrt{2x}] \le \exp(-x)$ [LM00, Lemma 1], and C_1 depends on d and σ . For the lower bound, applying (60) and Lemma 11 with $L = \lceil (m+1)^{1/d} \rceil$ and $\delta = \sqrt{\frac{\pi}{\sigma(L-1)}}$, we obtain

$$\begin{aligned} \mathcal{E}^{\star}(m, N(0, \sigma^{2} I_{d}), \mathsf{TV}) &\geq \frac{(2\pi)^{d}}{2L^{d}} \exp\left(-\frac{d\delta^{2}(L-1)^{2}}{2}\right) \min_{\theta \in [-\pi,\pi]^{d}} \frac{1}{(2\pi)^{d/2}(\delta\sigma)^{d}} e^{-\frac{||\theta||^{2}}{2\delta^{2}\sigma^{2}}} \\ &= \frac{(2\pi)^{\frac{d}{2}}}{2(L\delta\sigma)^{d}} \exp\left(-\frac{d\delta^{2}(L-1)^{2}}{2} - \frac{d\pi^{2}}{2\delta^{2}\sigma^{2}}\right) \\ &\geq \frac{1}{2(L\sigma)^{\frac{d}{2}}} \exp\left(-\frac{d\pi(L-1)}{\sigma}\right) \\ &\geq \frac{1}{2^{\frac{d}{2}+1}m^{\frac{1}{2}}\sigma^{\frac{d}{2}}} \exp\left(-\frac{d\pi m^{1/d}}{\sigma}\right). \end{aligned}$$

D Proof of the convergence rates of NPMLE

We introduce some notations for the metric entropy. For $\epsilon > 0$, an ϵ -net of a set \mathcal{F} with respect to a metric d is a set \mathcal{N} such that for all $f \in \mathcal{F}$, there exists $g \in \mathcal{N}$ such that $d(g, f) \leq \epsilon$. The minimum cardinality of ϵ -nets is denoted by $N(\epsilon, \mathcal{F}, d)$. Given a family \mathcal{P} of distributions, define $\mathcal{F}_{\mathcal{P}} \triangleq \{f_P : P \in \mathcal{P}\}.$

The following lemma bounds the L_{∞} -metric entropy of the Gaussian mixture densities with compactly supported and sub-Weibull mixing distributions. Note that this is an improvement of the previous result [Zha09, Lemma 2] which deals with truncated L_{∞} -norm on a compact interval. Combined with our main result in Theorem 1, this also improves the untruncated L_{∞} entropy estimate in [GvdV07b, Lemma 2] which gives $\log N(\epsilon, \mathcal{F}_{\mathcal{P}_M^{\text{Bdd}}}, L_{\infty}) \lesssim M \log \frac{1}{\epsilon} \log \frac{M}{\epsilon}$. The key idea is to use the smoothness of the Gaussian mixtures to relate L_{∞} -error (pointwise) to L_1 -error (TV), the latter of which can be further bounded by moment matching.

Lemma 12. Let $0 < \epsilon < 1$. There exists a universal constant C such that

$$\log N(\epsilon, \mathcal{F}_{\mathcal{P}_{M}^{Bdd}}, L_{\infty}) \leq Cm^{\star}(\epsilon, \mathcal{P}_{M}^{Bdd}, \mathsf{TV}) \log \frac{1}{\epsilon}$$

Additionally, there exists a constant C_{α} depending only on α such that

$$\log N(\epsilon, \mathcal{F}_{\mathcal{P}_{\alpha}(\beta)}, L_{\infty}) \le C_{\alpha} m^{\star}(\epsilon, \mathcal{P}_{\alpha}(\beta), \mathsf{TV}) \log \frac{1}{\epsilon}.$$

Proof. First, we prove the result for $\mathcal{P} = \mathcal{P}_M^{\mathsf{Bdd}}$. Let $m = m^*(\epsilon, \mathcal{P}, \mathsf{TV})$ and t = M. Let $\mathcal{S}_m \triangleq \{(p_1, \ldots, p_m) | \sum_{i=1}^m p_i = 1, p_i \ge 0\}$ and $\mathcal{N}_m \subseteq \mathcal{S}_m$ be the smallest ϵ -net of \mathcal{S}_m under the L_1 -distance. Let $\mathcal{L} \triangleq \{0, \pm \epsilon, \ldots, \pm \lfloor \frac{t}{\epsilon} \rfloor \epsilon\}$. Define the following set of finite mixture densities

$$\mathcal{C} \triangleq \left\{ \sum_{j=1}^m w_j N(\theta_j, 1) : (w_1, \dots, w_m) \in \mathcal{N}_m, \theta_1 \leq \dots \leq \theta_m, \{\theta_j\}_{j=1}^m \subseteq \mathcal{L} \right\}.$$

The cardinality of \mathcal{C} is upper bounded by

$$|\mathcal{C}| \le \binom{m+|\mathcal{L}|-1}{m} |\mathcal{N}_m| \stackrel{\text{(a)}}{\le} \frac{\left(\frac{2t}{\epsilon}+m\right)^m}{m!} 2m \left(1+\frac{1}{\epsilon}\right)^{m-1} \le 2m \left(\frac{2t}{\epsilon}+m\right)^m \left(\frac{2e}{\epsilon m}\right)^m,$$

where (a) holds by $\binom{n}{m} \leq \frac{n^m}{m!}$ and the fact that $|\mathcal{N}_m| \leq 2m \left(1 + \frac{1}{\epsilon}\right)^{m-1}$ [PW24, Corollary 27.4]. Then, we prove the covering property of \mathcal{C} . By definition, for any $P \in \mathcal{P}$, there exists

Then, we prove the covering property of \mathcal{C} . By definition, for any $P \in \mathcal{P}$, there exists $P_m = \sum_{i=1}^m w_j \delta_{\theta_j}$ with $\theta_1 \leq \cdots \leq \theta_m$ and $|\theta_j| \leq t$ for all j such that $\mathsf{TV}(f_{P_m}, f_P) \leq \epsilon$. Let $\theta'_j = \theta_j \frac{\lfloor |\theta_j|/\epsilon \rfloor}{|\theta_j|/\epsilon} \in \mathcal{L}$ and consider $P'_m = \sum_{i=1}^m w_j \delta_{\theta'_j}$. By Jensen's inequality and the fact that $\mathsf{TV}(N(0,1), N(\epsilon, 1)) \leq \sqrt{\frac{1}{2\pi}} \epsilon$, we have $\mathsf{TV}(f_{P_m}, f_{P'_m}) \leq \sqrt{\frac{1}{2\pi}} \epsilon$. Finally, we construct $P''_m = \sum_{j=1}^m w'_j \delta_{\theta'_j} \in \mathcal{C}$ by choosing $w' \in \mathcal{N}$ so such $L_1(w, w') \leq \epsilon$. By triangle inequality, $\mathsf{TV}(f_{P'_m}, f_{P''_m}) \leq \frac{1}{2}L_1(w, w') \leq \frac{1}{2}\epsilon$. Hence, \mathcal{C} is a 2ϵ -net of $\mathcal{F}_{\mathcal{P}}$ under TV , and thus

$$\log N(2\epsilon, \mathcal{F}_{\mathcal{P}}, \mathsf{TV}) \le m \log \left(\frac{4et}{\epsilon^2 m} + \frac{2e}{\epsilon}\right) + \log 2m \stackrel{(a)}{\le} O\left(m \log \frac{1}{\epsilon}\right),\tag{61}$$

where (a) holds since $m \gtrsim M = t$ by Theorem 7.

Finally, since $||f'_P||_{\infty} \leq ||\phi'||_{\infty} = (2\pi e)^{-1/2}$, $L_{\infty}(f_P, f_Q) \geq (\frac{8}{\pi e})^{1/4} \sqrt{\epsilon}$ implies $\mathsf{TV}(f_P, f_Q) \geq 2\epsilon$. Then,

$$\log N((8/(\pi e))^{1/4} \sqrt{\epsilon}, \mathcal{F}_{\mathcal{P}}, L_{\infty}) \le \log N(2\epsilon, \mathcal{F}_{\mathcal{P}}, \mathsf{TV}) \le O\left(m \log \frac{1}{\epsilon}\right),$$
(62)

and equivalently, $\log \mathcal{N}(\epsilon, \mathcal{F}_{\mathcal{P}}, L_{\infty}) \leq O\left(m \log \frac{1}{\epsilon}\right)$.

The proof for $\mathcal{P} = \mathcal{P}_{\alpha}(\beta)$ is analogous, where we need to choose t depending on m and β as in the proof of Theorem 4. Specifically, let $t = t(m,\beta) \asymp_{\alpha} m^{\frac{2}{2+\alpha}} \beta^{\frac{\alpha}{2+\alpha}}$ if $m^{\alpha-2} \lesssim_{\alpha} \beta^{2\alpha}$, and $t \asymp_{\alpha} \beta \left(m \log \frac{m^{\alpha-2}}{\beta^{2\alpha}} \right)^{\frac{1}{\alpha}}$ otherwise. Plugging $t = t(m,\beta)$ into (61) and noting that $m \gtrsim_{\alpha} \beta$ by Theorem 7, we likewise obtain that $\log \mathcal{N}(\epsilon, \mathcal{F}_{\mathcal{P}}, L_{\infty}) \lesssim_{\alpha} (m \log \frac{1}{\epsilon})$.

Proof of Theorem 8. We first prove the result for $P^* \in \mathcal{P}_M^{\mathsf{Bdd}}$. Let $\epsilon = n^{-2(c\vee 1)}$, $\delta = s\epsilon_n$. Theorem 1 implies that $\epsilon_n^2 \asymp \frac{m^*}{n} \log \frac{1}{\epsilon}$, where $m^* \triangleq m^*(\epsilon, \mathcal{P}, \mathsf{TV})$. Consider the smallest ϵ -net denoted by \mathcal{N} of

$$\mathcal{F} \triangleq \{f_P : P \in \mathcal{P}, H(f_P, f_{P^*}) \ge \delta\} \subseteq \mathcal{F}_{\mathcal{P}}$$

under the L_{∞} -distance. Without loss of generality, assume $\mathcal{N} \subseteq \mathcal{F}$ [Ver18, Exercise 4.2.9]. By Lemma 12, we have $H_{\epsilon} = \log |\mathcal{N}| \leq m^{\star} \log \frac{1}{\epsilon}$. It follows from the definition of the ϵ -net that, if $H(f_{\hat{P}}, f_{P^{\star}}) \geq \delta$, then there exists $g \in \mathcal{N}$ such that $f_{\hat{P}}(x) \leq g(x) + \epsilon$ for all $x \in \mathbb{R}$. By the optimality of \hat{P} ,

$$0 \le \sum_{i=1}^{n} \log \frac{f_{\hat{P}}(X_i)}{f_{P^{\star}}(X_i)} \le \max_{g \in \mathcal{N}} \sum_{i=1}^{n} \log \frac{g+\epsilon}{f_{P^{\star}}}(X_i).$$

$$(63)$$

For a fixed function $g \in \mathcal{N}$, applying Chernoff bound yields that

$$\mathbb{P}\left[\sum_{i=1}^{n}\log\frac{g+\epsilon}{f_{P^{\star}}}(X_i) \ge 0\right] \le \exp\left(n\log\mathbb{E}\sqrt{\frac{g+\epsilon}{f_{P^{\star}}}}\right),$$

where

$$\mathbb{E}\sqrt{\frac{g+\epsilon}{f_{P^{\star}}}} \le \mathbb{E}\sqrt{\frac{g}{f_{P^{\star}}}} + \sqrt{\epsilon}\mathbb{E}\sqrt{\frac{1}{f_{P^{\star}}}} = 1 - \frac{1}{2}H^2(g, f_{P^{\star}}) + \sqrt{\epsilon}\int_{\mathbb{R}}\sqrt{f_{P^{\star}}(x)}\mathrm{d}x.$$
(64)

Since $g \in \mathcal{N}$, we have $H(g, f_{P^*}) \geq \delta$. Additionally, applying Cauchy-Schwarz inequality yields $\mathbb{E}\sqrt{1/f_{P^*}} = (\int_{|x|\leq 2M} + \int_{|x|>2M})\sqrt{f_{P^*}(x)} dx \leq c_0(\sqrt{M}\vee 1)$ for some universal constant c_0 . Then, we have $\mathbb{E}\sqrt{(g+\epsilon)/f_{P^*}} \leq 1 - \frac{\delta^2}{2} + c_0\sqrt{\epsilon}(\sqrt{M}\vee 1)$. Also note that $n\epsilon_n^2 \asymp m^*\log\frac{1}{\epsilon} \gtrsim M \vee 1 \geq \sqrt{M} \vee 1$ according to Theorem 7. Hence, by the union bound, there exist absolute constants $c_1, s^* > 0$ such that for any $s > s^*$,

$$\mathbb{P}\left[\max_{g\in\mathcal{N}}\sum_{i=1}^{n}\log\frac{g+\epsilon}{f_{P^{\star}}}(X_{i})\geq 0\right]\leq \exp\left(-n\left(\frac{\delta^{2}}{2}-c_{0}\sqrt{\epsilon}(\sqrt{M}\vee1)\right)+H_{\epsilon}\right)\\\leq \exp\left(-c_{1}s^{2}m^{\star}\log n\right).$$
(65)

Consequently, we conclude (29).

The proof for $P^* \in \mathcal{P}_{\alpha}(\beta)$ follows from a similar argument. By using ϵ_n specified in (31), the condition $\epsilon_n^2 \gtrsim \frac{m^*}{n} \log \frac{1}{\epsilon}$ is satisfied by Theorem 2. To bound the integral $\int_{\mathbb{R}} \sqrt{f_P(x)} dx$, note that $\int_{|x| \leq \beta} \sqrt{f_P(x)} dx \leq \sqrt{2\beta}$ and

$$\begin{split} \int_{|x|>\beta} \sqrt{f_P(x)} \mathrm{d}x &\leq \int_{|x|>\beta} \sqrt{\mathbb{E}\phi(x-\theta) \left(\mathbf{1}\{|x-\theta|\leq |x|/2\} + \mathbf{1}\{|x-\theta|>|x|/2\}\right)} \mathrm{d}x \\ &\leq \int_{|x|>\beta} \sqrt{\phi(0)\mathbb{P}[|\theta|\geq |x|/2]} + \sqrt{\phi(x/2)} \mathrm{d}x \leq c_\alpha(\beta \vee 1), \end{split}$$

where the last inequality holds by the tail probability bound (6) for some $c_{\alpha} > 0$ that depends on α . Then, Theorem 7 implies that $m^* \log \frac{1}{\epsilon} \gtrsim \beta \lor 1 \gtrsim_{\alpha} \int \sqrt{f_P(x)} dx$. Following the similar derivation, (65) holds for some $c_1 > 0$ and $s > s^*$, where c_1, s^* may depend on α .

Remark 4. The preceding proof is simpler than existing arguments (e.g., the proofs of Zha09, Theorem 1] and its multivariate extension [SG20, Theorem 2.1]), which rely on truncated $L_{\infty}([-M,M])$ metric entropy of the mixture density. As such, in (63) one needs to take into account the contribution from those $|X_i| \geq M$. In comparison, we directly apply the global L_{∞} entropy in Lemma 12, which avoids truncating the sample, and directly bound the contribution of $\int \sqrt{f_{P^*}}$ in (64) which can be afforded since the mixing distributions here are light-tailed.

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