# Complementary Graph Entropy, AND Product, and Disjoint Union of Graphs 

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#### Abstract

In the zero-error Slepian-Wolf source coding problem, the optimal rate is given by the complementary graph entropy $\bar{H}$ of the characteristic graph. It has no single-letter formula, except for perfect graphs, for the pentagon graph with uniform distribution $G_{5}$, and for their disjoint union. We consider two particular instances, where the characteristic graphs respectively write as an AND product $\wedge$, and as a disjoint union $\sqcup$. We derive a structural result that equates $\bar{H}(\wedge \cdot)$ and $\bar{H}(\sqcup \cdot)$ up to a multiplicative constant, which has two consequences. First, we prove that the cases where $\bar{H}(\wedge \cdot)$ and $\bar{H}(\sqcup \cdot)$ can be linearized coincide. Second, we determine $\bar{H}$ in cases where it was unknown: products of perfect graphs; and $G_{5} \wedge G$ when $G$ is a perfect graph, using Tuncel et al.'s result for $\bar{H}\left(G_{5} \sqcup G\right)$. The graphs in these cases are not perfect in general.


## I. Introduction

We study the zero-error variant of Slepian and Wolf source coding problem depicted in Figure 11 where the estimate $\widehat{X}^{n}$ must be equal to $X^{n}$ with probability one. This problem is also called "restricted inputs" in Alon and Orlitsky's work [1].

## A. Characteristic graphs and optimal rate $\bar{H}$

An adequate probabilistic graph $G$ (i.e. a graph with an underlying probability distribution on its vertices) can be associated to a given instance of zero-error source coding problem in Figure 1, as in Witsenhausen's work [2]. This graph is called "characteristic graph" of the problem, as it encompasses the problem data in its structure: the vertices are the source alphabet, with the source probability distribution $P_{X}$ on these vertices, and two source symbols $x x^{\prime}$ are adjacent if they are "confusable", i.e. $P_{X, Y}(x, y) P_{X, Y}\left(x^{\prime}, y\right)>0$ for some side information symbol $y$. By construction, the encoder must map adjacent symbols in $G$ to different codewords in order to prevent any decoding error: the colorings of the graph $G$ directly correspond to zero-error encoding mappings.

The best rate that can be achieved in the problem of Figure 1 with $n=1$ is the minimal entropy of the colorings of $G$, as shown in [1]. This quantity is called chromatic entropy and is denoted by

$$
\begin{equation*}
H_{\chi}(G) \doteq \inf \{H(c(V)) \mid c \text { is a coloring of } G\} \tag{1}
\end{equation*}
$$

The asymptotic optimal rate in the problem of Figure 1 is characterized by

$$
\begin{equation*}
\bar{H}(G)=\lim _{n \rightarrow \infty} \frac{1}{n} H_{\chi}\left(G^{\wedge n}\right) \tag{2}
\end{equation*}
$$



Fig. 1. Zero-error Slepian-Wolf source coding problem.
where $G^{\wedge n}$ is the $n$-iterated AND product of the characteristic graph $G$, see [1]. As shown in [3], it is equal to the complementary graph entropy defined in [4].

A single-letter formula for $\bar{H}$ is not known, except for perfect graphs [5]; and for $G_{5} \sqcup G$ and its complement, for all perfect graph $G$ [6], where $G_{5}$ is the pentagon graph with uniform distribution.

## B. Characteristic graph structure in particular instances

Since determining $\bar{H}$ is difficult, let us consider particular instances of the problem in Figure 11 depicted in Figure 2. Both settings have a characteristic graph with a specific structure. Thanks to the side information at the encoder in Figure 2]a, the characteristic graph is the disjoint union ( $\sqcup$ ) of a family of auxiliary probabilistic graphs $\left(G_{z}\right)_{z \in \mathcal{Z}}$; and in Figure 2]b the characteristic graph is the AND product $(\wedge)$ of the $\left(G_{z}\right)_{z \in \mathcal{Z}}$. Both $\sqcup$ and $\wedge$ are binary operators on probabilistic graphs that play a central role in this study. A natural question arises in the context of Figure 2) can we determine the optimal rates if we only know $\bar{H}\left(G_{z}\right)$ for all $z \in \mathcal{Z}$ ? With the subadditivity results in [6, Theorem 2], we know that $\bar{H}\left(\bigsqcup_{z \in \mathcal{Z}}^{P_{g(Y)}} G_{z}\right) \leq \sum_{z \in \mathcal{Z}} P_{g(Y)} \bar{H}\left(G_{z}\right)$ and $\bar{H}\left(\bigwedge_{z \in \mathcal{Z}} G_{z}\right) \leq \sum_{z \in \mathcal{Z}} \bar{H}\left(G_{z}\right)$ holds in general, however characterizing the cases where equality holds is an open problem.

## C. Related work

If the decoder wants to recover a function $f(X, Y)$ instead of $X$, the setting of Figure 1 becomes the zero-error variant of the "coding for computing" problem [7]. Charpenay et al. study in [8] the variant with side information at the encoder, i.e. the setting from Figure 2 a with $f(X, Y)$ requested by the decoder. In [9], Ravi and Dey study a setting with a bidirectional relay. In [10], Malak introduces a fractional version of chromatic entropy in a lossless coding for computing scenario.
a.

b.


Fig. 2. Two particular instances of zero-error Slepian-Wolf source coding problem, where $g: \mathcal{Y} \rightarrow \mathcal{Z}$ is deterministic, $\left(X_{z}^{\prime n}, Y_{z}^{\prime n}\right) \sim P_{X, Y \mid g(Y)=z}^{n}$ for all $z \in \mathcal{Z}$, and the pairs $\left(\left(X_{z}^{\prime n}, Y_{z}^{\prime n}\right)\right)_{z \in \mathcal{Z}}$ are mutually independent. For all $z \in \mathcal{Z}$, the auxiliary graph $G_{z}$ is Witsenhausen's characteristic graph for the pair $\left(X_{z}^{\prime}, Y_{z}^{\prime}\right)$.

Another important problem is the Shannon capacity $\Theta$ of a graph [11], which characterizes the optimal rate in the zeroerror channel coding scenario. Marton has shown in [12] that $\bar{H}(G)+C(G, P)=H(P)$, where $P$ is the underlying probability distribution of $G$, and $C(G, P)$ is the graph capacity relative to $P$. The same questions on linearization arise for $\Theta$ : for which $G, G^{\prime}$ do we have $\Theta\left(G \wedge G^{\prime}\right)=\Theta(G) \Theta\left(G^{\prime}\right)$ ? A counterexample is shown by Haemers in [13], using an upper-bound on $\Theta$ based on the rank of the adjacency matrix. Refinements of Haemers bound are developed in [14] by Bukh and Cox, and in [15] by Gao et al. Recently in [16], Schrijver shows that $\Theta\left(G \wedge G^{\prime}\right)=\Theta(G) \Theta\left(G^{\prime}\right)$ is equivalent to $\Theta\left(G \sqcup G^{\prime}\right)=\Theta(G)+\Theta\left(G^{\prime}\right)$. The computability of $\Theta$ is investigated in [17] by Boche and Deppe. An asymptotic expression for $\Theta$ using semiring homomorphisms is given by Zuiddam et al. in [18]. In [19], Gu and Shayevitz study the two-way channel case. An extension of $\Theta$ for secure communication is developed in [20] by Wiese et al.

## D. Contributions

In this paper we link the complementary graph entropies of a disjoint union of probabilistic graphs with that of their product, i.e. $\bar{H}(\sqcup \cdot)$ and $\bar{H}(\wedge \cdot)$. First, we show a structural result on the complementary graph entropy of a disjoint union w.r.t. a type $P_{A}$, that makes use of $\wedge$ instead of $\sqcup$. This enables us to equate $\bar{H}(\sqcup \cdot)$ and $\bar{H}(\wedge \cdot)$ up to a multiplicative constant. This formula has several consequences.
Firstly, we can derive with it a single-letter formula $\bar{H}$ of products of perfect graphs. This case was unsolved as a product of perfect graphs is not perfect in general. However, a disjoint union of perfect graphs is perfect, this is why studying disjoint unions is the key. Finally, it enables us to show that the linearizations of $\bar{H}(\sqcup \cdot)$ and $\bar{H}(\wedge \cdot)$ are equivalent; i.e. if equality holds for either equation in Tuncel et al.'s subadditivity results [6, Theorem 2], then equality also holds for the other one. We use this result to determine the complementary graph entropy of the non-perfect probabilistic graph $G_{5} \wedge G$ when $G$ is perfect.

In Section [II] we define the graph-theoretic concepts we need to formulate our main theorems in Section IIII and their


Fig. 3. An empty graph $G_{1}=\left(N_{3},\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)\right)$ and a complete graph $G_{2}=$ $\left(K_{2},\left(\frac{1}{3}, \frac{2}{3}\right)\right)$, along with their AND product $G_{1} \wedge G_{2}$ and their disjoint union $G_{1} \sqcup G_{2}$ w.r.t. $\left(\frac{1}{4}, \frac{3}{4}\right)$.
consequences in Section IV An example of application for these theorems is given in Section $\bar{\square}$, and the main proofs are developed in Section VI Section VII and Section VIII

## II. Notations and definitions

We denote sequences by $x^{n}=\left(x_{1}, \ldots, x_{n}\right)$.
The set of probability distributions over $\mathcal{X}$ is denoted by $\Delta(\mathcal{X}) ; P_{X} \in \Delta(\mathcal{X})$ is the distribution of a random variable $X$. The uniform distribution is denoted by Unif. The conditional distribution of $X$ knowing $Y$ is denoted by $P_{X \mid Y}$.

A probabilistic graph $G$ is a tuple $\left(\mathcal{V}, \mathcal{E}, P_{V}\right)$, where $(\mathcal{V}, \mathcal{E})$ is a graph and $P_{V} \in \Delta(\mathcal{V})$. A subset $\mathcal{S} \subseteq \mathcal{V}$ is independent in $G$ if for all $x, x^{\prime} \in \mathcal{S}, x x^{\prime} \notin \mathcal{E}$. A mapping $c: \mathcal{V} \rightarrow \mathcal{C}$ is a coloring if $c^{-1}(i)$ is independent for all $i \in \mathcal{C}$. The cycle, complete, and empty graphs with $n$ vertices are respectively denoted by $C_{n}, K_{n}, N_{n}$.

Definition II. 1 (AND product $\wedge$ ) The AND product of $G_{1}=$ $\left(\mathcal{V}_{1}, \mathcal{E}_{1}, P_{V_{1}}\right)$ and $G_{2}=\left(\mathcal{V}_{2}, \mathcal{E}_{2}, P_{V_{2}}\right)$ is a probabilistic graph denoted by $G_{1} \wedge G_{2}$ with:

- $\mathcal{V}_{1} \times \mathcal{V}_{2}$ as set of vertices,
- $P_{V_{1}} P_{V_{2}}$ as probability distribution on the vertices,
- $\left(v_{1} v_{2}\right),\left(v_{1}^{\prime} v_{2}^{\prime}\right)$ are adjacent if $v_{1} v_{1}^{\prime} \in \mathcal{E}_{1}$ AND $v_{2} v_{2}^{\prime} \in \mathcal{E}_{2}$; with the convention of self-adjacency for all vertices.
We denote by $G_{1}^{\wedge n}$ the $n-$ th $A N D$ power: $G_{1}^{\wedge n} \doteq G_{1} \wedge \ldots \wedge G_{1}$.
Definition II. 2 (Disjoint union $\sqcup$ of probabilistic graphs)
Let $\mathcal{A}$ be a finite set, and let $P_{A} \in \Delta(\mathcal{A})$. For all $a \in \mathcal{A}$, let $G_{a}=\left(\mathcal{V}_{a}, \mathcal{E}_{a}, P_{V_{a}}\right)$ be a probabilistic graph, their disjoint union w.r.t. $P_{A}$ is a probabilistic graph $\left(\mathcal{V}, \mathcal{E}, P_{V}\right)$ denoted by $\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}$ and defined by:
$\mathcal{V}=\bigsqcup_{a \in \mathcal{A}} \mathcal{V}_{a}$ is the disjoint union of the sets $\left(\mathcal{V}_{a}\right)_{a \in \mathcal{A}}$;
- For all $v, v^{\prime} \in \mathcal{V}, v v^{\prime} \in \mathcal{E}$ iff they both belong to the same $\mathcal{V}_{a}$ and $v v^{\prime} \in \mathcal{E}_{a}$;
- $P_{V}=\sum_{a \in \mathcal{A}} P_{A}(a) P_{V_{a}}$; note that the $\left(P_{V_{a}}\right)_{a \in \mathcal{A}}$ have disjoint support in $\mathcal{V}$.

Remark II. 3 The disjoint union $\sqcup$ that we consider here is also called "sum of graphs" by Tuncel et al. in [6]. Note that $\sqcup$ is the disjoint union over the vertices: it differs in nature from the union over the edges $\cup$ that is already studied in the literature, in particular in [21], [5] and [12].

An example of AND product and disjoint union is given in Figure 3

## III．Main result

In this section， $\mathcal{A}$ is a finite set，$P_{A}$ is a distribution from $\Delta(\mathcal{A})$ and $\left(G_{a}\right)_{a \in \mathcal{A}}$ is a family of probabilistic graphs．

In Theorem 【II．2 we give an expression for the comple－ mentary graph entropy of a disjoint union w．r．t．a type；the proof is given in Section 【II－A．With Corollary III．3 we equate $\bar{H}(\sqcup \cdot)$ and $\bar{H}(\wedge \cdot)$ up to a multiplicative constant when $P_{A}=\operatorname{Unif}(\mathcal{A})$ ．

Definition III． 1 （Type of a sequence）Let $a^{k} \in \mathcal{A}^{k}$ ，its type $T_{a^{k}}$ is its empirical distribution．The set of types of sequences from $\mathcal{A}^{k}$ is denoted by $\Delta_{k}(\mathcal{A}) \subset \Delta(\mathcal{A})$ ．

Theorem III． 2 If $P_{A} \in \Delta_{k}(\mathcal{A})$ for some $k \in \mathbb{N}^{\star}$ then

$$
\begin{equation*}
\bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)=\frac{1}{k} \bar{H}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge k P_{A}(a)}\right) \tag{3}
\end{equation*}
$$

Corollary III． $3 \bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{\mathrm{Unif}(\mathcal{A})} G_{a}\right)=\frac{1}{|\mathcal{A}|} \bar{H}\left(\bigwedge_{a \in \mathcal{A}} G_{a}\right)$ ．

## A．Proof of Theorem III． 2

In order to complete the proof，we need Lemma 1 it is the cornerstone of the connection between $\bar{H}(\sqcup \cdot)$ and $\bar{H}(\wedge \cdot)$ ． The main reasons why $\wedge$ appears in（4）are the AND powers used in $\bar{H}$ ，and the distributivity of $\wedge$ w．r．t．$\sqcup$（see Lemma2）． The proof of Lemma 1 is developed in Section VI

Lemma 1 Let $\left(\bar{a}_{n}\right)_{n \in \mathbb{N}^{\star}} \in \mathcal{A}^{\mathbb{N}^{\star}}$ be any sequence such that $T_{\bar{a}^{n}} \rightarrow P_{A}$ when $n \rightarrow \infty$ ．Then we have

$$
\begin{equation*}
\bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)=\lim _{n \rightarrow \infty} \frac{1}{n} H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge n T_{\bar{a}^{n}}(a)}\right) \tag{4}
\end{equation*}
$$

Now let us prove Theorem 【II．2 Let $\left(\bar{a}_{n}\right)_{n \in \mathbb{N}^{\star}}$ be a $k$－ periodic sequence such that $T_{\bar{a}^{k}}=P_{A}$ ，then $T_{\bar{a}^{n k}}=T_{\bar{a}^{k}}$ for all $n \in \mathbb{N}^{\star}$ ，and $T_{\bar{a}^{n}} \underset{n \rightarrow \infty}{\rightarrow} P_{A}$ ．We can use Lemma 1 and consider every $k$－th term in the limit：

$$
\begin{aligned}
\bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right) & =\lim _{n \rightarrow \infty} \frac{1}{k n} H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge k n T_{\bar{a}^{k n}}(a)}\right) \\
& =\lim _{n \rightarrow \infty} \frac{1}{k n} H_{\chi}\left(\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge k T_{a^{k}}(a)}\right)^{\wedge n}\right) \\
& =\frac{1}{k} \bar{H}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge k P_{A}(a)}\right)
\end{aligned}
$$

## IV．CONSEQUENCES

A．Single－letter formula of $\bar{H}$ for products of perfect graphs
With the exceptions of $G_{5}=\left(C_{5}, \operatorname{Unif}(\{1, \ldots, 5\})\right)$ and $G_{5} \sqcup G$ and its complement when $\bar{H}(G)$ is known，the only cases where $\bar{H}$ is known are perfect graphs with any under－ lying distribution：it is given by the Körner graph entropy， defined below．We extend the known cases with TheoremIV． 6 which gives a single－letter expression for $\bar{H}$ for AND products of perfect graphs．This case was not solved before，as a product of perfect graphs is not perfect in general（see Figure 4 for a counterexample）．The proof of Theorem IV．6 is developed in Section VIII．

Definition IV． 1 （Induced subgraph）The subgraph induced in a graph $G$ by a subset of vertices $\mathcal{S}$ is the graph ob－ tained from $G$ by keeping only the vertices in $\mathcal{S}$ and the edges between them，and is denoted by $G[\mathcal{S}]$ ．When $G$ is a probabilistic graph，we give it the underlying probability distribution $P_{V} / P_{V}(\mathcal{S})$ ．

Definition IV． 2 （Perfect graph）A graph $G=(\mathcal{V}, \mathcal{E})$ is per－ fect if $\forall \mathcal{S} \subset \mathcal{V}, \chi(G[\mathcal{S}])=\omega(G[\mathcal{S}])$ ；where $\omega$ is the size of the largest clique（i．e．complete induced subgraph）；and $\chi(G[\mathcal{S}])$ is the smallest $|\mathcal{C}|$ such that there exists a coloring $c: \mathcal{S} \rightarrow \mathcal{C}$ of $G[\mathcal{S}]$ ．By extension，we call perfect a probabilistic graph $\left(\mathcal{V}, \mathcal{E}, P_{V}\right)$ if $(\mathcal{V}, \mathcal{E})$ is perfect．

Definition IV． 3 （Körner graph entropy $H_{\kappa}$ ）For all $G=$ $\left(\mathcal{V}, \mathcal{E}, P_{V}\right)$ ，let $\Gamma(G)$ be the collection of independent sets of vertices in $G$ ．The Körner graph entropy of $G$ is defined by

$$
\begin{equation*}
H_{\kappa}(G)=\min _{V \in W \in \Gamma(G)} I(W ; V) \tag{5}
\end{equation*}
$$

where the minimum is taken over all distributions $P_{W \mid V} \in$ $\Delta(\mathcal{W})^{\mathcal{V}}$ ，with $\mathcal{W}=\Gamma(G)$ and with the constraint that the random vertex $V$ belongs to the random independent set $W$ with probability one，i．e．$V \in W \in \Gamma(G)$ in（5）．

Theorem IV． 4 （Strong perfect graph theorem，from［22］） A graph $G$ is perfect if and only if neither $G$ nor its complement have an induced odd cycle of length at least 5 ．

Theorem IV． 5 （from［5］）Let $G$ be a perfect probabilistic graph，then $\bar{H}(G)=H_{\kappa}(G)$ ．

Theorem IV． 6 When $\left(G_{a}\right)_{a \in \mathcal{A}}$ is a family of perfect prob－ abilistic graphs，the following single－letter characterizations hold：

$$
\begin{equation*}
\bar{H}\left(\bigwedge_{a \in \mathcal{A}} G_{a}\right)=\sum_{a \in \mathcal{A}} \bar{H}\left(G_{a}\right)=\sum_{a \in \mathcal{A}} H_{\kappa}\left(G_{a}\right), \tag{6}
\end{equation*}
$$



Fig．4．This is the AND product of two perfect graphs $C_{6}$ and $C_{8}$ ．The thick edges represent an induced subgraph $C_{7}$ ，which makes $C_{6} \wedge C_{8}$ non perfect by the strong perfect graph Theorem（see Theorem IV．4．

$$
\begin{equation*}
\bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)=\sum_{a \in \mathcal{A}} P_{A}(a) \bar{H}\left(G_{a}\right)=\sum_{a \in \mathcal{A}} P_{A}(a) H_{\kappa}\left(G_{a}\right) . \tag{7}
\end{equation*}
$$

## B. Linearization of the complementary graph entropy

In their subadditivity result [6, Theorem 2], Tuncel et al. show that for all probabilistic graphs $G_{1}, G_{2}$ and $\alpha \in(0,1)$,

$$
\begin{align*}
& \bar{H}\left(G_{1} \stackrel{(\alpha, 1-\alpha)}{\sqcup} G_{2}\right) \leq \alpha \bar{H}\left(G_{1}\right)+(1-\alpha) \bar{H}\left(G_{2}\right),  \tag{8}\\
& \bar{H}\left(G_{1} \wedge G_{2}\right) \leq \bar{H}\left(G_{1}\right)+\bar{H}\left(G_{2}\right) \tag{9}
\end{align*}
$$

We show in Theorem $\nabla .7$ that the cases where equality holds in (8) and (9) coincide.

Theorem IV. 7 For all probabilistic graphs $G_{1}, G_{2}$, for all $\alpha \in(0,1)$, we have:

$$
\begin{align*}
& \bar{H}\left(G_{1} \stackrel{(\alpha, 1-\alpha)}{\sqcup} G_{2}\right)=\alpha \bar{H}\left(G_{1}\right)+(1-\alpha) \bar{H}\left(G_{2}\right)  \tag{10}\\
\Longleftrightarrow & \bar{H}\left(G_{1} \wedge G_{2}\right)=\bar{H}\left(G_{1}\right)+\bar{H}\left(G_{2}\right) . \tag{11}
\end{align*}
$$

We prove and use the more general formula stated in Theorem IV.8. The proof is given in Section VII

Theorem IV. 8 Let $P_{A} \in \Delta(\mathcal{A})$ with full-support, then the following equivalence holds

$$
\begin{align*}
\bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right) & =\sum_{a \in \mathcal{A}} P_{A}(a) \bar{H}\left(G_{a}\right)  \tag{12}\\
\Longleftrightarrow \bar{H}\left(\bigwedge_{a \in \mathcal{A}} G_{a}\right) & =\sum_{a \in \mathcal{A}} \bar{H}\left(G_{a}\right) . \tag{13}
\end{align*}
$$

A case where equality holds in (12) is developed by Tuncel et al. in [6, Lemma 3]: $G_{5} \doteq\left(C_{5}, \operatorname{Unif}(\{1, \ldots, 5\})\right)$ along with any perfect graph. We provide a single-letter formula for $\bar{H}\left(G_{5} \wedge G\right)$ when $G$ is perfect; while $G_{5} \wedge G$ is not perfect as $G_{5} \wedge G$ contains an induced $C_{5}$ (see Theorem IV.4). The proof of the following Corollary is given in Appendix A.

Corollary IV. 9 For all perfect probabilistic graph $G$,

$$
\begin{equation*}
\bar{H}\left(G \wedge G_{5}\right)=\bar{H}(G)+\bar{H}\left(G_{5}\right)=H_{\kappa}(G)+\frac{1}{2} \log 5 . \tag{14}
\end{equation*}
$$

## V. Example

In this section, for all $i \in \mathbb{N}^{\star}, G_{i}$ denotes the cycle graph with $i$ vertices uniform distribution, i.e. $G_{i}=$ $\left(C_{i}, \operatorname{Unif}(\{0, \ldots, i-1\})\right)$. Both $G_{6}$ and $G_{8}$ are perfect, and as shown in Figure 4, $G_{6} \wedge G_{8}$ is not a perfect graph. We have:

$$
\begin{align*}
H_{\kappa}\left(G_{6}\right) & =H\left(V_{6}\right)-\max _{V_{6} \in W_{6} \in \Gamma\left(G_{6}\right)} H\left(V_{6} \mid W_{6}\right)  \tag{15}\\
& =1+\log 3-\log 3=1 \tag{16}
\end{align*}
$$

as $H\left(V_{6} \mid W_{6}\right)$ in (15) is maximized by taking $W_{6}=\{0,2,4\}$ when $V_{6} \in\{0,2,4\}$, and $W_{6}=\{1,3,5\}$ otherwise.

Similarly, $H_{\kappa}\left(G_{8}\right)=1$.
We can use Theorem IV.5 to find $\bar{H}\left(G_{6} \wedge G_{8}\right)$ :

$$
\begin{equation*}
\bar{H}\left(G_{6} \wedge G_{8}\right)=H_{\kappa}\left(G_{6}\right)+H_{\kappa}\left(G_{8}\right)=2 \tag{17}
\end{equation*}
$$

We can build an optimal coloring of $G_{6} \wedge G_{8}, c^{*}$ : $\left(v_{6}, v_{8}\right) \mapsto\left(\mathbb{1}_{v_{6}}\right.$ is even, $\mathbb{1}_{v_{8}}$ is even $)$.

## VI. Proof of Lemma 1

## A. Preliminary results

Lemma 2 establishes the distributivity of $\wedge$ w.r.t. $\sqcup$ for probabilistic graphs, similarly as in [18] for graphs without underlying distribution. Lemma 3 states that $\bar{H}$ can be computed with subgraphs induced by sets that have an asymptotic probability one, in particular we will use it with typical sets of vertices. The proofs of Lemma 2 and Lemma 3 are respectively given in Appendix C and Appendix D .

Lemma 2 Let $\mathcal{A}, \mathcal{B}$ be finite sets, let $P_{A} \in \Delta(\mathcal{A})$ and $P_{B} \in$ $\Delta(\mathcal{B})$. For all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, let $G_{a}=\left(\mathcal{V}_{a}, \mathcal{E}_{a}, P_{V_{a}}\right)$ and $G_{b}=\left(\mathcal{V}_{b}, \mathcal{E}_{b}, P_{V_{b}}\right)$ be probabilistic graphs. Then

$$
\begin{equation*}
\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right) \wedge\left(\bigsqcup_{b \in \mathcal{B}}^{P_{B}} G_{b}\right)=\bigsqcup_{(a, b) \in \mathcal{A} \times \mathcal{B}}^{P_{A} P_{B}} G_{a} \wedge G_{b} \tag{18}
\end{equation*}
$$

Lemma 3 Let $G=\left(\mathcal{V}, \mathcal{E}, P_{V}\right)$, and $\left(\mathcal{S}^{n}\right)_{n \in \mathbb{N}^{\star}}$ be a sequence of sets such that for all $n \in \mathbb{N}^{\star}, \mathcal{S}^{n} \subset \mathcal{V}^{n}$, and $P_{V}^{n}\left(\mathcal{S}^{n}\right) \rightarrow 1$ when $n \rightarrow \infty$. Then $\bar{H}(G)=\lim _{n \rightarrow \infty} \frac{1}{n} H_{\chi}\left(G^{\wedge n}\left[\mathcal{S}^{n}\right]\right)$.

Definition VI. 1 (Isomorphic probabilistic graphs) Let
$G_{1}=\left(\mathcal{V}_{1}, \mathcal{E}_{1}, P_{V_{1}}\right)$ and $G_{2}=\left(\mathcal{V}_{2}, \mathcal{E}_{2}, P_{V_{2}}\right)$. We say that $G_{1}$ is isomorphic to $G_{2}$ if there exists an isomorphism between them, i.e. a bijection $\psi: \mathcal{V}_{1} \rightarrow \mathcal{V}_{2}$ such that:

- For all $v_{1}, v_{1}^{\prime} \in \mathcal{V}_{1}, v_{1} v_{1}^{\prime} \in \mathcal{E}_{1} \Longleftrightarrow \psi\left(v_{1}\right) \psi\left(v_{1}^{\prime}\right) \in \mathcal{E}_{2}$,
- For all $v_{1} \in \mathcal{V}_{1}, P_{V_{1}}\left(v_{1}\right)=P_{V_{2}}\left(\psi\left(v_{1}\right)\right)$.

Lemma 4 (from [8]) Let $\mathcal{B}$ be a finite set, let $P_{B} \in \Delta(\mathcal{B})$ and let $\left(G_{b}\right)_{b \in \mathcal{B}}$ be a family of isomorphic probabilistic graphs, then $H_{\chi}\left(\bigsqcup_{b^{\prime} \in \mathcal{B}}^{P_{B}} G_{b^{\prime}}\right)=H_{\chi}\left(G_{b}\right)$ for all $b \in \mathcal{B}$.

## B. Main proof of Lemma $\mathbb{Z}$

For all $a \in \mathcal{A}$, let $G_{a}=\left(\mathcal{V}_{a}, \mathcal{E}_{a}, P_{V_{A}}\right)$, and let $G=$ $\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}$. Let $P_{A} \in \Delta(\mathcal{A})$, and let $\left(\bar{a}_{n}\right)_{n \in \mathbb{N}^{\star}} \in \mathcal{A}^{\mathbb{N}^{\star}}$ be a sequence such that $T_{\bar{a}^{n}} \rightarrow P_{A}$ when $n \rightarrow \infty$.

Let $\epsilon>0$, and for all $n \in \mathbb{N}^{\star}$ let

$$
\begin{align*}
& \mathcal{T}_{\epsilon}^{n}\left(P_{A}\right) \doteq\left\{a^{n} \in \mathcal{A}^{n} \mid\left\|T_{a^{n}}-P_{A}\right\|_{\infty} \leq \epsilon\right\}  \tag{19}\\
& P^{\prime n} \doteq \frac{P_{A}^{n}}{P_{A}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(P_{A}\right)\right)}, \quad \mathcal{S}_{\epsilon}^{n} \doteq \bigsqcup_{a^{n} \in \mathcal{T}_{\epsilon}^{n}\left(P_{A}\right)} \prod_{t \leq n} \mathcal{V}_{a_{t}} .
\end{align*}
$$

By Lemma 3 we have

$$
\begin{equation*}
\bar{H}(G)=\lim _{n \rightarrow \infty} \frac{1}{n} H_{\chi}\left(G^{\wedge n}\left[\mathcal{S}_{\epsilon}^{n}\right]\right) \tag{20}
\end{equation*}
$$

as $P_{V}^{n}\left(\mathcal{S}_{\epsilon}^{n}\right) \rightarrow 1$ when $n \rightarrow \infty$. Let us study the limit in (20). For all $n$ large enough, $\bar{a}^{n} \in \mathcal{T}_{\epsilon}^{n}\left(P_{A}\right)$ as $T_{\bar{a}^{n}} \rightarrow P_{A}$. Therefore, for all $a^{n} \in \mathcal{T}_{\epsilon}^{n}\left(P_{A}\right)$ and $a^{\prime} \in \mathcal{A}$,

$$
\begin{equation*}
\left|T_{\bar{a}^{n}}\left(a^{\prime}\right)-T_{a^{n}}\left(a^{\prime}\right)\right| \leq 2 \epsilon . \tag{21}
\end{equation*}
$$

We have on one hand

$$
\begin{align*}
& H_{\chi}\left(\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)^{\wedge n}\left[\mathcal{S}_{\epsilon}^{n}\right]\right) \\
= & H_{\chi}\left(\left(\bigsqcup_{a^{n} \in \mathcal{A}^{n}}^{P_{n}^{n}} \bigwedge_{t \leq n} G_{a_{t}}\right)\left[\mathcal{S}_{\epsilon}^{n}\right]\right) \tag{22}
\end{align*}
$$

$$
\begin{align*}
& =H_{\chi}\left(\bigsqcup_{a^{n} \in \mathcal{T}_{\epsilon}^{n}\left(P_{A}\right)}^{P^{\prime \prime}} \bigwedge_{t \leq n} G_{a_{t}}\right)  \tag{23}\\
& =H_{\chi}\left(\bigsqcup_{a^{n} \in \mathcal{T}_{\epsilon}^{n}\left(P_{A}\right)}^{P^{\prime \prime}} \bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\wedge n T_{a^{n}}\left(a^{\prime}\right)}\right)  \tag{24}\\
& \leq H_{\chi}\left(\bigsqcup_{a^{n} \in \mathcal{T}_{\epsilon}^{n}\left(P_{A}\right)}^{P^{\prime \prime}} \bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\wedge n T_{\bar{a}^{n}}\left(a^{\prime}\right)+\lceil 2 n \epsilon\rceil}\right)  \tag{25}\\
& =H_{\chi}\left(\bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\wedge n T_{\bar{a}^{n}}\left(a^{\prime}\right)+\lceil 2 n \epsilon\rceil}\right)  \tag{26}\\
& \leq H_{\chi}\left(\bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\left.\wedge n T_{\bar{a}^{n}\left(a^{\prime}\right)}\right)+H_{\chi}\left(\bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\wedge\lceil 2 n \epsilon\rceil}\right)}\right.  \tag{27}\\
& \leq H_{\chi}\left(\bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\left.\wedge n T_{\bar{a}^{n}\left(a^{\prime}\right)}\right)+\lceil 2 n \epsilon\rceil|\mathcal{A}| \log |\mathcal{V}| ;}\right. \tag{28}
\end{align*}
$$

where (22) comes from Lemma2, (23) comes from the definition of $\mathcal{S}_{\epsilon}^{n}$ and $P^{\prime n}$ in (19); (24) is a rearrangement of the terms inside the product; (25) comes from (21); (26) follows from Lemma 4 the graphs $\left(\bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\wedge n T_{\bar{a}^{n}}\left(a^{\prime}\right)+\lceil 2 n \epsilon\rceil}\right)_{a^{n} \in \mathcal{T}_{\epsilon}^{n}\left(P_{A}\right)}$ are isomorphic as they do not depend on $a^{n}$; 27) follows from the subadditivity of $H_{\chi}$; and (28) is the upper bound on $H_{\chi}$ given by the highest entropy of a coloring.

On the other hand, we obtain with similar arguments

$$
\begin{align*}
& H_{\chi}\left(\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)^{\wedge n}\left[\mathcal{S}_{\epsilon}^{n}\right]\right) \\
\geq & H_{\chi}\left(\bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\wedge n T_{\bar{a}^{n}}\left(a^{\prime}\right)-\lceil 2 n \epsilon\rceil}\right)  \tag{29}\\
\geq & H_{\chi}\left(\bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\wedge n T_{\bar{a}^{n}}\left(a^{\prime}\right)}\right)-H_{\chi}\left(\bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\wedge\lceil 2 n \epsilon\rceil}\right),  \tag{30}\\
\geq & H_{\chi}\left(\bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\wedge n T_{\bar{a}^{n}}\left(a^{\prime}\right)}\right)-\lceil 2 n \epsilon\rceil|\mathcal{A}| \log |\mathcal{V}| . \tag{31}
\end{align*}
$$

Note that (30) also comes from the subadditivity of $H_{\chi}$ : $H_{\chi}\left(G_{2}\right) \geq H_{\chi}\left(G_{1} \wedge G_{2}\right)-H_{\chi}\left(G_{1}\right)$ for all $G_{1}, G_{2}$.
By combining (28) and (31) we obtain

$$
\begin{align*}
& \left|\lim _{n \rightarrow \infty} \frac{1}{n} H_{\chi}\left(G^{\wedge n}\left[\mathcal{S}_{\epsilon}^{n}\right]\right)-\lim _{n \rightarrow \infty} \frac{1}{n} H_{\chi}\left(\bigwedge_{a^{\prime} \in \mathcal{A}} G_{a^{\prime}}^{\wedge n T_{\bar{a}^{n}}\left(a^{\prime}\right)}\right)\right| \\
& \leq 2 \epsilon|\mathcal{A}| \log |\mathcal{V}| . \tag{32}
\end{align*}
$$

As this holds for all $\epsilon>0$, combining (20) and (32) yields the desired result.

## VII. Proof of Theorem IV. 8

## A. Preliminary results

In Lemma [5 we give regularity properties of $P_{A} \mapsto$ $\bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)$. The proof of Lemma 5 is developed in Appendix E Lemma6states that if a convex function $\gamma$ of $\Delta(\mathcal{A})$ meets the linear interpolation of the $\left(\gamma\left(\mathbb{1}_{a}\right)\right)_{a \in \mathcal{A}}$ at an interior point, then $\gamma$ is linear. We use it for proving the equivalence in Theorem IV.8, by considering $\gamma=P_{A} \mapsto \bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)$. The proof of Lemma 6 is given in Appendix F

Lemma 5 The function $P_{A} \mapsto \bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)$ is convex and $\left(\log \max _{a}\left|\mathcal{V}_{a}\right|\right)$-Lipschitz.

Lemma 6 Let $\mathcal{A}$ be a finite set, and $\gamma: \Delta(\mathcal{A}) \rightarrow \mathbb{R}$ be a convex function. Then the following holds:

$$
\begin{align*}
& \exists P_{A} \in \operatorname{int}(\Delta(\mathcal{A})), \gamma\left(P_{A}\right)=\sum_{a \in \mathcal{A}} P_{A}(a) \gamma\left(\mathbb{1}_{a}\right)  \tag{33}\\
\Longleftrightarrow & \forall P_{A} \in \Delta(\mathcal{A}), \gamma\left(P_{A}\right)=\sum_{a \in \mathcal{A}} P_{A}(a) \gamma\left(\mathbb{1}_{a}\right) \tag{34}
\end{align*}
$$

where $\operatorname{int}(\Delta(\mathcal{A}))$ is the interior of $\Delta(\mathcal{A})$ (i.e. the full-support distributions on $\mathcal{A})$.

## B. Main proof of Theorem IV. 8

$(\Longrightarrow)$ Assume that $\bar{H}\left(\bigwedge_{a \in \mathcal{A}} G_{a}\right)=\sum_{a \in \mathcal{A}} \bar{H}\left(G_{a}\right)$.
We can use Corollary III.3. $\bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{\operatorname{Unif}(\mathcal{A})} G_{a}\right)=$ $\sum_{a \in \mathcal{A}} \frac{1}{|\mathcal{A}|} \bar{H}\left(G_{a}\right)$. Thus, the function $P_{A} \mapsto \bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)$ is convex by Lemma 5, and satisfies (33) with the interior point $P_{A}=\operatorname{Unif}(\mathcal{A})$ : by Lemma 6 we have

$$
\begin{equation*}
\forall P_{A} \in \Delta(\mathcal{A}), \bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)=\sum_{a \in \mathcal{A}} P_{A}(a) \bar{H}\left(G_{a}\right) \tag{35}
\end{equation*}
$$

$(\Longleftarrow)$ Conversely, assume (35), then $P_{A} \mapsto \bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)$ is linear. We can use Corollary III.3 and we have $\bar{H}\left(\bigwedge_{a \in \mathcal{A}} G_{a}\right)=|\mathcal{A}| \bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{\mathrm{Unif}(\mathcal{A})} G_{a}\right)=\sum_{a \in \mathcal{A}} \bar{H}\left(G_{a}\right)$.

## VIII. Proof of Theorem IV. 6

## A. Preliminary results

Lemma 7 comes from [23, Corollary 3.4], and states that the function $P_{A} \mapsto H_{\kappa}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)$, defined analogously to $P_{A} \mapsto \bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)$, is always linear. We give a proof of Lemma 7 in Appendix $G$ for the sake of completeness. The proof of Lemma 8 is given in Appendix $[\mathbf{H}$

Lemma 7 For all probabilistic graphs $\left(G_{a}\right)_{a \in \mathcal{A}}$ and $P_{A} \in$ $\Delta(\mathcal{A})$, we have $H_{\kappa}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)=\sum_{a \in \mathcal{A}} P_{A}(a) H_{\kappa}\left(G_{a}\right)$.

Lemma 8 The probabilistic graph $\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}$ is perfect if and only if $G_{a}$ is perfect for all $a \in \mathcal{A}$.

## B. Main proof of Theorem IV. 6

For all $a \in \mathcal{A}$, let $G_{a}=\left(\mathcal{V}_{a}, \mathcal{E}_{a}, P_{V_{a}}\right)$ be a perfect probabilistic graph. By Lemma $8, \bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}$ is also perfect; and we have $\bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)=H_{\kappa}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)$ by Theorem IV. 5 We also have $H_{\kappa}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)=\sum_{a \in \mathcal{A}} P_{A}(a) H_{\kappa}\left(G_{a}\right)=$ $\sum_{a \in \mathcal{A}} P_{A}(a) \bar{H}\left(G_{a}\right)$ by Lemma 7 and Theorem IV.5 used on the perfect graphs $\left(G_{a}\right)_{a \in \mathcal{A}}$.

Therefore (12) is satisfied by the graphs $\left(G_{a}\right)_{a \in \mathcal{A}}$ and $P_{A}$ : by Theorem IV.8, it follows that $\bar{H}\left(\bigwedge_{a \in \mathcal{A}} G_{a}\right)=$ $\sum_{a \in \mathcal{A}} \bar{H}\left(G_{a}\right)=\sum_{a \in \mathcal{A}} H_{\kappa}\left(G_{a}\right)$, where the last equality comes from Theorem [V.5]

## IX. Conclusion

Theorem III. 2 shows that $\bar{H}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)=$ $\frac{1}{k} \bar{H}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge k P_{A}(a)}\right)$ holds for all $P_{A} \in \Delta_{k}(\mathcal{A})$. The consequences of this result are stated in Theorem IV.6 Theorem IV. 8 and Corollary IV. 9 . We provide a single-letter formula for $\bar{H}$ for a new class of graphs. By (2), this allows to characterize optimal rates for the two source coding problems depicted in Figure 2

Proposition IX. 1 The optimal rates in the settings from Figure $2 \sqrt{2}$ and Figure $2 b$ are respectively given by $\bar{H}\left(\bigsqcup_{z \in \mathcal{Z}}^{P_{g(Y)}} G_{z}\right)$ and $\bar{H}\left(\bigwedge_{z \in \mathcal{Z}} G_{z}\right)$.

## References

[1] N. Alon and A. Orlitsky, "Source coding and graph entropies," IEEE Transactions on Information Theory, vol. 42, no. 5, pp. 1329-1339, 1996.
[2] H. Witsenhausen, "The zero-error side information problem and chromatic numbers (corresp.)," IEEE Transactions on Information Theory, vol. 22, no. 5, pp. 592-593, 1976.
[3] P. Koulgi, E. Tuncel, S. L. Regunathan, and K. Rose, "On zero-error source coding with decoder side information," IEEE Transactions on Information Theory, vol. 49, no. 1, pp. 99-111, 2003.
[4] J. Korner and G. Longo, "Two-step encoding for finite sources," IEEE Transactions on Information Theory, vol. 19, no. 6, pp. 778-782, 1973.
[5] I. Csiszár, J. Körner, L. Lovász, K. Marton, and G. Simonyi, "Entropy splitting for antiblocking corners and perfect graphs," Combinatorica, vol. 10, no. 1, pp. 27-40, 1990.
[6] E. Tuncel, J. Nayak, P. Koulgi, and K. Rose, "On complementary graph entropy," IEEE transactions on information theory, vol. 55, no. 6, pp. 2537-2546, 2009.
[7] A. Orlitsky and J. R. Roche, "Coding for computing," in Proceedings of IEEE 36th Annual Foundations of Computer Science. IEEE, 1995, pp. 502-511.
[8] N. Charpenay, M. 1. Treust, and A. Roumy, "Zero-error coding for computing with encoder side-information," arXiv preprint arXiv:2211.03649 2022.
[9] J. Ravi and B. K. Dey, "Zero-error function computation through a bidirectional relay," in 2015 IEEE Information Theory Workshop (ITW). IEEE, 2015, pp. 1-5.
[10] D. Malak, "Fractional graph coloring for functional compression with side information," arXiv preprint arXiv:2204.11927 2022.
[11] C. Shannon, "The zero error capacity of a noisy channel," IRE Transactions on Information Theory, vol. 2, no. 3, pp. 8-19, 1956.
[12] K. Marton, "On the shannon capacity of probabilistic graphs," Journal of Combinatorial Theory, Series B, vol. 57, no. 2, pp. 183-195, 1993.
[13] W. Haemers et al., "On some problems of lovász concerning the shannon capacity of a graph," IEEE Transactions on Information Theory, vol. 25, no. 2, pp. 231-232, 1979.
[14] B. Bukh and C. Cox, "On a fractional version of haemers' bound," IEEE Transactions on Information Theory, vol. 65, no. 6, pp. 33403348, 2018.
[15] L. Gao, S. Gribling, and Y. Li, "On a tracial version of haemers bound," IEEE Transactions on Information Theory, 2022.
[16] A. Schrijver, "On the shannon capacity of sums and products of graphs," Indagationes Mathematicae, vol. 34, no. 1, pp. 37-41, 2023.
[17] H. Boche and C. Deppe, "Computability of the zero-error capacity of noisy channels," in 2021 IEEE Information Theory Workshop (ITW). IEEE, 2021, pp. 1-6.
[18] J. Zuiddam et al., Algebraic complexity, asymptotic spectra and entanglement polytopes. Institute for Logic, Language and Computation, 2018.
[19] Y. Gu and O. Shayevitz, "On the non-adaptive zero-error capacity of the discrete memoryless two-way channel," Entropy, vol. 23, no. 11, p. 1518, 2021.
[20] M. Wiese, T. J. Oechtering, K. H. Johansson, P. Papadimitratos, H. Sandberg, and M. Skoglund, "Secure estimation and zero-error secrecy capacity," IEEE Transactions on Automatic Control, vol. 64, no. 3, pp. 1047-1062, 2018.
[21] J. Körner and K. Marton, "Graphs that split entropies," SIAM journal on discrete mathematics, vol. 1, no. 1, pp. 71-79, 1988.
[22] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, "The strong perfect graph theorem," Annals of mathematics, pp. 51-229, 2006.
[23] G. Simonyi, "Graph entropy: a survey," Combinatorial Optimization, vol. 20, pp. 399-441, 1995.

## Appendix A Proof of Corollary IV. 9

By [6] Lemma 3], if $G$ is perfect then

$$
\begin{equation*}
\bar{H}\left(G \stackrel{(\alpha, 1-\alpha)}{\sqcup} G_{5}\right)=\alpha \bar{H}(G)+(1-\alpha) \bar{H}\left(G_{5}\right) . \tag{36}
\end{equation*}
$$

By Theorem IV.8, we have $\bar{H}\left(G \wedge G_{5}\right)=\bar{H}(G)+\bar{H}\left(G_{5}\right)=$ $H_{\kappa}(G)+\frac{\log 5}{2}$; where the last equality comes from [3] Example 1] which states that $\bar{H}\left(G_{5}\right)=\frac{\log 5}{2}$, and from Theorem IV. 6

## Appendix B <br> Proof dependencies

An illustration of the dependencies between the results can be found in Figure 5


Fig. 5. An arrow from A to B means that A is used in the proof of B. Results from the literature are represented with a dashed outline.

## Appendix C <br> PROOF OF LEMMA 2

The probabilistic graphs in both sides of (18) have

$$
\begin{equation*}
\left(\bigsqcup_{a \in \mathcal{A}} \mathcal{V}_{a}\right) \times\left(\bigsqcup_{b \in \mathcal{B}} \mathcal{V}_{b}\right)=\bigsqcup_{(a, b) \in \mathcal{A} \times \mathcal{B}} \mathcal{V}_{a} \times \mathcal{V}_{b} \tag{37}
\end{equation*}
$$

as set of vertices, with underlying distribution

$$
\begin{align*}
& \left(\sum_{a \in \mathcal{A}} P_{A}(a) P_{V_{a}}\right)\left(\sum_{b \in \mathcal{B}} P_{B}(b) P_{V_{b}}\right) \\
& =\sum_{(a, b) \in \mathcal{A} \times \mathcal{B}} P_{A}(a) P_{B}(b) P_{V_{a}} P_{V_{b}} . \tag{38}
\end{align*}
$$

Now let us show that these two graphs have the same edges. Let $\left(v_{\mathcal{A}}, v_{\mathcal{B}}\right),\left(v_{\mathcal{A}}^{\prime}, v_{\mathcal{B}}^{\prime}\right) \in\left(\bigsqcup_{a \in \mathcal{A}} \mathcal{V}_{a}\right) \times\left(\bigsqcup_{b \in \mathcal{B}} \mathcal{V}_{b}\right)$; let $a_{*}, a_{*}^{\prime} \in$ $\mathcal{A}$ and $b_{*}, b_{*}^{\prime} \in \mathcal{B}$ be the unique indexes such that

$$
\begin{equation*}
\left(v_{\mathcal{A}}, v_{\mathcal{B}}\right) \in \mathcal{V}_{a_{*}} \times \mathcal{V}_{b_{*}} \quad \text { and } \quad\left(v_{\mathcal{A}}^{\prime}, v_{\mathcal{B}}^{\prime}\right) \in \mathcal{V}_{a_{*}^{\prime}} \times \mathcal{V}_{b_{*}^{\prime}} \tag{39}
\end{equation*}
$$

We have:

$$
\begin{equation*}
\left(v_{\mathcal{A}}, v_{\mathcal{B}}\right),\left(v_{\mathcal{A}}^{\prime}, v_{\mathcal{B}}^{\prime}\right) \text { are adjacent in }\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right) \wedge\left(\bigsqcup_{b \in \mathcal{B}}^{P_{B}} G_{b}\right) \tag{40}
\end{equation*}
$$

$$
\begin{gather*}
\Longleftrightarrow v_{\mathcal{A}}, v_{\mathcal{A}}^{\prime} \text { adjacent in } \bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a} \text { and } \\
\quad v_{\mathcal{B}}, v_{\mathcal{B}}^{\prime} \text { adjacent in } \bigsqcup_{b \in \mathcal{B}}^{P_{B}} G_{b} \tag{41}
\end{gather*}
$$

$\Longleftrightarrow a_{*}=a_{*}^{\prime}$ and $v_{\mathcal{A}} v_{\mathcal{A}}^{\prime} \in \mathcal{E}_{a_{*}}$ and $b_{*}=b_{*}^{\prime}$ and $v_{\mathcal{B}} v_{\mathcal{B}}^{\prime} \in \mathcal{E}_{b_{*}}$
$\Longleftrightarrow\left(a_{*}, b_{*}\right)=\left(a_{*}^{\prime}, b_{*}^{\prime}\right)$ and
$\left(v_{\mathcal{A}}, v_{\mathcal{B}}\right),\left(v_{\mathcal{A}}^{\prime}, v_{\mathcal{B}}^{\prime}\right)$ are adjacent in $G_{a_{*}} \wedge G_{b_{*}}$

$$
\begin{equation*}
\bigsqcup_{(a, b) \in \mathcal{A} \times \mathcal{B}}^{P_{A} P_{B}} G_{a} \wedge G_{b} . \tag{43}
\end{equation*}
$$

## Appendix D

Proof of Lemma 3

## A. Preliminary results

In Lemma 9 we give upper and lower bounds on the chromatic entropy of an induced subgraph $G[\mathcal{S}]$, using the chromatic entropy of the whole graph $G$ and the probability $P_{V}(\mathcal{S})$. The core idea is that if $P_{V}(\mathcal{S})$ is close to 1 and $H_{\chi}(G)$ is big, then $H_{\chi}(G[\mathcal{S}])$ is close to $H_{\chi}(G)$. The proof of Lemma 9 is given in Appendix [1]

Lemma 9 Let $G=\left(\mathcal{V}, \mathcal{E}, P_{V}\right)$ and $\mathcal{S} \subset \mathcal{V}$, then

$$
\begin{equation*}
H_{\chi}(G)-1-\left(1-P_{V}(\mathcal{S})\right) \log |\mathcal{V}| \leq H_{\chi}(G[\mathcal{S}]) \leq \frac{H_{\chi}(G)}{P_{V}(\mathcal{S})} \tag{45}
\end{equation*}
$$

Remark D. $1 H_{\chi}(G[\mathcal{S}])$ can be greater than $H_{\chi}(G)$, even if $G[\mathcal{S}]$ has less vertices and inherits the structure of $G$. This stems from the normalized distribution $P_{V} / P_{V}(\mathcal{S})$ on the vertices of $G[\mathcal{S}]$ which gives more weight to the vertices in $\mathcal{S}$. For example, consider

$$
G=\left(N_{5}, \operatorname{Unif}(\{1, \ldots, 5\})\right) \stackrel{(1-\epsilon, \epsilon)}{\sqcup}\left(K_{5}, \operatorname{Unif}(\{1, \ldots, 5\})\right),
$$

with $\mathcal{S}$ being the vertices in the connected component $K_{5}$ in $G$. Then $H_{\chi}(G)=\epsilon \log 5$ and $H_{\chi}(G[\mathcal{S}])=\log 5$.

## B. Main proof of Lemma 3

By Lemma 9 we have for all $n \in \mathbb{N}^{\star}$ :

$$
\begin{align*}
& H_{\chi}\left(G^{\wedge n}\right)-1-\left(1-P_{V}^{n}\left(\mathcal{S}^{n}\right)\right) \log |\mathcal{V}| \\
\leq & H_{\chi}\left(G^{\wedge n}\left[\mathcal{S}^{n}\right]\right) \leq \frac{H_{\chi}\left(G^{\wedge n}\right)}{P_{V}^{n}\left(\mathcal{S}^{n}\right)} . \tag{46}
\end{align*}
$$

Since $P_{V}^{n}\left(\mathcal{S}^{n}\right) \rightarrow 1$, and $H_{\chi}\left(G^{\wedge n}\right)=n \bar{H}(G)+o(n)$ when $n \rightarrow \infty$, the desired results follows immediately by normalization and limit.

## Appendix E

Proof of Lemma 5

## A. Preliminary results

Lemma 10 is a generalization for infinite sequences of the following observation: if $T_{\bar{a}^{n}}=P_{A} \in \Delta_{n}(\mathcal{A})$ satisfies $P_{A}=$ $\frac{i}{n} P_{A}^{\prime}+\frac{n-i}{n} P_{A}^{\prime \prime}$ with $P_{A}^{\prime} \in \Delta_{i}(\mathcal{A})$ and $P_{A}^{\prime \prime} \in \Delta_{n-i}(\mathcal{A})$, then $\bar{a}^{n}$ can be separated into two subsequences $a^{\prime i}$ and $a^{\prime \prime n-i}$ such that $T_{a^{\prime i}}=P_{A}^{\prime}$ and $T_{a^{\prime \prime n-i}}=P_{A}^{\prime \prime}$. The proof is given in Appendix J]

Lemma 10 (Type-splitting lemma) Let $\left(\bar{a}_{n}\right)_{n \in \mathbb{N}^{\star}} \in \mathcal{A}^{\mathbb{N}^{\star}}$ be a sequence such that $T_{\bar{a}^{n}} \rightarrow P_{A} \in \Delta(\mathcal{A})$ when $n \rightarrow \infty$, let $\beta \in(0,1)$ and $P_{A}^{\prime}, P_{A}^{\prime \prime} \in \Delta(\mathcal{A})$ such that

$$
\begin{equation*}
P_{A}=\beta P_{A}^{\prime}+(1-\beta) P_{A}^{\prime \prime} \tag{47}
\end{equation*}
$$

Then there exists a sequence $\left(b_{n}\right)_{n \in \mathbb{N}^{\star}} \in\{0,1\}^{\mathbb{N}^{\star}}$ such that the two extracted sequences $a^{\prime} \doteq\left(\bar{a}_{n}\right)_{\substack{n \in \mathbb{N}^{\star} \\ b_{n}=0}}$, and $a^{\prime \prime} \doteq$ $\left(\bar{a}_{n}\right)_{\substack{n \in \mathbb{N}^{\star} \\ b_{n}=1}}$, satisfy

$$
\begin{align*}
& T_{b^{n}} \underset{n \rightarrow \infty}{\rightarrow}(\beta, 1-\beta),  \tag{48}\\
& T_{a^{\prime n}} \underset{n \rightarrow \infty}{\rightarrow} P_{A}^{\prime},  \tag{49}\\
& T_{a^{\prime \prime n}} \underset{n \rightarrow \infty}{\rightarrow} P_{A}^{\prime \prime} .
\end{align*}
$$

## B. Main proof of Lemma 5

( $\eta$ Lipschitz) Let us first prove that $\eta$ is Lipschitz. For all $P_{A}, P_{A}^{\prime} \in \Delta(\mathcal{A})$ we need to bound the quantity $\mid \eta\left(P_{A}\right)-$ $\eta\left(P_{A}^{\prime}\right) \mid$; by Lemma 1 this is equivalent to bounding

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n}\left|H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge n T_{\bar{a}^{n}}(a)}\right)-H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge n T_{\bar{a}^{\prime n}}(a)}\right)\right| \tag{50}
\end{equation*}
$$

where $\left(T_{\bar{a}^{n}}, T_{\bar{a}^{\prime n}}\right) \rightarrow\left(P_{A}, P_{A}^{\prime}\right)$ when $n \rightarrow \infty$.
Fix $n \in \mathbb{N}^{\star}$, we assume that the quantity inside $|\cdot|$ in (50) is positive; the other case can be treated with the same arguments by symmetry of the roles. We have

$$
\left.\begin{array}{rl} 
& H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge n T_{\bar{a}} n}(a)\right.
\end{array}\right)-H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge n T_{\bar{a}^{\prime} n}(a)}\right),
$$

$$
\begin{align*}
= & H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge n \min \left(T_{\bar{a}^{n}}(a), T_{\bar{a}^{\prime n}}(a)\right)} \bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge n\left|T_{\bar{a}^{n}}(a)-T_{\bar{a}^{\prime n}}(a)\right|_{+}}\right) \\
& -H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge n \min \left(T_{\bar{a}^{n}}(a), T_{\bar{a}^{\prime n}}(a)\right)}\right) \\
\leq & H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge n\left|T_{\bar{a}^{n}}(a)-T_{\bar{a}^{\prime n}}(a)\right|_{+}}\right) \tag{54}
\end{align*}
$$

$\leq \log \left(\max _{a}\left|\mathcal{V}_{a}\right|\right) \sum_{a \in \mathcal{A}} n\left|T_{\bar{a}^{n}}(a)-T_{\bar{a}^{\prime n}}(a)\right|_{+}$
$\leq n \log \left(\max _{a}\left|\mathcal{V}_{a}\right|\right)\left\|T_{\bar{a}^{n}}-T_{\bar{a}^{\prime n}}\right\|_{1}$,
where $|\cdot|_{+}=\max (\cdot, 0)$ and $\left\|T_{\bar{a}^{n}}-T_{\bar{a}^{\prime n}}\right\|_{1}=\sum_{a \in \mathcal{A}} \mid T_{\bar{a}^{n}}(a)-$ $T_{\bar{a}^{\prime n}}(a) \mid ;$ (52) follows from the removal of terms in the second product, as $H_{\chi}\left(G \wedge G^{\prime}\right) \geq H_{\chi}(G)$ for all probabilistic graphs $G, G^{\prime}$; (53) is an arrangement of the terms in the first product, as $\min (s, t)+\max (s-t, 0)=s$ for all real numbers $s, t$; (54) comes from the subadditivity of $H_{\chi}$; (55) follows from $H_{\chi}\left(G_{a}\right) \leq \log \max _{a^{\prime}}\left|\mathcal{V}_{a^{\prime}}\right|$ for all $a \in \mathcal{A}$; 56 results from $\left|T_{\bar{a}^{n}}(a)-T_{\bar{a}^{\prime n}}(a)\right|_{+} \leq\left|T_{\bar{a}^{n}}(a)-T_{\bar{a}^{\prime n}}(a)\right|$ for all $a \in \mathcal{A}$.

By normalization and limit, it follows that

$$
\begin{align*}
\left|\eta\left(P_{A}\right)-\eta\left(P_{A}^{\prime}\right)\right| & \leq \lim _{n \rightarrow \infty} \log \left(\max _{a}\left|\mathcal{V}_{a}\right|\right) \cdot\left\|T_{\bar{a}^{n}}-T_{\bar{a}^{\prime n}}\right\|_{1}  \tag{57}\\
& =\log \left(\max _{a}\left|\mathcal{V}_{a}\right|\right) \cdot\left\|P_{A}-P_{A}^{\prime}\right\|_{1} . \tag{58}
\end{align*}
$$

Hence $\eta$ is $\left(\log \max _{a}\left|\mathcal{V}_{a}\right|\right)$-Lipschitz.
( $\eta$ convex) Let us now prove that $\eta$ is convex. Let $P_{A}^{\prime}, P_{A}^{\prime \prime} \in$ $\Delta(\mathcal{A})$, and $\beta \in(0,1)$, we have by Lemma 1

$$
\begin{equation*}
\eta\left(\beta P_{A}^{\prime}+(1-\beta) P_{A}^{\prime \prime}\right)=\lim _{n \rightarrow \infty} \frac{1}{n} H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge n T_{\bar{a}^{n}}(a)}\right) \tag{59}
\end{equation*}
$$

where $T_{\bar{a}^{n}} \rightarrow \beta P_{A}^{\prime}+(1-\beta) P_{A}^{\prime \prime}$ when $n \rightarrow \infty$. By Lemma 10 there exists $\left(b_{n}\right)_{n \in \mathbb{N}^{\star}} \in\{0,1\}^{\mathbb{N}^{\star}}$ such that the decomposition of $\left(\bar{a}_{n}\right)_{n \in \mathbb{N}^{\star}}$ into two subsequences $a^{\prime} \doteq\left(\bar{a}_{n}\right)_{\substack{n \in \mathbb{N}^{\star} \\ b_{n}=0}}$, and $a^{\prime \prime} \doteq\left(\bar{a}_{n}\right)_{\substack{n \in \mathbb{N}^{\star} \\ b_{n}=1}}$, satisfies

$$
\begin{array}{ll}
T_{b^{n}} \underset{n \rightarrow \infty}{\rightarrow}(\beta, 1-\beta), \\
T_{a^{\prime n}} \underset{n \rightarrow \infty}{\rightarrow} P_{A}^{\prime}, & T_{a^{\prime \prime n}} \underset{n \rightarrow \infty}{\rightarrow} P_{A}^{\prime \prime} . \tag{61}
\end{array}
$$

For all $n \in \mathbb{N}^{\star}$, let $\Xi(n) \doteq n T_{b^{n}}(0)$, we have

$$
\begin{align*}
& \eta\left(\beta P_{A}^{\prime}+(1-\beta) P_{A}^{\prime \prime}\right) \\
= & \lim _{n \rightarrow \infty} \frac{1}{n} H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge \Xi(n) T_{a^{\prime} \Xi(n)}(a)+(n-\Xi(n)) T_{a^{\prime \prime n}-\Xi(n)}(a)}\right) \\
\leq & \lim _{n \rightarrow \infty} \frac{\Xi(n)}{n} \frac{1}{\Xi(n)} H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge \Xi(n) T_{\left.a^{\prime \prime( }\right)}(a)}\right) \\
& +\frac{n-\Xi(n)}{n} \frac{1}{n-\Xi(n)} H_{\chi}\left(\bigwedge_{a \in \mathcal{A}} G_{a}^{\wedge(n-\Xi(n)) T_{a^{\prime \prime n}-\Xi(n)}(a)}\right) \tag{65}
\end{align*}
$$

$=\beta \eta\left(P_{A}^{\prime}\right)+(1-\beta) \eta\left(P_{A}^{\prime \prime}\right) ;$
where (63) comes from (59); (65) follows from the subadditivity of $H_{\chi}$; (66) comes from (60), (61) and Lemma11 Since (66) holds for all $P_{A}^{\prime}, P_{A}^{\prime \prime} \in \Delta(\mathcal{A})$ and $\beta \in(0,1)$, we have that $\eta$ is convex.

## Appendix F <br> Proof of Lemma 6

It can be easily observed that

$$
\begin{align*}
& \exists P_{A} \in \operatorname{int}(\Delta(\mathcal{A})), \gamma\left(P_{A}\right)=\sum_{a \in \mathcal{A}} P_{A}(a) \gamma\left(\mathbb{1}_{a}\right)  \tag{67}\\
\Longleftarrow & \forall P_{A} \in \Delta(\mathcal{A}), \gamma\left(P_{A}\right)=\sum_{a \in \mathcal{A}} P_{A}(a) \gamma\left(\mathbb{1}_{a}\right) . \tag{68}
\end{align*}
$$

Now let us prove (67) $\Rightarrow$ (68). Let $P_{A}^{*} \in \operatorname{int} \Delta(\mathcal{A})$ such that $\gamma\left(P_{A}^{*}\right)=\sum_{a \in \mathcal{A}} P_{A}^{*}(a) \gamma\left(\mathbb{1}_{a}\right)$. Let $m: \Delta(\mathcal{A}) \rightarrow \mathbb{R}$ linear such that $m\left(P_{A}^{*}\right)=\gamma\left(P_{A}^{*}\right)$ and $\forall P_{A} \in \Delta(\mathcal{A}), m\left(P_{A}\right) \leq \gamma\left(P_{A}\right)$. We have

$$
\begin{equation*}
0=\gamma\left(P_{A}^{*}\right)-m\left(P_{A}^{*}\right)=\sum_{a \in \mathcal{A}} P_{A}^{*}(a)\left(\gamma\left(\mathbb{1}_{a}\right)-m\left(\mathbb{1}_{a}\right)\right) \tag{69}
\end{equation*}
$$

and therefore $\gamma\left(\mathbb{1}_{a}\right)=m\left(\mathbb{1}_{a}\right)$ for all $a \in \mathcal{A}$, as $\gamma-m \geq 0$ and $P_{A}^{*}(a)>0$ for all $a \in \mathcal{A}$. For all $P_{A} \in \Delta(\mathcal{A})$, we have

$$
\begin{equation*}
f\left(P_{A}\right) \leq \sum_{a \in \mathcal{A}} P_{A}(a) \gamma\left(\mathbb{1}_{a}\right) \tag{70}
\end{equation*}
$$

$$
\begin{equation*}
=\sum_{a \in \mathcal{A}} P_{A}(a) m\left(\mathbb{1}_{a}\right)=m\left(P_{A}\right) \tag{71}
\end{equation*}
$$

hence $\gamma=m$ and $\gamma$ is linear.

## Appendix G

## Proof of Lemma 7

Let $G_{a}=\left(\mathcal{V}_{a}, \mathcal{E}_{a}, P_{V_{a}}\right)$, and $G=\left(\mathcal{V}, \mathcal{E}, P_{V}\right)$ such that $G=\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}$. Let $A$ be the random variable with distribution $P_{A}$ such that $V=V_{A}$, i.e. $P_{V \mid A=a}=P_{V_{a}}$.

## Achievability

For all $a \in \mathcal{A}$, let $W_{a}^{*}$ be a minimizer of

$$
\begin{equation*}
\min _{V_{a} \in W_{a} \in \Gamma\left(G_{a}\right)} I\left(V_{a} ; W_{a}\right) \tag{72}
\end{equation*}
$$

Let $W^{*}$ be the random variable defined as follows: for all $\mathcal{S} \in \Gamma(G), a \in \mathcal{A}$ and $v_{a} \in \mathcal{V}_{a}$,

$$
\begin{equation*}
P_{W^{*} \mid A=a, V=v_{a}}(\mathcal{S}) \doteq P_{W_{a}^{*} \mid V_{a}=v_{a}}\left(\mathcal{S}_{a}\right) \prod_{\substack{a^{\prime} \in \mathcal{A} \\ a^{\prime} \neq a}} P_{W_{a^{\prime}}^{*}}\left(\mathcal{S}_{a^{\prime}}\right) \tag{73}
\end{equation*}
$$

where $\mathcal{S}$ is uniquely decomposed as $\bigsqcup_{a \in \mathcal{A}} \mathcal{S}_{a}$, with $\mathcal{S}_{a} \in$ $\Gamma\left(G_{a}\right)$ for all $a \in \mathcal{A}$. The random variable $W^{*}$ takes its values in $\Gamma(G)$, as

$$
\begin{equation*}
P_{W^{*}}(\mathcal{S})>0 \Longrightarrow \forall a \in \mathcal{A}, P_{W_{a}^{*}}\left(\mathcal{S}_{a}\right)>0 \tag{74}
\end{equation*}
$$

The conditional distribution w.r.t. $(A=a)$ writes:

$$
\begin{align*}
P_{W^{*} \mid A=a}(\mathcal{S}) & =\sum_{v_{a} \in \mathcal{V}_{a}} P_{V_{a}}\left(v_{a}\right) P_{W^{*} \mid A=a, V=v_{a}}(\mathcal{S})  \tag{75}\\
& =\sum_{v_{a} \in \mathcal{V}_{a}} P_{V_{a}}\left(v_{a}\right) P_{W_{a}^{*} \mid V_{a}=v_{a}}\left(\mathcal{S}_{a}\right) \prod_{\substack{a^{\prime} \in \mathcal{A} \\
a^{\prime} \neq a}} P_{W_{a^{\prime}}^{*}}\left(\mathcal{S}_{a^{\prime}}\right) \\
& =\prod_{a^{\prime} \in \mathcal{A}} P_{W_{a^{\prime}}^{*}}\left(\mathcal{S}_{a^{\prime}}\right) \tag{76}
\end{align*}
$$

It follows that the random variable $W^{*}$ is independent of $A$ as the expression (77) does not depend on $a$. Note that $P_{W^{*}}$ defined in (73) is a probability distribution, as

$$
\begin{align*}
\sum_{\mathcal{S} \in \Gamma(G)} P_{W^{*}}(\mathcal{S}) & =\sum_{\mathcal{S} \in \Gamma(G)} \prod_{a^{\prime} \in \mathcal{A}} P_{W_{a^{\prime}}^{*}}\left(\mathcal{S}_{a^{\prime}}\right)  \tag{78}\\
& =\sum_{\left(\mathcal{S}_{a^{\prime}}\right)_{a^{\prime} \in \mathcal{A}} \in \prod_{a^{\prime} \in \mathcal{A}} \Gamma\left(G_{a^{\prime}}\right)} \prod_{a^{\prime} \in \mathcal{A}} P_{W_{a^{\prime}}^{*}}\left(\mathcal{S}_{a^{\prime}}\right) \tag{79}
\end{align*}
$$

$$
\begin{equation*}
=1 \tag{80}
\end{equation*}
$$

where (78) comes from (77); (79) follows from $\Gamma(G)=$ $\left\{\bigsqcup_{a \in \mathcal{A}} \mathcal{S}_{a} \mid \forall a \in \mathcal{A}, \mathcal{S}_{a} \in \Gamma\left(G_{a}\right)\right\} ;$ and 80 holds as $W_{a}^{*}$ takes its values in $\Gamma\left(G_{a}\right)$ for all $a \in \mathcal{A}$.
Now, let us show that $V \in W^{*}$ with probability one. For all $a \in \mathcal{A}$ and $v_{a} \in \mathcal{V}_{a}$,

$$
\begin{equation*}
\left\{\mathcal{S} \cap \mathcal{V}_{a} \mid \mathcal{S} \in \operatorname{supp} P_{W^{*} \mid V=v_{a}}\right\}=\operatorname{supp} P_{W_{a}^{*} \mid V_{a}=v_{a}} \tag{81}
\end{equation*}
$$

where supp denotes the support of a probability distribution. Since $V_{a} \in W_{a}^{*}$ with probability one, all the sets in
$\operatorname{supp} P_{W_{a}^{*} \mid V_{a}=v_{a}}$ contain $v_{a}$, hence all sets in $\operatorname{supp} P_{W^{*} \mid V=v_{a}}$ also contain $v_{a}: V \in W^{*}$ with probability one.

Now let us combine the results on $W^{*}$ :

$$
\begin{align*}
H_{\kappa}(G) & \leq I\left(V ; W^{*}\right)  \tag{82}\\
& =I\left(V, A ; W^{*}\right)  \tag{83}\\
& =I\left(A ; W^{*}\right)+\sum_{a \in \mathcal{A}} P_{A}(a) I\left(V ; W^{*} \mid A=a\right)  \tag{84}\\
& =\sum_{a \in \mathcal{A}} P_{A}(a) I\left(V_{a} ; W_{a}^{*}\right)  \tag{85}\\
& =\sum_{a \in \mathcal{A}} P_{A}(a) H_{\kappa}\left(G_{a}\right) ; \tag{86}
\end{align*}
$$

where (82) holds as $W^{*}$ takes its values in $\Gamma(G)$ and $V \in W^{*}$ with probability one; (83) holds as $A$ is a deterministic function of $V$; 84) comes from the decomposition $I(V, A ; W)=I(V ; W \mid A)+I(A ; W)$; 85) follows from the independence of $A$ and $W^{*}$; (86) comes from the fact that $W_{a}^{*}$ minimizes (72).

## Converse

$$
\begin{align*}
H_{\kappa}\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right) & =\min _{V \in W \in \Gamma(G)} I(V, A ; W)  \tag{87}\\
& \geq \min _{V \in W \in \Gamma(G)} \sum_{a \in \mathcal{A}} P_{A}(a) I(V ; W \mid A=a)  \tag{88}\\
& =\sum_{a \in \mathcal{A}} P_{A}(a) \min _{V \in W \in \Gamma(G)} I(V ; W \mid A=a)  \tag{89}\\
& =\sum_{a \in \mathcal{A}} P_{A}(a) \min _{V_{a} \in W \in \Gamma\left(G_{a}\right)} I\left(V_{a} ; W\right)  \tag{90}\\
& =\sum_{a \in \mathcal{A}} P_{A}(a) H_{\kappa}\left(G_{a}\right) ; \tag{91}
\end{align*}
$$

where (87) holds as $A$ is a deterministic function of $V$; 88) follows from the decomposition $I(V, A ; W)=I(V ; W \mid A)+$ $I(A ; W)$; and 90) holds as $V=V_{A}$.

## Appendix H

## Proof of Lemma 8

$(\Longrightarrow)$ Let $G=\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}$ be a perfect probabilistic graph. Let $a^{\prime} \in \mathcal{A}$ and $\mathcal{S}_{a^{\prime}} \subset \mathcal{V}_{a^{\prime}}$. We have $\chi\left(\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)\left[\mathcal{S}_{a^{\prime}}\right]\right)=$ $\omega\left(\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)\left[\mathcal{S}_{a^{\prime}}\right]\right)$ since $G$ is perfect, and therefore $\chi\left(G_{a^{\prime}}\left[\mathcal{S}_{a^{\prime}}\right]\right)=\omega\left(G_{a^{\prime}}\left[\mathcal{S}_{a^{\prime}}\right]\right)$, as $\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)\left[\mathcal{S}_{a^{\prime}}\right]=G_{a^{\prime}}\left[\mathcal{S}_{a^{\prime}}\right]$. Thus all the graphs $\left(G_{a}\right)_{a \in \mathcal{A}}$ are perfect.
$(\Longleftarrow)$ Conversely, assume that for all $a \in \mathcal{A}, G_{a}=$ $\left(\mathcal{V}_{a}, \mathcal{E}_{a}, P_{V_{a}}\right)$ is perfect. Then for all $\mathcal{S} \subset \bigsqcup_{a \in \mathcal{A}} \mathcal{V}_{a}, \mathcal{S}$ can be written as $\bigsqcup_{a \in \mathcal{A}} \mathcal{S}_{a}$ where $\mathcal{S}_{a} \subset \mathcal{V}_{a}$ for all $a \in \mathcal{A}$, and we have for all $P_{A} \in \Delta(\mathcal{A})$ :

$$
\begin{align*}
\chi\left(\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)[\mathcal{S}]\right) & =\chi\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\left[\mathcal{S}_{a}\right]\right)  \tag{92}\\
& =\max _{a \in \mathcal{A}} \chi\left(G_{a}\left[\mathcal{S}_{a}\right]\right)  \tag{93}\\
& =\max _{a \in \mathcal{A}} \omega\left(G_{a}\left[\mathcal{S}_{a}\right]\right) \tag{94}
\end{align*}
$$

and similarly, $\omega\left(\left(\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}\right)[\mathcal{S}]\right)=\max _{a \in \mathcal{A}} \omega\left(G_{a}\left[\mathcal{S}_{a}\right]\right)$. Hence $\bigsqcup_{a \in \mathcal{A}}^{P_{A}} G_{a}$ is also perfect.

## Appendix I

## Proof of Lemma 9

Let $c^{*}: \mathcal{V} \rightarrow \mathcal{C}$ and $c_{\mathcal{S}}^{*}: \mathcal{S} \rightarrow \mathcal{C}$ be the optimal colorings of $G$ and $G[\mathcal{S}]$, respectively. Consider the coloring $c: \mathcal{V} \rightarrow \mathcal{C} \sqcup \mathcal{V}$ of $G$ defined by $c(v)=c_{\mathcal{S}}^{*}$ if $v \in \mathcal{S}, c(v)=v$ otherwise.
(Lower bound) On one hand, we have

$$
\begin{align*}
H_{\chi}(G) \leq & H\left(c(V), \mathbb{1}_{V \in \mathcal{S}}\right)  \tag{95}\\
= & H\left(\mathbb{1}_{V \in \mathcal{S}}\right)+P_{V}(\mathcal{S}) H(c(V) \mid V \in \mathcal{S}) \\
& +\left(1-P_{V}(\mathcal{S})\right) H(c(V) \mid V \notin \mathcal{S})  \tag{96}\\
\leq & 1+H\left(c_{\mathcal{S}}^{*}(V) \mid V \in \mathcal{S}\right)+\left(1-P_{V}(\mathcal{S})\right) \log |\mathcal{V}|  \tag{97}\\
= & H_{\chi}(G[\mathcal{S}])+1+\left(1-P_{V}(\mathcal{S})\right) \log |\mathcal{V}| \tag{98}
\end{align*}
$$

where (95) comes from the fact that $c$ is a coloring of $G$; (96) is a decomposition using conditional entropies; (97) comes from the construction of $c:\left.c\right|_{\mathcal{S}}=c_{\mathcal{S}}^{*}$; (98) follows from the optimality of $c_{\mathcal{S}}^{*}$ as a coloring of $G[\mathcal{S}]$.
(Upper bound) On the other hand,

$$
\begin{align*}
& H_{\chi}(G[\mathcal{S}]) \\
\leq & H\left(c^{*}(V) \mid V \in \mathcal{S}\right) \\
= & \frac{1}{P_{V}(\mathcal{S})}\left(H\left(c^{*}(V) \mid \mathbb{1}_{V \in \mathcal{S}}\right)-\left(1-P_{V}(\mathcal{S})\right) H\left(c^{*}(V) \mid V \notin \mathcal{S}\right)\right) \tag{100}
\end{align*}
$$

$\leq \frac{H\left(c^{*}(V)\right)}{P_{V}(\mathcal{S})}=\frac{H_{\chi}(G)}{P_{V}(\mathcal{S})}$
where (99) comes from the fact that $c^{*}$ induces a coloring of $G[\mathcal{S}]$; 100) is a decomposition using conditional entropies; (101) results from the elimination of negative terms and the optimality of $c^{*}$.

## Appendix J

## Proof of Lemma 10

Let $\left(\bar{a}_{n}\right)_{n \in \mathbb{N}^{\star}} \in \mathcal{A}^{\mathbb{N}^{\star}}$ be a sequence such that $T_{\bar{a}^{n}} \rightarrow P_{A}=$ $\beta P_{A}^{\prime}+(1-\beta) P_{A}^{\prime \prime}$ when $n \rightarrow \infty$.
Consider a sequence $\left(B_{n}\right)_{n \in \mathbb{N} \star}$ of independent Bernoulli random variables such that for all $n \in \mathbb{N}^{\star}$,

$$
\begin{equation*}
\operatorname{Pr}\left(B_{n}=0\right)=\frac{\beta P_{A}^{\prime}\left(\bar{a}_{n}\right)}{P_{A}\left(\bar{a}_{n}\right)} \tag{102}
\end{equation*}
$$

By the strong law of large numbers,

$$
\begin{equation*}
\operatorname{Pr}\left(T_{B^{n}, \bar{a}^{n}} \underset{n \rightarrow \infty}{\rightarrow}\left(\beta P_{A}^{\prime},(1-\beta) P_{A}^{\prime \prime}\right)\right)=1 . \tag{103}
\end{equation*}
$$

Therefore, there exists at least one realization $\left(b_{n}\right)_{n \in \mathbb{N}^{*}}$ of $\left(B_{n}\right)_{n \in \mathbb{N}^{\star}}$ such that $T_{b^{n}, \bar{a}^{n}}$ converges to $\left(\beta P_{A}^{\prime},(1-\beta) P_{A}^{\prime \prime}\right)$. The convergences of marginal and conditional types yield

$$
\begin{align*}
& T_{b^{n}} \underset{n \rightarrow \infty}{\rightarrow}(\beta, 1-\beta),  \tag{104}\\
& T_{a^{\prime n}} \underset{n \rightarrow \infty}{\rightarrow} P_{A}^{\prime}, \quad T_{a^{\prime \prime n}} \underset{n \rightarrow \infty}{\rightarrow} P_{A}^{\prime \prime}, \tag{105}
\end{align*}
$$

where $a^{\prime} \doteq\left(\bar{a}_{n}\right)_{\substack{n \in \mathbb{N}^{\star} \\ b_{n}=0}}$, and $a^{\prime \prime} \doteq\left(\bar{a}_{n}\right)_{\substack{n \in \mathbb{N}^{\star} \\ b_{n}=1}}$, are the extracted
sequences.

