Non-deterministic Multi-valued Matrices for First-order Logics of Formal Inconsistency

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Abstract

Paraconsistent logic is the study of contradictory yet non-trivial theories. One of the best-known approaches to designing useful paraconsistent logics is da Costa's approach, which has led to the family of Logics of Formal Inconsistency (LFIs), where the notion of inconsistency is expressed at the object level. In this paper we use non-deterministic matrices, *a generalization of standard multi-valued matrices, to provide simple and modular finite-valued semantics for a large family of first-order LFIs. The modular approach provides new insights into the semantic role of each of the studied axioms and the dependencies between them. We also prove the* effectiveness of our semantics, *a crucial property for constructing counterexamples, and apply it to show a non-trivial proof-theoretical property of the studied LFIs.*

1. Introduction

In classical logic any proposition can be inferred from an inconsistent set of assumptions. Thus classical logic fails to capture that information systems which contain some inconsistent information may still produce useful answers. For such cases one needs a *paraconsistent logic* ([5, 6]), which is a logic that allows contradictory yet non-trivial theories. There are several approaches to the problem of designing a useful paraconsistent logic. One of the best known is da Costa's approach ([9, 7]), which has led to the family of *Logics of Formal Inconsistency* (LFIs). This family is based on two main ideas. First of all, propositions are divided into two sorts: the "normal" (or "consistent") and the "abnormal" (or "inconsistent") ones. The second idea is to express the meta-theoretical notions of consistency/inconsistency at the object language level, by adding to the language a new connective $\bullet$, with the intended meaning of $\bullet\varphi$ being "φ is inconsistent". (Sometimes the dual connective $\circ$, expressing consistency is used, see e.g. [8]). Using the inconsistency operator, one can limit the applicability of the rule $\varphi, \neg\varphi \vdash \psi$ (which amounts to "a single contradiction entails everything" and leads to trivialization in case of contradictions in classical logic) to the case when φ is consistent (i.e., $\varphi, \neg\varphi, \neg\bullet\varphi \vdash \psi$).

Although the syntactic formulations of LFIs are relatively simple, already on the propositional level the problem of finding semantic interpretations for them is rather complicated: the vast majority of LFIs cannot be characterized by means of finite multi-valued matrices. Moreover, for the majority of them no useful infinite-valued matrices are known. Thus other types of semantics, like bivaluations semantics and possible translations semantics have been proposed ([7]). However, it is not clear how to extend these types of semantics to the first-order level.

An alternative framework for providing semantics for propositional paraconsistent logics was used in [2, 1]. This framework is based on a generalization of the standard multi-valued matrices, called *non-deterministic matrices* (Nmatrices). Nmatrices are multi-valued structures, in which the value assigned by a valuation to a complex formula can be chosen *non-deterministically* out of a certain nonempty set of options. The framework of Nmatrices has a number of attractive properties. First of all, the semantics provided by Nmatrices is *modular*: the main effect of each of the rules of a proof system is reducing the degree of non-determinism of operations, by forbidding some options. The semantics of a proof system is obtained by combining the semantic constraints imposed by its rules in a rather straightforward way. Secondly, this semantics is *effective*[1], i.e. any partial valuation closed under subformulas can be extended to a full valuation. This property is crucial for the usefulness of semantics, in particular for constructing counterexamples.

The main goal of this paper is to extend the modular se-

[1] No general theorem of effectiveness is available for the semantics of bivaluations or for possible translations semantics described in [7] and has to be proven from scratch for any instance of these types of semantics.

mantic framework of Nmatrices to the first-order level. The first steps in this direction were taken in [3] for a very restricted family of paraconsistent logics (no consistency operators), and in [10, 4], where finite non-deterministic semantics was provided for a family of LFIs with the consistency operator $\circ$. In this paper we study the semantic effects of 18 new axioms, capturing de Morgan principles and inconsistency propagation both for the propositional connectives and quantifiers. We provide five-valued non-deterministic semantics (which are reduced to four and three values in some cases) for a large family of first-order LFIs using the inconsistency operator $\bullet$[2]. This family includes the system LFI1*, designed in [7] for treating inconsistent information in evolutionary databases. It is one of the few LFIs which can be characterized by a deterministic three-valued matrix. We will see that the matrix given in [7] coincides with the characteristic Nmatrix defined for it in this paper. The modularity of the semantic framework, preserved on the first-order level, provides new insights into the semantic effect of each of the axioms the dependencies between them. For instance, we show that four of the schemata in the axiomatization of LFI1* of [7] are derivable from the rest of its axioms.

One of the well-known properties of LFIs is their lack of the principle of intersubstitutability of provable equivalents (IPE), which holds in classical logic. In a system $\mathbf{S}$, in which the IPE principle holds, two equivalent sentences are logically indistinguishable, i.e. the provability of $A \leftrightarrow B$ in $\mathbf{S}$ entails the provability of $\psi(A) \leftrightarrow \psi(B)$ in $\mathbf{S}$ for any ψ. Unfortunately, this principle does not hold for any of the LFIs in this paper: already on the propositional level one cannot infer $\neg(A \wedge B) \leftrightarrow \neg(B \wedge A)$ from $(A \wedge B) \leftrightarrow (B \wedge A)$ in these systems. This abnormality becomes really harmful on the first-order level. Even the α-conversion principle[3] does not hold: although $\forall x p(x) \leftrightarrow \forall y p(y)$ is provable in the first-order LFIs discussed in this paper, $\neg\forall x p(x) \leftrightarrow \neg\forall y p(y)$ is not, which is of course unacceptable in any reasonable logical system. A similar problem arises in the case of vacuous quantification: the provability of $\forall x \exists y p(x) \leftrightarrow \forall x p(x)$ in these systems does not imply the provability of $\neg(\forall x \exists y p(x)) \leftrightarrow \neg(\forall x p(x))$. The straightforward solution proposed by da Costa ([9]) for the last two problems is adding an extra-postulate capturing these principles. In a similar way, one can use other natural extra-postulates, e.g. for capturing the commutativity of $\wedge$ (i.e., $A \wedge B$ and $B \wedge A$ are intersubstitutable).

The second goal of this paper is to formalize these ideas by incorporating the extra-postulates into the semantic framework of Nmatrices in a modular way, so that new postulates can easily be added in accordance to the intended applications of the system. As a case-study we consider seven basic extra-postulates, including α-conversion, vacuous quantification, commutativity and idempotency of $\wedge$ and $\vee$. Incorporating these postulates complicates the semantics, and as a result their effectiveness is less evident. Nevertheless, we formulate necessary and sufficient conditions for the effectiveness of the semantics for each of the extra-postulates, and show that all of the semantics defined in this paper are effective. Finally, we apply their effectiveness to prove a non-trivial proof-theoretical property of the first-order LFIs in this paper.

[2] We chose to use the $\bullet$ operator for technical reasons, in paricular to be able to prove corollary 11 in the sequel.

[3] The principle identifies syntactic objects differing only in the names of their bound variables.

2 Preliminaries

2.1. A taxonomy of first-order LFIs

In what follows, L is a first-order language. We denote by Frm_L the set of wffs of L, by Frm_L^{cl} the set of L-sentences. $\mathcal{L}_{\mathrm{C}}$ is a first-order language over $\{\bullet, \neg, \wedge, \vee, \supset, \forall, \exists\}$.

Definition 1 *Let $\mathbf{HCL}^+$ be some Hilbert-type system which has Modus Ponens as the only inference rule, and is sound and strongly complete for the positive fragment of classical propositional logic. The first-order system $\mathbf{HCL}^+_{FOL}$ is obtained from $\mathbf{HCL}^+$ by adding the axioms $\forall x\psi \supset \psi\{t/x\}$ and $\psi\{t/x\} \supset \exists x\psi$, where t is any term free for x in ψ, and the inference rules $\frac{(\varphi \supset \psi)}{(\varphi \supset \forall x\psi)}$ and $\frac{(\psi \supset \varphi)}{(\exists x\psi \supset \varphi)}$, where t is free for x in ψ and $x \notin Fv[\varphi]$. The system $\mathbf{QB}$[4] is obtained from $\mathbf{HCL}^+_{FOL}$ by adding the schemata* **(t)** $\neg\varphi \vee \varphi$ *and* **(b)** $\neg \bullet \varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi)$.

We obtain a large family of first-order LFIs by adding to the basic system $\mathbf{QB}$ different combinations of the following schemata:

Definition 2 *The set Ax consists of*[5]*:*

$(\mathbf{c})\ \neg\neg\varphi \supset \varphi$ $(\mathbf{e})\ \varphi \supset \neg\neg\varphi$ $(\mathbf{i_1})\ \bullet\varphi \supset \varphi$ $(\mathbf{i_2})\ \bullet\varphi \supset \neg\varphi$
$(\mathbf{Dm^1_\forall})\ \neg\forall x\psi \supset \exists x\neg\psi$ $(\mathbf{Dm^2_\forall})\ \exists x\neg\psi \supset \neg\forall x\psi$
$(\mathbf{Dm^1_\exists})\ \neg\exists x\psi \supset \forall x\neg\psi$ $(\mathbf{Dm^2_\exists})\ \forall x\neg\psi \supset \neg\exists x\psi$
$(\mathbf{Dm^1_\wedge})\ \neg(\psi \wedge \varphi) \supset (\neg\psi \vee \neg\varphi)$
$(\mathbf{Dm^2_\wedge})\ (\neg\psi \vee \neg\varphi) \supset \neg(\psi \wedge \varphi)$
$(\mathbf{Dm^1_\vee})\ \neg(\psi \vee \varphi) \supset (\neg\psi \wedge \neg\varphi)$
$(\mathbf{Dm^2_\vee})\ (\neg\psi \wedge \neg\varphi) \supset \neg(\psi \vee \varphi)$
$(\mathbf{J^1_\wedge})\ \bullet(\psi \wedge \varphi) \supset ((\bullet\psi \wedge \varphi) \vee (\bullet\varphi \wedge \psi))$
$(\mathbf{J^2_\wedge})\ ((\bullet\psi \wedge \varphi) \vee (\bullet\varphi \wedge \psi)) \supset (\bullet(\psi \wedge \varphi))$
$(\mathbf{J^1_\vee})\ \bullet(\psi \vee \varphi) \supset ((\bullet\psi \wedge \neg\varphi) \vee (\bullet\varphi \wedge \neg\psi))$

[4] In [4] the name **QB** is used for a slightly different first-order system.

[5] The schemata **(c)**, **(e)**, **(i$_1$)** and **(i$_2$)** were used in [1, 10] (with the dual operator $\circ$). The rest are studied in context of Nmatrices for the first time. The letters **Dm** stand for 'De Morgan'. The names of the axioms are taken from [7].

$(\mathbf{J^2_\vee})$ $((\bullet\psi \wedge \neg\varphi) \vee (\bullet\varphi \wedge \neg\psi)) \supset \bullet(\psi \vee \varphi)$
$(\mathbf{J^1_\supset})$ $\bullet(\psi \supset \varphi) \supset (\psi \wedge \bullet\varphi)$
$(\mathbf{J^2_\supset})$ $(\psi \wedge \bullet\varphi) \supset \bullet(\psi \supset \varphi)$
$(\mathbf{J^1_\forall})$: $\bullet\forall x\psi \supset (\exists x \bullet \psi \wedge \forall x\psi)$
$(\mathbf{J^2_\forall})$: $(\exists x \bullet \psi \wedge \forall x\psi) \supset \bullet\forall x\psi$
$(\mathbf{J^1_\exists})$: $\bullet\exists x\psi \supset (\exists x \bullet \psi \wedge \forall x\neg\psi)$
$(\mathbf{J^2_\exists})$: $(\exists x \bullet \psi \wedge \forall x\neg\psi) \supset \bullet\exists x\psi$
For $X \subseteq \mathbf{Ax}$, the system $\mathbf{QB}[X]$ is obtained from $\mathbf{QB}$ by adding the schemata in X.

Notation: We denote $\mathbf{QB}[X]$ by $\mathbf{QB}s$, where s is a string consisting of the names of the schemata in X (thus we write $\mathbf{QBce}$ rather than $\mathbf{QB}[\{(\mathbf{c}), (\mathbf{e})\}]$. If both $(\mathbf{i_1})$ and $(\mathbf{i_2})$ are in X we abbreviate it by $\mathbf{i}$. Also, if $\mathbf{x^i_y}$ is in X for every $\mathbf{y} \in \{\supset, \wedge, \vee, \forall, \exists\}$, $\mathbf{i} \in \{1, 2\}$ and some $\mathbf{x} \in \{\mathbf{J}, \mathbf{Dm}\}$, we simply write $\mathbf{x}$.
Remark: $\mathbf{QBcieJDm_\forall Dm_\exists}$ is the first-order system LFI1* designed in [7] for handling evolutionary databases.

2.2. Nmatrices for first-order languages

Our main semantic tool is the following generalization of a multi-valued matrix ([2, 1, 3, 10]):

Definition 3 (Non-deterministic matrix) *A non-deterministic matrix (Nmatrix) for a language L is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$ and $\mathcal{O}$ includes the following interpretation functions: (i) $\tilde{\diamond}_\mathcal{M} : \mathcal{V}^n \to P^+(\mathcal{V})$ for every n-ary connective $\diamond$, and (ii) $\tilde{Q}_\mathcal{M} : P^+(\mathcal{V}) \to P^+(\mathcal{V})$ for every quantifier Q.*

The notion of an *L-structure* for an Nmatrix is defined standardly (see, e.g. [3]). For an L-structure $S = \langle D, I\rangle$, $L(D)$ is the language obtained from L by adding to it the set of *individual constants* $\{\overline{a} \mid a \in D\}$. I is extended to $L(D)$ as follows: $I[\overline{a}] = a$.
Given an L-structure S, define the relation $\sim^S$ on $L(D)$-sentences: $\psi \sim^S \psi'$ if ψ can be obtained from ψ by any number of replacements of a closed term $\mathbf{t}$ for a closed term $\mathbf{t}'$, such that $I[\mathbf{t}] = I[\mathbf{t}']$.

Definition 4 (S-valuation) *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. An S-valuation $v : Frm^{\mathsf{cl}}_{L(D)} \to \mathcal{V}$ is* legal in $\mathcal{M}$ *if it satisfies: (i) if $\psi \sim^S \psi'$, then $v[\psi] = v[\psi']$, (ii) $v[p(t_1, ..., t_n)] = I[p][I[t_1], ..., I[t_n]]$, (iii) $v[\diamond(\psi_1, ..., \psi_n)] \in \tilde{\diamond}_\mathcal{M}[v[\psi_1], ..., v[\psi_n]]$, and (iv) $v[Qx\psi] \in \tilde{Q}_\mathcal{M}[\{v[\psi\{\overline{a}/x\}] \mid a \in D\}]$ for $Q \in \{\forall, \exists\}$.*

Definition 5 (Semantics) *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. An $\mathcal{M}$-legal S-valuation v is* a model of a formula ψ in $\mathcal{M}$, *denoted by $S, v \models_\mathcal{M} \psi$, if $v[\psi'] \in \mathcal{D}$ for every closed instance ψ' of ψ in $L(D)$. A formula ψ is $\mathcal{M}$-valid in S if for every $\mathcal{M}$-legal S-valuation v, $S, v \models_\mathcal{M} \psi$. ψ is $\mathcal{M}$-valid if ψ is $\mathcal{M}$-valid in every L-structure for $\mathcal{M}$. The consequence relation $\vdash_\mathcal{M}$ between sets of L-formulas and L-formulas is defined as follows: $\Gamma \vdash_\mathcal{M} \psi$ if for every L-structure S and every $\mathcal{M}$-legal S-valuation v: $S, v \models_\mathcal{M} \Gamma$ implies that $S, v \models_\mathcal{M} \psi$.*

Definition 6 (Refinement) *Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1\rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2\rangle$ be Nmatrices for L. $\mathcal{M}_2$ is a* refinement *of $\mathcal{M}_1$ if $\mathcal{V}_2 \subseteq \mathcal{V}_1$, $\mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{V}_2$, $\tilde{\diamond}_{\mathcal{M}_2}[a_1, ..., a_n] \subseteq \tilde{\diamond}_{\mathcal{M}_1}[a_1, ...a_n]$ for every n-ary connective $\diamond$ of L and every $a_1, ..., a_n \in \mathcal{V}_2$ and $\tilde{Q}_{\mathcal{M}_2}[H] \subseteq \tilde{Q}_{\mathcal{M}_1}[H]$ for every quantifier Q of L and every $H \subseteq \mathcal{V}_2$.*

3 Non-deterministic semantics for LFIs

In this section we provide non-deterministic semantics for the first-order LFIs obtained from the basic system $\mathbf{QB}$ by adding various combinations of schemata from Ax. The results in this section are an extension and generalization of the results of [1, 10].
The system $\mathbf{QB}$ treats the connectives $\wedge, \vee, \supset$ and the quantifiers $\forall, \exists$ similarly to classical logic. The treatment of $\bullet$ and $\neg$ is different: intuitively, the truth/falsity of $\neg\psi$ or $\bullet\psi$ is not completely determined by the truth/falsity of ψ. More data is needed for it. The central idea is to include all the relevant data concerning a sentence ψ in the truth-value from $\mathcal{V}$ which is assigned to ψ. In our case the relevant data beyond the truth/falsity of ψ is the truth/falsity of $\neg\psi$ and of $\bullet\psi$. This leads to the use of elements from $\{0, 1\}^3$ as truth-values, where the intended meaning of $v[\psi] = \langle x, y, z\rangle$ is as follows: $x = 1$ iff $v[\psi] \in \mathcal{D}$, $y = 1$ iff $v[\neg\psi] \in \mathcal{D}$ and $z = 1$ iff $v[\bullet\psi] \in \mathcal{D}$. Note that because of the schema $(\mathbf{t})$, not all tuples can be used as legal truth values. The schema $(\mathbf{t})$ means that at least one of $\varphi, \neg\varphi$ must be true. Thus, the truth values $\langle 0, 0, 0\rangle$ and $\langle 0, 0, 1\rangle$ are rejected. The schema $(\mathbf{b})$ means that if φ and $\neg\varphi$ are true, then $\neg\bullet\varphi$ must be false. Since for every $v \in \tilde{\bullet}[\langle 1, 1, 0\rangle]$, $v = \langle 0, x, y\rangle$ (recall that the third element specifies the truth/falsity of $\bullet\psi$), it means that x must be 0, which yields an illegal truth value, and thus $\langle 1, 1, 0\rangle$ is also rejected. We are left with the following five truth values: $f = \langle 0, 1, 0\rangle$, $f_I = \langle 0, 1, 1\rangle$, $t = \langle 1, 0, 0\rangle$, $t_I = \langle 1, 0, 1\rangle$, $I = \langle 1, 1, 1\rangle$.

Definition 7 *The Nmatrix $\mathcal{QM}_5 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ is defined as follows: $\mathcal{V}_5 = \{t, f, I, t_I, f_I\}$, $\mathcal{D} = \{t, t_I, I\}$ and $\mathcal{F} = \mathcal{V} - \mathcal{D}$. The operations in $\mathcal{O}$ are defined as follows:*

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases} \qquad a \tilde{\supset} b = \begin{cases} \mathcal{D} & a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D} & \text{otherwise} \end{cases} \qquad \tilde{\neg} a = \begin{cases} \mathcal{F} & a \in \{t, t_I\} \\ \mathcal{D} & a \in \{f, f_I, I\} \end{cases}$$

$$\tilde{\bullet} a = \begin{cases} \mathcal{F} & a \in \{t, f\} \\ \mathcal{D} & a \in \{t_I, f_I\} \\ \{t, t_I\} & a = I \end{cases}$$

$$\tilde{\forall}[H] = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases} \quad \tilde{\exists}[H] = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

Note that the definition of $\tilde{\bullet}_{\mathcal{QM}_5}$ is a direct consequence of the schema **(b)**, according to which $\neg \bullet \varphi$, φ and $\neg\varphi$ cannot all be true at the same time. This is guaranteed by the condition $\tilde{\bullet}[I] \in \{t_I, t\}$.

Theorem 8 *For a set $\mathcal{L}_C$-formulas $\Gamma \cup \{\psi\}$: $\Gamma \vdash_{\mathcal{QM}_5} \psi$ iff $\Gamma \vdash_{\mathbf{QB}} \psi$.*

The proof involves a rather standard Henkin construction and is omitted here.

Next we study the semantic effects of extending our basic system $\mathbf{QB}$ with the schemata from Ax. The obtained semantics is modular: the addition of a schema leads to a certain refinement of the basic Nmatrix $\mathcal{QM}_5$, and the semantics of a system is obtained by simply combining all relevant refinements. The refining conditions for the schemata from Ax are:

Definition 9 $\mathbf{Cond(c)}$*: $a \in \{f, f_I\} \Rightarrow \tilde{\neg}[a] \subseteq \{t, t_I\}$.*
$\mathbf{Cond(e)}$ $\tilde{\neg}[I] \subseteq \{I\}$
$\mathbf{Cond(i_1)}$*: delete f_I;* $\mathbf{Cond(i_2)}$*: delete t_I.*
$\mathbf{Cond(Dm^1_\wedge)}$*: if $a, b \in \{t, t_I\}$, then $\tilde{\wedge}[a, b] \subseteq \{t, t_I\}$.*
$\mathbf{Cond(Dm^2_\wedge)}$*: if a or b are in $\{f, f_I, I\}$, then $\tilde{\wedge}[a, b] \subseteq \{f, f_I, I\}$*
$\mathbf{Cond(Dm^1_\vee)}$*: if a or b are in $\{t, t_I\}$, then $\tilde{\vee}[a, b] \subseteq \{t, t_I\}$.*
$\mathbf{Cond(Dm^2_\vee)}$*: if $a, b \in \{f, f_I, I\}$, then $\tilde{\vee}[a, b] \subseteq \{f, f_I, I\}$*
$\mathbf{Cond(Dm^1_\forall)}$*: if $H \cap \{f, f_I, I\} = \emptyset$, then $\tilde{\forall}[H] \subseteq \{t, t_I\}$.*
$\mathbf{Cond(Dm^1_\exists)}$*: if $H \cap \{t, t_I\} \neq \emptyset$, then $\tilde{\exists}[H] \subseteq \{t, t_I\}$.*
$\mathbf{Cond(Dm^2_\exists)}$*: if $H \cap \{t, t_I\} = \emptyset$, then $\tilde{\exists}[H] \subseteq \{f, f_I, I\}$.*
$\mathbf{Cond(J^1_\wedge)}$*: if $[a \in \{t, f\}$ or $b \in \{f, f_I\}]$ and $[b \in \{t, f\}$ or $a \in \{f, f_I\}]$, then $\tilde{\wedge}[a, b] \subseteq \{t, f\}$.*
$\mathbf{Cond(J^2_\wedge)}$*: if $[a \in \{t_I, f_I, I\}$ and $b \in \{t, I, t_I\}]$ or $[b \in \{t_I, f_I, I\}$ and $a \in \{t, I, t_I\}]$, then $\tilde{\wedge}[a, b] \subseteq \{I, f_I, t_I\}$.*
$\mathbf{Cond(J^1_\vee)}$*: if $[a \in \{t, f\}$ or $b \in \{t, t_I\}]$ and $[b \in \{t, f\}$ or $a \in \{t, t_I\}]$, then $\tilde{\vee}[a, b] \subseteq \{t, f\}$.*
$\mathbf{Cond(J^2_\vee)}$*: if $[a \in \{t_I, I, f_I\}$ and $b \in \{f, f_I, I\}]$ or $[b \in \{t_I, I, f_I\}$ and $a \in \{f, f_I, I\}]$, then $\tilde{\vee}[a, b] \subseteq \{t_I, f_I, I\}$.*
$\mathbf{Cond(J^1_\supset)}$*: if $a \in \{t, f\}$ or $b \in \{f, f_I\}$, then $\tilde{\supset}[b, a] \subseteq \{t, f\}$.*
$\mathbf{Cond(J^2_\supset)}$*: if $a \in \{t, t_I, I\}$ and $b \in \{f_I, t_I, I\}$, then $\tilde{\supset}[b, a] \subseteq \{I, t_I, f_I\}$.*
$\mathbf{Cond(J^1_\forall)}$*: if $H \subseteq \{t, f\}$ or $H \cap \{f, f_I\} \neq \emptyset$, then $\tilde{\forall}[H] \subseteq \{t, f\}$.*
$\mathbf{Cond(J^2_\forall)}$*: for $H \subseteq \{t, t_I, I\}$, such that $t_I \in H$ or $I \in H$: $\tilde{\forall}[H] \subseteq \{I, f_I, t_I\}$.*
$\mathbf{Cond(J^1_\exists)}$*: if $H \cap \{t, t_I\} \neq \emptyset$ or $H \subseteq \{t, f\}$, then $\tilde{\exists}[H] \subseteq \{t, f\}$.*
$\mathbf{Cond(J^2_\exists)}$*: if $H \subseteq \{I, f, f_I\}$ and $\{I, t_I, f_I\} \cap H \neq \emptyset$, then $\tilde{\exists}[H] \subseteq \{I, t_I, f_I\}$.*

For $X \subseteq Ax$, $\mathcal{QM}_5[X]$ is the weakest refinement of $\mathcal{QM}_5$ which satisfies the refining conditions of the schemata from X.

It is easy to see that for every $\mathbf{X} \subseteq Ax$ the conditions in $\mathbf{X}$ are coherent, the interpretations of the connectives and quantifiers in $\mathcal{QM}_5[\mathbf{X}]$ are not empty and so $\mathcal{QM}_5[\mathbf{X}]$ is well-defined.

Example 1: The interpretations of $\neg, \bullet$ in $\mathcal{QM}_5\mathbf{c}$ are:

	$\mathbf{f}$	$\mathbf{f_I}$	$\mathbf{I}$	$\mathbf{t}$	$\mathbf{t_I}$
$\tilde{\neg}$	$\{t, t_I\}$	$\{t, t_I\}$	$\{I, t, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$
$\tilde{\bullet}$	$\{t, t_I, I\}$	$\{t, t_I, I\}$	$\{t, t_I\}$	$\{f, f_I\}$	$\{t, t_I, I\}$

Example 2: The interpretations of $\forall$ and $\exists$ in $\mathcal{QM}_5\mathbf{iJ^1_\forall J^1_\exists}$ are [6]:

$\mathbf{H}$	$\tilde{\forall}[\mathbf{H}]$	$\tilde{\exists}[\mathbf{H}]$
$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t, I\}$	$\{t, I\}$
$\{t, f\}$	$\{f\}$	$\{t\}$
$\{t, I\}$	$\{t, I\}$	$\{t, I\}$
$\{f, I\}$	$\{f\}$	$\{t\}$
$\{t, f, I\}$	$\{f\}$	$\{t\}$

Example 3: In [7] it is shown that LFI1* can be characterized by a deterministic three-valued matrix. Note that the Nmatrix $\mathcal{QM}_5\mathbf{cieJDm_\forall Dm_\exists}$ is indeed completely deterministic and matches the three-valued semantics of [7] (where the truth-values $0, \frac{1}{2}, 1$ are used instead of f, I, t respectively).

Theorem 10 *For a set of $\mathcal{L}_C$-formulas $\Gamma \cup \{\psi\}$ and some $X \subseteq Ax$: $\Gamma \vdash_{\mathbf{QB}[X]} \psi$ iff $\Gamma \vdash_{\mathcal{QM}_5[X]} \psi$.*

The proof is quite similar to the proof of Thm. 29 in [4] and is omitted here.

The modular approach of Nmatrices provides some important insights into the semantic role of each of the above schemata. For instance, it is easy to see that for every $\mathbf{x} \in \{\wedge, \vee, \forall, \exists\}$ and $\mathbf{j} \in \{1, 2\}$, the semantic effects of the conditions $\mathbf{Cond(J^j_x)}$ and $\mathbf{Cond(Dm^j_x)}$ on $\mathcal{QM}_5$ differ only in their behavior for the truth-values t_I and f_I. In the presence of **(i**$_1$**)** and **(i**$_2$**)**, t_I and f_I are deleted and their semantic effects on $\mathcal{QM}_5\mathbf{i}$ coincide. This leads to the following observation:

Corollary 11 *For any $\mathbf{x} \in \{\wedge, \vee, \forall, \exists\}$ and $\mathbf{i} \in \{1, 2\}$: $\vdash_{\mathbf{QBiJ^i_x}} \mathbf{Dm^i_x}$ and $\vdash_{\mathbf{QBiDm^i_x}} \mathbf{J^i_x}$.*

Thus $\mathbf{Dm^i_x}$ and $\mathbf{J^i_x}$ are equivalent in any extension of $\mathbf{QBi}$, and so the axioms $\mathbf{Dm^1_Q}$ and $\mathbf{Dm^2_Q}$ for $\mathbf{Q} \in \{\forall, \exists\}$ in the axiomatization of LFI1* of [8] are derivable from the rest of the axioms.

[6] By **Cond(i)**, f_I and t_I are deleted and we are left with only three truth-values: t, f, I.

4 Extra-postulates and effectiveness

The IPE principle does not hold for the family of LFIs discussed in this paper, i.e. two equivalent sentences are not necessarily logically indistinguishable[7]. For instance, from $\forall xp(x) \leftrightarrow \forall yp(y)$ one cannot infer $\neg\forall xp(x) \leftrightarrow \neg\forall yp(y)$ in $\mathbf{QB}$ and so the α-conversion principle does not hold. A similar situation can be observed for vacuous quantification and the (less evident) principles of commutativity and idempotency of $\wedge$ and $\vee$. da Costa's straightforward solution (to the first two problems, see e.g. [9]) is adding explicit extra-postulates to capture the desired principles. We extend this idea to the rest of the principles as follows.

Definition 12 *For a language L, the set $\mathbf{CNG}_L$ includes the following binary relations over Frm_L. For $\diamond \in \{\wedge, \vee\}$: (i) $R_L^{C_\diamond} = \{\langle (A \diamond B), (B \diamond A)\rangle \mid A, B \in Frm_L\}$ and (ii) $R_L^{I_\diamond} = \{\langle (A \diamond A), A\rangle \mid A \in Frm_L\}$. For $Q \in \{\forall, \exists\}$: (iii) $R_L^{\alpha} = \{\langle A, B\rangle \mid A \equiv_\alpha B, A, B \in Frm_L\}$, and (iv) $R_L^{v_Q} = \{\langle QxA, A\rangle \mid x \notin Fv(A), A \in Frm_L\}$.*
For $Z \subseteq \mathbf{CNG}_L$, R_Z is the minimal congruence relation on Frm_L, such that for every $R \in Z$: $R \subseteq \mathsf{R}_Z$.

Note that in da Costa's first-order C-systems (denote their language by L_{dc}), the congruence relation $\mathsf{R}_{\{R^{\alpha}_{L_{dc}}, R^{v_\forall}_{L_{dc}}, R^{v_\forall}_{L_{dc}}\}}$ is explicitly used ([9]).

Definition 13 *For $X \subseteq Ax$ and $Z \subseteq \mathbf{CNG}_{\mathcal{L}_C}$, the system $\mathbf{QB}[X][Z]$ is obtained from the system $\mathbf{QB}[X]$ by adding the extra-postulate (Z) $\psi \supset \psi'$ for any $\psi, \psi' \in Frm_{\mathcal{L}_C}$, such that $\mathsf{R}_Z(\psi, \psi')$.*

In order to provide semantics for the new class of LFIs defined above, we refine the notion of a consequence relation induced by an Nmatrix (see Defn. 5). Let $Z \subseteq \mathbf{CNG}_L$, such that $Z = \{R_L^{x_1}, ..., R_L^{x_n}\}$, where $x_1, ..., x_n \in \{C_\wedge, C_\vee, I_\wedge, I_\vee, \alpha, v_\forall, v_\exists\}$. Given an L-structure $S = \langle D, I\rangle$, we denote by Z_D the extension of Z to the language $L(D)$: $Z_D = \{R_{L(D)}^{x_1}, ..., R_{L(D)}^{x_n}\}$. (Consequently, R_{Z_D} is the minimal congruence relation over $Frm_{L(D)}$ including all the relations from Z_D.)

Definition 14 *Let S be an L-structure and $\mathcal{M}$ an Nmatrix for L, and let $Z \subseteq \mathbf{CNG}_L$. An S-valuation v is* R_Z-legal *in $\mathcal{M}$ if (i) it is legal in $\mathcal{M}$, and (ii) it respects the R_{Z_D} relation, i.e. for every two $L(D)$-sentences ψ, ψ', such that $\mathsf{R}_{Z_D}(\psi, \psi')$: $v[\psi] = v[\psi']$.*
For a set $\Gamma \cup \{\psi\}$ of L-formulas, $\Gamma \vdash_{\mathcal{M}}^{\mathsf{R}_Z} \psi$ if for every L-structure S and every S-valuation v which is R_Z-legal in $\mathcal{M}$: if $S, v \models_{\mathcal{M}} \Gamma$, then $S, v \models_{\mathcal{M}} \psi$.

[7]Recall that two sentences A and B are *logically indistinguishable* in a system $\mathbf{S}$ if $\varphi(A) \vdash_{\mathbf{S}} \varphi(B)$ and $\varphi(B) \vdash_{\mathbf{S}} \varphi(A)$ for every sentence $\varphi(\psi)$ in the language of $\mathbf{S}$.

Theorem 15 *For a set of L-formulas $\Gamma \cup \{\psi\}$, $X \subseteq Ax$ and $Z \subseteq \mathbf{CNG}_{\mathcal{L}_C}$: $\Gamma \vdash_{\mathbf{QB}[X][Z]} \psi$ iff $\Gamma \vdash^{\mathsf{R}_Z}_{\mathcal{QM}_5(X)} \psi$.*

The proof is a straighforward adaptation of the proof of Thm. 10 and is left to the reader.
Perhaps the most important property of the semantic framework of Nmatrices, crucial for constructing counterexamples, is *effectiveness*: for determining whether $\Gamma \vdash_{\mathcal{M}} \varphi$ it always suffices to check only *partial* valuations, defined only on *subformulas* of $\Gamma \cup \{\varphi\}$. On the first-order level, this can be formalized as follows.

Definition 16 *Let S be an L-structure and $\mathcal{M}$ an Nmatrix for L. Let $W_S \subseteq Frm^{\mathsf{cl}}_{L(D)}$ be a set closed under subformulas[8]. For $Z \subseteq \mathbf{CNG}_L$, a partial S-valuation on W_S is* R_Z-legal in *$\mathcal{M}$ if it is legal in $\mathcal{M}$ and respects R_{Z_D}.*

Definition 17 (Effectiveness) *Let $Z \subseteq \mathbf{CNG}_L$. An Nmatrix $\mathcal{M}$ for L is* effective *(*effective for R_Z*) if for every L-structure S and every set of $L(D)$-sentences W_S closed under subformulas: if v_p is a partial S-valuation on W_S which is legal (R_Z-legal) in $\mathcal{M}$, then it can be extended to a full S-valuation legal (R_Z-legal) in $\mathcal{M}$.*

Effectiveness is a trivial property for deterministic multi-valued matrices. The proof of effectiveness for Nmatrices in the propositional case is also very simple (see Prop. 2 in[1]). However, effectiveness becomes much less evident when congruence relations are involved. In fact, given $Z \subseteq \mathbf{CNG}_L$, an Nmatrix is not necessarily effective for R_Z. Consider, for instance, an Nmatrix $\mathcal{M}_1 = \langle \{t, f\}, \{t\}, \mathcal{O}\rangle$, with the following non-standard interpretation of $\forall$: $\tilde{\forall}[\{H\}] = \{t\}$ for every $H \subseteq P^+(\{t, f\})$. Let $Z = \{R_L^{v_\forall}\}$. Let $S = \langle \{a\}, I\rangle$ be an L-structure, such that $I[c] = a$ and $I[p][a] = f$. Then no partial valuation on $\{p(c)\}$ which is R_Z-legal in $\mathcal{M}$ can be extended to a full $\mathcal{M}$-legal valuation v, respecting both the R_Z relation and the interpretation of $\forall$. Below we formulate the required conditions for the effectiveness of an Nmatrix $\mathcal{M}$ for R_Z.

Definition 18 *For $Z \subseteq \mathbf{CNG}_L$, an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ for L is R_Z-suitable if: (i) $R_L^{v_Q} \in Z$ implies that for every $a \in \mathcal{V}$: $a \in \tilde{Q}_{\mathcal{M}}[\{a\}]$, (ii) $R_L^{C_\diamond} \in Z$ implies that for every $a, b \in \mathcal{V}$: $\tilde{\diamond}_{\mathcal{M}}[a, b] = \tilde{\diamond}_{\mathcal{M}}[b, a]$, (iii) $R_L^{I_\diamond} \in Z$ implies that for every $a \in \mathcal{V}$: $a \in \tilde{\diamond}_{\mathcal{M}}[a, a]$.*

Note that any Nmatrix is effective[9] for $R_{\{R_L^{\alpha}\}}$.
For instance, the Nmatrix $\mathcal{M}_1$ defined above is not suitable for $R_{\{R_L^{v_\forall}\}}$, since $f \notin \tilde{\forall}_{\mathcal{M}_1}[\{f\}]$.

[8]For an L-structure $S = \langle D, I\rangle$, we say that a set of sentences $W_S \subseteq Frm^{\mathsf{cl}}_{L(D)}$ is *closed under subformulas* if: (i) $\psi_1, ..., \psi_n \in W_S$ whenever $\diamond(\psi_1, ..., \psi_n) \in W_S$, and (ii) for every $a \in D$: $\psi\{\overline{a}/x\} \in W_S$ whenever $Qx\psi \in W_S$.

[9]This intuitively implies that the α-equivalence principle is more basic than the rest of the principles studied here, since the latter depend on the semantic interpretation of the connectives and quantifiers of L.

Theorem 19 *For $Z \subseteq \mathbf{CNG}_L$, an Nmatrix $\mathcal{M}$ for L is effective for R_Z iff it is R_Z-suitable.*

Corollary 20 *For every $X \subseteq \mathbf{Ax}$ and every $Z \subseteq \mathbf{CNG}_{\mathcal{L}_C}$, $\mathcal{QM}_5[X]$ is effective for R_Z.*

Note that the Nmatrix $\mathcal{M}_5{}^f_{Bl}$ defined in [10] (which is a characteristic Nmatrix for da Costa's C_1^*) is *not* effective for $\mathsf{R}_{\{R_L^C\wedge\}}$.

Next we apply the effectiveness of our semantics to prove the following proof-theoretical property of LFIs:

Theorem 21 *Let $\mathbf{S}$ be a system over a language L including $\{\supset, \neg\}$, s.t. $A \vdash_{\mathbf{S}} B$ whenever $\mathsf{R}_Z(A, B)$. Let $Z \subseteq \mathbf{CNG}_L$. If $\mathbf{QBieJ}[Z]$ is an extension of $\mathbf{S}$, then two L-sentences ψ, φ are logically indistinguishable in $\mathbf{S}$ iff $\mathsf{R}_Z(\psi, \varphi)$.*

Proof: Assume that $\mathsf{R}_Z(\psi, \varphi)$ does not hold. Let S be an L-structure and W_S - the minimal set of $L(D)$-sentences closed under subformulas, such that $\neg\neg\neg(\varphi \supset \varphi) \in W_S$. Define a partial S-valuation v on W_S, such that: $v[\varphi \supset \varphi] = t, v[\neg(\varphi \supset \varphi)] = f, v[\neg\neg(\varphi \supset \varphi)] = I, v[\neg\neg\neg(\varphi \supset \varphi)] = I$. Extend v, so that: $v[\psi \supset \psi] = t$, $v[\neg(\psi \supset \psi)] = f$, $v[\neg\neg(\psi \supset \psi)] = t$, $v(\neg\neg\neg(\psi \supset \psi)) = f$. It is easy to see that v is R_Z-legal in $\mathcal{QM}_5\mathbf{ieJ}$. By Cor. 20, v can be extended to a full valuation which is R_Z-legal in $\mathcal{QM}_5\mathbf{ieJ}$. By Thm. 15, $\neg\neg\neg(\varphi \supset \varphi) \not\vdash_{\mathbf{S}} \neg\neg\neg(\psi \supset \psi)$ Hence ψ and φ are not logically indistinguishable in $\mathbf{S}$. The other direction is trivial.$\square$

This theorem extends the results in [1, 10] and Remark 4.8 in [7] (concerning the propositional fragment of **QBieJDm**, called LFI1 there) by covering all the *first-order* systems between **QBie** and **QBieJ** and adding all combinations of the extra-postulates.

5 Summary

Non-deterministic multi-valued matrices are an attractive semantic framework due to their modularity and effectiveness. In this paper we have used Nmatrices to provide simple modular finite-valued non-deterministic semantics for a large useful family of first-order LFIs. The modular approach provides new insights into the semantic roles of each of the studied schemata and the dependencies between them. We have shown that for any $\mathbf{j} \in \{1, 2\}$ and $\mathbf{x} \in \{\forall, \exists, \wedge, \vee\}$, $\mathbf{J}^{\mathbf{j}}_{\mathbf{x}}$ is equivalent to $\mathbf{Dm}^{\mathbf{j}}_{\mathbf{x}}$ in any extension of $\mathbf{QBi}$. Then we have formalized the notion of adding extra-postulates to the LFIs to deal with their lack of the IPE principle. We have used seven natural extra-postulates capturing α-conversion, vacuous quantification and commutativity and idempotency of $\wedge$ and $\vee$. However, it is clear from our case study that this method can be extended to other natural postulates chosen according to the intended applications of the system. We have shown that in the presence of the extra-postulates, the effectiveness of the semantics becomes problematic. Nevertheless, all of the semantics considered here were shown to be effective. Finally, effectiveness was applied to prove an important proof-theoretical property of some of the studied LFIs.

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