

A Qualitative Modal Representation of Quantum Register Transformations (Extended Version)

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Abstract

We introduce two modal natural deduction systems that are suitable to represent and reason about transformations of quantum registers in an abstract, qualitative, way. Quantum registers represent quantum systems, and can be viewed as the structure of quantum data for quantum operations. Our systems provide a modal framework for reasoning about operations on quantum registers (unitary transformations and measurements) in terms of possible worlds (as abstractions of quantum registers) and accessibility relations between these worlds. We give a Kripke-style semantics that formally describes quantum register transformations, and prove the soundness and completeness of our systems with respect to this semantics.

1. Introduction

Quantum computing defines an alternative computational paradigm, based on a quantum model [4] rather than a classical one. The basic units of the quantum model are the *quantum bits*, or *qubits* for short (mathematically, normalized vectors of the Hilbert Space \mathbb{C}^2). Qubits represent informational units and can assume both classical values 0 and 1, and all their superpositional values.

A *quantum register* is a generalization of the qubit: a generic quantum register is the representation of a quantum state of n qubits (mathematically, it is a normalized vector of the Hilbert space \mathbb{C}^{2^n}). In this paper, we are not interested in the structure of quantum registers, but rather in the way quantum registers are transformed. Hence, we will abstract away from the internals of quantum registers and represent them in a generic way in order to describe how operations transform a register into another one.

It is possible to modify a quantum register in two ways: by applying a unitary transformation or by measuring. Unitary transformations (corresponding to the so-called unitary operators of the Hilbert space) model the internal evolution of a quantum system, whereas measurements correspond to the results of the interaction between a quantum system and an observer. The outcome of an observation can be either the reduction to a quantum state or the reduction to a classical (non quantum) state. In particular, in this paper, we say that a quantum register w is *classical* iff w is idempotent with respect to measurement, i.e. each measurement of w has w as outcome. We call a measurement *total* when the outcome of the measurement is a classical register.

We propose to model measurement and unitary transformations by means of suitable modal operators. More specifically, the main contribution of this paper is the formalization of a modal natural deduction system [12, 14] in order to represent (in an abstract, qualitative, way) the fundamental operations on quantum registers: unitary transformations and total measurements. We call this system MSQR. We also formalize a variant of this system, called MSpQR, to represent the case of generic (not necessarily total) measurements.

It is important to observe that our logical systems are not a quantum logic. Since 1936 [5], various logics have been investigated as a means to formalize reasoning about propositions taking into account the principles of quantum theory, e.g. [7, 8]. In general, it is possible to view quantum logic as a logical axiomatization of quantum theory, which provides an adequate foundation for a theory of reversible quantum processes, e.g. [1, 2, 3, 10].

Our work moves from quite a different point of view: we do not aim to propose a general logical formalization of quantum theory, rather we describe how it is possible to use modal logic to reason in a simple way about quantum register transformations. Informally, in our proposal, a modal world represents (an abstraction of) a quantum register. The discrete temporal evolution of a quantum register is controlled and determined by a sequence of unitary transformations and measurements that can change the description of a quantum state into other descriptions. So, the evolution of a quantum register can be viewed as a graph, where the nodes are the (abstract) quantum registers and the arrows represent quantum transformations. The arrows give us the so-called accessibility relations of Kripke models and two nodes linked by an arrow represent two related quantum states: the target node is obtained from the source node by means of the operation specified in the decoration of the arrow.

Modal logic, as a logic of possible worlds, is thus a natural way to represent this description of a quantum system: the worlds model the quantum registers and the relations of accessibility between worlds model the dynamical behavior of the system, as a consequence of the application of measurements and unitary transformations. To emphasize this semantic view of modal logic, we give our deduction system in the style of *labelled deduction* [9, 13, 15], a framework for giving uniform presentations of different non-classical logics. The intuition behind labelled deduction is that the labelling (sometimes also called prefixing, annotating or subscripting) allows one to explicitly encode in the syntax additional information, of a semantic or proof-theoretical nature, that is otherwise implicit in the logic one wants to capture. Most notably, in the case of modal logic, this additional information comes from the underlying Kripke semantics: the labelled formula $x:A$ intuitively means that A holds at the world denoted by the label x within the underlying Kripke structure (i.e. model), and labels also allow one to specify at the syntactic level how the different worlds are related in the Kripke structures (e.g. the formula xRy specifies that the world denoted by y is accessible from that denoted by x).

We proceed as follows. In Section 2, we define the labelled modal natural deduction system MSQR, which contains two modal operators suitable to represent and reason about unitary transformations and total measurements of quantum registers. In Section 3, we give a possible worlds semantics that formally describes these quantum register transformations, and prove the soundness and completeness of MSQR with respect to this semantics. In Section 4, we formalize MSpQR, a variant of MSQR that provides a modal system representing all the possible (thus not necessarily total) measurements. We conclude in Section 5 with a brief summary and a discussion of future work. Full proofs of the technical results are given in the appendix.

2 The deduction system MSQR

Our labelled modal natural deduction system MSQR, which formally represents unitary transformations and total measurements of quantum registers, comprises of rules that derive formulas of two kinds: modal formulas and relational formulas. We thus define a modal language and a relational language.

The alphabet of the *relational language* consists of:

- the binary symbols \cup and M ,
- a denumerable set x_0, x_1, \dots of *labels*.

Metavariables x, y, z , possibly annotated with subscripts and superscripts, range over the set of labels. For brevity, we will sometimes speak of a “world” x meaning that the label x stands for a world $\mathcal{I}(x)$, where \mathcal{I} is an interpretation function mapping labels into worlds as formalized in Definition 2 below.

The set of *relational formulas* (*r-formulas*) is given by expressions of the form $x\cup y$ and $x\mathsf{M}y$.

The alphabet of the *modal language* consists of:

- a denumerable set r, r_0, r_1, \dots of *propositional symbols*,
- the standard *propositional connectives* \perp and \supset ,
- the unary *modal operators* \Box and \blacksquare .

The set of *modal formulas* (*m-formulas*) is the least set that contains \perp and the propositional symbols, and is closed under the propositional connectives and the modal operators. Metavariables A, B, C , possibly indexed, range over modal formulas. Other connectives can be defined in the usual manner, e.g. $\neg A \equiv A \supset \perp$, $A \wedge B \equiv \neg(A \supset \neg B)$, $A \leftrightarrow B \equiv (A \supset B) \wedge (B \supset A)$, $\Diamond A \equiv \neg \Box \neg A$, $\blacklozenge A \equiv \neg \blacksquare \neg A$, etc.

Let us give, in a rather informal way, the intuitive meaning of the modal operators of our language:

$$\begin{array}{c}
\frac{[x:A] \quad \vdots \quad x:B}{x:A \supset B} \supset I \quad \frac{x:A \supset B \quad x:B}{x:B} \supset E \quad \frac{[x:\neg A] \quad \vdots \quad y:\perp}{x:A} RAA \quad \frac{x:\perp}{\alpha} \perp E \quad \frac{[xRy] \quad \vdots \quad y:A}{x:\star A} \star I^* \quad \frac{x:\star A \quad xRy}{y:A} \star E \\
\\
\frac{}{xUx} U_{refl} \quad \frac{xUy}{yUx} U_{symm} \quad \frac{xUy \quad yUz}{xUz} U_{trans} \quad \frac{xMy}{xUy} UI \quad \frac{[xMy] \quad \vdots \quad \alpha}{\alpha} M_{ser}^* \quad \frac{xMy}{yMy} M_{srefl} \\
\\
\frac{\alpha(x) \quad xMx \quad xMy}{\alpha(y/x)} M_{sub1} \quad \frac{\alpha(y) \quad xMx \quad xMy}{\alpha(x/y)} M_{sub2}
\end{array}$$

In $\star I$, y is fresh: it is different from x and does not occur in any assumption on which $y:A$ depends other than xRy .

In M_{ser} , y is fresh: it is different from x and does not occur in α nor in any assumption on which α depends other than xMy .

Figure 1. The rules of MSQR

- $\Box A$ means: A is true after the application of any unitary transformation.
- $\blacksquare A$ means: A is true in each quantum register obtained by a total measurement.

A *labelled formula* (*l-formula*) is an expression $x:A$, where x is a label and A is an m-formula. A *formula* is either an r-formula or an l-formula. The metavariable α , possibly indexed, ranges over formulas. We write $\alpha(x)$ to denote that the label x occurs in the formula α , so that $\alpha(y/x)$ denotes the substitution of the label y for all occurrences of x in α .

Figure 1 shows the rules of MSQR, where the notion of *discharged/open assumption* is standard [12, 14], e.g. the formula $[x:A]$ is discharged in the rule $\supset I$:

Propositional rules: The rules $\supset I$, $\supset E$ and RAA are just the labelled version of the standard ([12, 14]) natural deduction rules for implication introduction and elimination and for *reductio ad absurdum*, where we do not enforce Prawitz’s side condition that $A \neq \perp$.¹ The “mixed” rule $\perp E$ allows us to derive a generic formula α whenever we have obtained a contradiction \perp at a world x .

Modal rules: We give the rules for a generic modal operator \star , with a corresponding generic accessibility relation R , since all the modal operators share the structure of these basic introduction/elimination rules; this holds because, for instance, we express $x:\Box A$ as the metalevel implication $xUy \implies y:A$ for an arbitrary y accessible from x . In particular:

- if \star is \Box then R is U ,
- if \star is \blacksquare then R is M .

Other rules:

- In order to axiomatize \Box , we add rules U_{refl} , U_{symm} , and U_{trans} , formalizing that U is an equivalence relation.
- In order to axiomatize \blacksquare , we add rules formalizing the following properties:
 - If xMy then there is specific unitary transformation (depending on x and y) that generates y from x : rule UI .
 - The total measurement process is serial: rule M_{ser} says that if from the assumption xMy we can derive α for a *fresh* y (i.e. y is different from x and does not occur in α nor in any assumption on which α depends other than xMy), then we can discharge the assumption (since there always is some y such that xMy) and conclude α .
 - The total measurement process is shift-reflexive: rule M_{srefl} .

¹See [15] for a detailed discussion on the rule RAA , which in particular explains how, in order to maintain the duality of modal operators like \Box and \Diamond , the rule must allow one to derive $x:A$ from a contradiction \perp at a possibly different world y , and thereby discharge the assumption $x:\neg A$.

- Invariance with respect to classical worlds: rules M_{sub1} and M_{sub2} say that, if xMx and xMy , then y must be equal to x and so we can substitute the one for the other in any formula α .

Definition 1 (Derivations and proofs). A derivation of a formula α from a set of formulas Γ in **MSQR** is a tree formed using the rules in **MSQR**, ending with α and depending only on a finite subset of Γ ; we then write $\Gamma \vdash \alpha$. A derivation of α in **MSQR** depending on the empty set, $\vdash \alpha$, is a proof of α in **MSQR** and we then say that α is a theorem of **MSQR**.

For instance, the following labelled formula schemata are all provable in **MSQR** (where, in parentheses, we give the intuitive meaning of each formula in terms of quantum register transformations):

1. $x:\Box A \supset A$
(the identity transformation is unitary).
2. $x:A \supset \Box \Diamond A$
(each unitary transformation is invertible).
3. $x:\Box A \supset \Box \Box A$
(unitary transformations are composable).
4. $x:\blacksquare A \supset \blacklozenge A$
(it is always possible to perform a total measurement of a quantum register).
5. $x:\blacksquare(A \leftrightarrow \blacksquare A)$
(it is always possible to perform a total measurement with a complete reduction of a quantum register to a classical one).
6. $x:\blacksquare A \supset \blacksquare \blacksquare A$
(total measurements are composable).

As concrete examples, Figure 2 contains the proofs of the formulas 5 and 6, where, for simplicity, here and in the following (cf. Figure 5), we employ the rules for equivalence ($\leftrightarrow I$) and for negation ($\neg I$ and $\neg E$), which are derived from the propositional rules as is standard. For instance,

$$\frac{\begin{array}{c} [x:A]^1 \\ \vdots \\ y:\perp \\ \hline x:\neg A \end{array} \neg I^1}{\text{abbreviates}} \frac{\begin{array}{c} [x:A]^1 \\ \vdots \\ y:\perp \\ \hline x:\perp \end{array} \perp E \text{ (or } RAA)}{x:A \supset \perp \supset I^1}$$

We can similarly derive rules about r -formulas. For instance, we can derive a rule for the transitivity of M as shown at the top of the proof of the formula 6 in Figure 2:

$$\frac{\frac{xMy \quad yMz}{xMz} Mtrans}{\text{abbreviates}} \frac{\frac{yMz \quad \frac{yMz}{zMz} Msrefl}{xMz} xMy}{xMz} Msub1$$

3. A semantics for unitary transformations and total measurements

We give a semantics that formally describes unitary transformations and total measurements of quantum registers, and then prove that **MSQR** is sound and complete with respect to this semantics. Together with the corresponding result for generic measurements in Section 4, this means that our modal systems indeed provide a representation of quantum registers and operations on them, which was the main goal of the paper.

Definition 2 (Frames, models, structures). A frame is a tuple $\mathcal{F} = \langle W, U, M \rangle$, where:

$$\begin{array}{c}
\frac{[y:A]^2 \quad \frac{[xMy]^1}{yMy} \text{ Msrefl} \quad [yMz]^3}{\frac{z:A}{y:\blacksquare A} \blacksquare I^3} \text{ Msub1} \quad \frac{[y:\blacksquare A]^4 \quad \frac{[xMy]^1}{yMy} \text{ Msrefl}}{y:A} \text{ Msub1} \\
\frac{y:A \supset \blacksquare A}{y:\blacksquare A \supset A} \supset I^2 \quad \frac{y:A}{y:\blacksquare A \supset A} \supset I^4 \\
\frac{y:A \leftrightarrow \blacksquare A}{x:\blacksquare(A \leftrightarrow \blacksquare A)} \blacksquare I^1 \\
\frac{[yMz]^3}{\frac{[yMz]^3}{zMz} \text{ Msrefl} \quad [xMy]^2} \text{ Msub1} \\
\frac{[x:\blacksquare A]^1}{xMz} \blacksquare E \\
\frac{z:A}{y:\blacksquare A} \blacksquare I^3 \\
\frac{y:\blacksquare A}{x:\blacksquare\blacksquare A} \blacksquare I^2 \\
\frac{x:\blacksquare\blacksquare A}{x:\blacksquare A \supset \blacksquare\blacksquare A} \supset I^1
\end{array}$$

Figure 2. Examples of proofs in MSQR

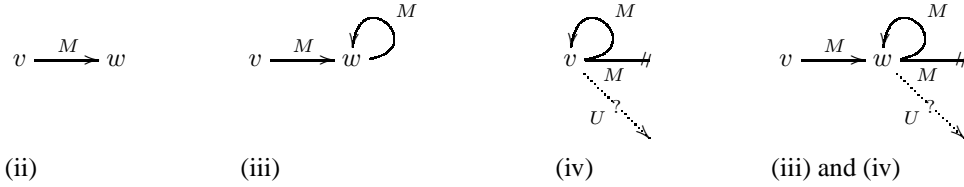


Figure 3. Some properties of the relation M

- W is a non-empty set of worlds
(representing abstractly the quantum registers);
- $U \subseteq W \times W$ is an equivalence relation
(vUw means that w is obtained by applying a unitary transformation to v ; U is an equivalence relation since identity is a unitary transformation, each unitary transformation must be invertible, and unitary transformations are composable);
- $M \subseteq W \times W$
(vMw means that w is obtained by means of a total measurement of v);

with the following properties:

- (i) $\forall v, w. vMw \implies vUw$
- (ii) $\forall v. \exists w. vMw$
- (iii) $\forall v, w. vMw \implies wMw$
- (iv) $\forall v, w. vMv \ \& \ vMw \implies v = w$

(i) means that although it is not true that measurement is a unitary transformation, locally for each v , if vMw then there is a particular unitary transformation, depending on v and w , that generates w from v ; the vice versa cannot hold, since in quantum theory measurements cannot be used to obtain the unitary evolution of a quantum system. (ii) means that each quantum register is totally measurable. (iii) and (iv) together mean that after a total measurement we obtain a classical world. Figure 3 shows properties (ii), (iii) and (iv), respectively, as well as the combination of (iii) and (iv).²

²Note that while (iv) says that v is idempotent with respect to M , a unitary transformation U could still be applied to v (and hence the dotted arrow decorated with a “?” for U).

$\supset I, \supset E, RAA, \perp E, \star I^*, \star E, Urefl, Usymm, Utrans,$

$$\frac{xPy}{xUy} \text{ PUI} \quad \frac{xPy \quad yPz}{xPz} \text{ Ptrans} \quad \frac{\begin{array}{c} [xPy][yPy] \\ \vdots \\ \frac{\alpha}{\alpha} \text{ class}^* \end{array}}{\alpha} \text{ Psub1} \quad \frac{\alpha(x) \quad xPx \quad xPy}{\alpha(y/x)} \text{ Psub2} \quad \frac{\alpha(y) \quad xPx \quad xPy}{\alpha(x/y)} \text{ Psub2}$$

In $\star I$, y is fresh: it is different from x and does not occur in any assumption on which $y:A$ depends other than xRy .

In class , y is fresh: it is different from x and does not occur in α nor in any assumption on which α depends other than xPy and yPy .

Figure 4. The rules of MSpQR

A model is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where \mathcal{F} is a frame and $V : W \rightarrow 2^{Prop}$ is an interpretation function mapping worlds into sets of formulas.

A structure is a pair $\mathcal{S} = \langle \mathcal{M}, \mathcal{I} \rangle$, where \mathcal{M} is a model and $\mathcal{I} : \text{Var} \rightarrow W$ is an interpretation function mapping variables (labels) into worlds in W , and mapping a relation symbol $R \in \{U, M\}$ into the corresponding frame relation $\mathcal{I}(R) \in \{U, M\}$. We extend \mathcal{I} to formulas and sets of formulas in the obvious way: $\mathcal{I}(x:A) = \mathcal{I}(x):A$, $\mathcal{I}(xRy) = \mathcal{I}(x)\mathcal{I}(R)\mathcal{I}(y)$, and $\mathcal{I}(\{\alpha_1, \dots, \alpha_n\}) = \{\mathcal{I}(\alpha_1), \dots, \mathcal{I}(\alpha_n)\}$.

Given this semantics, we can define what it means for formulas to be true, and then prove the soundness and completeness of MSQR.

Definition 3 (Truth). Truth for an m -formula in a model $\mathcal{M} = \langle W, U, M, V \rangle$ is the smallest relation \models satisfying:

$$\begin{array}{ll} \mathcal{M}, w \models r & \text{iff } r \in V(w) \\ \mathcal{M}, w \models A \supset B & \text{iff } \mathcal{M}, w \models A \implies \mathcal{M}, w \models B \\ \mathcal{M}, w \models \Box A & \text{iff } \forall w'. wUw' \implies \mathcal{M}, w' \models A \\ \mathcal{M}, w \models \blacksquare A & \text{iff } \forall w'. wMw' \implies \mathcal{M}, w' \models A \end{array}$$

Thus, for an m -formula A , we write $\mathcal{M} \models A$ iff $\mathcal{M}, w \models A$ for all w .

Truth for a formula α in a structure $\mathcal{S} = \langle \mathcal{M}, \mathcal{I} \rangle$ is then the smallest relation \models satisfying:

$$\begin{array}{ll} \mathcal{M}, \mathcal{I} \models xMy & \text{iff } \mathcal{I}(x)M\mathcal{I}(y) \\ \mathcal{M}, \mathcal{I} \models xUy & \text{iff } \mathcal{I}(x)U\mathcal{I}(y) \\ \mathcal{M}, \mathcal{I} \models x:A & \text{iff } \mathcal{M}, \mathcal{I}(x) \models A \end{array}$$

We will omit \mathcal{M} when it is not relevant, and we will denote $\mathcal{I} \models x:A$ also by $\models \mathcal{I}(x):A$ or even $\models w:A$ for $\mathcal{I}(x) = w$. By extension, $\mathcal{M}, \mathcal{I} \models \Gamma$ iff $\mathcal{M}, \mathcal{I} \models \alpha$ for all α in the set of formulas Γ . Thus, for a set of formulas Γ and a formula α ,

$$\begin{array}{ll} \Gamma \models \alpha & \text{iff } \forall \mathcal{M}, \mathcal{I}. \mathcal{M} \models \mathcal{I}(\Gamma) \implies \mathcal{M} \models \mathcal{I}(\alpha) \\ & \text{iff } \forall \mathcal{M}, \mathcal{I}. \mathcal{M}, \mathcal{I} \models \Gamma \implies \mathcal{M}, \mathcal{I} \models \alpha \end{array}$$

By adapting standard proofs (see, e.g., [9, 12, 13, 14, 15] and the proofs in the appendix), we have:

Theorem 1 (Soundness and completeness of MSQR). $\Gamma \vdash \alpha$ iff $\Gamma \models \alpha$.

4. Generic measurements

In quantum computing, not all measurements are required to be total: think, for example, of the case of observing only the first qubit of a quantum register. To this end, in this section, we formalize MSpQR, a variant of MSQR that provides a modal system representing all the possible (thus not necessarily total) measurements. We obtain MSpQR from MSQR by means of the following changes:

$$\begin{array}{c}
\frac{[x: \Box \neg(A \supset \Box A)]^2 \quad [xPy]^1}{y: \neg(A \supset \Box A)} \Box E \quad \frac{\frac{[y:A]^3 \quad [yPy]^1 \quad [yPz]^4}{z:A} P_{sub1} \quad \frac{z:A}{y: \Box A} \Box I^4}{y: A \supset \Box A} \supset I^3 \\
\frac{y: \neg(A \supset \Box A) \quad y: A \supset \Box A}{\neg E} \\
\frac{y: \perp}{x: \neg \Box \neg(A \supset \Box A)} \neg I^2 \\
\frac{x: \neg \Box \neg(A \supset \Box A)}{x: \neg \Box \neg(A \supset \Box A)} class^1
\end{array}$$

Figure 5. An example proof in MSpQR

- The alphabet of the modal language contains the unary modal operator \Box instead of \blacksquare , with corresponding \Diamond , where $\Box A$ intuitively means that A is true in each quantum register obtained by a measurement.
- The set of relational formulas contains expressions of the form xPy instead of xMy .
- The rules of MSpQR are given in Figure 4. In particular, \star is either \Box (as before) or \Diamond , for which then R is P , and whose properties are formalized by the following additional rules:
 - If xPy then there is a specific unitary transformation (depending on x and y) that generates y from x : rule PUI .
 - The measurement process is transitive: rule $Ptrans$.
 - There are (always reachable) classical worlds: $class$ says that y is a classical world reachable from world x by a measurement.
 - Invariance with respect to classical worlds for measurement: rules P_{sub1} and P_{sub2} .

Derivations and proofs in MSpQR are defined as for MSQR. For instance, in addition to the formulas for \Box already listed for MSQR, the following labelled formula schemata are all provable in MSpQR (as shown, e.g., for formula 3 in Figure 5):

1. $x: \Box A \supset \Diamond A$
(it is always possible to perform a measurement of a quantum register).
2. $x: \Box A \supset \Box \Box A$
(measurements are composable).
3. $x: \Diamond(A \supset \Box A)$, i.e. $x: \neg \Box \neg(A \supset \Box A)$
(it is always possible to perform a measurement with a complete reduction of a quantum register to a classical one).

The semantics is also obtained by simple changes with respect to the definitions of Section 3. A *frame* is a tuple $\mathcal{F} = \langle W, U, P \rangle$, where $P \subseteq W \times W$ and vPw means that w is obtained by means of a measurement of v , with the following properties:

- (i) $\forall v, w. vPw \implies vUw$
(as for (i) in Section 3).
- (ii) $\forall v, w', w''. vPw' \ \& \ w'Pw'' \implies vPw''$
(measurements are composable).
- (iii) $\forall v. \exists w. vPw \ \& \ wPw$
(each quantum register v can be reduced to a classical one w by means of a measurement).
- (iv) $\forall v, w. vPv \ \& \ vPw \implies v = w$
(each measurement of a classical register v has v as outcome).

Models and *structures* are defined as before, with $\mathcal{I}(P) = P$, while the *truth* relation now comprises the clauses

$$\begin{array}{ll}
\mathcal{M}, w \models \Box A & \text{iff } \forall w'. wPw' \implies \mathcal{M}, w' \models A \\
\mathcal{M}, \mathcal{I} \models xPy & \text{iff } \mathcal{I}(x)P\mathcal{I}(y)
\end{array}$$

Finally, MSpQR is also sound and complete.

Theorem 2 (Soundness and completeness of MSpQR). $\Gamma \vdash \alpha$ iff $\Gamma \models \alpha$.

5. Conclusions and future work

We have shown that our modal natural deduction systems MSQR and MSpQR provide suitable representations of quantum register transformations. As future work, we plan to investigate the proof theory of our systems (e.g. normalization, subformula property, (un)decidability), in view of a possible mechanization of reasoning in MSQR and MSpQR (e.g. encoding them into a logical framework [11]). We are also working at extending our approach to represent and reason about further quantum notions, such as entanglement.

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A Proof of soundness and completeness

Theorem 1 follows from Theorems 3 and 4 below.

Theorem 3 (Soundness of MSQR). $\Gamma \vdash \alpha$ implies $\Gamma \models \alpha$.

Proof. We let \mathcal{M} be an arbitrary model and prove that if $\Gamma \vdash \alpha$ then $\models \mathcal{J}(\Gamma)$ implies $\models \mathcal{J}(\alpha)$ for any \mathcal{J} . The proof proceeds by induction on the structure of the derivation of α from Γ . The base case, where $\alpha \in \Gamma$, is trivial. There is one step case for each rule of MSQR.

Consider an application of the rule RAA ,

$$\frac{\begin{array}{c} [x:\neg A] \\ \vdots \\ y:\perp \end{array}}{x:A} RAA$$

where $\Gamma' \vdash y:\perp$ with $\Gamma' = \Gamma \cup \{x:\neg A\}$. By the induction hypothesis, $\Gamma' \vdash y:\perp$ implies $\mathcal{J}(\Gamma') \models \mathcal{J}(y):\perp$ for any \mathcal{J} . We assume $\models \mathcal{J}(\Gamma)$ and prove $\models \mathcal{J}(x):A$. Since $\not\models w:\perp$ for any world w , from the induction hypothesis we obtain $\not\models \mathcal{J}(\Gamma')$, and thus $\not\models \mathcal{J}(x):\neg A$, i.e. $\models \mathcal{J}(x):A$ and $\not\models \mathcal{J}(x):\perp$.

Consider an application of the rule $\perp E$,

$$\frac{x:\perp}{\alpha} \perp E$$

with $\Gamma \vdash x:\perp$. By the induction hypothesis, $\Gamma \vdash x:\perp$ implies $\mathcal{J}(\Gamma) \models \mathcal{J}(x):\perp$ for any \mathcal{J} . We assume $\models \mathcal{J}(\Gamma)$ and prove $\models \mathcal{J}(\alpha)$ for an arbitrary formula α . If $\models \mathcal{J}(\Gamma)$ then $\models \mathcal{J}(x):\perp$ by the induction hypothesis. But since $\not\models w:\perp$ for any world w , then $\not\models \mathcal{J}(\Gamma)$ and thus $\models \mathcal{J}(\alpha)$ for any α .

Consider an application of the rule $\star I$

$$\frac{\begin{array}{c} [xRy] \\ \vdots \\ y:A \end{array}}{x:\star A} \star I$$

where $\Gamma' \vdash y:A$ with y fresh and with $\Gamma' = \Gamma \cup \{xRy\}$. By the induction hypothesis, for all interpretations \mathcal{J} , if $\models \mathcal{J}(\Gamma)$ then $\models \mathcal{J}(y):A$. We let \mathcal{J} be any interpretation such that $\models \mathcal{J}(\Gamma)$, and show that $\models \mathcal{J}(x):\star A$. Let w be any world such that $\mathcal{J}(x)\mathcal{J}(R)w$ where $\mathcal{J}(R) \in \{U, M\}$ depending on \star . Since \mathcal{J} can be trivially extended to another interpretation (still called \mathcal{J} for simplicity) by setting $\mathcal{J}(y) = w$, the induction hypothesis yields $\models \mathcal{J}(y):A$, i.e. $\models w:A$, and thus $\models \mathcal{J}(x):\star A$.

Consider an application of the rule $\star E$

$$\frac{x:\star A \quad xRy}{y:A} \star E$$

with $\Gamma_1 \vdash x:\star A$ and $\Gamma_2 \vdash xRy$, and $\Gamma \supseteq \Gamma_1 \cup \Gamma_2$. We assume $\models \mathcal{J}(\Gamma)$ and prove $\models \mathcal{J}(y):A$. By the induction hypothesis, for all interpretations \mathcal{J} , if $\models \mathcal{J}(\Gamma_1)$ then $\models \mathcal{J}(x):\star A$ and if $\models \mathcal{J}(\Gamma_2)$ then $\models \mathcal{J}(x)\mathcal{J}(R)\mathcal{J}(y)$, where $\mathcal{J}(R) \in \{U, M\}$ depending on \star . If $\models \mathcal{J}(\Gamma)$, then $\models \mathcal{J}(x):\star A$ and $\models \mathcal{J}(x)\mathcal{J}(R)\mathcal{J}(y)$, and thus $\models \mathcal{J}(y):A$.

The rules $Urefl$, $Usymm$, and $Utrans$ are sound by the properties of U .

The rule UI is sound by property (i) in Definition 2.

Consider an application of the rule $Mser$

$$\frac{\begin{array}{c} [xMy] \\ \vdots \\ \alpha \end{array}}{\alpha} Mser$$

with $\Gamma' = \Gamma \cup \{xMy\}$, for y fresh. By the induction hypothesis, $\Gamma' \vdash \alpha$ implies $\mathcal{J}(\Gamma') \models \mathcal{J}(\alpha)$ for any \mathcal{J} . Let us suppose that there is an \mathcal{J}' such that $\models \mathcal{J}'(\Gamma')$ and $\not\models \mathcal{J}'(\alpha)$. Let us consider an \mathcal{J}'' such that $\mathcal{J}''(z) = \mathcal{J}'(z)$ for all z such that $z \neq y$ and $\mathcal{J}''(y)$ is the world w such that $\mathcal{J}''(y)Mw$, which exists by property (ii) in Definition 2. Since y does not occur in Γ nor in α , we then have that $\models \mathcal{J}''(\Gamma')$ and $\not\models \mathcal{J}''(\alpha)$, contradicting the universality of the consequence of the induction hypothesis. Hence, $Mser$ is sound.

The rule $Mrefl$ is sound by property (iii) in Definition 2.

Consider an application of the rule *Msub1*

$$\frac{\alpha(x) \quad xMx \quad xMy}{\alpha(y/x)} \text{ Msub1}$$

with $\Gamma_1 \vdash \alpha(x)$, $\Gamma_2 \vdash xMx$, $\Gamma_3 \vdash xMy$, and $\Gamma \supseteq \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. We assume $\models \mathcal{J}(\Gamma)$ and prove $\models \mathcal{J}(\alpha(y/x))$. By the induction hypothesis, $\Gamma_1 \vdash \alpha(x)$ implies $\mathcal{J}(\Gamma_1) \models \mathcal{J}(\alpha(x))$, $\Gamma_2 \vdash xMx$ implies $\mathcal{J}(\Gamma_2) \models \mathcal{J}(x)M\mathcal{J}(x)$, and $\Gamma_3 \vdash xMy$ implies $\mathcal{J}(\Gamma_3) \models \mathcal{J}(x)M\mathcal{J}(y)$. By property (iv) in Definition 2, we then have $\mathcal{J}(x) = \mathcal{J}(y)$ and thus $\models \mathcal{J}(\alpha(y/x))$. The case for rule *Msub2* follows analogously. \square

To prove completeness (Theorem 4), we give some preliminary definitions and results. For simplicity, we will split each set of formulas Γ into a pair (LF, RF) of the subsets of l-formulas and r-formulas of Γ , and then prove $(LF, RF) \models \alpha$ implies $(LF, RF) \vdash \alpha$. We call (LF, RF) a *context* and, slightly abusing notation, we write $\alpha \in (LF, RF)$ whenever $\alpha \in LF$ or $\alpha \in RF$, and write $x \in (LF, RF)$ whenever the label x occurs in some $\alpha \in (LF, RF)$. We say that a context (LF, RF) is *consistent* iff $(LF, RF) \not\models x:\perp$ for every x , so that we have:

Fact 1. *If (LF, RF) is consistent, then for every x and every A , either $(LF \cup \{x:A\}, RF)$ is consistent or $(LF \cup \{x:\neg A\}, RF)$ is consistent.*

Let $\overline{(LF, RF)}$ be the *deductive closure* of (LF, RF) for r-formulas under the rules of MSQR, i.e.

$$\overline{(LF, RF)} \equiv \{xRy \mid (LF, RF) \vdash xRy\}$$

for $R \in \{U, M\}$. We say that a context (LF, RF) is *maximally consistent* iff

1. it is consistent,
2. it is deductively closed for r-formulas, i.e. $(LF, RF) = \overline{(LF, RF)}$, and
3. for every x and every A , either $x:A \in (LF, RF)$ or $x:\neg A \in (LF, RF)$.

Let us write $(LF, RF) \models^{\mathcal{S}^c} \alpha$ when $\mathcal{S}^c \models (LF, RF)$ implies $\mathcal{S}^c \models \alpha$. Completeness follows by a Henkin-style proof, where a canonical structure

$$\mathcal{S}^c = \langle \mathcal{M}^c, \mathcal{J}^c \rangle = \langle W^c, U^c, M^c, V^c, \mathcal{J}^c \rangle$$

is built to show that $(LF, RF) \not\models \alpha$ implies $(LF, RF) \not\models^{\mathcal{S}^c} \alpha$, i.e. $\mathcal{S}^c \models (LF, RF)$ and $\mathcal{S}^c \not\models \alpha$.

In standard proofs for unlabelled modal logics (e.g. [6]) and for other non-classical logics, the set W^c is obtained by progressively building maximally consistent sets of formulas, where consistency is locally checked within each set. In our case, given the presence of l-formulas and r-formulas, we modify the Lindenbaum lemma to extend (LF, RF) to one single maximally consistent context (LF^*, RF^*) , where consistency is “globally” checked also against the additional assumptions in RF .³ The elements of W^c are then built by partitioning LF^* and RF^* with respect to the labels, and the relations R between the worlds are defined by exploiting the information in RF^* .

In the Lindenbaum lemma for predicate logic, a maximally consistent and ω -complete set of formulas is inductively built by adding for every formula $\neg \forall x.A$ a *witness* to its truth, namely a formula $\neg A[c/x]$ for some new individual constant c . This ensures that the resulting set is ω -complete, i.e. that if, for every closed term t , $A[t/x]$ is contained in the set, then so is $\forall x.A$. A similar procedure applies here in the case of l-formulas of the form $x:\neg \star A$. That is, together with $x:\neg \star A$ we consistently add $y:\neg A$ and xRy for some new y , which acts as a *witness world* to the truth of $x:\neg \star A$. This ensures that the maximally consistent context (LF^*, RF^*) is such that if $xRz \in (LF^*, RF^*)$ implies $z:B \in (LF^*, RF^*)$ for every z , then $x:\star B \in (LF^*, RF^*)$, as shown in Lemma 2 below. Note that in the standard completeness proof for unlabelled modal logics, one instead considers a canonical model \mathcal{M}^c and shows that if $w \in W^c$ and $\mathcal{M}^c, w \models \neg \star A$, then W^c also contains a world w' accessible from w that serves as a witness world to the truth of $\neg \star A$ at w , i.e. $\mathcal{M}^c, w' \models \neg A$.

Lemma 1. *Every consistent context (LF, RF) can be extended to a maximally consistent context (LF^*, RF^*) .*

³We consider only consistent contexts. If (LF, RF) is inconsistent, then $LF, RF \vdash x:A$ for all $x:A$, and thus completeness immediately holds for l-formulas. Our language does not allow us to define inconsistency for a set of r-formulas, but, whenever (LF, RF) is inconsistent, the canonical model built in the following is nonetheless a counter-model to non-derivable r-formulas.

Proof. We first extend the language of MSQR with infinitely many new constants for witness worlds. Systematically let b range over labels, c range over the new constants for witness worlds, and a range over both. All these may be subscripted. Let l_1, l_2, \dots be an enumeration of all l-formulas in the extended language; when l_i is $a:A$, we write $\neg l_i$ for $a:\neg A$. Starting from $(LF_0, RF_0) = (LF, RF)$, we inductively build a sequence of consistent contexts by defining (LF_{i+1}, RF_{i+1}) to be:

- (LF_i, RF_i) , if $(LF_i \cup \{l_{i+1}\}, RF_i)$ is inconsistent; else
- $(LF_i \cup \{l_{i+1}\}, RF_i)$, if l_{i+1} is not $a:\neg \star A$; else
- $(LF_i \cup \{a:\neg \star A, c:\neg A\}, RF_i \cup \{aRc\})$ for a $c \notin (LF_i \cup \{a:\neg \star A\}, RF_i)$, if l_{i+1} is $a:\neg \star A$.

Every (LF_i, RF_i) is consistent. To show this we show that if $(LF_i \cup \{a:\neg \star A\}, RF_i)$ is consistent, then so is $(LF_i \cup \{a:\neg \star A, c:\neg A\}, RF_i \cup \{aRc\})$ for a $c \notin (LF_i \cup \{a:\neg \star A\}, RF_i)$; the other cases follow by construction. We proceed by contraposition. Suppose that

$$(LF_i \cup \{a:\neg \star A, c:\neg A\}, RF_i \cup \{aRc\}) \vdash a_j:\perp$$

where $c \notin (LF_i \cup \{a:\neg \star A\}, RF_i)$. Then, by RAA ,

$$(LF_i \cup \{a:\neg \star A\}, RF_i \cup \{aRc\}) \vdash c:A,$$

and $\star I$ yields

$$(LF_i \cup \{a:\neg \star A\}, RF_i) \vdash a:\star A.$$

Since also

$$(LF_i \cup \{a:\neg \star A\}, RF_i) \vdash a:\neg \star A,$$

by $\neg E$ we have

$$(LF_i \cup \{a:\neg \star A\}, RF_i) \vdash a:\perp,$$

i.e. $(LF_i \cup \{a:\neg \star A\}, RF_i)$ is inconsistent. Contradiction.

Now define

$$(LF^*, RF^*) = \left(\bigcup_{i \geq 0} LF_i, \bigcup_{i \geq 0} RF_i \right)$$

We show that (LF^*, RF^*) is maximally consistent, by showing that it satisfies the three conditions in the definition of maximal consistency. For the first condition, note that if

$$\left(\bigcup_{i \geq 0} LF_i, \bigcup_{i \geq 0} RF_i \right)$$

is consistent, then so is

$$\overline{\left(\bigcup_{i \geq 0} LF_i, \bigcup_{i \geq 0} RF_i \right)}.$$

Now suppose that (LF^*, RF^*) is inconsistent. Then for some finite (LF', RF') included in (LF^*, RF^*) there exists an a such that $(LF', RF') \vdash a:\perp$. Every l-formula $l \in (LF', RF')$ is in some (LF_j, RF_j) . For each $l \in (LF', RF')$, let i_l be the least j such that $l \in (LF_j, RF_j)$, and let $i = \max\{i_l \mid l \in (LF', RF')\}$. Then $(LF', RF') \subseteq (LF_i, RF_i)$, and (LF_i, RF_i) is inconsistent, which is not the case.

The second condition is satisfied by definition of (LF^*, RF^*) .

For the third condition, suppose that $l_{i+1} \notin (LF^*, RF^*)$. Then $l_{i+1} \notin (LF_{i+1}, RF_{i+1})$ and $(LF_i \cup \{l_{i+1}\}, RF_i)$ is inconsistent. Thus, by Fact 1, $(LF_i \cup \{\neg l_{i+1}\}, RF_i)$ is consistent, and $\neg l_{i+1}$ is consistently added to some (LF_j, RF_j) during the construction, and therefore $\neg l_{i+1} \in (LF^*, RF^*)$. \square

The following lemma states some properties of maximally consistent contexts.

Lemma 2. *Let (LF^*, RF^*) be a maximally consistent context. Then*

1. $(LF^*, RF^*) \vdash a_i R a_j$ iff $a_i R a_j \in (LF^*, RF^*)$.
2. $(LF^*, RF^*) \vdash u:A$ iff $a:A \in (LF^*, RF^*)$.

3. $a:B \supset C \in (LF^*, RF^*)$ iff $a:B \in (LF^*, RF^*)$ implies $a:C \in (LF^*, RF^*)$.

4. $a_i:\star B \in (LF^*, RF^*)$ iff $a_i Ra_j \in (LF^*, RF^*)$ implies $a_j:B \in (LF^*, RF^*)$ for all a_j .

Proof. 1 and 2 follow immediately by definition. We only treat 4 as 3 follows analogously. For the left-to-right direction, suppose that $a_i:\star B \in (LF^*, RF^*)$. Then, by (ii), $(LF^*, RF^*) \vdash a_i:\star B$, and, by $\star E$, we have $(LF^*, RF^*) \vdash a_i Ra_j$ implies $(LF^*, RF^*) \vdash a_j:B$ for all a_j . By 1 and 2, conclude $a_i Ra_j \in (LF^*, RF^*)$ implies $a_j:B \in (LF^*, RF^*)$ for all a_j . For the converse, suppose that $a_i:\star B \notin (LF^*, RF^*)$. Then $a_i:\neg\star B \in (LF^*, RF^*)$, and, by the construction of (LF^*, RF^*) , there exists an a_j such that $a_i Ra_j \in (LF^*, RF^*)$ and $a_j:B \notin (LF^*, RF^*)$. \square

We can now define the canonical structure

$$\mathcal{S}^c = \langle \mathcal{M}^c, \mathcal{I}^c \rangle = \langle W^c, U^c, M^c, V^c, \mathcal{I}^c \rangle$$

Definition 4. Given a maximal consistent context (LF^*, RF^*) , we define the canonical structure \mathcal{S}^c as follows:

- $W^c = \{a \mid a \in (LF^*, RF^*)\}$,
- $(a_i, a_j) \in U^c$ iff $a_i U a_j \in (LF^*, RF^*)$,
- $(a_i, a_j) \in M^c$ iff $a_i M a_j \in (LF^*, RF^*)$,
- $V^c(r) = a$ iff $a:r \in (LF^*, RF^*)$,
- $\mathcal{I}^c(a) = a$.

Note that the standard definition of R^c adopted for unlabelled modal logics, i.e.

$$(a_i, a_j) \in R^c \text{ iff } \{A \mid \Box A \in a_i\} \subseteq a_j,$$

is not applicable in our setting, since $\{A \mid \Box A \in a_i\} \subseteq a_j$ does *not* imply $\vdash a_i Ra_j$. We would therefore be unable to prove completeness for r-formulas, since there would be cases, e.g. when $RF = \{\}$, where $\not\vdash a_i Ra_j$ but $(a_i, a_j) \in R^c$ and thus $\mathcal{S}^c \models a_i Ra_j$. Hence, we instead define $(a_i, a_j) \in R^c$ iff $a_i Ra_j \in (LF^*, RF^*)$; note that therefore $a_i Ra_j \in (LF^*, RF^*)$ implies $\{A \mid \Box A \in a_i\} \subseteq a_j$. As a further comparison with the standard definition, note that in the canonical model the label a can be identified with the set of formulas $\{A \mid a:A \in (LF^*, RF^*)\}$. Moreover, we immediately have:

Fact 2. $a_i Ra_j \in (LF^*, RF^*)$ iff $(LF^*, RF^*) \models^{\mathcal{S}^c} a_i Ra_j$.

The deductive closure of (LF^*, RF^*) for r-formulas ensures not only completeness for r-formula, as shown in Theorem 4 below, but also that the conditions on R^c are satisfied, so that \mathcal{S}^c is really a structure for MSQR. More concretely:

- U^c is an equivalence relation by construction and rules $Urefl$, $Usymm$, and $Utrans$. For instance, for transitivity, consider an arbitrary context (LF, RF) from which we build \mathcal{S}^c . Assume $(a_i, a_j) \in U^c$ and $(a_j, a_k) \in U^c$. Then $a_i U a_j \in (LF^*, RF^*)$ and $a_j U a_k \in (LF^*, RF^*)$. Since (LF^*, RF^*) is deductively closed, by 1 in Lemma 2 and rule $Utrans$, we have $a_i U a_k \in (LF^*, RF^*)$. Thus, $(a_i, a_k) \in U^c$ and U^c is indeed transitive.
- $\forall v, w \in W^c. v M w \implies v U w$ holds by construction and rule UI .
- $\forall v \in W^c. \exists w \in W^c. v M w$ holds by construction and rule $Mser$. For the sake of contradiction, consider an arbitrary a_i and a variable a'_j that do not satisfy the property. Define $(LF', RF') = (LF^*, RF^*) \cup \{a_i M a'_j\}$. Then it cannot be the case that $(LF', RF') \vdash \alpha$, for otherwise $(LF^*, RF^*) \vdash \alpha$ would be derivable by an application of the rule $Mser$. Thus, $(LF', RF') \not\vdash \alpha$. But then (LF', RF') must be in the chain of contexts built in Lemma 2. So, by the maximality of (LF^*, RF^*) , we have that $(LF', RF') = (LF^*, RF^*)$, contradicting our assumption. Hence, for some a_j , the r-formula $a_i M a_j$ is in (LF^*, RF^*) , which is what we had to show.
- $\forall v, w \in W^c. v M w \implies w M w$ holds by construction and rule $Msrefl$.

- $\forall v, w \in W^c. vMv \ \& \ vMw \implies v = w$ holds by construction and rules *Msub1* and *Msub2* since v is a classical world. Consider an arbitrary context (LF, RF) from which we build \mathcal{S}^c and assume $(a_i, a_i) \in M^c$ and $(a_i, a_j) \in M^c$. Then $a_iMa_i \in (LF^*, RF^*)$ and $a_iMa_j \in (LF^*, RF^*)$. Thus, for each $a_i:A \in (LF^*, RF^*)$, we also have $a_j:A \in (LF^*, RF^*)$; otherwise, since (LF^*, RF^*) is deductively closed, we would have $a_j:\neg A \in (LF^*, RF^*)$ and also $a_j:A \in (LF^*, RF^*)$ by *I* in Lemma 2 and rule *Msub1*, and thus a contradiction. Similarly, if $a_j:A \in (LF^*, RF^*)$ then $a_i:A \in (LF^*, RF^*)$ by rule *Msub2*. Hence, for each m-formula A , we have that $a_i:A \in (LF^*, RF^*)$ iff $a_j:A \in (LF^*, RF^*)$, which means that a_i and a_j are equal with respect to m-formulas.

Under the same assumptions, we can similarly show that a_i and a_j are equal with respect to r-formulas, i.e. that whenever (LF^*, RF^*) contains an r-formula that includes a_i then it also contains the same r-formula with a_j substituted for a_i , and vice versa. To this end, we must consider 8 different cases corresponding to 8 different r-formulas.

- If $a_kUa_i \in (LF^*, RF^*)$ for some a_k , then from the assumption that $a_iMa_j \in (LF^*, RF^*)$ we have $a_iUa_j \in (LF^*, RF^*)$, by *I* in Lemma 2 and rule *UI*. Therefore, $a_kUa_j \in (LF^*, RF^*)$ by rule *Utrans*.
- We can reason similarly for $a_jUa_k \in (LF^*, RF^*)$ and also apply rules *UI* and *Utrans* to conclude that then also $a_iUa_k \in (LF^*, RF^*)$.
- If $a_iUa_k \in (LF^*, RF^*)$ for some a_k , then from the assumption that $a_iMa_j \in (LF^*, RF^*)$ we have $a_iUa_j \in (LF^*, RF^*)$, by *I* in Lemma 2 and rule *UI*, and thus $a_jUa_i \in (LF^*, RF^*)$, by rule *Usymm*. Therefore, $a_jUa_k \in (LF^*, RF^*)$ by rule *Utrans*.
- We can reason similarly for $a_kUa_j \in (LF^*, RF^*)$ and also apply rules *UI*, *Usymm*, and *Utrans* to conclude that then also $a_kUa_i \in (LF^*, RF^*)$.
- If $a_kMa_i \in (LF^*, RF^*)$ for some a_k , then from the assumption that $a_iMa_j \in (LF^*, RF^*)$ we have $a_kMa_j \in (LF^*, RF^*)$, by *I* in Lemma 2 and the derived rule *Mtrans*.
- We can reason similarly for $a_jMa_k \in (LF^*, RF^*)$ and also apply rule *Mtrans* to conclude that then also $a_iUa_k \in (LF^*, RF^*)$.
- If $a_iMa_k \in (LF^*, RF^*)$ for some a_k , then from the assumptions that $a_iMa_i \in (LF^*, RF^*)$ and $a_iMa_j \in (LF^*, RF^*)$ we have $a_jMa_k \in (LF^*, RF^*)$, by *I* in Lemma 2 and rule *Msub1*.
- We can reason similarly for $a_kMa_j \in (LF^*, RF^*)$ and apply rule *Msub2* to conclude that then also $a_kMa_i \in (LF^*, RF^*)$.

Hence, a_i and a_j are equal also with respect to r-formulas, and thus $a_i = a_j$ whenever $(a_i, a_i) \in M^c$ and $(a_i, a_j) \in M^c$, which is what we had to show.

By Lemma 2 and Fact 2, it follows that:

Lemma 3. $a:A \in (LF^*, RF^*)$ iff $(LF^*, RF^*) \models^{\mathcal{S}^c} a:A$.

Proof. We proceed by induction on the grade of $a:A$, and we treat only the step case where $a:A$ is $a_i:\star B$; the other cases follow analogously. For the left-to-right direction, assume $a_i:\star B \in (LF^*, RF^*)$. Then, by Lemma 2, $a_iRa_j \in (LF^*, RF^*)$ implies $a_j:B \in (LF^*, RF^*)$, for all a_j . Fact 2 and the induction hypothesis yield that $(LF^*, RF^*) \models^{\mathcal{S}^c} a_j:B$ for all a_j such that $(LF^*, RF^*) \models^{\mathcal{S}^c} a_i\mathcal{S}^c(R)a_j$, i.e. $(LF^*, RF^*) \models^{\mathcal{S}^c} a_i:\star B$ by Definition 3. For the converse, assume $a_i:\neg\star B \in (LF^*, RF^*)$. Then, by Lemma 2, $a_iRa_j \in (LF^*, RF^*)$ and $a_j:\neg B \in (LF^*, RF^*)$, for some a_j . Fact 2 and the induction hypothesis yield $(LF^*, RF^*) \models^{\mathcal{S}^c} a_iRa_j$ and $(LF^*, RF^*) \models^{\mathcal{S}^c} a_j:\neg B$, i.e. $(LF^*, RF^*) \models^{\mathcal{S}^c} a_i:\neg\star B$ by Definition 3. \square

We can now finally show:

Theorem 4 (Completeness of MSQR). $\Gamma \models \alpha$ implies $\Gamma \vdash \alpha$.

Proof. If $(LF, RF) \not\models b_iRb_j$, then $b_iRb_j \notin (LF^*, RF^*)$, and thus $(LF^*, RF^*) \not\models^{\mathcal{S}^c} b_iRb_j$ by Fact 2.

If $(LF, RF) \not\models b:A$, then $(LF \cup \{b:\neg A\}, RF)$ is consistent; otherwise there exists a b_i such that $(LF \cup \{b:\neg A\}, RF) \vdash b_i:\perp$, and then $(LF, RF) \vdash b:A$. Therefore, by Lemma 1, $(LF \cup \{b:\neg A\}, RF)$ is included in a maximally consistent context $((LF \cup \{b:\neg A\})^*, RF^*)$. Then, by Lemma 3, $((LF \cup \{b:\neg A\})^*, RF^*) \models^{M^c} b:\neg A$, i.e. $((LF \cup \{w:\neg A\})^*, RF^*) \not\models^{\mathcal{S}^c} b:A$, and thus $(LF, RF) \not\models^{\mathcal{S}^c} w:A$. \square

We can reason similarly to show the soundness and completeness of MSpQR with respect to the corresponding semantics: Theorem 2 follows from Theorems 5 and 6 below.

Theorem 5 (Soundness of MSpQR). $\Gamma \vdash \alpha$ implies $\Gamma \models \alpha$.

Proof. We let \mathcal{M} be an arbitrary model and prove that if $\Gamma \vdash \alpha$ then $\models \mathcal{J}(\Gamma)$ implies $\models \mathcal{J}(\alpha)$ for any \mathcal{J} . The proof proceeds by induction on the structure of the derivation of α from Γ . The base case, where $\alpha \in \Gamma$, is trivial. There is one step case for each rule of MSpQR, where the soundness of the rules $\supset I$, $\supset E$, RAA , $\perp E$, $Urefl$, $Usym$, $Utrans$ follows exactly like in the proof of Theorem 3.

The soundness of the rules $\star I$ and $\star E$ follows exactly like in the proof of Theorem 3, with the only difference that when \star is \Box then R is P .

The rule PUI is sound by property (i) in the definition of the semantics for MSpQR.

The rule $Ptrans$ is sound by property (ii) in the definition of the semantics for MSpQR.

The soundness of the rule *class* follows like for the soundness of the rule *Mser* in the proof of Theorem 3, this time exploiting property (iii) in the definition of the semantics for MSpQR.

The soundness of the rules *Psub1* and *Psub2* follows like for the soundness of the rules *Msub1* and *Msub2* in the proof of Theorem 3, this time exploiting property (iv) in the definition of the semantics for MSpQR. \square

To prove completeness (Theorem 4), we proceed like for the case of MSQR, mutatis mutandis in the construction of the canonical model. In particular, given a maximal consistent context (LF^*, RF^*) , we define the canonical structure $\mathcal{J}^c = \langle W^c, U^c, P^c, V^c, \mathcal{J}^c \rangle$ by setting

- $(a_i, a_j) \in P^c$ iff $a_i Pa_j \in (LF^*, RF^*)$.

To show that the conditions on R^c are satisfied, so that \mathcal{J}^c is really a structure for MSpQR, we reuse the results proved for MSQR and in addition show the following:

- $\forall v, w \in W^c. vPw \implies vUw$ holds by construction and rule *PUI*.
- $\forall v, w', w'' \in W^c. vPw' \ \& \ w'Pw'' \implies vPw''$ holds by construction and rule *Ptrans*.
- $\forall v \in W^c. \exists w \in W^c. vPw \ \& \ wPv$ holds by construction and rule *class*. For the sake of contradiction, consider an arbitrary a_i and a variable a'_j that do not satisfy the property. Define $(LF', RF') = (LF^*, RF^*) \cup \{a_i Pa'_j, a'_j Pa_i\}$. Then it cannot be the case that $(LF', RF') \vdash \alpha$, for otherwise $(LF^*, RF^*) \vdash \alpha$ would be derivable by an application of the rule *class*. Thus, $(LF', RF') \not\vdash \alpha$. But then (LF', RF') must be in the chain of contexts built in Lemma 2. So, by the maximality of (LF^*, RF^*) , we have that $(LF', RF') = (LF^*, RF^*)$, contradicting our assumption. Hence, for some a_j , the r-formulas $a_i Ma_j$ and $a_j Ma_i$ are both in (LF^*, RF^*) , which is what we had to show.
- $\forall v, w \in W^c. vPv \ \& \ vPw \implies v = w$ holds by construction and rules *Psub1* and *Psub2* since v is a classical world. Consider an arbitrary context (LF, RF) from which we build \mathcal{J}^c and assume $(a_i, a_i) \in P^c$ and $(a_i, a_j) \in P^c$. Then $a_i Pa_i \in (LF^*, RF^*)$ and $a_i Pa_j \in (LF^*, RF^*)$. Thus, for each $a_i : A \in (LF^*, RF^*)$, we also have $a_j : A \in (LF^*, RF^*)$; otherwise, since (LF^*, RF^*) is deductively closed, we would have $a_j : \neg A \in (LF^*, RF^*)$ and also $a_j : A \in (LF^*, RF^*)$ by 1 in Lemma 2 and rule *Psub1*, and thus a contradiction. Similarly, if $a_j : A \in (LF^*, RF^*)$ then $a_i : A \in (LF^*, RF^*)$ by rule *Psub2*. Hence, for each m-formula A , we have that $a_i : A \in (LF^*, RF^*)$ iff $a_j : A \in (LF^*, RF^*)$, which means that a_i and a_j are equal with respect to m-formulas.

Under the same assumptions, we can similarly show that a_i and a_j are equal with respect to r-formulas, i.e. that whenever (LF^*, RF^*) contains an r-formula that includes a_i then it also contains the same r-formula with a_j substituted for a_i , and vice versa. To this end, we must consider 8 different cases corresponding to 8 different r-formulas.

- If $a_k Ua_i \in (LF^*, RF^*)$ for some a_k , then from the assumption that $a_i Pa_j \in (LF^*, RF^*)$ we have $a_i Ua_j \in (LF^*, RF^*)$, by 1 in Lemma 2 and rule *PUI*. Therefore, $a_k Ua_j \in (LF^*, RF^*)$ by rule *Utrans*.
- We can reason similarly for $a_j Ua_k \in (LF^*, RF^*)$ and also apply rules *PUI* and *Utrans* to conclude that then also $a_i Ua_k \in (LF^*, RF^*)$.
- If $a_i Ua_k \in (LF^*, RF^*)$ for some a_k , then from the assumption that $a_i Pa_j \in (LF^*, RF^*)$ we have $a_i Ua_j \in (LF^*, RF^*)$, by 1 in Lemma 2 and rule *PUI*, and thus $a_j Ua_i \in (LF^*, RF^*)$, by rule *Usym*. Therefore, $a_j Ua_k \in (LF^*, RF^*)$ by rule *Utrans*.

- We can reason similarly for $a_k \cup a_j \in (LF^*, RF^*)$ and also apply rules *PUI*, *Usymm*, and *Utrans* to conclude that then also $a_k \cup a_i \in (LF^*, RF^*)$.
- If $a_k \text{Pa}_i \in (LF^*, RF^*)$ for some a_k , then from the assumption that $a_i \text{Pa}_j \in (LF^*, RF^*)$ we have $a_k \text{Pa}_j \in (LF^*, RF^*)$, by *I* in Lemma 2 and the rule *Ptrans*.
- We can reason similarly for $a_j \text{Pa}_k \in (LF^*, RF^*)$ and also apply rule *Ptrans* to conclude that then also $a_i \cup a_k \in (LF^*, RF^*)$.
- If $a_i \text{Pa}_k \in (LF^*, RF^*)$ for some a_k , then from the assumptions that $a_i \text{Pa}_i \in (LF^*, RF^*)$ and $a_i \text{Pa}_j \in (LF^*, RF^*)$ we have $a_j \text{Pa}_k \in (LF^*, RF^*)$, by *I* in Lemma 2 and rule *Psub1*.
- We can reason similarly for $a_k \text{Pa}_j \in (LF^*, RF^*)$ and apply rule *Psub2* to conclude that then also $a_k \text{Pa}_i \in (LF^*, RF^*)$.

Hence, a_i and a_j are equal also with respect to *r*-formulas, and thus $a_i = a_j$ whenever $(a_i, a_i) \in P^c$ and $(a_i, a_j) \in P^c$, which is what we had to show.

Proceeding like for MSQR, we then have:

Theorem 6 (Completeness of MSpQR). $\Gamma \models \alpha$ implies $\Gamma \vdash \alpha$.

△