A representation theorem for quantale valued sup-algebras

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Abstract—With this paper we hope to contribute to the theory of quantales and quantale-like structures. It considers the notion of Q-sup-algebra and shows a representation theorem for such structures generalizing the well-known representation theorems for quantales and sup-algebras. In addition, we present some important properties of the category of Q-sup-algebras.

Index Terms—sup-lattice, sup-algebra, quantale, *Q*-module, *Q*-order, *Q*-sup-lattice, *Q*-sup-algebra.

INTRODUCTION

Two equivalent structures, quantale modules [1] and Q-sup-lattices [13] were independently introduced and studied. Stubbe [11] constructed an isomorphism between the categories of right Qmodules and cocomplete skeletal Q-categories for a given unital quantale Q. Employing his results, Solovyov [10] obtained an isomorphism between the categories of Q-algebras and Q-valued quantales, where Q is additionally assumed to be commutative.

Resende introduced (many-sorted) sup-algebras that are certain partially ordered algebraic structures which generalize quantales, frames and biframes (pointless topologies) as well as various lattices of multiplicative ideals from ring theory and functional analysis (C*-algebras, von Neumann algebras). One-sorted case was studied e.g. by Zhang and Laan [12], Paseka [4], and, in the generalized form of Q-sup-algebras by Šlesinger [7], [8].

The paper is organized as follows. First we present several necessary algebraic concepts as suplattice, sup-algebra, quantale and quantale module. Using quantales as a base structure for valuating fuzzy concepts, we recall the notion of a Q-order –

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a fuzzified variant of partial order relations, and a Q-sup-lattice for a fixed unital commutative quantale Q.

We then recall Solovyov's isomorphism between the category of Q-sup-lattices and the category of Q-modules, and between the category of Q-supalgebras and the category of Q-module-algebras. This isomorphism provides a relation between quantale-valued sup-algebras, which are expressed through fuzzy concepts, and quantale modulealgebras, which is a notion expressed in terms of universal algebra.

In Section II we establish several important properties of the category of Q-sup-algebras, e.g., an adjoint situation between categories of Q-sup-algebras and Q-algebras, and the fact that the category of Qsup-algebras is a monadic construct.

In Section III we focus on the notion of a Q-MAnucleus for Q-module-algebras and state its main properties. In the last section we introduce a Q-MAnucleus on the free Q-sup-algebra and using it, we establish our main theorem for Q-sup-algebras that generalizes the well-known representation theorems for quantales and sup-algebras.

In this paper, we take for granted the concepts and results on quantales, category theory and universal algebra. To obtain more information on these topics, we direct the reader to [2], [3] and [6].

I. BASIC NOTIONS, DEFINITIONS AND RESULTS

A. Sup-lattices, sup-algebras, quantales and quantale modules

A sup-lattice A is a partially ordered set (complete lattice) in which every subset S has a join

(supremum) $\bigvee S$, and therefore also a meet (infimum) $\bigwedge S$. The greatest element is denoted by \top , the least element by \bot . A *sup-lattice homomorphism* f between sup-lattices A and B is a joinpreserving mapping from A to B, i.e., $f(\bigvee S) =$ $\bigvee \{f(s) \mid s \in S\}$ for every subset S of A. The category of sup-lattices will be denoted **Sup**. Note that a mapping $f: A \to B$ if a sup-lattice homomorphism if and only if it has a *right adjoint* $g: B \to A$, by which is meant a mapping g that satisfies

$$f(a) \le b \iff a \le g(b)$$

for all $a \in A$ and $b \in B$. We write $f \dashv g$ in order to state that g is a right adjoint to f (equivalently, f is a left adjoint to g).

A type is a set Ω of function symbols. To each $\omega \in \Omega$, a number $n \in \mathbb{N}_0$ is assigned, which is called the *arity* of ω (and ω is called an *n*-ary function symbol). Then for each $n \in \mathbb{N}_0$, $\Omega_n \subseteq \Omega$ will denote the subset of all *n*-ary function symbols from Ω .

Given a set Ω , an algebra of type Ω (shortly, an Ω -algebra) is a pair $\mathcal{A} = (A, \Omega)$ where for each $\omega \in \Omega$ with arity *n*, there is an *n*-ary operation $f_{\omega}: A^n \to A$.

A sup-algebra of type Ω (shortly, a sup-algebra) is a triple $\mathcal{A} = (A, \bigvee, \Omega)$ where (A, \bigvee) is a suplattice, (A, Ω) is an Ω -algebra, and each operation ω is join-preserving in any component, that is,

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1, \ldots, n\}$, $a_1, \ldots, a_n \in A$, and $B \subseteq A$.

A join-preserving mapping $\phi: A \to B$ from a sup-algebra (A, \bigvee, Ω) to a sup-algebra (B, \bigvee, Ω) is called a *sup-algebra homomorphism* if

$$\omega_B(\phi(a_1),\ldots,\phi(a_n))=\phi(\omega_A(a_1,\ldots,a_n))$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $a_1, \ldots, a_n \in A$, and

$$\omega_B = \phi(\omega_A)$$

for any $\omega \in \Omega_0$.

Common instances of sup-algebras include the following (operation arities that are evident from context are omitted):

1) sup-lattices with $\Omega = \emptyset$,

- 2) (commutative) quantales Q [6] with Ω = {·} such that · is an associative (and commutative) binary operation, and unital quantales Q with Ω = {·, 1} such that 1 is the unit of the associative binary operation ·,
- 3) *quantale modules* A [6] with $\Omega = \{q * \mid q \in Q\}$ such that Q is a quantale such that $(\bigvee S) * a = \bigvee_{s \in S} s * a$ and $p * (q * a) = (p \cdot q) * a$ for all $S \subseteq Q$, $p, q \in Q$ and $a \in A$.

For any element q of a quantale Q, the unary operation $q \cdot -: Q \to Q$ is join-preserving, therefore it has a (meet-preserving) right adjoint $q \to -: Q \to$ Q, characterized by $q \cdot r \leq s \iff r \leq q \to s$. Written explicitly, $q \to s = \bigvee \{r \in Q \mid q \cdot r \leq s\}.$

Similarly, there is a right adjoint $q \leftarrow -: Q \rightarrow Q$ for $-\cdot q$, characterized by $r \cdot q \leq s \iff r \leq q \leftarrow s$, and satisfying $q \leftarrow s = \bigvee \{r \in Q \mid r \cdot q \leq s\}$. If Q is commutative, the operations \rightarrow and \leftarrow clearly coincide, and we will keep denoting them \rightarrow .

Note that the real unit interval [0, 1] with standard partial order and multiplication of reals is a commutative quantale.

B. *Q*-sup-lattices

By a *base quantale* we mean a unital commutative quantale Q. The base quantale is the structure in which Q-orders and Q-subsets are to be evaluated. For developing the theory in the rest of this paper, let Q be an arbitrary base quantale that remains fixed from now on. Note that we do not require the multiplicative unit 1 of the base quantale to be its greatest element \top .

Let X be a set. A mapping $e: X \times X \rightarrow Q$ is called a Q-order if for any $x, y, z \in X$ the following are satisfied:

- 1) $e(x, x) \ge 1$ (reflexivity),
- 2) $e(x,y) \cdot e(y,z) \le e(x,z)$ (transitivity),
- 3) if $e(x, y) \ge 1$ and $e(y, x) \ge 1$, then x = y (antisymmetry).

The pair (X, e) is then called a *Q*-ordered set. For a *Q*-order *e* on *X*, the relation \leq_e defined as $x \leq_e y \iff e(x, y) \geq 1$ is a partial order in the usual sense. This means that any *Q*-ordered set can be viewed as an ordinary poset satisfying additional properties.

Vice versa, for a partial order \leq on a set X we can define a Q-order e_{\leq} by

$$e_{\leq}(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

A *Q*-subset of a set X is an element of the set Q^X .

For Q-subsets M, N of a set X, we define the subsethood degree of M in N as

$$sub_X(M,N) = \bigwedge_{x \in X} (M(x) \to N(x)).$$

In particular, (Q^X, sub_X) is a Q-ordered set.

Let M be a Q-subset of a Q-ordered set (X, e). An element s of X is called a Q-*join* of M, denoted |M| M if:

- 1) $M(x) \le e(x, s)$ for all $x \in X$, and
- 2) for all $y \in X$, $\bigwedge_{x \in X} (M(x) \to e(x, y)) \le e(s, y)$.

If $\bigsqcup M$ exists for any $M \in Q^X$, we call (X, e)*Q-join complete*, or a *Q-sup-lattice*.

Let X and Y be sets, and $f: X \to Y$ be a mapping. **Zadeh's forward power set operator** for f maps Q-subsets of X to Q-subsets of Y by

$$\vec{f_Q}(M)(y) = \bigvee_{x \in f^{-1}(y)} M(x).$$

Let (X, e_X) and (Y, e_Y) be Q-ordered sets. We say that a mapping $f: X \to Y$ is Q-join-preserving if for any Q-subset M of X such that $\bigsqcup M$ exists, $\bigsqcup_Y f_Q^{-}(M)$ exists and

$$f\left(\bigsqcup_X M\right) = \bigsqcup_Y f_Q^{\rightarrow}(M).$$

It is known that the category Q-Sup of Q-suplattices and Q-join-preserving mappings is isomorphic to the category Q-Mod of Q-modules (see [10], [11]). We have functors F and Q such that

- 1) $F: Q-\text{Mod} \to Q-\text{Sup}$, given a Q-module A: $e(a,b) = a \to_Q b$, and $\bigsqcup M = \bigvee_{a \in A} (M(a) * a)$
- 2) G: Q-Sup $\rightarrow Q$ -Mod, given a Q-sup-lattice A:

$$a \le b \iff 1 \le e(a,b), \forall S = \bigsqcup M_S^1, q * a = \bigsqcup M_a^q \text{ where } M_S^q(a) = \begin{cases} q & \text{if } x \in S, \\ \bot & \text{otherwise.} \end{cases}$$

3) A mapping that is a morphism in either category, becomes a morphism in the other one as well. 4) $G \circ F = 1_{Q-Mod}$ and $F \circ G = 1_{Q-Sup}$, i.e., the categories Q-Mod and Q-Sup are isomorphic.

Hence results on quantale modules can be directly transferred to *Q*-sup-lattices and conversely. We will speak about *Solovyov's isomorphism*.

C. Q-sup-algebras and Q-module-algebras

A *Q*-sup-algebra of type Ω (shortly, a *Q*-supalgebra) is a triple $\mathcal{A} = (A, \bigsqcup, \Omega)$ where (A, \bigsqcup) is a *Q*-sup-lattice, (A, Ω) is an Ω -algebra, and each operation ω is *Q*-join-preserving in any component, that is,

$$\omega\left(a_{1},\ldots,a_{j-1},\bigsqcup M,a_{j+1},\ldots,a_{n}\right) = \\ \bigsqcup \omega\left(a_{1},\ldots,a_{j-1},-,a_{j+1},\ldots,a_{n}\right)\overrightarrow{\rho}(M)$$

for any $n \in \mathbb{N}, \omega \in \Omega_n, j \in \{1, \ldots, n\}, a_1, \ldots, a_n \in A$, and $M \in Q^A$.

Let (A, \bigsqcup_A, Ω) and (B, \bigsqcup_B, Ω) be Q-sup algebras, and $\phi: A \to B$ be a Q-join-preserving mapping and a sup-algebra homomorphism. Then ϕ is called a Q-sup-algebra homomorphism.

As instances of Q-sup-algebras, we may typically encounter the Q-counterparts of those from examples of sup-algebras:

- 1) Q-sup-lattices ($\Omega = \emptyset$),
- Q-quantales (Ω = {·}) such · is an associative binary operation. Q-quantales correspond via Solovyov's isomorphism to quantale algebras [10].

By *Q*-module-algebra of type Ω (shortly, *Q*-module-algebra) we will denote the structure $\mathcal{A} = (A, \bigvee, *, \Omega)$ where $(A, \bigvee, *)$ is a *Q*-module, (A, Ω) is an Ω -algebra, and each operation ω is a *Q*-module homomorphism in any component, that is,

$$\omega (a_1, \dots, a_{j-1}, \bigvee B, a_{j+1}, \dots, a_n) = \bigvee \{ \omega (a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n) \mid b \in B \} \omega (a_1, \dots, a_{j-1}, q * b, a_{j+1}, \dots, a_n) = q * \omega (a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n)$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1, \ldots, n\}$, $a_1, \ldots, a_n, b \in A$, and $B \subseteq A$.

A mapping $\phi: A \to B$ from a Q-module-algebra $(A, \bigvee, *, \Omega)$ to a Q-module-algebra $(B, \bigvee, *, \Omega)$ is called a Q-module-algebra homomorphism if it is both a Q-module homomorphism and an Ω -algebra homomorphism.

For a given quantale Q and a type Ω , let Q-Sup- Ω -Alg denote the category of Q-supalgebras of type Ω with Q-sup-algebra homomorphisms, and Q-Mod- Ω -Alg the category of Qmodule-algebras of type Ω with Q-module-algebra homomorphisms. From [8, Theorem 3.3.15.] we know that Q-Sup- Ω -Alg and Q-Mod- Ω -Alg are isomorphic via Solovyov's isomorphism.

II. SOME CATEGORICAL PROPERTIES OF Q-SUP-ALGEBRAS

In this section we establish some categorical properties of the category Q-Sup- Ω -Alg needed in the sequel. We begin with a construction of a Qsup-algebra of type Ω from an Ω -algebra to obtain an adjoint situation. Using this result we prove that Q-Sup- Ω -Alg is a monadic construct (see [2]).

As shown e.g. in [8, Theorem 2.2.43], Q^X is the free Q-sup-lattice over a set X (and also a free Q-module over X). But we can state more.

Theorem II.1. Any Ω -algebra A gives rise to a Q-sup-algebra Q^A with operations defined by

$$\bigcup_{Q^A} (A_1, \dots, A_n)(a) = \\ \bigvee_{\omega_A(a_1, \dots, a_n) = a} A_1(a_1) \cdot \dots \cdot A_n(a_n),$$

given $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$, $A_1, \ldots, A_n \in Q^A$, $\omega \in \Omega_n$.

Proof. By Solovyov's isomorphism it is enough to check that Q^A is a Q-module-algebra of type Ω . We have

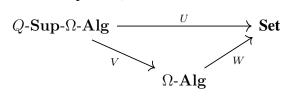
$$\begin{split} & \omega \left(A_1, \dots, A_{j-1}, \bigvee \Gamma, A_{j+1}, \dots, A_n \right) (a) = \\ & \bigvee_{\omega_A(a_1, \dots, a_n) = a} A_1(a_1) \cdot \dots \cdot A_{j-1}(a_{j-1}) \cdot \\ & \vee \Gamma(a_j) \cdot A_{j+1}(a_{j+1}) \cdot \dots \cdot A_n(a_n) = \\ & \vee \{ \bigvee_{\omega_A(a_1, \dots, a_n) = a} A_1(a_1) \cdot \dots \cdot A_{j-1}(a_{j-1}) \cdot \\ & B(a_j) \cdot A_{j+1}(a_{j+1}) \cdot \dots \cdot A_n(a_n)) \mid B \in \Gamma \} = \\ & \vee \{ \omega(A_1, \dots, A_{j-1}, B, A_{j+1}, \dots, A_n) \mid B \in \Gamma \} (a) \end{split}$$

and

$$\omega (A_{1}, \dots, A_{j-1}, q * B, a_{j+1}, \dots, A_{n}) (a) =
\bigvee_{\omega_{A}(a_{1}, \dots, a_{n})=a} A_{1}(a_{1}) \cdot \dots \cdot A_{j-1}(a_{j-1}) \cdot
(q \cdot B(a_{j})) \cdot A_{j+1}(a_{j+1}) \cdot \dots \cdot A_{n}(a_{n}) =
\bigvee_{\omega_{A}(a_{1}, \dots, a_{n})=a} q \cdot (A_{1}(a_{1}) \cdot \dots \cdot A_{j-1}(a_{j-1}) \cdot B(a_{j}) \cdot A_{j+1}(a_{j+1}) \cdot \dots \cdot A_{n}(a_{n})) =
q \cdot \bigvee_{\omega_{A}(a_{1}, \dots, a_{n})=a} A_{1}(a_{1}) \cdot \dots \cdot A_{j-1}(a_{j-1}) \cdot B(a_{j}) \cdot A_{j+1}(a_{j+1}) \cdot \dots \cdot A_{n}(a_{n}) =
q * \omega(A_{1}, \dots, A_{j-1}, B, A_{j+1}, \dots, A_{n})(a)$$

for any $a \in A$, $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1, \ldots, n\}$, $a_1, \ldots, a_n \in A$, $A_1, \ldots, A_n, B \in Q^A$, $\Gamma \subseteq Q^A$, and $q \in Q$.

Note that there exists the following commutative triangle of the obvious forgetful functors (notice that Set is the category of sets and mappings, and Ω -Alg is the category of algebras of type Ω and their homomorphisms):



Theorem II.2. The forgetful functor V: Q-Sup- Ω -Alg $\rightarrow \Omega$ -Alg has a left adjoint F_Q .

Proof. Let A be an Ω -algebra. Let us show that Q^A is a free Q-module-algebra over A. For every $a \in A$ there exists a map $\alpha_a \in Q^A$ defined by

$$\alpha_a(b) = \begin{cases} 1, & \text{if } a = b, \\ \bot, & \text{otherwise.} \end{cases}$$

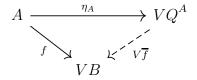
As in [10, Theorem 3.1] for quantale algebras, we obtain a Ω -algebra homomorphism $\eta_A: A \to VQ^A$ defined by $\eta_A(a) = \alpha_a$. Namely, $\eta_A(\omega(a_1, \ldots, a_n))(a) = \alpha_{\omega(a_1, \ldots, a_n)}(a) = 1$ if and only if $\omega(a_1, \ldots, a_n) = a$ (otherwise it is \bot), and

$$\bigvee_{\omega_A(b_1,\dots,b_n)=a} \eta_A(b_1) \cdot \dots \cdot \eta_A(b_n) = 1$$

if and only if $a_j = b_j$ for all $j \in \{1, ..., n\}$ and $\omega_A(b_1, ..., b_n) = a$ (otherwise it is \bot). Hence we obtain

$$\eta_A(\omega(a_1,\ldots,a_n)) = \omega(\eta_A(a_1),\ldots,\eta_A(a_n)).$$

It is easy to show that for every homomorphism $f: A \to VB$ in Ω -Alg there exists a unique homomorphism $\overline{f}: Q^A \to B$ in Q-Sup- Ω -Alg (given by $\overline{f}(\alpha) = \bigvee_{a \in A} \alpha(a) * f(a)$, where * is the module action on B given by Solovyov's isomorphism) such that the triangle



commutes. We will only check that

$$\overline{f}(\omega(\alpha_1,\ldots,\alpha_n)) = \omega_B(\overline{f}(\alpha_1),\ldots,\overline{f}(\alpha_n)).$$

Let us compute

$$\overline{f}(\omega(\alpha_1,\ldots,\alpha_n)) = \bigvee_{a \in A} \omega(\alpha_1,\ldots)(a) * f(a)$$

$$= \bigvee_{a \in A} \bigvee_{\omega_A(a_1,\ldots,a_n)=a} \alpha_1(a_1) \cdot \ldots \cdot \alpha_n(a_n) * f(a)$$

$$= \bigvee \{(\alpha_1(a_1) \cdot \ldots \cdot \alpha_n(a_n)) * f(\omega_A(a_1,\ldots,a_n)) \mid a_1,\ldots,a_n \in A\} =$$

$$\bigvee \{(\alpha_1(a_1) \cdot \ldots \cdot \alpha_n(a_n)) * \omega_B(f(a_1),\ldots)) \mid a_1,\ldots,a_n \in A\} =$$

$$\omega_B(\bigvee \{\alpha_1(a_1) * f(a_1) \mid a_1 \in A\},\ldots)) =$$

$$\omega_B(\overline{f}(\alpha_1),\ldots,\overline{f}(\alpha_n)).$$

The remaining properties of \overline{f} follow by the same considerations as in [10, Theorem 3.1].

Remark II.3. Note that similarly as in [10, Remark 3.2] we obtain an adjoint situation (η, ϵ) : $F_Q \dashv V$: Q-Sup- Ω -Alg $\rightarrow \Omega$ -Alg, where $F_Q(A) = Q^A$ for every Ω -algebra A of type Ω and ϵ_B : $F_QVB \rightarrow B$ is given by $\epsilon_B(\alpha) = \bigvee_{b \in B} \alpha(b) * b = \bigsqcup \alpha$ for every Q-sup-algebra B.

Since the functor $W: \Omega$ -Alg \rightarrow Set has a left adjoint (see [2]), we obtain the following.

Corollary II.4. U: Q-Sup- Ω -Alg \rightarrow Set has a left adjoint.

By the same categorical arguments as in [10] we obtain the following theorem and corollary.

Theorem II.5. The category Q-Sup- Ω -Alg of Q-sup-algebras of type Ω is a monadic construct.

Corollary II.6. The category Q-Sup- Ω -Alg is complete, cocomplete, wellpowered, extremally cowellpowered, and has regular factorizations. Moreover, monomorphisms are precisely those morphisms that are injective functions.

III. Q-NUCLEI IN Q-Mod- Ω -Alg and their properties

In this section we introduce the notion of a Q-MA-nucleus for Q-module-algebras and present its main properties.

Definition III.1. Let A be a Q-module-algebra of type Ω . A Q-module-algebra nucleus on A (shortly Q-MA-nucleus is a map $j: A \to A$ such that for any $n \in \mathbb{N}, \omega \in \Omega_n, a, b, a_1, \dots, a_n \in A$, and $q \in Q$:

(i) $a \le b$ implies $j(a) \le j(b)$; (ii) $a \le j(a)$; (iii) $j \circ j(a) \le j(a)$; (iv) $\omega(j(a_1), \dots, j(a_n)) \le j(\omega(a_1, \dots, a_n))$; (v) $q * j(a) \le j(q * a)$.

Note that by Solovyov's isomorphism *Q*-MAnuclei correspond to *Q*-ordered algebra nuclei introduced in [8].

By the same straightforward computations as in [10, Proposition 4.2 and Corollary 4.3] or [8, Proposition 3.3.6 and Proposition 3.3.8] we obtain the following.

Proposition III.2. Let A be a Q-module-algebra of type Ω and j be a Q-MA-nucleus on A. For any $n \in \mathbb{N}, \omega \in \Omega_n, a, a_1, \dots, a_n \in A, S \subseteq A$, and $q \in Q$ we have:

(i) $(j \circ j)(a) = j(a);$ (ii) $j(\bigvee S) = j(\bigvee j(S));$ (iii) $j(\omega(a_1, \dots, a_n)) = j(\omega(j(a_1), \dots, j(a_n)));$ (iv) j(q * a) = j(q * j(a)).

Corollary III.3. Let A be a Q-module-algebra of type Ω and j be a Q-MA-nucleus on A. Define $A_j = \{a \in A \mid j(a) = a\}$. Then $A_j = j(A)$ and, moreover, A_j is a Q-module-algebra of type Ω with the following structure:

- (i) $\bigvee_{A_j} S = j(\bigvee S)$ for every $S \subseteq A_j$;
- (ii) $\omega_{A_j}(a_1, \dots, a_n) = j(\omega(a_1, \dots, a_n))$ for every $n \in \mathbb{N}, \ \omega \in \Omega_n, \ a_1, \dots, a_n \in A_j;$
- (iii) $q *_{A_j} a = j(q * a)$ for every $a \in A_j$ and $q \in Q$.

IV. Representation theorem for
$$Q$$
-sup-algebras

Now we are ready to show a representation theorem for *Q*-sup-algebras.

Given a Q-module A, every $a \in A$ gives rise to the adjunction (in ordered sets) $A \xleftarrow{a \twoheadrightarrow \cdot}{ \swarrow \cdot * a} Q$ where $a \twoheadrightarrow b = \bigvee \{q \in Q \mid q * a \leq b\}.$

If moreover A is a Q-sup-algebra we use this adjunction to construct a nucleus on the (free) Q-sup-algebra Q^{VA} .

Proposition IV.1. Let A be a Q-sup-algebra of type Ω . There exists a Q-MA-nucleus j_A on Q^{VA} defined by $j_A(\alpha)(a) = a \twoheadrightarrow \epsilon_A(\alpha)$.

Proof. It is enough to check the conditions of Definition III.1. Conditions (i), (ii), (iii) and (v) follow

by the same consideration as in [10, Proposition 5.1]. Let us check condition (iv).

Let $n \in \mathbb{N}$, $\omega \in \Omega_n$, $a \in A$, $\alpha_1, \ldots, \alpha_n \in Q^{VA}$. We want that $\omega(j_A(\alpha_1), \ldots, j_A(\alpha_n))(a) \leq j_A(\omega(\alpha_1, \ldots, \alpha_n))(a) = a \twoheadrightarrow \epsilon_A(\omega(\alpha_1, \ldots, \alpha_n))$. We compute

$$\begin{split} &\omega(j_A(\alpha_1),\ldots,j_A(\alpha_n))(a) * a = \\ &\bigvee_{\omega_A(a_1,\ldots,a_n)=a} j_A(\alpha_1)(a_1) \cdot \ldots \cdot j_A(\alpha_n)(a_n) * a = \\ &\bigvee_{\omega_A(a_1,\ldots,a_n)=a} \omega_A((a_1 \twoheadrightarrow \epsilon_A(\alpha_1)) * a_1,\ldots) \leq \\ &\omega_A(\epsilon_A(\alpha_1),\ldots,\epsilon_A(\alpha_n)) \leq \epsilon_A(\omega(\alpha_1,\ldots,\alpha_n)). \end{split}$$

The last inequality is valid because ϵ_A is a homomorphism of Q-sup-algebras.

As in [10] we introduce, for any element $a \in A$ of a Q-sup-algebra A, a map $\beta_a \in Q^{VA}$ defined by $\beta_a(x) = x \rightarrow a$. By the same arguments as in [10, Lemma 5.2] we obtain the following.

Lemma IV.2. Let A be a Q-sup-algebra of type Ω . For every $a \in A$:

(i)
$$\epsilon_A(\beta_a) = a$$
, (ii) $\beta_a \in (Q^{VA})_{j_a}$

Using the above lemma, we can conclude with our main theorem.

Theorem IV.3. (Representation Theorem). Let A be a Q-sup-algebra of type Ω . The map $\rho_A: A \rightarrow (Q^{VA})_{j_A}$ defined by $\rho_A(a) = \beta_a$ is an isomorphism of Q-sup-algebras.

Proof. By mimicking the proof of [10, Theorem 5.3] we get that ρ_A is a bijective Q-module homomorphism, i.e., it is a bijective homomorphism of Q-sup-lattices. Let us prove that $\omega_{(Q^{VA})_{j_A}}(\rho_A(a_1),\ldots,\rho_A(a_n)) = \rho_A(\omega_A(a_1,\ldots,a_n))$ for all $n \in \mathbb{N}, \omega \in \Omega_n$, and $a_1,\ldots,a_n \in A$. We compute

$$\omega_{(Q^{VA})_{j_A}}(\rho_A(a_1),\ldots,\rho_A(a_n))(c) =$$

$$j_A(\omega(\rho_A(a_1),\ldots,\rho_A(a_n)))(c) =$$

$$c \twoheadrightarrow \epsilon_A(\omega(\rho_A(a_1),\ldots,\rho_A(a_n))) =$$

$$c \twoheadrightarrow \omega_A(\epsilon_A(\rho_A(a_1)),\ldots,\epsilon_A(\rho_A(a_n))) =$$

$$c \twoheadrightarrow \omega_A(a_1,\ldots,a_n) = \rho_A(\omega_A(a_1,\ldots,a_n))(c).$$

Hence ρ_A is an isomorphism of Q-sup-algebras.

Remark IV.4. Note that in case of Q = 2 our Theorem follows from [5, Theorem 2.2.30]. As noted by Solovyov in [10] β_a corresponds to the lower set $\downarrow a$ and $\epsilon_A : 2^A \to A$ is the join operation on A.

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