Hamiltonian-Laceability of Star Graphs

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Abstract

Suppose G is a bipartite graph with two partite sets of equal size. G is said to be strongly hamiltonian-laceable if there is a hamiltonian path between every two vertices that belong to different partite sets, and there is a path of (maximal) length $N-2$ between every two vertices that belong to the same partite set, where N is the order of G. The star graph is known to be bipartite. In this paper, we show that the n-dimensional star graph, where $n \geq 4$ is strongly hamiltonian-laceable.

1 Introduction

Usually when the hamiltonicity of a graph G is concerned, it is investigated whether G is hamiltonian or hamiltonian-connected. A cycle (path) in G is called a hamiltonian cycle (path) if it contains every vertex of G exactly once. G is said to be hamiltonian if it contains a hamiltonian cycle, and hamiltonian-connected if there exists a hamiltonian path between every two vertices of G. Since a bipartite graph is not hamiltonian-connected, Wong [5] has introduced the concept of hamiltonian-laceability for the class of bipartite graphs. A bipartite graph $G = (V1, V2, E)$ with $|V1| = |V2|$ is hamiltonian-laceable if there is a hamiltonian path between every vertex of V_1 and every vertex of V_2, where V_1 and V_2 are the two partite sets of G. We note that any path between two vertices of the same partite set has length at most $|V1|+|V2|-2$.

It is meaningful to extend the concept of hamiltonian-laceability so that the lengths of the paths between two vertices of the same partite set are specified and the edge faults are considered. We say that a hamiltonian-laceable graph $G = (V1, V2, E)$ is strongly if G additionally owns the property that there is a path of length $|V1|+|V2|-2$ between every two vertices of the same partite set. Further, G is k edge fault-tolerant strongly hamiltonian-laceable if it remains strongly hamiltonian-laceable after removing at most k edges. In other words, there is a longest path between every two vertices of a k edge fault-tolerant strongly hamiltonian-laceable graph G, even if at most k edges of G are removed. The longest path has length $|V1| + |V2| - 1$ if the two vertices belong to different partite sets, and $|V1| + |V2| - 2$ if the two vertices belong to the same partite set.

The star graph [1], which belongs to the class of Cayley graphs, has been recognized as an attractive alternative to the hypercube. It possesses many nice topological properties, e.g., recursiveness, vertex and edge symmetry, maximal fault tolerance, sublogarithmic degree and diameter, and strong resilience [1] [2], which are desirable when we are building an interconnection topology for parallel and distributed systems. In [3], Jwo, Lakshmivarahan, and Dhall have shown that the star graph is bipartite. Besides, its two partite sets have equal size. In this paper we show that the n-dimensional star graph is strongly hamiltonian-laceable when $n \geq 4$.

2 Prelimiaries

The *n-dimensional star graph*, denoted by S_n, is defined as follows.

Definition 1 *The vertex set of S_n is denoted by $\{a_1a_2...a_n | \ a_1a_2...a_n$ is a permutation of $\{1, 2, ..., n\}\}$. Vertex adjacency is defined as follows: $a_1a_2...a_n$ is adjacent to $a_ia_2...a_{i-1}a_1a_{i+1}...a_n$ for all $2 \leq i \leq n$. The vertices of S_n are $n!$ permutations of $\{1, 2, ..., n\}$, and there is an edge between two vertices of S_n if and only if they can be obtained from each other by swapping the leftmost number with one of the other $n-1$ numbers. For convenience we refer to the position of a_i in $a_1a_2...a_n$ as the ith dimension, and $(a_1a_2...a_n, a_ia_2...a_{i-1}a_1a_{i+1}...a_n)$ as the ith-dimensional edge.*

Definition 2 *There are embedded S_r's contained in S_n, where $1 \leq r \leq n$. An embedded S_r can be conveniently represented by $< s_1s_2...s_n >_r$, where $s_1 = *$, $s_i \in \{*, 1, 2, ..., n\}$ for all $2 \leq i \leq n$, and exactly r of s_1, s_2, ..., s_n are $*$ ($*$ denotes a "don't care" symbol).*

Definition 3 *An i-partition on $< s_1s_2...s_n >_r$ partitions $< s_1s_2...s_n >_r$ into r embedded $S'_{r-1}s$, denoted by $< s_1s_2...s_{i-1}qs_{i+1}...s_n >_{r-1}$, where $2 \leq i \leq n$, $s_i = *$, and $q \in \{1, 2, ..., n\} - \{s_1, s_2, ..., s_n\}$.*

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Definition 4 *An $(i_1, i_2, ..., i_m)$-partition on $< s_1 s_2 ... s_n >_r$ performs an i_1-partition, an i_2-partition, ..., an i_m-partition, sequentially, on $< s_1 s_2 ... s_n >_r$, where $i_1 i_2 ... i_m$ is a permutation of m elements from $\{2, 3, ..., n\}$.*

Definition 5 *Two embedded S_r's $< s_1 s_2 ... s_n >_r$ and $< t_1 t_2 ... t_n >_r$ are said to be* adjacent *if $s_j \neq *$, $t_j \neq *$, and $s_j \neq t_j$ for some $2 \leq j \leq n$, and $s_i = t_i$ for all $1 \leq i \leq n$ and $i \neq j$. Moreover, the position j is denoted by $dif(< s_1 s_2 ... s_n >_r, < t_1 t_2 ... t_n >_r)$.*

Definition 6 *Let A_1, A_2, ..., $A_{n(n-1)(n-2)\cdots(r+1)}$ represent those embedded S_r's that are obtained by executing an $(i_1, i_2, ..., i_{n-r})$-partition on S_n, where $1 \leq r \leq n-1$. They form an r-*path*, denoted by $P_r = [A_1, A_2, ..., A_{n(n-1)(n-2)\cdots(r+1)}]$, if A_i is adjacent to A_{i+1} for all $1 \leq i \leq n(n-1)(n-2)\cdots(r+1)-1$. Each vertex of P_r i.e., A_i, is called an r-vertex, and each edge of P_r, i.e., (A_i, A_{i+1}), is called an r-edge.*

Definition 7 *An* i-partition *on $P_r = [A_1, A_2, ..., A_{n(n-1)(n-2)\cdots(r+1)}]$ performs an i-partition on A_1, A_2, ..., $A_{n(n-1)(n-2)\cdots(r+1)}$, respectively, where $2 \leq i \leq n$ and $r \geq 2$. After an i-partition, each A_j is partitioned into r $(r-1)$-vertices, where $1 \leq j \leq n(n-1)(n-2)\cdots(r+1)$. Since every two of the r $(r-1)$-vertices are joined with an $(r-1)$-edge, each A_j can be viewed as a complete graph of r $(r-1)$-vertices. Throughout this paper, we refer to the complete graph as K_r^{r-1}. We note that each vertex of K_r^{r-1} is an $(r-1)$-vertex and each edge of K_r^{r-1} is an $(r-1)$-edge.*

3 Hamiltonian-Laceability of Star Graphs

In this section we show S_n with $n \geq 4$ is strongly hamiltonian-laceable.

Lemma 3.1 *Suppose $U = < u_1 u_2 ... u_n >_r$, $V = < v_1 v_2 ... v_n >_r$, and $W = < w_1 w_2 ... w_n >_r$ are arbitrary three consecutive r-vertices in a P_r, where $r \geq 2$. Let $p = dif(U, V)$ and $q = dif(V, W)$. If $u_p \neq w_q$, then after executing a partition on the P_r each $(r-1)$-vertex of V is connected to U or W.*

Proof: Without loss of generality, we assume that a j-partition is executed on the P_r, where $2 \leq j \leq n$. Hence, $u_j = v_j = w_j = *$. Since $p = dif(U, V) \neq 1$ and $q = dif(V, W) \neq 1$, we have $u_p \neq v_p$, $v_q \neq w_q$, $u_i = v_i$ for all $1 \leq i \leq n$ and $i \neq p$, and $v_i = w_i$ for all $1 \leq i \leq n$ and $i \neq q$. Suppose conversely $u_p \neq w_q$ and there exists an $(r-1)$-vertex, say $V_1 = < v_1 v_2 ... v_{j-1} z v_{j+1} ... v_n >_{r-1}$, of V which is not connected to either of U and W. Thus, $z = u_p$, for otherwise V_1 is adjacent to some $(r-1)$-vertex of U. Similarly, $z = w_q$. This implies $u_p = w_q$, which contradicts our assumption. **Q.E.D.**

Lemma 3.2 *Suppose u and v are arbitrary two distinct vertices of S_n with $n \geq 4$. There exists a P_{n-1} whose first $(n-1)$-vertex contains u and whose last $(n-1)$-vertex contains v.*

Proof: Suppose $u = u_1 u_2 ... u_n$ and $v = v_1 v_2 ... v_n$. Without loss of generality, we assume $u_j \neq v_j$ for some $2 \leq j \leq n$. After a j-partition, S_n is partitioned into n $(n-1)$-vertices, which form a K_n^{n-1}. Clearly, u and v belong to two different vertices, say U and V, of the K_n^{n-1}. The desired P_{n-1} can be constructed as a hamiltonian path from U to V in the K_n^{n-1}. **Q.E.D.**

In the rest of this paper, we suppose u and v are the beginning vertex and the ending vertex, respectively, of a path. We call an r-vertex the beginning r-vertex (ending r-vertex) if it contains u (v). Besides, a path from U to V is abbreviated to a $U - V$ path.

Lemma 3.3 *A P_{r-1} whose first $(r-1)$-vertex is the beginning $(r-1)$-vertex and whose last $(r-1)$-vertex is the ending $(r-1)$-vertex can be obtained from a P_r whose first r-vertex is the beginning r-vertex and whose last r-vertex is the ending r-vertex, where $4 \leq r \leq n-1$ and $n \geq 5$.*

Proof: Suppose $P_r = [A_1, A_2, ..., A_{n(n-1)(n-2)\cdots(r+1)}]$, where A_1 is the beginning r-vertex and $A_{n(n-1)(n-2)\cdots(r+1)}$ is the ending r-vertex. After executing a partition on the P_r, each A_i forms a K_r^{r-1}, where $1 \leq i \leq n(n-1)(n-2)\cdots(r+1)$. Since each A_i contains at least three $(r-1)$-vertices, we can select two distinct $(r-1)$-vertices, say X_i and Y_i, from each A_i so that X_1 is the beginning $(r-1)$-vertex, $Y_{n(n-1)(n-2)\cdots(r+1)}$ is the ending $(r-1)$-vertex, and for $2 \leq j \leq n(n-1)(n-2)\cdots(r+1)-1$, X_j and Y_j are adjacent to Y_{j-1} and X_{j+1}, respectively. Since there exists a hamiltonian $X_i - Y_i$ path in the K_r^{r-1} formed by A_i, the desired P_{r-1} can be obtained by concatenating all the hamiltonian paths interleaved with $(r-1)$-edges (Y_1, X_2), (Y_2, X_3), ..., $(Y_{n(n-1)(n-2)\cdots(r+1)-1}, X_{n(n-1)(n-2)\cdots(r+1)})$. **Q.E.D.**

In the rest of this paper, X_i and Y_i as specified above are referred to as the entry $(r-1)$-vertex and the exit $(r-1)$-vertex of A_i, respectively.

Lemma 3.4 *A P_5 whose first 5-vertex is the beginning 5-vertex and whose last 5-vertex is the ending 5-vertex can be obtained in S_n with $n > 5$.*

A $P_r = [A_1, A_2, ..., A_{n(n-1)(n-2)\cdots(r+1)}]$ in S_n, where $2 \leq r \leq n-1$, is said to be *good* if it satisfies the following three conditions.

(Cond. 1) A_1 and $A_{n(n-1)(n-2)\cdots(r+1)}$ are the beginning and ending r-vertices, respectively.

(Cond. 2) For arbitrary three consecutive r-vertices $X = < x_1 x_2 ... x_n >_r$, $Y = < y_1 y_2 ... y_n >_r$, and $Z = < z_1 z_2 ... z_n >_r$ in the P_r, $x_{dif(X,Y)} \neq z_{dif(Y,Z)}$ holds.

(Cond. 3) After executing a k-partition on the P_r for some $2 \leq k \leq n$, the beginning (ending) $(r-1)$-vertex in A_1 ($A_{n(n-1)(n-2)\cdots(r+1)}$) is not connected to A_2 ($A_{n(n-1)(n-2)\cdots(r+1)-1}$).

In the rest of this section we show that a good P_3 can be obtained in S_n. Given arbitrary two vertices of S_n, a longest path between them can be constructed from a good P_3.

Lemma 3.5 *A good P_4 can be obtained from a P_5 whose first 5-vertex is the beginning 5-vertex and whose last 5-vertex is the ending 5-vertex.*

Proof: We suppose $P_5 = [A_1, A_2, ..., A_{n(n-1)(n-2)\cdots 6}]$, where A_1 and $A_{n(n-1)(n-2)\cdots 6}$ are the beginning and ending 5-vertices. Without loss of generality, we assume that the P_5 is obtained from S_n by executing an $(a_1, a_2, ..., a_{n-5})$-partition, where $a_1 a_2 ... a_{n-5}$ is an arrangement out of $\{2, 3, ..., n\}$. Let $j \in \{2, 3, ..., n\} - \{a_1, a_2, ..., a_{n-5}\}$. First, a j-partition is executed on the P_5, and so each A_i forms a K_5^4, where $1 \leq i \leq n(n-1)(n-2)\cdots 6$. In the rest of the proof we construct a good P_4 from the P_5 by establishing a hamiltonian path for each K_5^4.

Suppose $u = u_1 u_2 ... u_n$ and $v = v_1 v_2 ... v_n$ are the beginning and ending vertices, respectively. A hamiltonian path for the K_5^4 formed by A_1 can be established as follows. Let $X_1 = < x_1 x_2 ... x_n >_4$ be the beginning 4-vertex (in A_1), T be the 4-vertex of A_1 that is not connected to A_2, and $W = < w_1 w_2 ... w_n >_4$ be a 4-vertex of A_1 which is different from X_1 and has $w_j = u_k$ for some $k \in \{2, 3, ..., n\} - \{j, a_1, a_2, ..., a_{n-5}\}$. Since there are four 4-edges between A_1 and A_2, there exists a 4-vertex $Y_1 \notin \{X_1, W\}$ which is connected to A_2. If $X_1 = T$ or ($X_1 \neq T$ and $T = W$), a hamiltonian $X_1 - Y_1$ path can be established as $(X_1, W) + P[W, Y_1]$, where $P[W, Y_1]$ denotes a $W - Y_1$ path passing all the vertices of the K_5^4 but X_1 exactly once. Otherwise, if $X_1 \neq T$ and $T \neq W$, a hamiltonian $X_1 - Y_1$ path can be established as $(X_1, W) + (W, T) + P[T, Y_1]$, where $P[T, Y_1]$ denotes a $T - Y_1$ path passing all the vertices of the K_5^4 but X_1 and W exactly once. Then we continue to establish a hamiltonian path for the K_5^4 formed by $A_{n(n-1)(n-2)\cdots 6}$. The construction of the hamiltonian path is similar to the situation of A_1. Let $Y_{n(n-1)(n-2)\cdots 6}$ be the ending 4-vertex (in $A_{n(n-1)(n-2)\cdots 6}$, C be the 4-vertex of $A_{n(n-1)(n-2)\cdots 6}$ that is not connected to $A_{n(n-1)(n-2)\cdots 6-1}$, and $D = < d_1 d_2 ... d_n >_4$ be the 4-vertex of $A_{n(n-1)(n-2)\cdots 6}$ that is different from $Y_{n(n-1)(n-2)\cdots 6}$ and has $d_j = v_k$ (here, k is identical with that k appearing in the situation of A_1). There exists a vertex $X_{n(n-1)(n-2)\cdots 6} \notin \{D, Y_{n(n-1)(n-2)\cdots 6}\}$ which is connected to $A_{n(n-1)(n-2)\cdots 6-1}$. If $Y_{n(n-1)(n-2)\cdots 6} = C$ or $Y_{n(n-1)(n-2)\cdots 6} \neq C$ and $C = D$), a hamiltonian $X_{n(n-1)(n-2)\cdots 6} - Y_{n(n-1)(n-2)\cdots 6}$ path can be established as $P[X_{n(n-1)(n-2)\cdots 6}, D] + (D, Y_{n(n-1)(n-2)\cdots 6})$, where $P[X_{n(n-1)(n-2)\cdots 6}, D]$ denotes an $X_{n(n-1)(n-2)\cdots 6} - D$ path passing all the vertices of the K_5^4 but $Y_{n(n-1)(n-2)\cdots 6}$ exactly once. Otherwise, if $Y_{n(n-1)(n-2)\cdots 6} \neq C$ and $C \neq D$, a hamiltonian $X_{n(n-1)(n-2)\cdots 6} - Y_{n(n-1)(n-2)\cdots 6}$ path can be established as $P[X_{n(n-1)(n-2)\cdots 6}, C] + (C, D) + (D, Y_{n(n-1)(n-2)\cdots 6})$, where $P[X_{n(n-1)(n-2)\cdots 6}, C]$ denotes an $X_{n(n-1)(n-2)\cdots 6} - C$ path passing all the vertices of the K_5^4 but D and $Y_{n(n-1)(n-2)\cdots 6}$ exactly once.

In the discussion above, X_1 and Y_1 ($X_{n(n-1)(n-2)\cdots 6}$ and $Y_{n(n-1)(n-2)\cdots 6}$ are the entry and exit 4-vertices of A_1 ($A_{n(n-1)(n-2)\cdots 6}$), respectively. Additionally, we use X_i and Y_i to denote the entry and exit 4-vertices of A_i, respectively, for $2 \leq i \leq n(n-1)(n-2)\cdots 6-1$. Let L_i (Q_i) be the 4-vertex of A_i that is not connected to A_{i-1} (A_{i+1}). A hamiltonian $X_i - Y_i$ path in the K_5^4 formed by A_i can be established according to the following four cases. Case 1. $Q_i = X_i$ and $L_i = Y_i$. A hamiltonian $X_i - Y_i$ path can be established easily. Case 2. $Q_i \neq X_i$ and $L_i = Y_i$. A hamiltonian $X_i - Y_i$ path can be established as $(X_i, Q_i) + P[Q_i, Y_i]$, where $P[Q_i, Y_i]$ denotes a $Q_i - Y_i$ path passing all the vertices of the K_5^4 but X_i exactly once. Case 3. $Q_i = X_i$ and $L_i \neq Y_i$. A hamiltonian $X_i - Y_i$ path can be established as $P[X_i, L_i] + (L_i, Y_i)$, where $P[X_i, L_i]$ denotes an $X_i - L_i$ path passing all the vertices of the K_5^4 but Y_i exactly once. Case 4. $Q_i = X_i$ and $L_i \neq Y_i$. If $Q_i = L_i$, a hamiltonian $X_i - Y_i$ path can be established as $(X_i, U_i, L_i, V_i, Y_i)$, where U_i and V_i are the other two 4-vertices of A_i than X_i, Y_i, and L_i. If $Q_i \neq L_i$, a hamiltonian $X_i - Y_i$ path can be established as $(X_i, Q_i, U_i, L_i, Y_i)$, where U_i is the other 4-vertex of A_i than X_i, Y_i, L_i, and Q_i.

Clearly the hamiltonian paths obtained above interleaved with 4-edges (Y_1, X_2), (Y_2, X_3), ..., $(Y_{n(n-1)(n-2)\cdots 6-1}, X_{n(n-1)(n-2)\cdots 6})$ form a P_4. Next, we show the P_4 good. (Cond. 1) holds because X_1 is the beginning 4-vertex and $Y_{n(n-1)(n-2)\cdots 6}$ is the ending 4-vertex. (Cond. 3) holds for the reason as follows. Recall that $w_j = u_k$ for some $k \in \{2, 3, ..., n\} - \{j, a_1, a_2, ..., a_{n-5}\}$. After executing a k-partition on the P_4, $X_1 = < x_1 x_2 ... x_n >_4$ forms a K_4^3. Since $x_k = u_k = w_j$, the beginning 3-vertex is not connected to W. Similarly, the ending 3-vertex is not connected to D. In the following, we show (Cond. 2) holds. Let $X = < x_1 x_2 ... x_n >_4$, $Y = < y_1 y_2 ... y_n >_4$, and $Z = < z_1 z_2 ... z_n >_4$ be arbitrary three consecutive 4-vertices in the P_4. Assuming $p = dif(X, Y)$ and $q = dif(Y, Z)$, we show $x_p \neq z_q$ according to three cases. If X is the exit 4-vertex of A_i for some $1 \leq i \leq n(n-1)(n-2)\cdots 6-1$, then Y is the entry 4-vertex of A_{i+1} and Z is the second 4-vertex in the hamiltonian path established for the K_5^4 formed by A_{i+1}. Besides, $p \neq j = q$. Suppose conversely $x_p = z_q$. Then, Z is not connected to A_i (recall that the pair of 4-vertices in A_i and A_{i+1} that are not adjacent are $< x_1 ... x_{q-1} z_p x_{q+1} ... x_n >_4$ and $< z_1 ... z_{q-1} x_p z_{q+1} ... z_n >_4$, respectively, where $x_q = z_p \neq x_p = z_q$ and $x_i = z_i$ for all $1 \leq i \leq n$ and $i \neq \{p, q\}$). According to our construction for the P_4, Z should be the third or fourth or fifth 4-vertex in the hamiltonian path established for the K_5^4 formed by A_{i+1}, which is a contradiction. If Z is the entry 4-vertex of A_i for some $1 < i \leq n(n-1)(n-2)\cdots 6$, then $x_p \neq z_q$ can be shown similarly. Otherwise, if X, Y, and Z belong to the same 4-vertex, then $p = dif(X, Y) = dif(X, Z) = dif(Y, Z) = q$. Since X and Z are different 4-vertices, we have $x_p \neq z_q$. This completes the proof. **Q.E.D.**

As with similar arguments to prove the above, we can show the following lemmas. Due to space limitation, the details are omitted.

Lemma 3.6 *A good P_3 can be obtained from a good P_4.*

Proof: We suppose $P_4 = [A_1, A_2, ..., A_{n(n-1)(n-2)\cdots 5}]$. Without loss of generality, we assume that the P_4 is obtained from S_n by executing an $(a_1, a_2, ..., a_{n-4})$-partition, where $a_1 a_2 ... a_{n-4}$ is an arrangement out of $\{2, 3, ..., n\}$. Since the P_4 is good, there exists $j \in \{2, 3, ..., n\} - \{a_1, a_2, ..., a_{n-4}\}$ so that after executing a j-partition on the P_4, the beginning (ending) 3-vertex in A_1 ($A_{n(n-1)(n-2)\cdots 5}$) is not adjacent to A_2 ($A_{n(n-1)(n-2)\cdots 5-1}$). Besides, each A_i forms a K_4^3, where $1 \leq i \leq n(n-1)(n-2)\cdots 5$. In the rest of the proof, we construct a good P_3 from the P_4 by establishing a hamiltonian path for each K_4^3. Suppose $u = u_1 u_2 ... u_n$ and $v = v_1 v_2 ... v_n$ are the beginning and ending vertices, respectively. We establish a hamiltonian path for the K_4^3 formed by A_1 as follows. Let X_1 be the beginning 3-vertex (in A_1) and $W = < w_1 w_2 ... w_n >_3$ be a 3-vertex in A_1 which is different from X_1 and has $w_j = u_k$ for some $k \in \{2, 3, ..., n\} - \{j, a_1, a_2, ..., a_{n-4}\}$. We note that X_1 is not connected to A_2. Since there are three 2-edges between A_1 and A_2, there is another 3-vertex $Y_1 \notin \{X_1, W\}$ in A_1 which is connected to A_2. A hamiltonian $X_1 - Y_1$ path can be established as $(X_1, W) + P[W, Y_1]$, where $P[W, Y_1]$ denotes a $W - Y_1$ path passing all the vertices of the K_4^3 but X_1 exactly once.

Then we continue to establish a hamiltonian path for the K_4^3 formed by $A_{n(n-1)(n-2)\cdots 5}$. Let $Y_{n(n-1)(n-2)\cdots 5}$ be the ending 3-vertex (in $A_{n(n-1)(n-2)\cdots 5}$) and $D = < d_1 d_2 ... d_n >_3$ be the 3-vertex in $A_{n(n-1)(n-2)\cdots 5}$ that is different from $Y_{n(n-1)(n-2)\cdots 5}$ and has $d_j = v_k$ (here, k is identical with that k appearing in the situation of A_1). There exists a 3-vertex $X_{n(n-1)(n-2)\cdots 5} \neq D$ in $A_{n(n-1)(n-2)\cdots 5}$ which is connected to $A_{n(n-1)(n-2)\cdots 5-1}$. A hamiltonian $X_{n(n-1)(n-2)\cdots 5} - Y_{n(n-1)(n-2)\cdots 5}$ path can be established as $P[X_{n(n-1)(n-2)\cdots 5}, D] + (D, Y_{n(n-1)(n-2)\cdots 5})$, where $P[X_{n(n-1)(n-2)\cdots 5}, D]$ denotes an $X_{n(n-1)(n-2)\cdots 5} - D$ path passing all the vertices of the K_4^3 but $Y_{n(n-1)(n-2)\cdots 5}$ exactly once. In the discussion above, X_1 and Y_1 ($X_{n(n-1)(n-2)\cdots 5}$ and $Y_{n(n-1)(n-2)\cdots 5}$) are the entry and exit 3-vertices of A_1 ($A_{n(n-1)(n-2)\cdots 5}$), respectively. By X_i and Y_i we denote the entry and exit 3-vertices of A_i, respectively, for $2 \leq i \leq n(n-1)(n-2)\cdots 5-1$. Let L_i (Q_i) be the 3-vertex in A_i that is not connected to A_{i-1} (A_{i+1}). A hamiltonian $X_i - Y_i$ path for the formed by A_i can be established according to the following four cases.

Case 1. $Q_i = X_i$ and $L_i = Y_i$. A hamiltonian $X_i - Y_i$ path can be established easily.

Case 2. $Q_i \neq X_i$ and $L_i = Y_i$. A hamiltonian $X_i - Y_i$ path can be established as $(X_i, Q_i) + P[Q_i, Y_i]$, where $P[Q_i, Y_i]$ denotes a $Q_i - Y_i$ path passing all the vertices of the K_4^3 but X_i exactly once.

Case 3. $Q_i = X_i$ and $L_i \neq Y_i$. A hamiltonian $X_i - Y_i$ path can be established as $P[X_i, L_i] + (L_i, Y_i)$, where $P[X_i, L_i]$ denotes an $X_i - L_i$ path passing all the vertices of the K_4^3 but Y_i exactly once.

Case 4. $Q_i \neq X_i$ and $L_i \neq Y_i$. Since the P_4 is good, Lemma 3.1 assures that each 3-vertex of A_i is connected to A_{i-1} or A_{i+1}. Hence, $Q_i \neq L_i$. A hamiltonian $X_i - Y_i$ path can be established as (X_i, Q_i, L_i, Y_i). The hamiltonian paths obtained above interleaved with 3-edges (Y_1, X_2), (Y_2, X_3), ..., $(Y_{n(n-1)(n-2)\cdots 5-1}, X_{n(n-1)(n-2)\cdots 5})$ form a P_3. Moreover, the P_3 is good, with the same arguments as the proof of Lemma 3.5. **Q.E.D.**

Lemma 3.7 *There is a good P_3 in S_5.*

Proof: Suppose $u = u_1 u_2 u_3 u_4 u_5$ and $v = v_1 v_2 v_3 v_4 v_5$ are the beginning and ending vertices, respectively. We assume $u_i \neq v_i$ for $i \in \{a_1, a_2, ..., a_k\} \subseteq \{1, 2, 3, 4, 5\}$ and $u_i = v_i$ otherwise, where $2 \leq k \leq 5$ and $a_1 < a_2 < \cdots < a_k$. First, an a_k-partition is executed on S_5, and so a K_5^4 results. We use U_4 and V_4 to denote the beginning and ending 4-vertices, respectively. In the following, we construct a good P_3 according to the values of k.

Case 1. $k = 2$. We assume $a_1 \neq 1$. The discussion for $a_1 = 1$ is very similar. For ease of explanation, we assume, without loss of generality, $a_1 = 2$ and $a_2 = 3$. We then arbitrarily select $l = 4$ from the set $\{2, 3, 4, 5\} - \{a_1, a_2\} = \{4, 5\}$, and let $S = < s_1 s_2 s_3 s_4 s_5 >_4 =< * * s_3 * * >_4$ be the vertex of the K_5^4 with $(s_{a_2} =) s_3 = u_4 (= u_l)$. Since there are five vertices in the K_5^4, we can find a 4-vertex $Z = < z_1 z_2 z_3 z_4 z_5 >_4 = < * * z_3 * * >_4 \notin \{U_4, S, V_4\}$ with $(z_{a_2} =) z_3 \neq v_1$. Let T be the other vertex than U_4, S, Z, and V_4 in the K_5^4. A hamiltonian path for the K_5^4 can be established as (U_4, S, T, Z, V_4), which constitutes a $P_4 = [U_4, S, T, Z, V_4]$. An l-partition is then executed on the P_4, and so each 4-vertex of the P_4 forms a K_4^3. By establishing a hamiltonian path for each K_4^3, a good P_3 can be obtained as follows.

First we establish a hamiltonian path for the K_4^3 formed by V_4. Let $V_3 = < * * v_3 v_4 * >_3$ be the ending 3-vertex (in V_4) and $D = < d_1 d_2 d_3 d_4 d_5 >_3 = < * * v_3 d_4 * >_3$ be the 3-vertex of V_4 that is not connected to Z. Since $s_{a_2} = u_l = v_l =) v_4 \neq z_3 (= z_{a_2})$, V_3 is connected to Z. So, $D \neq V_3$. Moreover, since there are three 3-edges between Z and V_4, there exists a 3-vertex $X \neq V_3$ in V_4 which is connected to Z. A hamiltonian path for the K_4^3 can be established as $P[X, D] + (D, V_3)$, where $P[X, D]$ denotes an $X - D$ path passing all the vertices of the K_4^3 but V_3 exactly once.

We then continue to establish a hamiltonian path for the K_4^3 formed by U_4. We have $d_l = v_r$ for some $r \in \{2, 3, 4, 5\} - \{a_2, l\} = \{2, 5\}$. We note $r \neq 1$ because D is the 3-vertex in V_4 that is not connected to Z (which implies $d_l = z_{a_2} \neq v_1$). Let $U_3 = < * * u_3 u_4 * >_3$ be the beginning 3-vertex (in

U_4) and $W = < w_1w_2w_3w_4w_5 >_3 = < **u_3w_4* >_3$ be the 3-vertex in U_4 that is different from U_3 and has $(w_l =)w_4 = u_r$. We note that U_3 is not connected to S because $(s_{a_2} =)s_3 = u_4(= u_l)$. So, there exists another 3-vertex $Y \notin \{U_3, W\}$ in U_4 which is connected to S. A hamiltonian path for the K_4^3 can be established as $(U_3, W) + P[W, Y]$, where $P[W, Y]$ denotes a $W - Y$ path passing all the vertices of the K_4^3 but U_3 exactly once.

Since there are three 3-edges between every two adjacent 4-vertices of the P_4, distinct entry and exit 3-vertices can be determined for S, T, and Z. Then, a hamiltonian path from the entry 3-vertex to the exit 3-vertex can be established for each K_4^3 formed by them, similar to the proof of Lemma 3.6, in order to satisfy (Cond. 2). The obtained hamiltonian paths interleaved with used 3-edges form a $P_3 = [A_1, A_2, ..., A_{20}]$, where $A_1 = U_3, A_2 = W, A_{19} = D$, and $A_{20} = V_3$. In the following we show that the P_3 is good.

Clearly, (Cond. 1) holds, and with the same arguments as the proof of Lemma 3.5, (Cond 2) also holds. After executing an r-partition on the P_3, each A_i forms a K_3^2, where $1 \leq i \leq 20$. Without loss of generality, we assume $r = 2$. Let $U_2 = < *u_2u_3u_4* >_2$ (in A_1) and $V_2 = < *v_2v_3v_4* >_2$ (in A_{20}) be the beginning and ending 2-vertices, respectively. Since $(u_r =)u_2 = w_4(= w_l = w_{dif(A_1,A_2)})$, U_2 is not connected to $W = A_2$. Similarly, since $(v_r =)v_2 = d_4(= d_l = d_{dif(A_{19},A_{20})})$, V_2 is not connected to $D = A_{19}$. Thus, (Cond. 3) holds.

Case 2. $k = 3$. The method for constructing a good P_3 is almost the same as Case 1, but k is changed to 3 and l is selected from the set $\{2, 3, 4, 5\} - \{a_1, a_2, a_3\}$.

Case 3. $k = 4$. We assume $u_l = v_l$, where $l \in \{1, 2, 3, 4, 5\} - \{a_1, a_2, a_3, a_4\}$. If $u_t \neq v_t$, $u_t \neq v_{a_4}$, and $v_t \neq u_{a_4}$ for some $t \in \{a_1, a_2, a_3\} - \{1\}$, then two 4-vertices $Q = < q_1q_2q_3q_4q_5 >_4$ and $H = < h_1h_2h_3h_4h_5 >_4$ with $q_{a_4} = u_t$ and $h_{a_4} = v_t$ are determined. A hamiltonian path for the K_5^4 can be established as (U_4, Q, T, H, V_4), where T is the other 4-vertex than U_4, Q, H, and V_4. The hamiltonian path forms a good $P_4 = [U_4, Q, T, H, V_4]$ for the following reasons. (Cond. 1) and (Cond 2) hold with the same reasons as Case 1. (Cond. 3) holds as a consequence of executing a t-partition on the P_4. By Lemma 3.6, a good P_3 can be obtained from the P_4.

Otherwise, if there exists no $t \in \{a_1, a_2, a_3\} - \{1\}$ satisfying $u_t \neq v_t$, $u_t \neq v_{a_4}$, and $v_t \neq u_{a_4}$, then $a_1 = 1$, which implies $l \neq 1$. The method for constructing a good P_3 is almost the same as Case 1, but k is changed to 4 and l is unique.

Case 4. $k = 5$. There exists a number $t \in \{a_1, a_2, a_3, a_4\} - \{1\}$ satisfying $u_t \neq v_t$, $u_t \neq v_{a_5}$, and $v_t \neq u_{a_5}$. A good P_3 can be obtained similar to Case 3. **Q.E.D.**

We note that S_3 forms a cycle of length six. The following two lemmas have been shown in [4].

Lemma 3.8 *[4] Suppose X and Y are two adjacent 3-vertices in a P_3, and let $(c_0, c_1, ..., c_5)$ denote the cycle formed by X. Then, the vertices of X that are connected to Y are c_j and $c_{(j+3)mod6}$ for some $0 \leq j \leq 5$.*

Lemma 3.9 *[4] Suppose $X = < x_1x_2...x_n >_3$, $Y = < y_1y_2...y_n >_3$, and $Z = < z_1z_2...z_n >_3$ are arbitrary three consecutive 3-vertices in a P_3. If $x_{dif(X,Y)} \neq z_{dif(Y,Z)}$, then the two vertices of Y that are connected to X are disjoint from the two of Y that are connected to Z.*

Lemma 3.10 *Suppose u and v are arbitrary two distinct vertices of S_n with $n \geq 4$. A longest $u - v$ path can be constructed from a good P_3. The longest path has length $n! - 1$ if $dist(u, v)$ is odd, and $n! - 2$ if $dist(u, v)$ is even, where $dist(u, v)$ is the distance between u and v.*

Proof: It is not difficult to check that this lemma holds for S_4 (recall that S_n is vertex symmetric). Hence, we assume $n \geq 5$. According to Lemmas ?? and 3.7, a good $P_3 = [A_1, A_2, ..., A_{n(n-1)(n-2)\cdots 4}]$ can be obtained in S_n. We use $(c_{i,0}, c_{i,1}, ..., c_{i,5})$ to denote the cycle formed by A_i, where $1 \leq i \leq n(n-1)(n-2)\cdots 4$. According to Lemma 3.8, two vertices $c_{1,j}$ and $c_{1,(j+3)mod6}$ ($c_{n(n-1)(n-2)\cdots 4,k}$ and $c_{n(n-1)(n-2)\cdots 4,(k+3)mod6}$) for some $0 \leq j \leq 5$ ($0 \leq k \leq 5$) are connected to A_2 ($A_{n(n-1)(n-2)\cdots 4-1}$). We have $u \neq \{c_{1,j}, c_{1,(j+3)mod6}\}$, for otherwise the beginning 2-vertex must be connected to A_2, which contradicts (Cond. 3). Similarly, $v \neq \{c_{n(n-1)(n-2)\cdots 4,k}, c_{n(n-1)(n-2)\cdots 4,(k+3)mod6}\}$. Since A_1 ($A_{n(n-1)(n-2)\cdots 4}$) forms a cycle of length 6, u (v) is adjacent to $c_{1,j}$ or $c_{1,(j+3)mod6}$ ($c_{n(n-1)(n-2)\cdots 4,k}$ or $c_{n(n-1)(n-2)\cdots 4,(k+3)mod6}$). Without loss of generality, we assume u is adjacent to $c_{1,j}$. We let $x_1 = u$ and $y_1 = c_{1,j}$, and select x_i and y_i, sequentially, for $i = 2, 3, ..., n(n-1)(n-2)\cdots 4-1$ from each A_i so that x_i is adjacent to both y_{i-1} and y_i, and $y_{n(n-1)(n-2)\cdots 4-1}$ is connected to $A_{n(n-1)(n-2)\cdots 4}$. Lemmas 3.8 and 3.9 assure the existence of x_i and y_i. Since A_1 contains a hamiltonian $u - y_1$ path and each A_i contains a hamiltonian $x_i - y_i$ path, a hamiltonian $u - y_{n(n-1)(n-2)\cdots 4-1}$ path (of length $n! - 6$) for $S_n - \{A_{n(n-1)(n-2)\cdots 4}\}$ thus results.

Next we augment the $u - y_{n(n-1)(n-2)\cdots 4-1}$ path with a longest $y_{n(n-1)(n-2)\cdots 4-1} - v$ path. Without loss of generality, we assume $y_{n(n-1)(n-2)\cdots 4-1}$ is adjacent to $c_{n(n-1)(n-2)\cdots 4,k}$. If $dist(u, v)$ is odd, any $u - v$ path has odd length because S_n is bipartite. So, $v \neq \{c_{n(n-1)(n-2)\cdots 4,(k+2)mod6}, c_{n(n-1)(n-2)\cdots 4,(k-2)mod6}\}$, for otherwise there exists a $u - v$ path of even length, which is a contradiction. Since we also have $v \neq \{c_{n(n-1)(n-2)\cdots 4,k}, c_{n(n-1)(n-2)\cdots 4,(k+3)mod6}\}$, v should be $c_{n(n-1)(n-2)\cdots 4,(k+1)mod6}$ or $c_{n(n-1)(n-2)\cdots 4,(k-1)mod6}$. In either case, there exists a hamiltonian $c_{n(n-1)(n-2)\cdots 4,k} - v$ path (of length 5) for $A_{n(n-1)(n-2)\cdots 4}$. Similarly, if $dist(u, v)$ is even, v should be $c_{n(n-1)(n-2)\cdots 4,(k+2)mod6}$ or $c_{n(n-1)(n-2)\cdots 4,(k-2)mod6}$. In either case, there exists a $c_{n(n-1)(n-2)\cdots 4,k} - v$ path of length 4 in $A_{n(n-1)(n-2)\cdots 4}$. This completes the proof. **Q.E.D.**

The following theorem holds as an immediate consequence of Lemma 3.10.

Theorem 3.11 *S_n with $n \geq 4$ is strongly hamiltonian-laceable.*

4 Concluding remarks

In this paper we have introduced the concept of strongly hamiltonian-laceability for star graphs. By extanding our results, we can show that the n-dimensional star graph, where $n \geq 6$, remains strongly hamiltonian-laceable, even if $n-4$ random edge faults happen, and show that the n-dimensional star graph, where $n \geq 6$, remains strongly hamiltonian-laceable, even if $n-3$ random edge faults happen, exclusive of two exceptions in which there are at most two vertices missing from the longest paths.

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