Reduced-Order Modeling Based on PRONY's and SHANK's Methods via the Bilinear Transformation

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Abstract

In this paper, we propose a new model-order reduction technique for linear dynamic systems. The idea behind this technique is to transform the dynamic system function from the s-domain into the z-domain via the bilinear transformation, then use Prony's [1, 2] or Shank's [3] least-squares approximation methods instead of the commonly employed Padé approximation method, and finally transform the reduced system back into the s-domain using the inverse bilinear transformation. Simulation results for large practical systems show that this technique based on Prony's and Shank's methods give much higher accuracy than the traditional Padé method, and result in lower-order approximations with negligible increase in simulation time.

1 Introduction

The computation of equivalent linear system models of large linear dynamic systems is a topic of considerable practical interest. This interest is motivated by the reduced complexity obtained by reducing the large linear subnetwork in a linear (or nonlinear) network. Ideally, linear analysis on these linear subnetworks is performed by first computing a state-space model, followed by the application of a suitable analysis method. However, the applicability of this method is limited since typical dynamic systems are represented by very large state matrices that require specialized large-scale eigen-analysis programs and computer resources. To avoid this practical limitation, model-order reduction methods are widely used in the solution of these systems. The basic idea behind model-order reduction is to replace the original system equations for the large linear network by an equivalent system with a much smaller state-space dimension such that the identified reduced-order model transfer function characteristics must approximate those of the full-order model.

In general, from approximation theory, there are four major categories of approximation methods that one can use depending on the overall accuracy, efficiency, and reliability desired [4]. The min-max methods rely on nonlinear optimization techniques which make them inefficient but highly accurate. Series expansion based methods are computationally efficient but may provide inaccurate results. Interpolation methods, on the other hand, are computationally efficient methods which agree exactly with the original system on sample points but, in general, give unpredictable accuracy at other points [5]. Finally, least-squares methods combine the accuracy feature of the min-max methods and the efficiency of the series expansion and interpolation methods by controlling the error between the original system function and the approximate function over all points and not just where the maximum error occurs (min-max) or at the sample data points (series expansion and interpolation).

The *s-domain* Padé approximation [6] is a combination of series expansion and interpolation methods that has been used in the asymptotic waveform evaluation (AWE) [7] algorithm in order to extract the dominant poles and residues of the system. Other Padé techniques based on Krylov-subspace methods — such as Padé via Lanczos (PVL) [8] and Arnoldi-based model-order reduction [9, 10] — provide *efficient* estimation of the original system response. However, the *accuracy* of these methods is limited by the order of the Padé approximation (i.e. the number of moments matched). The problem with the Padé approximation is that it equates the approximating transfer function to the original system function to obtain as many equations as there are unknowns [5, 6]. This equating technique is the major limitation in the Padé approximation because the resulting transfer function must contain a large number of poles and zeros in order for it to be sufficiently close to the original system.

In this paper, we propose a novel model-order reduction technique for obtaining a reduced-order transfer function approximation. The technique is based on three steps. First, the original large system transfer function is transformed from the *s*-domain into the *z*-domain via the bilinear transformation. It is well known that the bilinear transformation always preserves system stability, and can always be made to preserve the system frequency response characteristics for a specified frequency range [2, 11, 12]. In addition, working in the *z*-domain results in better approximations than pure *s*-domain approaches since the parameter α in the bilinear transformation places more emphasis on the

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frequency range of interest. Second, we propose to apply Prony's [1] or Shank's [3] least-squares approximation methods to reduce the order of the *transformed* system function. Third, the reduced-order system is transformed back to the s-domain using the inverse bilinear transformation. We show that Prony's and Shank's approximation methods perform significantly better than the traditional Padé approximation method. Furthermore, the derivation of this model-order reduction technique enables the different Krylov-subspace algorithms [8–10], traditionally applied for the Padé approximation, to be used with Prony's and Shank's methods to obtain efficient and accurate moment approximations.

The remainder of the paper is organized as follows. In Section 2, we describe the s-to-z system transformation together with moment computation techniques. We also describe the connection of the different Krylov-subspace algorithms in this new representation. In Section 3, we explain Prony's and Shank's least-squares approximation methods and compare them to the traditional Padé approximation. We corroborate the derived results in Section 4 with simulations of large practical systems and show the accuracy of our technique. Finally, Section 5 provides some concluding remarks.

2 s-to-z System Transformation

In this section, we briefly describe the bilinear transformation and then apply this transformation to represent a linear, time-invariant (LTI) system in the z-domain. We then describe three methods for computing the moments of the z-domain system function.

2.1 The bilinear transformation

The bilinear transformation is a linear fractional transformation given by $\beta : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$\beta : \; s \mapsto z = \frac{\alpha + s}{\alpha - s}, \tag{1}$$

where $\alpha \in \mathbb{R}$ is a constant equal to twice the sampling rate. This mapping has the property that it transforms the $j\Omega$-axis in the s-plane onto the unit circle ($z = e^{j\omega}$) in the z-plane [11]. Moreover, the left-half s-plane ($\mathrm{Re}(s) < 0$) is mapped inside the unit circle in the z-plane and the right-half s-plane ($\mathrm{Im}(s) > 0$) is mapped outside the unit circle in the z-plane, thus preserving system stability. The inverse bilinear transformation β^{-1} is given by $\beta(-\alpha/z)$.

The main advantage of using the bilinear transformation over other transformations such as the impulse- and step-invariant transformations, is that it preserves the magnitude characteristics of the transfer function. This follows from the fact that the parameter α provides a one degree of freedom that can be used to make the frequency response characteristics of the z-domain system function $\mathbf{H_d}(z)$ approximate those of the s-domain system function $\mathbf{H_c}(s)$ [12].

2.2 z-domain system representation

An LTI system of equations can be used to model a dynamic system as follows:

$$\begin{aligned} C\dot{x}(t) + Gx(t) + bu(t) &= 0, \\ y(t) &= l^T x(t), \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ represents the state variables of the system, $y \in \mathbb{R}^p$ are the p outputs of the system defined using $l \in \mathbb{R}^{n\times p}$, G and $C \in \mathbb{R}^{n\times n}$, and $b \in \mathbb{R}^{n\times m}$ represents excitations from m independent sources. It can be shown that the resulting system transfer function of (2) is given by

$$\mathbf{H_c}(s) = -l^T (G + sC)^{-1} b. \tag{3}$$

Applying the bilinear transformation in (1) to (3) for an appropriately chosen α to preserve the magnitude characteristics, the z-domain transfer function becomes

$$\begin{aligned} \mathbf{H_d}(z) &\triangleq \mathbf{H_c}(s)\big|_{s=\alpha\frac{1-z^{-1}}{1+z^{-1}}} \\ &= l^T (1+z^{-1})(I - z^{-1}A)^{-1} r, \end{aligned} \tag{4}$$

where $A = -(G+\alpha C)^{-1}(G - \alpha C)$, and $r = -(G+\alpha C)^{-1} b$. In order to invert the term $(I - z^{-1}A)$ in (4), we need to diagonalize A, which is numerically expensive and impractical to perform for typical large dynamic systems. Instead, one resorts to using Neumann's expansion [13]:

$$(I - z^{-1}A)^{-1} = I + z^{-1}A + z^{-2}A^2 + \cdots. \tag{5}$$

Substituting (5) in (4), the z-domain transfer function $\mathbf{H_d}(\mathbf{z})$ becomes

$$\mathbf{H_d}(z) = (1 + z^{-1}) \sum_{i=0}^{\infty} m_{c,i} z^{-i}, \tag{6}$$

where $m_{c,i} = l^T A^i r$ are the *s-domain moments* of $\mathbf{H_c}(s)$. $\mathbf{H_d}(\mathbf{z})$ can also be written in a more convenient form by defining the *z-domain moments*

$$m_{d,i} = m_{c,i-1} + m_{c,i}, \tag{7}$$

as

$$\mathbf{H_d}(z) = \sum_{i=0}^{\infty} m_{d,i} z^{-i}. \tag{8}$$

2.3 Computing the moments

It is usually sufficient to compute the first $w+1$ moments in (6) or (8), where w depends on the choice of the approximation algorithm used in the second step of the proposed model-reduction technique. In AWE [7], the moments $m_{c,i}$ are explicitly computed by first recursively solving the linear system of equations

$$(G + \alpha C) u_i = -C u_{i-1}, \quad i = 1, 2, \cdots, w,$$

for u_i with $u_0 = r$, and then $m_{c,i} = l^T u_i$. However, due to finite machine precision, this approach is numerically ill-conditioned; see [8]. A better alternative is to use Krylov-subspace methods which enable a more stable computation

of the moments. For instance, in the Lanczos algorithm, these moments are obtained by

$$m_{c,i} = l^T r e_1^T T_w^i e_1, \quad i = 0, 1, \cdots, w,$$

where e_1 is the first unit vector in $\mathbb{R}^w$ and T_w is a tridiagonal matrix [8]. Similarly, in the Arnoldi process the $m_{c,i}$ moments are given by

$$m_{c,i} = \|r\|_2 l^T V_w H_w^i e_1, \quad i = 0, 1, \cdots, w,$$

where $V_w \in \mathbb{R}^{n\times w}$, and H_w is a $w \times w$ upper Hessenberg matrix whose scaler entries are generated by the Arnoldi algorithm [14]. Finally, the z-domain moments are easily computed from the $m_{c,i}$ moments using (7).

3 Reduced-Order Modeling

The model-order reduction problem can now be stated as follows: Given the *transformed* z-domain system response as described by equation (8), find an approximate z-domain transfer function of the form

$$\tilde{\mathbf{H}}_\mathbf{d}(z) = \frac{\sum_{k=0}^{q-1} b_k z^{-k}}{1 + \sum_{k=1}^{q} a_k z^{-k}} \equiv \sum_{i=0}^{w} \tilde{m}_{d,i} z^{-i}, \tag{9}$$

whose response $\tilde{m}_{d,i}$ accurately approximate the response $m_{d,i}$ of $\mathbf{H}_\mathbf{d}(z)$. Once this approximate system is identified, it is transformed back into the s-domain using the inverse bilinear transformation β^{-1} in order to obtain the reduced-order s-domain transfer function.

The rational representation of the approximate transfer function given in equation (9) has $2q$ unknowns, namely, the coefficients $\{a_k\}_{k=1}^{q}$ and $\{b_k\}_{k=0}^{q-1}$. Existing techniques employ the Padé approximation to determine these coefficients [7–9]. The drawback of this technique is that it equates the $w+1$ terms of $\tilde{\mathbf{H}}_\mathbf{d}(z)$ of (9) to the first $w+1$ moments of $\mathbf{H}_\mathbf{d}(z)$ in (8) to get as many equations as there are unknowns. This in turn requires a large-order rational approximate function to achieve acceptable accuracy. To solve this problem, we propose using least-squares approximation methods, such as Prony's or Shank's methods, which determine the moments of $\tilde{\mathbf{H}}_\mathbf{d}(z)$ over a large number of moments of $\mathbf{H}_\mathbf{d}(z)$ but using a low-order rational approximate function.

3.1 Padé approximation

We will first briefly present the z-domain Padé approximation which is similar to the commonly used s-domain Padé approximation [7–9] in order to compare with our proposed methods. In this approximation, the parameter w is set to $2q-1$ and the responses $\tilde{m}_{d,i}$ and $m_{d,i}$ of the two transfer functions are equated for $0 \le i \le 2q-1$, in order to obtain as many equations as there are unknowns. This results in the following system of linear equations

$$m_{d,i} = \begin{cases} -\sum_{k=1}^{q} a_k m_{d,i-k} + b_i, & 0 \le i \le q-1, \quad \text{(10a)} \\ -\sum_{k=1}^{q} a_k m_{d,i-k}, & q \le i \le 2q-1, \quad \text{(10b)} \end{cases}$$

where it is assumed that moments with negative indices are zero. The coefficients $\{a_k\}$ are first obtained by solving q equations in (10b), which are then used to determine the coefficients $\{b_k\}$ in (10a). Thus, the Padé approximation method results in a perfect match between the $\tilde{m}_{d,i}$ and the original $m_{d,i}$ for the first $2q$ values of the impulse response. However, for $i \ge 2q$ there is no bound on the error between the two responses. This is the major limitation with the Padé approximation, and hence the resulting transfer function must contain a large number of poles and zeros in order for its impulse response to be sufficiently close to the response of the original transfer function. Furthermore, the Padé approximation results in a perfect match with the original moments $m_{d,i}$ only when the original system is rational and we have prior knowledge of its number of poles and zeros. However, this is not usually the case since we only know the impulse response data as given in (6).

3.2 Prony's method

The main problem with the Padé approximation is that it sets $w = 2q-1$ in order to get as many equations as there are unknowns. In Prony's least-squares method [1, 2], this constraint is removed and w can be a large number ($\gg 2q-1$) without necessarily resulting in a large-order rational function. This method tries to minimize the linear least-squares error between the two responses

$$\varepsilon = \sum_{i=0}^{w} |m_{d,i} - \tilde{m}_{d,i}|^2, \tag{11}$$

with respect to the coefficients $\{a_k\}$ of the approximate transfer function $\tilde{\mathbf{H}}_\mathbf{d}(z)$. This linear least-squares optimization minimizes the linear prediction error, and not the original minus approximate squared response error. The idea behind the linear prediction part is that instead of equating $\tilde{m}_{d,i} = m_{d,i}$ for all i, recursively compute $\tilde{m}_{d,i}$ using a linear predictor defined by the following set of linear equations

$$\tilde{m}_{d,i} = \begin{cases} -\sum_{k=1}^{q} a_k m_{d,i-k} + b_i, & 0 \le i \le q-1, \quad \text{(12a)} \\ -\sum_{k=1}^{q} a_k m_{d,i-k}, & q \le i \le w. \quad \text{(12b)} \end{cases}$$

The coefficients $\{a_k\}$ are then chosen so as to minimize the squared-prediction-error defined by

$$\varepsilon = \sum_{i=q}^{w} \left| m_{d,i} + \sum_{k=1}^{q} a_k m_{d,i-k} \right|^2, \quad l = 1, \cdots, q.$$

Setting $\partial\varepsilon/\partial a_l$ to zero, results in the following set of linear equations

$$\sum_{k=1}^{q} a_k \sum_{i=q}^{w} m_{d,i-k} m_{d,i-l} = -\sum_{i=q}^{w} m_{d,i} m_{d,i-l}, \quad l = 1, \cdots, q.$$

This set of equations can equivalently be written as

$$\sum_{k=1}^{q} a_k Y_{k,l} = -y_l, \quad l = 1, \cdots, q, \tag{13}$$

where, by definition, $Y_{k,l} = \sum_{i=q}^{w} m_{d,i-k} m_{d,i-l}$ and $y_l = \sum_{i=q}^{w} m_{d,i} m_{d,i-l}$. Equation (13) can be used to determine the coefficients $\{a_k\}$ which are then used to determine the coefficients $\{b_k\}$ using

$$b_k = \tilde{m}_{d,k} + \sum_{i=1}^{q} a_i m_{d,k-i}, \text{ for } k = 0, 1, \cdots, q-1.$$

3.3 Shank's method

Rather than obtaining the numerator coefficients $\{b_k\}$ using an exact fit as was the case in Padé and Prony's methods, one might use least-squares minimization. Hence, both the coefficients $\{a_k\}$ and $\{b_k\}$ are obtained using the least-squares minimization technique. This technique is known as Shank's method which approximates a rational transfer function by an all-pole approximate transfer function in *series* with an all-zero approximate transfer function. The method starts by using an all-pole approximate transfer function of the form

$$\mathbf{\tilde{H}_{d1}}(z) = \frac{b_0}{1 + \sum_{k=1}^{q} a_k z^{-k}} \equiv \sum_{i=0}^{w} \tilde{m}_{d1,i} z^{-i}. \qquad (14)$$

Similar to Prony's method, Shank's method computes the coefficients $\{a_k\}$ by minimizing the least-squares error between $m_{d,i}$ and $\tilde{m}_{d1,i}$. The result is

$$\sum_{k=1}^{q} a_k Y_{k,l} = -y_l, \quad l = 1, \cdots, q, \qquad (15)$$

where, by definition, $Y_{k,l} = \sum_{i=0}^{w} m_{d,i-k} m_{d,i-l}$ and $y_l = \sum_{i=0}^{w} m_{d,i} m_{d,i-l}$. Solving (15) we can obtain the $\{a_k\}$ coefficients. The impulse response $\tilde{m}_{d1,i}$ of the all-pole system can now be obtained as

$$\tilde{m}_{d1,i} = -\sum_{k=1}^{q} a_k \tilde{m}_{d1,i-k} + \delta_i, \quad i \geq 0.$$

If this response is now used to excite an all-zero approximate transfer function of the form

$$\mathbf{\tilde{H}_{d2}}(z) = \sum_{k=0}^{q-1} b_k z^{-k} \equiv \sum_{i=0}^{w} \tilde{m}_{d2,i} z^{-i},$$

its response $\tilde{m}_{d2,i}$ must approximate the response of the original system $m_{d,i}$. Therefore, the coefficients $\{b_k\}$ can now be computed by minimizing the least-squares errors of

$$\varepsilon = \sum_{i=0}^{w} \left| m_{d,i} - \tilde{m}_{d2,i} \right|^2 = \sum_{i=0}^{w} \left| m_{d,i} - \sum_{k=0}^{q-1} b_k \tilde{m}_{d1,i-k} \right|^2.$$

Setting $\partial\varepsilon/\partial b_k$ to zero, results in the following set of linear equations

$$\sum_{k=0}^{q-1} b_k X_{k,l} = x_l, \quad l = 0, 1, \cdots, q-1, \qquad (16)$$

where $X_{k,l} = \sum_{i=0}^{w} \tilde{m}_{d1,i-k} \tilde{m}_{d1,i-l}$ and $x_l = \sum_{i=0}^{w} m_{d,i} \tilde{m}_{d1,i-l}$. Solving (16) gives the coefficients $\{b_k\}$.

4 Applications

In this section, we apply the proposed model-order reduction technique using Prony's and Shank's methods on five large practical systems and compare their resulting responses with the corresponding original system response. We also show the responses resulting from using the commonly employed Padé method. The simulation times for the five examples are shown in Table 1. From these examples we show how Prony's and Shank's methods can produce lower-order approximations than the Padé approximation while maintaining sufficient accuracy and negligible increase in simulation time.

4.1 A fifth-order Chebyshev filter

Consider a fifth-order Chebyshev filter with system function given in (17) where $\omega_c = 2\pi \times 10^4$ rad/sec. The circuit implementation of this filter is shown in Figure 1. It consists of two second-order LCR resonators and a first-order op amp-RC circuit. Real 741 op amp circuits were used in the design. Using SPICE, we performed an AC analysis over the linear frequency range of 1 Hz to 300 MHz. The original system was of order 720. Using modified nodal analysis (MNA) stamps extracted from SPICE, we performed the three approximation procedures and the resulting system responses are plotted in Figure 2. As seen, a 6th-order Padé approximation of the resulting frequency response is a relatively poor approximation to the original response. However, Prony's and Shank's methods perform significantly better using only a 4th-order approximation. By increasing the order of the Padé approximation to a 7th-order we finally obtain a good match with the original Chebyshev filter.

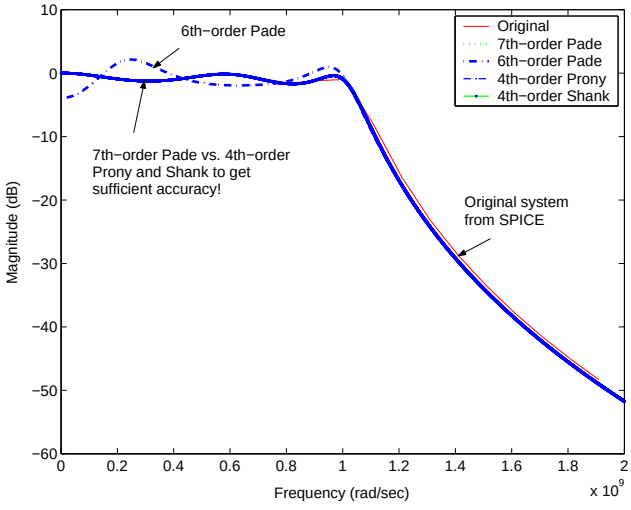


Figure 2. Chebyshev filter response.

4.2 Clamped beam

The clamped beam model of [15] has 348 states. The input represents the force applied to the structure at the free end, and the output is the resulting displacement. We plot in Figure 3 the magnitude response of the original system together with that of the reduced models. The Padé method re-

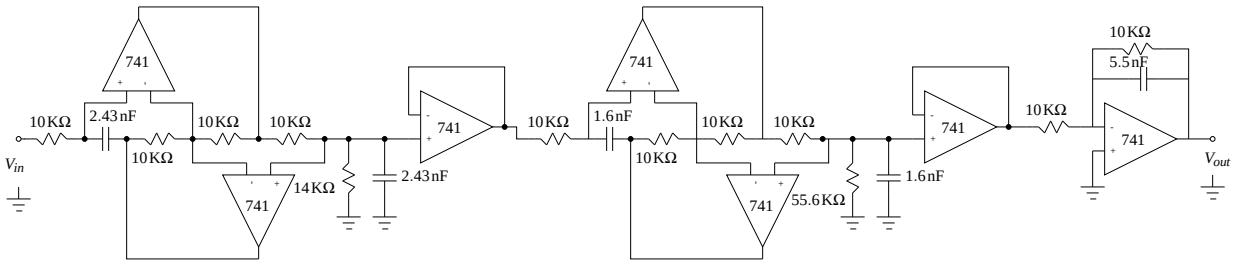


Figure 1. Fifth-order Chebyshev filter circuit schematic implemented using real 741 op amps.

$$\mathbf{H_c}(s) = \frac{\omega_c^5}{8.1408(s+0.2895\omega_c)(s^2+0.4684\omega_c s+0.4293\omega_c^2)(s^2+0.1789\omega_c s+0.9883\omega_c^2)} \tag{17}$$

quires a system of order 54 to obtain an exact match, whereas Prony's requires an order of 40 and Shank's an order of 38.

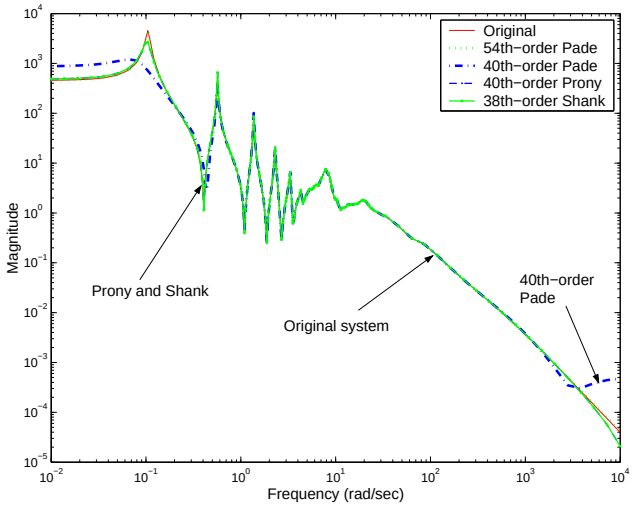


Figure 3. Clamped beam system response.

4.3 A large-order system

This example is similar in spirit to the example proposed in [16]. The system is of order 3018, generated using block matrices. Spikes in the system magnitude plots are artificially generated by explicitly placing the system eigenvalues of the G matrix as following: $\lambda(G) = \{-1 \pm j10, -1 \pm j700, -1 \pm j1200, -1 \pm j1400, -1 \pm j2600, -1 \pm j5800, -1 \pm j7000, -1 \pm j12000, -1 \pm j42000, -1, -2, \cdots, -3000\}$. As can be seen in Figure 4, using Prony's method a 35th-order system accurately approximates the magnitude waveform of the original system. Shank's method was able to reduce the order to 32 at the expense of increased simulation time. In contrast, a 35th-order Padé approximation fails to capture the magnitude response for frequencies less than 200 rad/sec.

4.4 International space-station

This example is a structural model of component 1r (Russian service module) of the International Space Station (ISS)

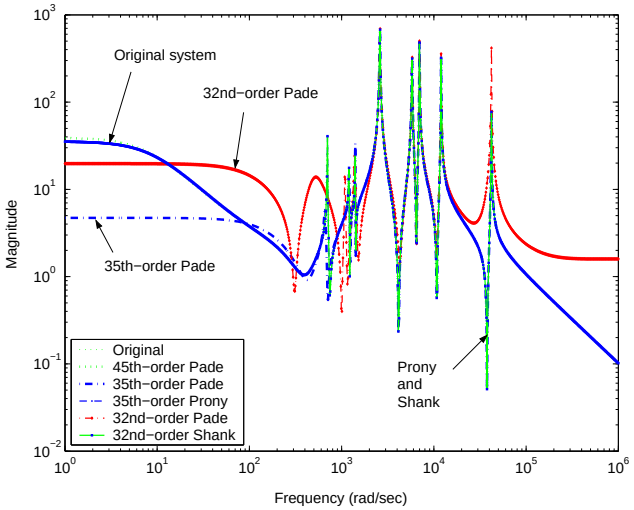


Figure 4. Large-order system response.

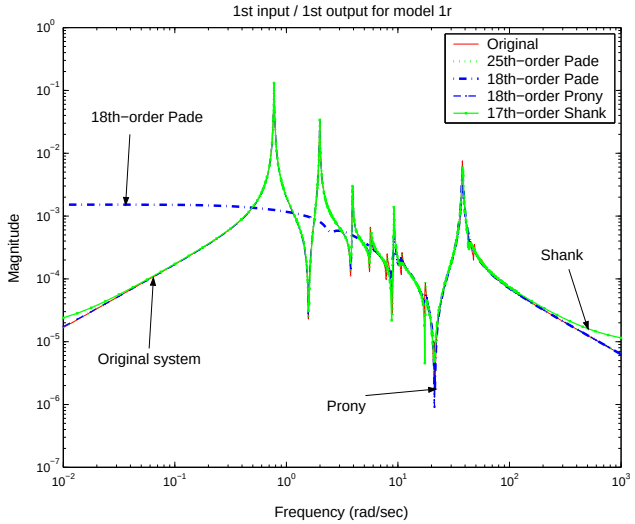
[17]. We plot in Figure 5 the magnitude responses for $\mathbf{H_{c,11}}(\mathbf{s})$ and $\mathbf{H_{c,12}}(\mathbf{s})$ of the original system as well as the reduced models. As shown in the plots, Prony's and Shank's methods accurately model the original system with lower orders than the Padé method.

4.5 PEEC Model

The partial element equivalent circuit (PEEC) model given in [9] contains 2100 capacitors, 172 inductors, and 6990 mutual inductors. As shown in Figure 6, a 45th-order Padé approximation performs poorly, in contrast to Prony's and Shank's approximations of the same degree.

5 Conclusion

In this paper, we have presented a new model-order reduction technique for approximating large dynamic systems. This technique is based on three steps: (1) transform the dynamic system function from the s-domain into the z-domain via the bilinear transformation, (2) use Prony's or Shank's approximation methods instead of the commonly employed Padé approximation method, and (3) transform the reduced



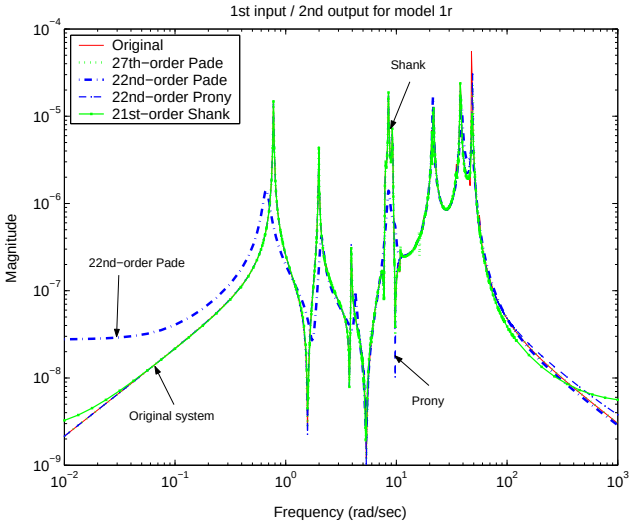


Figure 5. Magnitude response for the international space-station example.

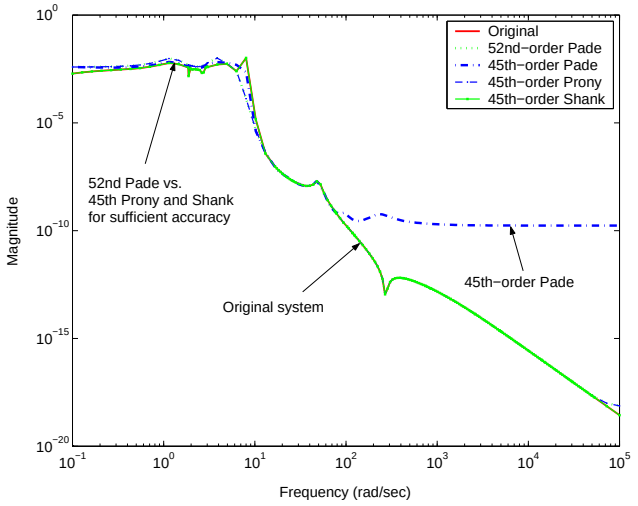


Figure 6. PEEC model response.

system back into the *s*-domain using the inverse bilinear transformation. We have shown through simulations of large practical systems the effectiveness of this technique in terms of accuracy and simulation time.

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Table 1. Comparison in simulation times.§

Example	Padé		Prony		Shank	
	w	time (s)	*w*	time (s)	*w*	time (s)
Chebyshev filter♮	13	1.07	50	1.43	50	2.63
Clamped beam†	107	0.54	120	0.84	120	1.34
Large system†	89	2.27	120	3.38	160	5.85
ISS 1-to-1‡	49	0.29	80	0.78	80	1.20
ISS 1-to-2‡	53	0.29	80	0.85	80	1.32
PEEC model†	103	1.54	120	1.92	120	3.67

§ Simulated with MATLAB on Sun Ultra 10 workstation.
Moments computed using:♮ Arnoldi process [10], † Lanczos process [8], ‡ AWE [7].