# Non-asymptotic Equipartition Properties for Independent and Identically Distributed Sources 

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[^0]
#### Abstract

Given an independent and identically distributed source $X=\left\{X_{i}\right\}_{i=1}^{\infty}$ with finite Shannon entropy or differential entropy (as the case may be) $H(X)$, the non-asymptotic equipartition property (NEP) with respect to $H(X)$ is established, which characterizes, for any finite block length $n$, how close $-\frac{1}{n} \ln p\left(X_{1} X_{2} \cdots X_{n}\right)$ is to $H(X)$ by determining the information spectrum of $X_{1} X_{2} \cdots X_{n}$, i.e., the distribution of $-\frac{1}{n} \ln p\left(X_{1} X_{2} \cdots X_{n}\right)$. Non-asymptotic equipartition properties (with respect to conditional entropy, mutual information, and relative entropy) in a similar nature are also established. These non-asymptotic equipartition properties are instrumental to the development of non-asymptotic coding (including both source and channel coding) results in information theory in the same way as the asymptotic equipartition property to all asymptotic coding theorems established so far in information theory. As an example, the NEP with respect to $H(X)$ is used to establish a non-asymptotic fixed rate source coding theorem, which reveals, for any finite block length $n$, a complete picture about the tradeoff between the minimum rate of fixed rate coding of $X_{1} \cdots X_{n}$ and error probability when the error probability is a constant, or goes to 0 with block length $n$ at a sub-polynomial $n^{-\alpha}, 0<\alpha<1$, polynomial $n^{-\alpha}$, $\alpha \geq 1$, or sub-exponential $e^{-n^{\alpha}}, 0<\alpha<1$, speed. In particular, it is shown that for any finite block length $n$, the minimum rate (in nats per symbol) of fixed rate coding of $X_{1} X_{2} \cdots X_{n}$ with error probability $\Theta\left(\frac{n^{-\alpha}}{\sqrt{\ln n}}\right)$ is $H(X)+\sqrt{\sigma_{H}^{2}(X)(2 \alpha)} \sqrt{\frac{\ln n}{n}}+O\left(\frac{\ln n}{n}\right)$, where $\alpha>0$ and $\sigma_{H}^{2}(X)=\mathbf{E}\left[-\ln p\left(X_{1}\right)\right]^{2}-H^{2}(X)$ is the information variance of $X$. With the help of the NEP with respect to other information quantities, non-asymptotic channel coding theorems of similar nature will be established in a separate paper.


## Index Terms

Asymptotic equipartition property (AEP), conditional entropy, entropy, fixed rate coding, information spectrum, mutual information, non-asymptotic equipartition property (NEP).

## I. Introduction

Consider an independent and identically distributed (IID) source $X=\left\{X_{i}\right\}_{i=1}^{\infty}$ with source alphabet $\mathcal{X}$ and finite entropy $H(X)$, where $H(X)$ is the Shannon entropy of $X_{i}$ if $\mathcal{X}$ is discrete, and the differential entropy of $X_{i}$ if $\mathcal{X}$ is the real line and each $X_{i}$ is a continuous random variable. Let $p(x)$ be the probability mass function (pmf) or probability density function (pdf) (as the case may be) of $X_{i}$. The asymptotic equipartition property (AEP) for $X$ is the assertion that

$$
\begin{equation*}
-\frac{1}{n} \ln p\left(X_{1} X_{2} \cdots X_{n}\right) \rightarrow H(X) \tag{1.1}
\end{equation*}
$$

either in probability or with probability one as $n$ goes to $\infty$. It implies that for sufficiently large $n$, with high probability, the outcomes of $X_{1} X_{2} \cdots X_{n}$ are approximately equiprobable with their respective probability ranging from $e^{-n(H(X)+\epsilon)}$ to $e^{-n(H(X)-\epsilon)}$, where $\epsilon>0$ is a small fixed number. Here and throughout the rest of the paper, ln stands for the logarithm with base $e$, and all information quantities are measures in nats.

The AEP is fundamental to information theory. It is not only instrumental to lossless source coding theorems, but also behind almost all asymptotic coding (including source, channel, and multi-user coding) theorems through the concepts of typical sets and typical sequences [1].

However, in the non-asymptotic regime where one wants to establish non-asymptotic coding results for finite block length $n$, the AEP in its current form can not be applied in general. In this paper, we aim to establish the non-asymptotic counterpart of the AEP, which is broadly referred to as the non-asymptotic equipartition property (NEP), so that the NEP can be applied to finite block length $n$. Specifically, with respect to $H(X)$, we first characterize, for any finite block length $n$, how close $-\frac{1}{n} \ln p\left(X_{1} X_{2} \cdots X_{n}\right)$ is to $H(X)$ by determining the information spectrum of $X_{1} X_{2} \cdots X_{n}$, i.e., the distribution of $-\frac{1}{n} \ln p\left(X_{1} X_{2} \cdots X_{n}\right)$; such a property is referred to as the NEP with respect to $H(X)$. For any IID source pair $(X, Y)=\left\{\left(X_{i}, Y_{i}\right)\right\}_{i=1}^{\infty}$ with finite conditional entropy $H(X \mid Y)$ and mutual information $I(X ; Y)$, where $H(X \mid Y)$ is the Shannon conditional entropy of $X_{i}$ given $Y_{i}$ if $X$ is discrete, and the conditional differential entropy of $X_{i}$ given $Y_{i}$ if $X$ is continuous, we then examine, for any finite block length $n$, how close $-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right)\left(-\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{p\left(Y^{n}\right)}\right.$, respectively) is to $H(X \mid Y)(I(X ; Y)$, respectively) by determining the distribution of $-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right)\left(-\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{p\left(Y^{n}\right)}\right.$, respectively), where $p\left(x^{n} \mid y^{n}\right)$ ( $p\left(y^{n} \mid x^{n}\right.$ ), respectively) is the conditional pmf or pdf (as the case may be) of $x^{n}=x_{1} x_{2} \cdots x_{n}$
( $y^{n}=y_{1} y_{2} \cdots y_{n}$, respectively) given $y^{n}$ ( $x^{n}$, respectively); these properties are referred to as the NEP with respect to $H(X \mid Y)$ and $I(X ; Y)$, respectively.

In the same way as the AEP plays an important role in establishing the asymptotic coding (including source, channel, and multi-user coding) results in information theory, our established NEP is also instrumental to the development of non-asymptotic source and channel coding results. Using the NEP with respect to $H(X)$, we further establish a non-asymptotic fixed rate source coding theorem, which reveals, for any finite block length $n$, a complete picture about the tradeoff between the minimum rate of fixed rate coding of $X_{1} \cdots X_{n}$ and error probability when the error probability is a constant, or goes to 0 with block length $n$ at a sub-polynomial $n^{-\alpha}$, $0<\alpha<1$, polynomial $n^{-\alpha}, \alpha \geq 1$, or sub-exponential $e^{-n^{\alpha}}, 0<\alpha<1$, speed. In particular, it is shown that for any finite block length $n$, the minimum rate (in nats per symbol) of fixed rate coding of $X_{1} X_{2} \cdots X_{n}$ with error probability $\Theta\left(\frac{n^{-\alpha}}{\sqrt{\ln n}}\right)$ is $H(X)+\sqrt{\sigma_{H}^{2}(X)(2 \alpha)} \sqrt{\frac{\ln n}{n}}+O\left(\frac{\ln n}{n}\right)$, where $\alpha>0$ and $\sigma_{H}^{2}(X)=\mathbf{E}\left[-\ln p\left(X_{1}\right)\right]^{2}-H^{2}(X)$ is the information variance of $X$. In a separate paper [3], non-asymptotic channel coding theorems of similar nature will be established with the help of the NEP with respect to other information quantities; in particular, it is shown [3] that for any binary input memoryless channel with uniform capacity achieving input $X$, random linear codes of block length $n$ can reach within $\sqrt{\sigma_{H}^{2}(X \mid Y)(2 \alpha)} \sqrt{\frac{\ln n}{n}}+O\left(\frac{\ln n}{n}\right)$ of the channel capacity while maintaining word error probability $\Theta\left(\frac{n^{-\alpha}}{\sqrt{\ln n}}\right)$, where $\alpha>0$ and $\sigma_{H}^{2}(X \mid Y)=\mathbf{E}[-\log p(X \mid Y)]^{2}-H^{2}(X \mid Y)$ is the conditional information variance of $X$ given $Y$ with $Y$ being the output of the channel in response to the input $X$.

The rest of the paper is organized as follows. Section $\Pi$ is devoted to the NEP with respect to $H(X)$. All results in Section $\Pi$ are then extended to the case of $H(X \mid Y)$ in Section III, thereby establishing the NEP with respect to $H(X \mid Y)$. In Section IV, we analyze the NEP with respect to the mutual information and relative entropy. Finally, in Section V, we apply the NEP with respect to $H(X)$ to investigate the performance of optimal fixed rate coding of $X_{1} X_{2} \cdots X_{n}$.

## II. NEP With Respect to Entropy

Define

$$
\begin{equation*}
\lambda^{*}(X) \stackrel{\Delta}{=} \sup \left\{\lambda \geq 0: \int p^{-\lambda+1}(x) d x<\infty\right\} \tag{2.1}
\end{equation*}
$$

where $\int d x$ is understood throughout this paper to be the summation over the source alphabet of $X$ if $X$ is discrete. Suppose that

$$
\begin{equation*}
\lambda^{*}(X)>0 \tag{2.2}
\end{equation*}
$$

Let

$$
\begin{equation*}
\sigma_{H}^{2}(X) \triangleq \int p(x)[-\ln p(x)]^{2} d x-H^{2}(X) \tag{2.3}
\end{equation*}
$$

which will be referred to as the information variance of $X$. It is not hard to see that under the assumption (2.2),

$$
\begin{equation*}
\int \frac{p^{-\lambda+1}(x)}{\left[\int p^{-\lambda+1}(y) d y\right]}|-\ln p(x)|^{k} d x<\infty \tag{2.4}
\end{equation*}
$$

and

$$
\int p^{-\lambda+1}(x) d x<\infty
$$

for any $\lambda \in\left(0, \lambda^{*}(X)\right)$ and any positive integer $k$. Further assume that

$$
\begin{equation*}
\sigma_{H}^{2}(X)>0 \text { and } \int p(x)|\ln p(x)|^{3} d x<\infty \tag{2.5}
\end{equation*}
$$

Then we have the following result, which will be referred to as the weak right NEP with respect to $H(X)$.

Theorem 1 (Weak Right NEP). For any $\delta \geq 0$, let

$$
r_{X}(\delta) \triangleq \sup _{\lambda \geq 0}\left[\lambda(H(X)+\delta)-\ln \int p^{-\lambda+1}(x) d x\right]
$$

Then the following hold:
(a) For any positive integer n,

$$
\begin{equation*}
\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n}\right)>H(X)+\delta\right\} \leq e^{-n r_{X}(\delta)} \tag{2.6}
\end{equation*}
$$

where $X^{n}=X_{1} X_{2} \cdots X_{n}$.
(b) Under the assumptions (2.2) and (2.5), there exists a $\delta^{*}>0$ such that for any $\delta \in\left(0, \delta^{*}\right]$ and any positive integer $n$,

$$
\begin{equation*}
r_{X}(\delta)=\frac{1}{2 \sigma_{H}^{2}(X)} \delta^{2}+O\left(\delta^{3}\right) \tag{2.7}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n}\right)>H(X)+\delta\right\} \leq e^{-n\left(\frac{\delta^{2}}{2 \sigma_{H}^{2}(x)}+O\left(\delta^{3}\right)\right)} \tag{2.8}
\end{equation*}
$$

Proof of Theorem 7. The inequality (2.6) follows from the Chernoff bound. To see this is indeed the case, note that

$$
\begin{align*}
\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X_{1} X_{2} \cdots X_{n}\right)>H(X)+\delta\right\} & =\operatorname{Pr}\left\{-\ln p\left(X_{1} X_{2} \cdots X_{n}\right)>n(H(X)+\delta)\right\} \\
& \leq \inf _{\lambda \geq 0} \frac{\mathbf{E}\left[e^{-\lambda \ln p\left(X_{1} X_{2} \cdots X_{n}\right)}\right]}{e^{n \lambda(H(X)+\delta)}} \\
& =\inf _{\lambda \geq 0} e^{-n\left[\lambda(H(X)+\delta)-\ln \mathbf{E}\left[p^{-\lambda}\left(X_{1}\right)\right]\right.} \\
& =\inf _{\lambda \geq 0} e^{-n\left[\lambda(H(X)+\delta)-\ln \int p^{-\lambda+1}(x) d x\right]} \\
& =e^{-n r_{X}(\delta)} \tag{2.9}
\end{align*}
$$

To show (2.7) and (2.8), we first analyze the property of $r_{X}(\delta)$ as a function of $\delta$ over the region $\delta \geq 0$. It is easy to see that $r_{X}(\delta)$ is convex and non-decreasing. For any $\lambda \in\left[0, \lambda^{*}(X)\right)$, define

$$
\begin{equation*}
\delta(\lambda) \triangleq \int \frac{p^{-\lambda+1}(x)}{\left[\int p^{-\lambda+1}(y) d y\right]}[-\ln p(x)] d x-H(X) \tag{2.10}
\end{equation*}
$$

which, in view of (2.4), is well defined. Using a similar argument as in [4, Properties 1 to 3], it is not hard to show that under the assumption (2.2), $\delta(\lambda)$ as a function of $\lambda$ is continuously differentiable up to any order over $\lambda \in\left(0, \lambda^{*}(X)\right)$. Taking the first order derivative of $\delta(\lambda)$ yields

$$
\begin{align*}
\delta^{\prime}(\lambda) & =\int \frac{p^{-\lambda+1}(x)}{\left[\int p^{-\lambda+1}(y) d y\right]}[-\ln p(x)]^{2} d x-\left[\int \frac{p^{-\lambda+1}(x)}{\left[\int p^{-\lambda+1}(y) d y\right]}[-\ln p(x)] d x\right]^{2} \\
& >0 \tag{2.11}
\end{align*}
$$

where the last inequality is due to (2.5). It is also easy to see that $\delta(0)=0$ and $\delta^{\prime}(0)=\sigma_{H}^{2}(X)$. Therefore, $\delta(\lambda)$ is strictly increasing over $\lambda \in\left[0, \lambda^{*}(X)\right)$. On the other hand, it is not hard to verify that under the assumption (2.2), the function $\lambda(H(X)+\delta)-\ln \int p^{-\lambda+1}(x) d x$ as a function of $\lambda$ is continuously differentiable over $\lambda \in\left[0, \lambda^{*}(X)\right)$ with its derivative equal to

$$
\begin{equation*}
\delta-\delta(\lambda) \tag{2.12}
\end{equation*}
$$

To continue, we distinguish between two cases: (1) $\lambda^{*}(X)=\infty$, and (2) $\lambda^{*}(X)<\infty$. In case (1), since $\delta(\lambda)$ is strictly increasing over $\lambda \in[0, \infty)$, it follows that for any $\delta=\delta(\lambda)$ for some $\lambda \in\left[0, \lambda^{*}(X)\right)$, the supremum in the definition of $r_{X}(\delta)$ is actually achieved at that particular $\lambda$, i.e.,

$$
\begin{equation*}
r_{X}(\delta(\lambda))=\lambda(H(X)+\delta(\lambda))-\ln \int p^{-\lambda+1}(x) d x \tag{2.13}
\end{equation*}
$$

In case (2), we have that for any $\delta=\delta(\lambda)$ for some $\lambda \in\left[0, \lambda^{*}(X)\right)$,

$$
\begin{equation*}
\beta(H(X)+\delta(\lambda))-\ln \int p^{-\beta+1}(x) d x<\lambda(H(X)+\delta(\lambda))-\ln \int p^{-\lambda+1}(x) d x \tag{2.14}
\end{equation*}
$$

for any $\beta \in\left[0, \lambda^{*}(X)\right)$ with $\beta \neq \lambda$. In view of the definition of $\lambda^{*}(X)$, 2.14) remains valid for any $\beta>\lambda^{*}(X)$ since then the left side of (2.14) is $-\infty$. What remains to check is when $\beta=\lambda^{*}(X)$. If

$$
\int p^{-\lambda^{*}(X)+1}(x) d x=\infty
$$

it is easy to see that (2.14) holds as well when $\beta=\lambda^{*}(X)$. Suppose now

$$
\int p^{-\lambda^{*}(X)+1}(x) d x<\infty .
$$

In this case, it follows from the dominated convergence theorem that

$$
\lim _{\beta \uparrow \lambda^{*}(X)} \int p^{-\beta+1}(x) d x=\int p^{-\lambda^{*}(X)+1}(x) d x
$$

and hence by letting $\beta$ go to $\lambda^{*}(X)$ from the left, we see that 2.14 holds as well when $\beta=$ $\lambda^{*}(X)$. Putting all cases together, we always have that for any $\delta=\delta(\lambda)$ for some $\lambda \in\left[0, \lambda^{*}(X)\right)$,

$$
\begin{equation*}
r_{X}(\delta(\lambda))=\lambda(H(X)+\delta(\lambda))-\ln \int p^{-\lambda+1}(x) d x \tag{2.15}
\end{equation*}
$$

Let

$$
\Delta^{*}(X) \triangleq \lim _{\lambda \uparrow \lambda^{*}(X)} \delta(\lambda)
$$

Since both $\delta(\lambda)$ and $\ln \int p^{-\lambda+1}(x) d x$ are continuously differentiable with respect to $\lambda \in\left(0, \lambda^{*}(X)\right)$ up to any order, it follows from (2.15) that $r_{X}(\delta)$ is also continuously differentiable with respect to $\delta \in\left(0, \Delta^{*}(X)\right)$ up to any order. (At $\delta=0, r_{X}(\delta)$ is continuously differentiable up to at least the third order inclusive.) Taking the first and second order derivatives of $r_{X}(\delta)$ with respect to $\delta$, we have

$$
\begin{align*}
r_{X}^{\prime}(\delta) & =\frac{d r_{X}(\delta)}{d \delta} \\
& =\frac{d r_{X}(\delta(\lambda))}{d \lambda} \frac{d \lambda}{d \delta} \\
& =\frac{d r_{X}(\delta(\lambda))}{d \lambda} \frac{1}{\delta^{\prime}(\lambda)} \\
& =\frac{1}{\delta^{\prime}(\lambda)}\left[H(X)+\delta(\lambda)+\lambda \delta^{\prime}(\lambda)-\int \frac{p^{-\lambda+1}(x)}{\left[\int p^{-\lambda+1}(y) d y\right]}[-\ln p(x)] d x\right] \\
& =\lambda \tag{2.16}
\end{align*}
$$

and

$$
\begin{align*}
r_{X}^{\prime \prime}(\delta) & =\frac{d \lambda}{d \delta} \\
& =\frac{1}{\delta^{\prime}(\lambda)} \tag{2.17}
\end{align*}
$$

where $\delta=\delta(\lambda)$. Therefore, $r_{X}(\delta)$ is convex, strictly increasing, and continuously differentiable up to at least the third order (inclusive) over $\delta \in\left[0, \Delta^{*}(X)\right.$ ). Note that from (2.16) and 2.17), we have $r_{X}^{\prime}(0)=0$ and $r_{X}^{\prime \prime}(0)=1 / \sigma_{H}^{2}(X)$. Expanding $r_{X}(\delta)$ at $\delta=0$ by the Taylor expansion, we then have that there exists a $\delta^{*}>0$ such that

$$
\begin{equation*}
r_{X}(\delta)=\frac{1}{2 \sigma_{H}^{2}(X)} \delta^{2}+O\left(\delta^{3}\right) \tag{2.18}
\end{equation*}
$$

for $\delta \in\left(0, \delta^{*}\right]$. The inequality (2.8) now follows immediately from 2.6) and 2.18). This completes the proof of Theorem 1.

Having analyzed the function $r_{X}(\delta)$, we are now ready for a stronger version of the right NEP. For any $\lambda \in\left[0, \lambda^{*}(X)\right)$, define

$$
\begin{gather*}
f_{\lambda}(x) \triangleq \frac{p^{-\lambda}(x)}{\int p^{-\lambda+1}(y) d y}  \tag{2.19}\\
\sigma_{H}^{2}(X, \lambda) \triangleq \int f_{\lambda}(x) p(x)|-\ln p(x)-(H(X)+\delta(\lambda))|^{2} d x  \tag{2.20}\\
M_{H}(X, \lambda) \triangleq \int f_{\lambda}(x) p(x)|-\ln p(x)-(H(X)+\delta(\lambda))|^{3} d x \tag{2.21}
\end{gather*}
$$

and

$$
\begin{equation*}
f_{\lambda}\left(x^{n}\right) \triangleq \prod_{i=1}^{n} f_{\lambda}\left(x_{i}\right) \tag{2.22}
\end{equation*}
$$

where $\delta(\lambda)$ is defined in (2.10). Write $M_{H}(X, 0)$ as $M_{H}(X)$. It is easy to see that $\sigma_{H}^{2}(X, 0)=$ $\sigma_{H}^{2}(X), \sigma_{H}^{2}(X, \lambda)=\delta^{\prime}(\lambda)$, and

$$
\begin{equation*}
\left.M_{H}(X)=\int p(x) \mid-\ln p(x)-H(X)\right)\left.\right|^{3} d x \tag{2.23}
\end{equation*}
$$

Then we have the following stronger result.

Theorem 2 (Strong Right NEP). Under the assumptions (2.2) and (2.5), the following hold:
(a) For any $\delta \in\left(0, \Delta^{*}(X)\right)$ and any positive integer $n$

$$
\begin{equation*}
\bar{\xi}_{H}(X, \lambda, n) e^{-n r_{X}(\delta)} \geq \operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n}\right)>H(X)+\delta\right\} \geq \underline{\xi}_{H}(X, \lambda, n) e^{-n r_{X}(\delta)} \tag{2.24}
\end{equation*}
$$

where $\lambda=r_{X}^{\prime}(\delta)>0$,

$$
\begin{align*}
& \bar{\xi}_{H}(X, \lambda, n)=\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)} \\
& \quad+e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}}\left[Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)-Q\left(\rho^{*}+\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)\right]  \tag{2.25}\\
& \underline{\xi}_{H}(X, \lambda, n)=e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\rho_{*}+\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right) \tag{2.26}
\end{align*}
$$

with $Q\left(\rho^{*}\right)=\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}$ and $Q\left(\rho_{*}\right)=\frac{1}{2}-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}, Q(t)=\frac{1}{\sqrt{2 \pi}} \int_{t}^{\infty} e^{-u^{2} / 2} d u$ and $C<1$ is the universal constant in the central limit theorem of Berry and Esseen.
(b) For any $\delta \leq c \sqrt{\frac{\ln n}{n}}$, where $c<\sigma_{H}(X)$ is a constant,

$$
\begin{align*}
Q\left(\frac{\delta \sqrt{n}}{\sigma_{H}(X)}\right)-\frac{C M_{H}(X)}{\sqrt{n} \sigma_{H}^{3}(X)} & \leq \operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n}\right)>H(X)+\delta\right\} \\
& \leq Q\left(\frac{\delta \sqrt{n}}{\sigma_{H}(X)}\right)+\frac{C M_{H}(X)}{\sqrt{n} \sigma_{H}^{3}(X)} \tag{2.27}
\end{align*}
$$

Proof of Theorem 2. From (2.15), it follows that with $\lambda=r_{X}^{\prime}(\delta)$

$$
\begin{equation*}
r_{X}(\delta)=\lambda(H(X)+\delta)-\ln \int p^{-\lambda+1}(x) d x \tag{2.28}
\end{equation*}
$$

Then it is not hard to verify that

$$
\begin{aligned}
\operatorname{Pr} & \left\{-\frac{1}{n} \ln p\left(X^{n}\right)>H(X)+\delta\right\} \\
& =\int_{-\frac{1}{n} \ln p\left(x^{n}\right)>H(X)+\delta} p\left(x^{n}\right) d x^{n} \\
& =\int_{-\frac{1}{n} \ln p\left(x^{n}\right)>H(X)+\delta} f_{\lambda}^{-1}\left(x^{n}\right) f_{\lambda}\left(x^{n}\right) p\left(x^{n}\right) d x^{n} \\
& =\int_{-\frac{1}{n} \ln p\left(x^{n}\right)>H(X)+\delta} e^{-n\left[-\frac{1}{n} \lambda \ln p\left(x^{n}\right)-\ln \int p^{-\lambda+1}(y) d y\right]} f_{\lambda}\left(x^{n}\right) p\left(x^{n}\right) d x^{n} \\
& =\int_{-\frac{1}{n} \ln p\left(x^{n}\right)>H(X)+\delta} e^{-n\left[-\frac{1}{n} \lambda \ln p\left(x^{n}\right)-\lambda(H(X)+\delta)+r_{X}(\delta)\right]} f_{\lambda}\left(x^{n}\right) p\left(x^{n}\right) d x^{n} \\
= & e^{-n r_{X}(\delta)} \int_{-\frac{1}{n} \ln p\left(x^{n}\right)>H(X)+\delta} e^{-n \lambda\left[-\frac{1}{n} \ln p\left(x^{n}\right)-(H(X)+\delta)\right]} f_{\lambda}\left(x^{n}\right) p\left(x^{n}\right) d x^{n} \\
= & e^{-n r_{X}(\delta)} \int_{-\frac{1}{n} \ln p\left(x^{n}\right)>H(X)+\delta} e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \frac{\ln p\left(x^{n}\right)-n(H(X)+\delta)}{\sqrt{n} \sigma_{H}(X, \lambda)}} f_{\lambda}\left(x^{n}\right) p\left(x^{n}\right) d x^{n}
\end{aligned}
$$

$$
\begin{align*}
& =e^{-n r_{X}(\delta)} \int_{\rho>0} \int_{\substack{\frac{\ln p\left(x^{n}\right)-n(H(X)+\delta)}{\sqrt{n} \sigma_{H}(X, \lambda)}=\rho}} e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho} f_{\lambda}\left(x^{n}\right) p\left(x^{n}\right) d x^{n} d \rho \\
& =e^{-n r_{X}(\delta)} \int_{0}^{+\infty} e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho} d\left(1-\bar{F}_{n}(\rho)\right) \\
& =e^{-n r_{X}(\delta)}\left[\bar{F}_{n}(0)-\int_{0}^{+\infty} \sqrt{n} \lambda \sigma_{H}(X, \lambda) e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho} \bar{F}_{n}(\rho) d \rho\right] \tag{2.29}
\end{align*}
$$

where the last equality is due to integration by parts,

$$
\begin{aligned}
\bar{F}_{n}(\rho) & \triangleq \operatorname{Pr}\left\{\frac{-\ln p\left(Z^{n}\right)-n(H(X)+\delta)}{\sqrt{n} \sigma_{H}(X, \lambda)}>\rho\right\} \\
& =\operatorname{Pr}\left\{\sum_{i=1}^{n} \frac{-\ln p\left(Z_{i}\right)-(H(X)+\delta)}{\sqrt{n} \sigma_{H}(X, \lambda)}>\rho\right\}
\end{aligned}
$$

and $\left\{Z_{i}\right\}_{i=1}^{n}$ are IID random variables with pmf or pdf (as the case may be) $f_{\lambda}(x) p(x)$. Let

$$
\begin{align*}
\xi_{n} & \triangleq \bar{F}_{n}(0)-\int_{0}^{+\infty} \sqrt{n} \lambda \sigma_{H}(X, \lambda) e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho} \bar{F}_{n}(\rho) d \rho  \tag{2.30}\\
& =\int_{0}^{+\infty} \sqrt{n} \lambda \sigma_{H}(X, \lambda) e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho}\left[\bar{F}_{n}(0)-\bar{F}_{n}(\rho)\right] d \rho \tag{2.31}
\end{align*}
$$

At this point, we invoke the following central limit theorem of Berry and Esseen [2], Theorem 1.2].

Lemma 1. Let $V_{1}, V_{2}, \cdots$ be independent real random variables with zero means and finite third moments, and set

$$
\sigma_{n}^{2}=\sum_{i=1}^{n} \mathbf{E} V_{i}^{2}
$$

Then there exists a universal constant $C<1$ such that for any $n \geq 1$,

$$
\sup _{-\infty<t<+\infty}\left|\operatorname{Pr}\left\{\sum_{i=1}^{n} V_{i}>\sigma_{n} t\right\}-Q(t)\right| \leq C \sigma_{n}^{-3} \sum_{i=1}^{n} \mathbf{E}\left|V_{i}\right|^{3} .
$$

Towards evaluating $\xi_{n}$, we can bound $\bar{F}_{n}(\rho)$ in terms of $Q(\rho)$, by applying Lemma 1 to
$\left\{-\ln p\left(Z_{i}\right)-(H(X)+\delta)\right\}_{i=1}^{n}$. Then for $\rho>0$, we have

$$
\begin{align*}
\bar{F}_{n}(0) & \leq Q(0)+\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)} \\
& =\frac{1}{2}+\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}  \tag{2.32}\\
\bar{F}_{n}(\rho) & \geq\left[Q(\rho)-\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right]^{+} \tag{2.33}
\end{align*}
$$

and

$$
\begin{align*}
\bar{F}_{n}(0)-\bar{F}_{n}(\rho) & \geq\left[Q(0)-\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}-\left(Q(\rho)+\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right)\right]^{+} \\
& =\left[\frac{1}{2}-Q(\rho)-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right]^{+} \tag{2.34}
\end{align*}
$$

where $[x]^{+}=\max \{x, 0\}$. Now plugging (2.32) and (2.33) into (2.30) yields

$$
\begin{align*}
\xi_{n} & \leq \frac{1}{2}+\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}-\int_{0}^{+\infty} \sqrt{n} \lambda \sigma_{H}(X, \lambda) e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho}\left[Q(\rho)-\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right]^{+} d \rho \\
& =\frac{1}{2}+\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}-\int_{0}^{\rho^{*}} \sqrt{n} \lambda \sigma_{H}(X, \lambda) e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho}\left[Q(\rho)-\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right] d \rho \\
& =\frac{1}{2}+\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}-\int_{0}^{\rho^{*}}\left[Q(\rho)-\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right] d\left(-e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho}\right) \\
& =\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}+\int_{0}^{\rho^{*}} \frac{1}{\sqrt{2 \pi}} e^{-\frac{\rho^{2}}{2}} e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho} d \rho \\
& =\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}+\int_{0}^{\rho^{*}} \frac{1}{\sqrt{2 \pi}} e^{-\frac{\left(\rho+\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)^{2}}{2}+\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} d \rho \\
& =\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}+e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}}\left[Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)-Q\left(\rho^{*}+\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)\right] \\
& =\bar{\xi}_{H}(X, \lambda, n) \tag{2.35}
\end{align*}
$$

where $Q\left(\rho^{*}\right)=\frac{C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}$, and meanwhile plugging (2.34) into (2.31) yields

$$
\xi_{n} \geq \int_{0}^{+\infty} \sqrt{n} \lambda \sigma_{H}(X, \lambda) e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho}\left[\frac{1}{2}-Q(\rho)-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right]^{+} d \rho
$$

$$
\begin{align*}
& =\int_{\rho_{*}}^{+\infty} \sqrt{n} \lambda \sigma_{H}(X, \lambda) e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho}\left[\frac{1}{2}-Q(\rho)-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right] d \rho \\
& =\int_{\rho_{*}}^{+\infty}\left[\frac{1}{2}-Q(\rho)-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right] d\left(-e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho}\right) \\
& =\int_{\rho_{*}}^{+\infty} \frac{1}{\sqrt{2 \pi}} e^{-\frac{\rho^{2}}{2}} e^{-\sqrt{n} \lambda \sigma_{H}(X, \lambda) \rho} d \rho \\
& =e^{\frac{\lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\rho_{*}+\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right) \\
& =\xi_{H}(X, \lambda, n) \tag{2.36}
\end{align*}
$$

where $Q\left(\rho_{*}\right)=\frac{1}{2}-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}$. Combining (2.29) with 2.35) and 2.36) completes the proof of part (a) of Theorem 2.

Applying Lemma 1 to the IID sequence $\left\{-\ln p\left(X_{i}\right)-H(X)\right\}_{i=1}^{n}$, we get (2.27). This completes the proof of Theorem 2.

Remark 1. Note that $\lambda=r_{X}^{\prime}(\delta)=\Theta(\delta)$. When $\lambda=\Omega(1)$ with respect to $n$, it can be easily verified that $\bar{\xi}_{H}(X, \lambda, n)$ and $\underline{\xi}_{H}(X, \lambda, n)$ are both on the order of $\frac{1}{\sqrt{n}}$, by applying well-known inequality

$$
\frac{1}{t+t^{-1}} \frac{1}{\sqrt{2 \pi}} e^{-\frac{t^{2}}{2}} \leq Q(t) \leq \frac{1}{t} \frac{1}{\sqrt{2 \pi}} e^{-\frac{t^{2}}{2}} .
$$

Meanwhile, on one hand, it is easy to see that

$$
\bar{\xi}_{H}(X, \lambda, n) \leq e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)+\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)} .
$$

On the other hand,

$$
\begin{aligned}
\underline{\xi}_{H}(X, \lambda, n) & =e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)-e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} \int_{\sqrt{n} \lambda \sigma_{H}(X, \lambda)}^{\rho_{*}+\sqrt{n} \lambda \sigma_{H}(X, \lambda)} \frac{1}{\sqrt{2 \pi}} e^{-\frac{\rho^{2}}{2}} d \rho \\
& =e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)-e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} \int_{0}^{\rho_{*}} \frac{1}{\sqrt{2 \pi}} e^{-\frac{\left(\rho+\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)^{2}}{2}} d \rho \\
& =e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)-\int_{0}^{\rho_{*}} \frac{1}{\sqrt{2 \pi}} e^{-\frac{\rho^{2}+2 \rho_{\sqrt{n} \lambda \sigma_{H}(X, \lambda)}^{2}}{2}} d \rho
\end{aligned}
$$

$$
\begin{aligned}
& \geq e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)-\int_{0}^{\rho_{*}} \frac{1}{\sqrt{2 \pi}} e^{-\frac{\rho^{2}}{2}} d \rho \\
& =e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}
\end{aligned}
$$

To further shed light on $\bar{\xi}_{H}(X, \lambda, n)$ and $\underline{\xi}_{H}(X, \lambda, n)$, we observe that

$$
\frac{1}{\sqrt{2 \pi} \sqrt{n} \lambda \sigma_{H}(X, \lambda)+\frac{1}{\sqrt{2 \pi} \sqrt{n} \lambda \sigma_{H}(X, \lambda)}} \leq e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right) \leq \frac{1}{\sqrt{2 \pi} \sqrt{n} \lambda \sigma_{H}(X, \lambda)} .
$$

And therefore, whenever $\lambda=o(1)$ and $\lambda=\omega\left(n^{-1}\right)$,

$$
e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)=\Theta\left(\frac{1}{\sqrt{n} \lambda}\right)=\omega\left(\frac{1}{\sqrt{n}}\right)
$$

which further implies

$$
\begin{aligned}
& \bar{\xi}_{H}(X, \lambda, n)=e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)(1+o(1)) \\
& \underline{\xi}_{H}(X, \lambda, n)=e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X, \lambda)}{2}} Q\left(\sqrt{n} \lambda \sigma_{H}(X, \lambda)\right)(1-o(1)) .
\end{aligned}
$$

Remark 2. Another interesting observation from the proof of Theorem 2, especially (2.29), is the recursive relation between

$$
\begin{aligned}
\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n}\right)>H(X)+\delta\right\} & =\operatorname{Pr}\left\{\frac{-\ln p\left(X^{n}\right)-n H(X)}{\sqrt{n} \sigma_{H}(X)}>\frac{\delta}{\sqrt{n} \sigma_{H}(X)}\right\} \\
& \triangleq \bar{F}_{X, n}\left(\frac{\delta}{\sqrt{n} \sigma_{H}(X)}\right)
\end{aligned}
$$

and

$$
\bar{F}_{Z, n}(\rho) \triangleq \bar{F}_{n}(\rho)=\operatorname{Pr}\left\{\frac{-\ln p\left(Z^{n}\right)-n(H(X)+\delta)}{\sqrt{n} \sigma_{H}(X, \lambda)}>\rho\right\} .
$$

As shown in the proof, a proper bound on $\bar{F}_{Z, n}(\rho)$ (using Berry-Esseen Central Limit Theorem) results in a bound (2.24) on $\bar{F}_{X, n}\left(\frac{\delta}{\sqrt{n} \sigma_{H}(X)}\right)$. To continue, we can apply this bound (2.24) on $\bar{F}_{Z, n}(\rho)$ to get another bound on $\bar{F}_{X, n}\left(\frac{\delta}{\sqrt{n} \sigma_{H}(X)}\right)$. Numerically, we can keep tightening the bound on $\bar{F}_{X, n}\left(\frac{\delta}{\sqrt{n} \sigma_{H}(X)}\right)$ in this recursive manner until no significant improvement can be made.

The probability that $-\frac{1}{n} \ln p\left(X^{n}\right)$ is away from $H(X)$ to the left can be bounded similarly. Define

$$
\begin{equation*}
\lambda_{-}^{*}(X) \triangleq \sup \left\{\lambda \geq 0: \int p^{\lambda+1}(x) d x<\infty\right\} \tag{2.37}
\end{equation*}
$$

Suppose that

$$
\begin{equation*}
\lambda_{-}^{*}(X)>0 \tag{2.38}
\end{equation*}
$$

Define for any $\delta \geq 0$

$$
r_{X,-}(\delta) \stackrel{\Delta}{=} \sup _{\lambda \geq 0}\left[\lambda(\delta-H(X))-\ln \int p^{\lambda+1}(x) d x\right]
$$

and for any $\lambda \in\left[0, \lambda_{-}^{*}(X)\right)$

$$
\delta_{-}(\lambda) \triangleq \int \frac{p^{\lambda+1}(x)}{\left[\int p^{\lambda+1}(y) d y\right]}[\ln p(x)] d x+H(X)
$$

Then under the assumption (2.5), $\delta_{-}(\lambda)$ is strictly increasing over $\lambda \in\left[0, \lambda_{-}^{*}(X)\right)$ with $\delta_{-}(0)=0$. Let

$$
\Delta_{-}^{*}(X)=\lim _{\lambda \uparrow \lambda_{-}^{*}(X)} \delta_{-}(\lambda)
$$

Following the proof of Theorem 1, we have that $r_{X,-}(\delta)$ is strictly increasing, convex, and continuously differentiable up to at least the third order inclusive over $\delta \in\left[0, \Delta_{-}^{*}(X)\right)$, and furthermore

$$
r_{X,-}(\delta)=\lambda(\delta-H(X))-\ln \int p^{\lambda+1}(x) d x
$$

with $\lambda=r_{X,-}^{\prime}(\delta)$ satisfying

$$
\delta_{-}(\lambda)=\delta
$$

Define

$$
\sigma_{H,-}^{2}(X, \lambda) \triangleq \int \frac{p^{\lambda+1}(x)}{\left[\int p^{\lambda+1}(y) d y\right]}\left|-\ln p(x)-\left(H(X)-\delta_{-}(\lambda)\right)\right|^{2} d x
$$

and

$$
M_{H,-}(X, \lambda) \triangleq \int \frac{p^{\lambda+1}(x)}{\left[\int p^{\lambda+1}(y) d y\right]}\left|-\ln p(x)-\left(H(X)-\delta_{-}(\lambda)\right)\right|^{3} d x
$$

In parallel with Theorems 1 and 2, we have the following result, which is referred to as the left NEP with respect to $H(X)$ and can be proved similarly.

Theorem 3 (Left NEP). For any positive integer n,

$$
\begin{equation*}
\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n}\right) \leq H(X)-\delta\right\} \leq e^{-n r_{X,-}(\delta)} \tag{2.39}
\end{equation*}
$$

Furthermore, under the assumptions (2.38) and (2.5), the following also hold:
(a) There exists a $\delta^{*}>0$ such that for any $\delta \in\left(0, \delta^{*}\right]$ and any positive integer $n$,

$$
\begin{equation*}
r_{X,-}(\delta)=\frac{1}{2 \sigma_{H}^{2}(X)} \delta^{2}+O\left(\delta^{3}\right) \tag{2.40}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n}\right) \leq H(X)-\delta\right\} \leq e^{-n\left(\frac{\delta^{2}}{2 \sigma_{H}^{2}(X)}+O\left(\delta^{3}\right)\right)} \tag{2.41}
\end{equation*}
$$

(b) For any $\delta \in\left(0, \Delta_{-}^{*}(X)\right)$ and any positive integer $n$

$$
\begin{equation*}
\bar{\xi}_{H,-}(X, \lambda, n) e^{-n r_{X,-}(\delta)} \geq \operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n}\right) \leq H(X)-\delta\right\} \geq \underline{\xi}_{H,-}(X, \lambda, n) e^{-n r_{X,-}(\delta)} \tag{2.42}
\end{equation*}
$$

where $\lambda=r_{X,-}^{\prime}(\delta)>0$, and

$$
\begin{align*}
& \bar{\xi}_{H,-}(X, \lambda, n)=\frac{2 C M_{H,-}(X, \lambda)}{\sqrt{n} \sigma_{H,-}^{3}(X, \lambda)} \\
& \quad+e^{\frac{n \lambda^{2} \sigma_{H,-}^{2}(X, \lambda)}{2}}\left[Q\left(\sqrt{n} \lambda \sigma_{H,-}(X, \lambda)\right)-Q\left(\rho^{*}+\sqrt{n} \lambda \sigma_{H,-}(X, \lambda)\right)\right]  \tag{2.43}\\
& \underline{\xi}_{H,-}(X, \lambda, n)=e^{\frac{n \lambda^{2} \sigma_{H,-}^{2}(X, \lambda)}{2}} Q\left(\rho_{*}+\sqrt{n} \lambda \sigma_{H,-}(X, \lambda)\right) \tag{2.44}
\end{align*}
$$

with $Q\left(\rho^{*}\right)=\frac{C M_{H,-}(X, \lambda)}{\sqrt{n} \sigma_{H,-}^{3}(X, \lambda)}$ and $Q\left(\rho_{*}\right)=\frac{1}{2}-\frac{2 C M_{H,-}(X, \lambda)}{\sqrt{n} \sigma_{H,-}^{3}(X, \lambda)}$.
(c) For any $\delta \leq c \sqrt{\frac{\ln n}{n}}$, where $c<\sigma_{H}(X)$ is a constant,

$$
\begin{align*}
Q\left(\frac{\delta \sqrt{n}}{\sigma_{H}(X)}\right)-\frac{C M_{H}(X)}{\sqrt{n} \sigma_{H}^{3}(X)} & \leq \operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n}\right) \leq H(X)-\delta\right\} \\
& \leq Q\left(\frac{\delta \sqrt{n}}{\sigma_{H}(X)}\right)+\frac{C M_{H}(X)}{\sqrt{n} \sigma_{H}^{3}(X)} \tag{2.45}
\end{align*}
$$

Remarks similar to those (Remark 1/and 2) following Theorem 2 can be drawn here concerning Theorem 3.

## III. NEP With Respect to Conditional Entropy

Consider now an IID source pair $(X, Y)=\left\{\left(X_{i}, Y_{i}\right)\right\}_{i=1}^{\infty}$ with finite conditional entropy $H(X \mid Y)$, where $H(X \mid Y)$ is the Shannon conditional entropy of $X_{i}$ given $Y_{i}$ if $X$ is discrete, and the conditional differential entropy of $X_{i}$ given $Y_{i}$ if $X$ is continuous. Let $p(x \mid y)$ be the conditional pmf or conditional pdf (as the case may be) of $X_{i}$ given $Y_{i}$, and $p(y)$ the pmf or pdf (as the case may be) of $Y_{i}$. By replacing $-\frac{1}{n} \ln p\left(X^{n}\right)$ with $-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right)$, all results and arguments in Section $\Pi$ can be carried over to this conditional case, yielding the NEP with respect to $H(X \mid Y)$.

Specifically, define

$$
\begin{equation*}
\lambda^{*}(X \mid Y) \triangleq \stackrel{\Delta}{=} \sup \left\{\lambda \geq 0: \int p(y)\left[\int p^{-\lambda+1}(x \mid y) d x\right] d y<\infty\right\} \tag{3.1}
\end{equation*}
$$

where $\int d y$ is understood throughout this paper to be the summation over the source alphabet of $Y$ if $Y$ is discrete. Suppose that

$$
\begin{equation*}
\lambda^{*}(X \mid Y)>0 \tag{3.2}
\end{equation*}
$$

Let

$$
\begin{equation*}
\sigma_{H}^{2}(X \mid Y) \triangleq \iint p(y) p(x \mid y)[-\ln p(x \mid y)]^{2} d x d y-H^{2}(X \mid Y) \tag{3.3}
\end{equation*}
$$

which will be referred to as the conditional information variance of $X$ given $Y$. It is not hard to see that under the assumption (3.2),

$$
\begin{equation*}
\iint \frac{p(y) p^{-\lambda+1}(x \mid y)}{\left[\iint p(v) p^{-\lambda+1}(u \mid v) d u d v\right]}|-\ln p(x \mid y)|^{k} d x d y<\infty \tag{3.4}
\end{equation*}
$$

and

$$
\iint p(y) p^{-\lambda+1}(x \mid y) d x d y<\infty
$$

for any $\lambda \in\left(0, \lambda^{*}(X \mid Y)\right)$ and any positive integer $k$. Further assume that

$$
\begin{equation*}
\sigma_{H}^{2}(X \mid Y)>0 \text { and } \iint p(y) p(x \mid y)|\ln p(x \mid y)|^{3} d x d y<\infty \tag{3.5}
\end{equation*}
$$

Define for any $\delta \geq 0$

$$
\begin{equation*}
r_{X \mid Y}(\delta) \stackrel{\Delta}{=} \sup _{\lambda \geq 0}\left[\lambda(H(X \mid Y)+\delta)-\ln \iint p(y) p^{-\lambda+1}(x \mid y) d x d y\right] \tag{3.6}
\end{equation*}
$$

and for any $\lambda \in\left[0, \lambda^{*}(X \mid Y)\right)$

$$
\begin{equation*}
\delta(\lambda) \triangleq \iint \frac{p(y) p^{-\lambda+1}(x \mid y)}{\left[\iint p(v) p^{-\lambda+1}(u \mid v) d u d v\right]}[-\ln p(x \mid y)] d x d y-H(X \mid Y) \tag{3.7}
\end{equation*}
$$

(Throughout this section, $\delta(\lambda)$ should be understood with its above definition.) Then under the assumptions (3.2) and (3.5), $\delta(\lambda)$ is strictly increasing over $\lambda \in\left[0, \lambda^{*}(X \mid Y)\right)$ with $\delta(0)=0$. Let

$$
\Delta^{*}(X \mid Y) \triangleq \lim _{\lambda \uparrow \lambda^{*}(X \mid Y)} \delta(\lambda)
$$

By an argument similar to that in the proof of Theorem 1, it can be shown that $r_{X \mid Y}(\delta)$ is strictly increasing, convex and continuously differentiable up to at least the third order inclusive over $\delta \in\left[0, \Delta^{*}(X \mid Y)\right.$ ), and furthermore $r_{X \mid Y}(\delta)$ has the following parametric expression

$$
\begin{equation*}
r_{X \mid Y}(\delta(\lambda))=\lambda(H(X \mid Y)+\delta(\lambda))-\ln \iint p(y) p^{-\lambda+1}(x \mid y) d x d y \tag{3.8}
\end{equation*}
$$

with $\delta(\lambda)$ defined in (3.7) and $\lambda=r_{X \mid Y}^{\prime}(\delta)$. For any $\lambda \in\left[0, \lambda^{*}(X \mid Y)\right)$, define

$$
\begin{equation*}
f_{\lambda}(x, y) \triangleq \frac{p^{-\lambda}(x \mid y)}{\iint p(v) p^{-\lambda+1}(u \mid v) d u d v} \tag{3.9}
\end{equation*}
$$

$$
\begin{align*}
& \sigma_{H}^{2}(X \mid Y, \lambda) \triangleq \iint f_{\lambda}(x, y) p(y) p(x \mid y)|-\ln p(x \mid y)-(H(X \mid Y)+\delta(\lambda))|^{2} d x d y  \tag{3.10}\\
& M_{H}(X \mid Y, \lambda) \triangleq \iint f_{\lambda}(x, y) p(y) p(x \mid y)|-\ln p(x \mid y)-(H(X \mid Y)+\delta(\lambda))|^{3} d x d y \tag{3.11}
\end{align*}
$$

where $\delta(\lambda)$ is defined in (3.7). Write $M_{H}(X \mid Y, 0)$ as $M_{H}(X \mid Y)$. It is easy to see that $\sigma_{H}^{2}(X \mid Y, 0)=$ $\sigma_{H}^{2}(X \mid Y), \sigma_{H}^{2}(X \mid Y, \lambda)=\delta^{\prime}(\lambda)$, and

$$
\begin{equation*}
\left.M_{H}(X \mid Y)=\iint p(y) p(x \mid y) \mid-\ln p(x \mid y)-H(X \mid Y)\right)\left.\right|^{3} d x d y \tag{3.12}
\end{equation*}
$$

In parallel with Theorems 1 and 2, we have the following result, which is referred to as the right NEP with respect to $H(X \mid Y)$ and can be proved similarly.

Theorem 4 (Right NEP With Respect to $H(X \mid Y)$ ). For any positive integer $n$,

$$
\begin{equation*}
\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right)>H(X \mid Y)+\delta\right\} \leq e^{-n r_{X \mid Y}(\delta)} \tag{3.13}
\end{equation*}
$$

where $X^{n}=X_{1} X_{2} \cdots X_{n}$ and $Y^{n}=Y_{1} Y_{2} \cdots Y_{n}$. Moreover, under the assumptions (3.2) and (3.5), the following also hold:
(a) There exists a $\delta^{*}>0$ such that for any $\delta \in\left(0, \delta^{*}\right]$ and any positive integer $n$,

$$
\begin{equation*}
r_{X \mid Y}(\delta)=\frac{1}{2 \sigma_{H}^{2}(X \mid Y)} \delta^{2}+O\left(\delta^{3}\right) \tag{3.14}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right)>H(X \mid Y)+\delta\right\} \leq e^{-n\left(\frac{\delta^{2}}{2 \sigma_{H}^{2}(X \mid Y)}+O\left(\delta^{3}\right)\right)} . \tag{3.15}
\end{equation*}
$$

(b) For any $\delta \in\left(0, \Delta^{*}(X \mid Y)\right)$ and any positive integer $n$

$$
\begin{align*}
\underline{\xi}_{H}(X \mid Y, \lambda, n) e^{-n r_{X \mid Y}(\delta)} & \leq \operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(Y^{n} \mid X^{n}\right)>H(X \mid Y)+\delta\right\} \\
& \leq \bar{\xi}_{H}(X \mid Y, \lambda, n) e^{-n r_{X \mid Y}(\delta)} \tag{3.16}
\end{align*}
$$

where $\lambda=r_{X \mid Y}^{\prime}(\delta)>0$, and

$$
\begin{align*}
& \bar{\xi}_{H}(X \mid Y, \lambda, n)=\frac{2 C M_{H}(X \mid Y, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X \mid Y, \lambda)} \\
& \quad+e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X \mid Y, \lambda)}{2}}\left[Q\left(\sqrt{n} \lambda \sigma_{H}(X \mid Y, \lambda)\right)-Q\left(\rho^{*}+\sqrt{n} \lambda \sigma_{H}(X \mid Y, \lambda)\right)\right]  \tag{3.17}\\
& \underline{\xi}_{H}(X \mid Y, \lambda, n)=e^{\frac{n \lambda^{2} \sigma_{H}^{2}(X \mid Y, \lambda)}{2}} Q\left(\rho_{*}+\sqrt{n} \lambda \sigma_{H}(X \mid Y, \lambda)\right) \tag{3.18}
\end{align*}
$$

with $Q\left(\rho^{*}\right)=\frac{C M_{H}(X \mid Y, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X \mid Y, \lambda)}$ and $Q\left(\rho_{*}\right)=\frac{1}{2}-\frac{2 C M_{H}(X \mid Y, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X \mid Y, \lambda)}$.
(c) For any $\delta \leq c \sqrt{\frac{\ln n}{n}}$, where $c<\sigma_{H}(X \mid Y)$ is a constant,

$$
\begin{align*}
Q\left(\frac{\delta \sqrt{n}}{\sigma_{H}(X \mid Y)}\right)-\frac{C M_{H}(X \mid Y)}{\sqrt{n} \sigma_{H}^{3}(X \mid Y)} & \leq \operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right)>H(X \mid Y)+\delta\right\} \\
& \leq Q\left(\frac{\delta \sqrt{n}}{\sigma_{H}(X \mid Y)}\right)+\frac{C M_{H}(X \mid Y)}{\sqrt{n} \sigma_{H}^{3}(X \mid Y)} \tag{3.19}
\end{align*}
$$

The probability that $-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right)$ is away from $H(X \mid Y)$ to the left can be bounded similarly. For completeness, we state the result without proof again. Define

$$
\begin{equation*}
\lambda_{-}^{*}(X \mid Y) \triangleq \stackrel{\Delta}{=} \sup \left\{\lambda \geq 0: \iint p(y) p^{\lambda+1}(x \mid y) d x d y<\infty\right\} \tag{3.20}
\end{equation*}
$$

Suppose that

$$
\begin{equation*}
\lambda_{-}^{*}(X \mid Y)>0 . \tag{3.21}
\end{equation*}
$$

Define for any $\delta \geq 0$

$$
r_{X \mid Y,-}(\delta) \triangleq \sup _{\lambda \geq 0}\left[\lambda(\delta-H(X \mid Y))-\ln \iint p(y) p^{\lambda+1}(x \mid y) d x d y\right]
$$

and for any $\lambda \in\left[0, \lambda_{-}^{*}(X \mid Y)\right)$

$$
\delta_{-}(\lambda) \triangleq \iint \frac{p(y) p^{\lambda+1}(x \mid y)}{\left[\iint p(v) p^{\lambda+1}(u \mid v) d u d v\right]}[\ln p(x \mid y)] d x d y+H(X \mid Y) .
$$

(Throughout this section, $\delta_{-}(\lambda)$ should be understood with its above definition.) Then under the assumption (3.5), $\delta_{-}(\lambda)$ is strictly increasing over $\lambda \in\left[0, \lambda_{-}^{*}(X \mid Y)\right)$ with $\delta_{-}(0)=0$. Let

$$
\Delta_{-}^{*}(X \mid Y)=\lim _{\lambda \uparrow \lambda_{-}^{*}(X \mid Y)} \delta_{-}(\lambda) .
$$

By using an argument similar to that in the proof of Theorem 1 , it can be shown that $r_{X \mid Y,-}(\delta)$ is strictly increasing, convex, and continuously differentiable up to at least the third order inclusive over $\delta \in\left[0, \Delta_{-}^{*}(X \mid Y)\right)$, and furthermore $r_{X \mid Y,-}(\delta)$ has the following parametric expression

$$
r_{X \mid Y,-}\left(\delta_{-}(\lambda)\right)=\lambda\left(\delta_{-}(\lambda)-H(X \mid Y)\right)-\ln \iint p(y) p^{\lambda+1}(x \mid y) d x d y
$$

with $\lambda=r_{X \mid Y,-}^{\prime}(\delta)$ satisfying

$$
\delta_{-}(\lambda)=\delta
$$

Define

$$
\sigma_{H,-}^{2}(X \mid Y, \lambda) \triangleq \iint \frac{p(y) p^{\lambda+1}(x \mid y)}{\left[\iint p(v) p^{\lambda+1}(u \mid v) d u d v\right]}\left|-\ln p(x \mid y)-\left(H(X \mid Y)-\delta_{-}(\lambda)\right)\right|^{2} d x d y
$$

and

$$
M_{H,-}(X \mid Y, \lambda) \triangleq \iint \frac{p(y) p^{\lambda+1}(x \mid y)}{\left[\iint p(v) p^{\lambda+1}(u \mid v) d u d v\right]}\left|-\ln p(x \mid y)-\left(H(X \mid Y)-\delta_{-}(\lambda)\right)\right|^{3} d x d y .
$$

In parallel with Theorem 3, we have the following result, which is referred to as the left NEP with respect to $H(X \mid Y)$ and can be proved similarly.

Theorem 5 (Left NEP With Respect to $H(X \mid Y)$ ). For any positive integer $n$,

$$
\begin{equation*}
\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right) \leq H(X \mid Y)-\delta\right\} \leq e^{-n r_{X \mid Y,-}(\delta)} \tag{3.22}
\end{equation*}
$$

Furthermore, under the assumptions (3.21) and (3.5), the following also hold:
(a) There exists a $\delta^{*}>0$ such that for any $\delta \in\left(0, \delta^{*}\right]$ and any positive integer $n$,

$$
\begin{equation*}
r_{X \mid Y,-}(\delta)=\frac{1}{2 \sigma_{H}^{2}(X \mid Y)} \delta^{2}+O\left(\delta^{3}\right) \tag{3.23}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right) \leq H(X \mid Y)-\delta\right\} \leq e^{-n\left(\frac{\delta^{2}}{2 \sigma_{H}^{2}(X \mid Y)}+O\left(\delta^{3}\right)\right)} . \tag{3.24}
\end{equation*}
$$

(b) For any $\delta \in\left(0, \Delta_{-}^{*}(X \mid Y)\right)$ and any positive integer $n$

$$
\begin{align*}
\underline{\xi}_{H,-}(X \mid Y, \lambda, n) e^{-n r_{X \mid Y,-}(\delta)} & \leq \operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(Y^{n} \mid X^{n}\right) \leq H(X \mid Y)-\delta\right\} \\
& \leq \bar{\xi}_{H,-}(X \mid Y, \lambda, n) e^{-n r_{X \mid Y,-}(\delta)} \tag{3.25}
\end{align*}
$$

where $\lambda=r_{X \mid Y,-}^{\prime}(\delta)>0$, and

$$
\begin{align*}
& \bar{\xi}_{H,-}(X \mid Y, \lambda, n)=\frac{2 C M_{H,-}(X \mid Y, \lambda)}{\sqrt{n} \sigma_{H,-}^{3}(X \mid Y, \lambda)} \\
& \quad+e^{\frac{n \lambda^{2} \sigma_{H,-}^{2}(X \mid Y, \lambda)}{2}}\left[Q\left(\sqrt{n} \lambda \sigma_{H,-}(X \mid Y, \lambda)\right)-Q\left(\rho^{*}+\sqrt{n} \lambda \sigma_{H,-}(X \mid Y, \lambda)\right)\right]  \tag{3.26}\\
& \underline{\xi}_{H,-}(X \mid Y, \lambda, n)=e^{\frac{n \lambda^{2} \sigma_{H,-}^{2}(X \mid Y, \lambda)}{2}} Q\left(\rho_{*}+\sqrt{n} \lambda \sigma_{H,-}(X \mid Y, \lambda)\right) \tag{3.27}
\end{align*}
$$

with $Q\left(\rho^{*}\right)=\frac{C M_{H,-}(X \mid Y, \lambda)}{\sqrt{n} \sigma_{H,-}^{3}(X \mid Y, \lambda)}$ and $Q\left(\rho_{*}\right)=\frac{1}{2}-\frac{2 C M_{H,-}(X \mid Y, \lambda)}{\sqrt{n} \sigma_{H,-}^{3}(X \mid Y, \lambda)}$.
(c) For any $\delta \leq c \sqrt{\frac{\ln n}{n}}$, where $c<\sigma_{H}(X \mid Y)$ is a constant,

$$
\begin{align*}
Q\left(\frac{\delta \sqrt{n}}{\sigma_{H}(X \mid Y)}\right)-\frac{C M_{H}(X \mid Y)}{\sqrt{n} \sigma_{H}^{3}(X \mid Y)} & \leq \operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right) \leq H(X \mid Y)-\delta\right\} \\
& \leq Q\left(\frac{\delta \sqrt{n}}{\sigma_{H}(X \mid Y)}\right)+\frac{C M_{H}(X \mid Y)}{\sqrt{n} \sigma_{H}^{3}(X \mid Y)} \tag{3.28}
\end{align*}
$$

Remarks similar to those (Remark 1 and 2) following Theorem 2 can be drawn here concerning Theorem 4 and 5.

Theorem 4] will be used in [3] to show that for any binary input memoryless channel with uniform capacity achieving input $X$, random linear codes of block length $n$ with either Elias' generator ensembles or Gallager's parity check ensembles can reach within $\delta+r_{X \mid Y}(\delta)+\frac{\ln n}{2 n}-$ $\frac{\ln \frac{2(1-C) M_{H}(X \mid Y, \lambda)}{\sigma_{H}^{3}(X \mid Y, \lambda)}}{n}$ of the channel capacity while maintaining word error probability upper bounded by $\left(\bar{\xi}_{H}(X \mid Y, \lambda, n)+\frac{2(1-C) M_{H}(X \mid Y, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X \mid Y, \lambda)}\right) e^{-n r_{X \mid Y}(\delta)}$. In particular, when $\delta=\sqrt{2 \alpha \sigma_{H}^{2}(X \mid Y)} \sqrt{\frac{\ln n}{n}}$, the word error probability is upper bounded by $\frac{1}{2 \sqrt{\pi \alpha \ln n}} n^{-\alpha}+O\left(n^{-\alpha \ln n} \sqrt{n}\right)$ and the achievable rate (in nats) of random linear codes of block length $n$ with either Elias' generator ensembles or Gallager's parity check ensembles is within $\sqrt{2 \alpha \sigma_{H}^{2}(X \mid Y)} \sqrt{\frac{\ln n}{n}}+\left(\alpha+\frac{1}{2}\right) \frac{\ln n}{n}+O\left(\frac{\ln \ln n}{n}\right)$ of the channel capacity; when $\delta=\frac{c}{\sqrt{n}}$ for any $c$, the word error probability is upper bounded by $Q\left(\frac{c}{\sigma_{H}(X \mid Y)}\right)+\frac{M_{H}(X \mid Y)}{\sigma_{H}^{3}(X \mid Y) \sqrt{n}}$ and the achievable rate (in nats) is within $\frac{c}{\sqrt{n}}+\frac{\ln n}{2 n}-\frac{1}{n} \ln \frac{(1-C) M_{H}(X \mid Y)}{\sigma_{H}^{3}(X \mid Y)}$ of the channel capacity.

We conclude this section by illustrating $r_{X \mid Y}(\delta)$ and $\sigma_{H}^{2}(X \mid Y)$ when $X$ and $Y$ are the uniform input and the corresponding output of the binary symmetric channel (BSC) and the binary input Gaussian channel.

Example 1 (BSC): Combining (3.7) and (3.8), it is not hard to verify that

$$
\begin{align*}
r_{X \mid Y}(\delta(\lambda)) & =\iint p(x, y) f_{\lambda}(x, y) \ln f_{\lambda}(x, y) d x d y \\
& =\iint p(x, y) f_{\lambda}(x, y) \ln \frac{p(x \mid y) f_{\lambda}(x, y)}{p(x \mid y)} d x d y \\
& \triangleq D\left(p(x \mid y) f_{\lambda}(x, y) \| p(x \mid y)\right) \tag{3.29}
\end{align*}
$$

For BSC, simple calculation reveals that

$$
p(x \mid y)= \begin{cases}1-p & \text { if } x=y  \tag{3.30}\\ p & \text { otherwise }\end{cases}
$$

and

$$
p(x \mid y) f_{\lambda}(x, y)= \begin{cases}\frac{(1-p)^{-\lambda+1}}{p^{-\lambda+1}+(1-p)^{-\lambda+1}} & \text { if } x=y  \tag{3.31}\\ \frac{p^{-\lambda+1}}{p^{-\lambda+1}+(1-p)^{-\lambda+1}} & \text { otherwise }\end{cases}
$$

By defining

$$
D(q \| p) \triangleq(1-q) \ln \frac{1-q}{1-p}+q \ln \frac{q}{p}
$$

and (3.29), we have

$$
\begin{align*}
r_{X \mid Y}(\delta(\lambda)) & =D\left(\frac{p^{-\lambda+1}}{p^{-\lambda+1}+(1-p)^{-\lambda+1}} \| p\right) \\
& =D\left(p+\frac{p(1-p)\left(p^{-\lambda}-(1-p)^{-\lambda}\right)}{p^{-\lambda+1}+(1-p)^{-\lambda+1}} \| p\right) \tag{3.32}
\end{align*}
$$

On the other hand, by substituting (3.30) and (3.31) into (3.7),

$$
\begin{equation*}
\delta(\lambda)=\frac{p(1-p)\left(p^{-\lambda}-(1-p)^{-\lambda}\right)}{p^{-\lambda+1}+(1-p)^{-\lambda+1}} \ln \frac{1-p}{p} \tag{3.33}
\end{equation*}
$$

and eventually, we have

$$
\begin{equation*}
r_{X \mid Y}(\delta)=D\left(p+\frac{\delta}{\ln \frac{1-p}{p}} \| p\right) \tag{3.34}
\end{equation*}
$$

and plugging (3.30) into (3.10) with $\lambda=0$ yields

$$
\begin{align*}
\sigma_{H}^{2}(X \mid Y) & =(1-p) \ln ^{2}(1-p)+p \ln ^{2} p-[-p \ln p-(1-p) \ln (1-p)]^{2} \\
& =p(1-p) \ln ^{2} \frac{1-p}{p} \tag{3.35}
\end{align*}
$$

Moreover, as $\mathcal{X}$ and $\mathcal{Y}$ are both finite alphabets, it is easy to show that $\lambda^{*}(X \mid Y)=\infty$, where $\lambda^{*}(X \mid Y)$ is defined in (3.1). Then

$$
\begin{equation*}
\Delta^{*}(X \mid Y)=\lim _{\lambda \uparrow+\infty} \delta(\lambda)=(1-p) \ln \frac{1-p}{p} \tag{3.36}
\end{equation*}
$$

and

$$
\begin{equation*}
r_{\max } \triangleq \lim _{\delta \uparrow \Delta^{*}(X \mid Y)} r_{X \mid Y}(\delta)=-\ln p \tag{3.37}
\end{equation*}
$$

Based on Theorem 4, $\Delta^{*}(X \mid Y)$ and $r_{\text {max }}$ can be interpreted in the following way. As

$$
\max _{x^{n}, y^{n}}-\frac{1}{n} \ln p\left(x^{n} \mid y^{n}\right)=-\ln p,
$$

then

$$
\begin{aligned}
\lim _{\delta \rightarrow \Delta^{*}(X \mid Y)} \operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right)>H(X \mid Y)+\delta\right\} & =\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right)=-\ln p\right\} \\
& =p^{n}=e^{n \ln p}=e^{-n r_{\max }}
\end{aligned}
$$

In addition, for $\delta \geq \Delta^{*}(X \mid Y)$,

$$
\operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right)>H(X \mid Y)+\delta\right\}=0
$$



Fig. 1. $r_{X \mid Y}(\delta)$ for BSC

By adopting the convention that $0 \ln 0=0$ and $e^{-\infty}=0$,

$$
r_{X \mid Y}(\delta)= \begin{cases}D\left(p+\frac{\delta}{\ln \frac{1-p}{p}} \| p\right) & \text { if } \delta \in\left[0, \Delta^{*}(X \mid Y)\right)  \tag{3.38}\\ +\infty & \text { if } \delta \geq \Delta^{*}(X \mid Y)\end{cases}
$$

A sample plot of $r_{X \mid Y}(\delta)$ is provided in Figure 1 when $p=0.10$.
Example 2 (Binary Input Gaussian Channel): Without loss of generality, we assume that the input of channel is modulated to $\{+1,-1\}$, and therefore

$$
\begin{equation*}
p(y \mid x)=\frac{1}{\sqrt{2 \pi} \sigma} e^{-\frac{|y-x|^{2}}{2 \sigma^{2}}} \tag{3.39}
\end{equation*}
$$

for $x=\{+1,-1\}$, where $\sigma^{2}$ is the variance of the noise. Calculation of $r_{X \mid Y}(\delta)$ and $\sigma_{H}^{2}(X \mid Y)$ is much more involved than that for BSC. Tedious evaluation is omitted here with results presented as follows. Let $U$ be a standard Gaussian random variable, i.e.

$$
p(u)=\frac{1}{\sqrt{2 \pi}} e^{-\frac{|u|^{2}}{2}}
$$

and define

$$
g(x) \stackrel{\Delta}{=} 1+e^{-2 x}
$$

Then

$$
\begin{align*}
\delta(\lambda) & =\frac{\mathbf{E}\left[g^{\lambda}\left(\frac{\sigma U+1}{\sigma^{2}}\right) \ln g\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}{\mathbf{E}\left[g^{\lambda}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}-\mathbf{E}\left[\ln g\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]  \tag{3.40}\\
r_{X \mid Y}(\delta(\lambda)) & =\lambda \frac{\mathbf{E}\left[g^{\lambda}\left(\frac{\sigma U+1}{\sigma^{2}}\right) \ln g\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}{\mathbf{E}\left[g^{\lambda}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}-\ln \left\{\mathbf{E}\left[g^{\lambda}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]\right\} \tag{3.41}
\end{align*}
$$

and

$$
\begin{equation*}
\sigma_{H}^{2}(X \mid Y)=\mathbf{E}\left[\ln ^{2} g\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]-\left\{\mathbf{E}\left[-\ln g\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]\right\}^{2} \tag{3.42}
\end{equation*}
$$

To get better understanding of those quantities, let us first determine $\lambda^{*}(X \mid Y)$ and $\Delta^{*}(X \mid Y)$. In fact, we can show that $\lambda^{*}(X \mid Y)=\infty$ by verifying that

$$
\int p(y)\left[\sum_{x \in \mathcal{X}} p^{-\lambda+1}(x \mid y)\right] d y<\infty
$$

for any finite $\lambda \geq 0$. Towards this, observe that

$$
\int p(y)\left[\sum_{x \in \mathcal{X}} p^{-\lambda+1}(x \mid y)\right] d y
$$

is an increasing function with respect to $\lambda$ since $p(x \mid y) \leq 1$ for any $x$ and $y$. Therefore,

$$
\begin{aligned}
\int p(y)\left[\sum_{x \in \mathcal{X}} p^{-\lambda+1}(x \mid y)\right] d y & =\mathbf{E}\left[g^{\lambda}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right] \\
& \leq \mathbf{E}\left[g^{[\lambda]}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]<\infty
\end{aligned}
$$

as

$$
\mathbf{E}\left[e^{s U}\right]=e^{\frac{s^{2}}{2}}<\infty
$$

for any finite $s$. Now let us show the claim $\Delta^{*}(X \mid Y)=\infty$. According to 3.40,

$$
\begin{aligned}
\delta(\lambda) & =\frac{\mathbf{E}\left[g^{\lambda}\left(\frac{\sigma U+1}{\sigma^{2}}\right) \ln g\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}{\mathbf{E}\left[g^{\lambda}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}-\mathbf{E}\left[\ln g\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right] \\
& =\frac{d}{d \lambda} \ln \mathbf{E}\left[g^{\lambda}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]-H(X \mid Y)
\end{aligned}
$$

As $H(X \mid Y)$ is a constant and always less than $\ln 2$, the claim $\Delta^{*}(X \mid Y)=\infty$ is equivalent to show

$$
\frac{d}{d \lambda} \ln \mathbf{E}\left[g^{\lambda}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]
$$

is unbounded when $\lambda \rightarrow \infty$. By the fact that $\delta(\lambda)$ is an increasing function of $\lambda$, which also implies that so is

$$
\frac{d}{d \lambda} \ln \mathbf{E}\left[g^{\lambda}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]
$$

we only have to verify that

$$
\frac{\ln \mathbf{E}\left[g^{k+1}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]-\ln \mathbf{E}\left[g^{k}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}{k+1-k}=\ln \frac{\mathbf{E}\left[g^{k+1}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}{\mathbf{E}\left[g^{k}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}
$$

or simply

$$
\frac{\mathbf{E}\left[g^{k+1}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}{\mathbf{E}\left[g^{k}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}
$$

is unbounded when $k \rightarrow \infty$, which is indeed the case as

$$
\begin{aligned}
\frac{\mathbf{E}\left[g^{k+1}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]}{\mathbf{E}\left[g^{k}\left(\frac{\sigma U+1}{\sigma^{2}}\right)\right]} & =\frac{\sum_{i=0}^{k+1}\binom{k+1}{i} e^{\frac{2 i^{2}-2 i}{\sigma^{2}}}}{\sum_{i=0}^{k}\binom{k}{i} e^{\frac{2 i^{2}-2 i}{\sigma^{2}}}} \\
& =\frac{\Theta\left(e^{\frac{2(k+1)^{2}-2(k+1)}{\sigma^{2}}}\right)}{\Theta\left(e^{\frac{2 k^{2}-2 k}{\sigma^{2}}}\right)} \\
& =\Theta\left(e^{\frac{4 k}{\sigma^{2}}}\right) \rightarrow \infty
\end{aligned}
$$

as $k \rightarrow \infty$. And consequently, it is not hard to see that

$$
r_{X \mid Y}(\delta) \rightarrow \infty
$$

as $\delta \rightarrow \infty$. The interpretation based on Theorem 4 is as follows:

$$
-\frac{1}{n} \ln p\left(x^{n} \mid y^{n}\right)-H(X \mid Y)
$$

can approach $\infty$ for proper choice of $x^{n}$ and $y^{n}$, but

$$
\lim _{\delta \rightarrow \infty} \operatorname{Pr}\left\{-\frac{1}{n} \ln p\left(X^{n} \mid Y^{n}\right)>H(X \mid Y)+\delta\right\}=e^{-\infty}=0
$$

Figure 2 shows a sample plot of $r_{X \mid Y}(\delta)$ for BIGC with $\sigma=1.0$.

## IV. NEP With Respect to Mutual Information and Relative Entropy

Consider now an IID source pair $(X, Y)=\left\{\left(X_{i}, Y_{i}\right)\right\}_{i=1}^{\infty}$ with finite mutual information $I(X ; Y)>0$. Let $p(y \mid x)$ be the conditional pmf or pdf (as the case may be) of $Y_{i}$ given $X_{i}$. In this section, we extend the NEP to $I(X ; Y)$ and relative entropy.


Fig. 2. $\quad r_{X \mid Y}(\delta)$ for BIGC

## A. NEP With Respect to $I(X ; Y)$

We begin with the left NEP with respect to $I(X ; Y)$. Define

$$
\begin{equation*}
\lambda_{-}^{*}(X ; Y) \triangleq \stackrel{\Delta}{=} \sup \left\{\lambda \geq 0: \iint p(x, y)\left[\frac{p(y \mid x)}{p(y)}\right]^{-\lambda} d x d y<\infty\right\} \tag{4.1}
\end{equation*}
$$

Suppose that

$$
\begin{equation*}
\lambda_{-}^{*}(X ; Y)>0 \tag{4.2}
\end{equation*}
$$

Let

$$
\begin{equation*}
\sigma_{I}^{2}(X ; Y) \triangleq \iint p(x, y)\left[\ln \frac{p(y \mid x)}{p(y)}\right]^{2} d x d y-I^{2}(X ; Y) \tag{4.3}
\end{equation*}
$$

which will be referred to as the mutual information variance of $X$ and $Y$. It is not hard to see that under the assumption (4.2),

$$
\begin{equation*}
\iint \frac{p(x, y)\left[\frac{p(y \mid x)}{p(y)}\right]^{-\lambda}}{\left[\iint p(u, v)\left[\frac{p(v \mid u)}{p(v)}\right]^{-\lambda} d u d v\right]}\left|-\ln \frac{p(y \mid x)}{p(y)}\right|^{k} d x d y<\infty \tag{4.4}
\end{equation*}
$$

and

$$
\iint p(x, y)\left[\frac{p(y \mid x)}{p(y)}\right]^{-\lambda} d x d y<\infty
$$

for any $\lambda \in\left(0, \lambda_{-}^{*}(X ; Y)\right)$ and any positive integer $k$. Further assume that

$$
\begin{equation*}
\sigma_{I}^{2}(X ; Y)>0 \text { and } \iint p(x, y)\left|\ln \frac{p(y \mid x)}{p(y)}\right|^{3} d x d x<\infty \tag{4.5}
\end{equation*}
$$

Define for any $\delta \geq 0$

$$
\begin{equation*}
r_{X ; Y,-}(\delta) \triangleq \sup _{\lambda \geq 0}\left[\lambda(\delta-I(X ; Y))-\ln \iint p(x, y)\left[\frac{p(y \mid x)}{p(y)}\right]^{-\lambda} d x d y\right] \tag{4.6}
\end{equation*}
$$

and for any $\lambda \in\left[0, \lambda_{-}^{*}(X ; Y)\right)$

$$
\begin{gather*}
f_{-\lambda}(x, y) \triangleq \frac{\left[\frac{p(y \mid x)}{p(y)}\right]^{-\lambda}}{\iint p(u, v)\left[\frac{p(v \mid u)}{p(v)}\right]^{-\lambda} d u d v}  \tag{4.7}\\
\delta_{-}(\lambda) \triangleq \iint p(x, y) f_{-\lambda}(x, y)\left[-\ln \frac{p(y \mid x)}{p(y)}\right] d x d y+I(X ; Y) . \tag{4.8}
\end{gather*}
$$

(Throughout this section, $\delta_{-}(\lambda)$ should be understood with its above definition.) Then under the assumptions (4.2) and (4.5), $\delta_{-}(\lambda)$ is strictly increasing over $\lambda \in\left[0, \lambda_{-}^{*}(X ; Y)\right)$ with $\delta_{-}(0)=0$. Let

$$
\Delta_{-}^{*}(X ; Y) \triangleq \lim _{\lambda \uparrow \lambda_{-}^{*}(X ; Y)} \delta_{-}(\lambda) .
$$

By an argument similar to that in the proof of Theorem 1, it can be shown that $r_{X ; Y,-}(\delta)$ is strictly increasing, convex and continuously differentiable up to at least the third order inclusive over $\delta \in\left[0, \Delta_{-}^{*}(X ; Y)\right)$, and furthermore $r_{X ; Y,-}(\delta)$ has the following parametric expression

$$
\begin{equation*}
r_{X ; Y,-}\left(\delta_{-}(\lambda)\right)=\lambda\left(\delta_{-}(\lambda)-I(X ; Y)\right)-\ln \iint p(x, y)\left[\frac{p(y \mid x)}{p(y)}\right]^{-\lambda} d x d y \tag{4.9}
\end{equation*}
$$

with $\lambda=r_{X ; Y,-}^{\prime}(\delta)$ satisfying

$$
\delta_{-}(\lambda)=\delta
$$

Further define for any $\lambda \in\left[0, \lambda_{-}^{*}(X ; Y)\right)$

$$
\begin{align*}
& \sigma_{I,-}^{2}(X ; Y, \lambda) \triangleq \iint f_{-\lambda}(x, y) p(x, y)\left|\ln \frac{p(y \mid x)}{p(y)}-\left(I(X ; Y)-\delta_{-}(\lambda)\right)\right|^{2} d x d y  \tag{4.10}\\
& M_{I,-}(X ; Y, \lambda) \triangleq \iint f_{-\lambda}(x, y) p(x, y)\left|\ln \frac{p(y \mid x)}{p(y)}-\left(I(X ; Y)-\delta_{-}(\lambda)\right)\right|^{3} d x d y \tag{4.11}
\end{align*}
$$

Write $M_{I,-}(X ; Y, 0)$ simply as $M_{I}(X ; Y)$. It is easy to see that $\sigma_{I,-}^{2}(X ; Y, 0)=\sigma_{I}^{2}(X ; Y)$, $\sigma_{I,-}^{2}(X ; Y, \lambda)=\delta_{-}^{\prime}(\lambda)$, and

$$
\begin{equation*}
\left.M_{I}(X ; Y)=\iint p(x, y) \left\lvert\, \ln \frac{p(y \mid x)}{p(y)}-I(X ; Y)\right.\right)\left.\right|^{3} d x d y \tag{4.12}
\end{equation*}
$$

In parallel with Theorems 3 and 5, we have the following result, which is referred to as the left NEP with respect to $I(X ; Y)$ and can be proved similarly.

Theorem 6 (Left NEP With Respect to $I(X ; Y)$ ). For any positive integer n,

$$
\begin{equation*}
\operatorname{Pr}\left\{\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{p\left(Y^{n}\right)} \leq I(X ; Y)-\delta\right\} \leq e^{-n r_{X ; Y,-}(\delta)} \tag{4.13}
\end{equation*}
$$

Furthermore, under the assumptions (4.2) and (4.5), the following also hold:
(a) There exists a $\delta^{*}>0$ such that for any $\delta \in\left(0, \delta^{*}\right]$ and any positive integer $n$,

$$
\begin{equation*}
r_{X ; Y,-}(\delta)=\frac{1}{2 \sigma_{I}^{2}(X ; Y)} \delta^{2}+O\left(\delta^{3}\right) \tag{4.14}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\operatorname{Pr}\left\{\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{p\left(Y^{n}\right)} \leq I(X ; Y)-\delta\right\} \leq e^{-n\left(\frac{\delta^{2}}{2 \sigma_{I}^{2}(X ; Y)}+O\left(\delta^{3}\right)\right)} \tag{4.15}
\end{equation*}
$$

(b) For any $\delta \in\left(0, \Delta_{-}^{*}(X ; Y)\right)$ and any positive integer $n$

$$
\begin{align*}
\underline{\xi}_{I,-}(X ; Y, \lambda, n) e^{-n r_{X ; Y,-}(\delta)} & \leq \operatorname{Pr}\left\{\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{p\left(Y^{n}\right)} \leq I(X ; Y)-\delta\right\} \\
& \leq \bar{\xi}_{I,-}(X ; Y, \lambda, n) e^{-n r_{X ; Y,-}(\delta)} \tag{4.16}
\end{align*}
$$

where $\lambda=r_{X ; Y,-}^{\prime}(\delta)>0$, and

$$
\begin{align*}
& \bar{\xi}_{I,-}(X ; Y, \lambda, n)=\frac{2 C M_{I,-}(X ; Y, \lambda)}{\sqrt{n} \sigma_{I,-}^{3}(X ; Y, \lambda)} \\
& \quad+e^{\frac{n \lambda^{2} \sigma_{I,-}^{2}(X ; Y, \lambda)}{2}}\left[Q\left(\sqrt{n} \lambda \sigma_{I,-}(X ; Y, \lambda)\right)-Q\left(\rho^{*}+\sqrt{n} \lambda \sigma_{I,-}(X ; Y, \lambda)\right)\right]  \tag{4.17}\\
& \underline{\xi}_{I,-}(X ; Y, \lambda, n)=e^{\frac{n \lambda^{2} \sigma_{I,-}^{2}(X ; Y, \lambda)}{2}} Q\left(\rho_{*}+\sqrt{n} \lambda \sigma_{I,-}(X ; Y, \lambda)\right) \tag{4.18}
\end{align*}
$$

with $Q\left(\rho^{*}\right)=\frac{C M_{I,-}(X ; Y, \lambda)}{\sqrt{n} \sigma_{I,-}^{3}(X ; Y, \lambda)}$ and $Q\left(\rho_{*}\right)=\frac{1}{2}-\frac{2 C M_{I,-}(X ; Y, \lambda)}{\sqrt{n} \sigma_{I,-}^{3}(X ; Y, \lambda)}$.
(c) For any $\delta \leq c \sqrt{\frac{\ln n}{n}}$, where $c<\sigma_{I}(X ; Y)$ is a constant,

$$
\begin{align*}
Q\left(\frac{\delta \sqrt{n}}{\sigma_{I}(X ; Y)}\right)-\frac{C M_{I}(X ; Y)}{\sqrt{n} \sigma_{I}^{3}(X ; Y)} & \leq \operatorname{Pr}\left\{\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{p\left(Y^{n}\right)} \leq I(X ; Y)-\delta\right\} \\
& \leq Q\left(\frac{\delta \sqrt{n}}{\sigma_{I}(X ; Y)}\right)+\frac{C M_{I}(X ; Y)}{\sqrt{n} \sigma_{I}^{3}(X ; Y)} \tag{4.19}
\end{align*}
$$

The probability that $\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{p\left(Y^{n}\right)}$ is away from $I(X ; Y)$ to the right can be bounded in a similar manner. For completeness, we state these bounds again without proof. Define

$$
\begin{equation*}
\lambda^{*}(X ; Y) \stackrel{\Delta}{=} \sup \left\{\lambda \geq 0: \iint p(x, y)\left[\frac{p(y \mid x)}{p(y)}\right]^{\lambda} d x d y<\infty\right\} \tag{4.20}
\end{equation*}
$$

Suppose that

$$
\begin{equation*}
\lambda^{*}(X ; Y)>0 . \tag{4.21}
\end{equation*}
$$

Define for any $\delta \geq 0$

$$
\begin{equation*}
r_{X ; Y}(\delta) \triangleq \sup _{\lambda \geq 0}\left[\lambda(I(X ; Y)+\delta)-\ln \iint p(x, y)\left[\frac{p(y \mid x)}{p(y)}\right]^{\lambda} d x d y\right] \tag{4.22}
\end{equation*}
$$

and for any $\lambda \in\left[0, \lambda^{*}(X ; Y)\right)$

$$
\begin{gather*}
f_{\lambda}(x, y) \triangleq \frac{\left[\frac{p(y \mid x)}{p(y)}\right]^{\lambda}}{\iint p(u, v)\left[\frac{p(v \mid u)}{p(v)}\right]^{\lambda} d u d v}  \tag{4.23}\\
\delta(\lambda) \triangleq \iint p(x, y) f_{\lambda}(x, y)\left[\ln \frac{p(y \mid x)}{p(y)}\right] d x d y-I(X ; Y) . \tag{4.24}
\end{gather*}
$$

(Throughout this section, $\delta(\lambda)$ should be understood with its above definition.) Then under the assumptions (4.21) and (4.5), $\delta(\lambda)$ is strictly increasing over $\lambda \in\left[0, \lambda^{*}(X ; Y)\right)$ with $\delta(0)=0$. Let

$$
\Delta^{*}(X ; Y) \triangleq \lim _{\lambda \uparrow \lambda^{*}(X ; Y)} \delta(\lambda)
$$

By an argument similar to that in the proof of Theorem 1, it can be shown that $r_{X ; Y}(\delta)$ is strictly increasing, convex and continuously differentiable up to at least the third order over $\delta \in\left[0, \Delta^{*}(X ; Y)\right)$, and furthermore $r_{X ; Y}(\delta)$ has the following parametric expression

$$
\begin{equation*}
r_{X ; Y}(\delta(\lambda))=\lambda(I(X ; Y)+\delta(\lambda))-\ln \iint p(x, y)\left[\frac{p(y \mid x)}{p(y)}\right]^{\lambda} d x d y \tag{4.25}
\end{equation*}
$$

with $\lambda=r_{X ; Y}^{\prime}(\delta)$ satisfying

$$
\delta(\lambda)=\delta
$$

Further define for any $\lambda \in\left[0, \lambda^{*}(X ; Y)\right)$

$$
\begin{align*}
\sigma_{I}^{2}(X ; Y, \lambda) & \triangleq \iint f_{\lambda}(x, y) p(x, y)\left|\ln \frac{p(y \mid x)}{p(y)}-(I(X ; Y)+\delta(\lambda))\right|^{2} d x d y  \tag{4.26}\\
M_{I}(X ; Y, \lambda) & \triangleq \iint f_{\lambda}(x, y) p(x, y)\left|\ln \frac{p(y \mid x)}{p(y)}-(I(X ; Y)+\delta(\lambda))\right|^{3} d x d y \tag{4.27}
\end{align*}
$$

It is easy to see that $\sigma_{I}^{2}(X ; Y, 0)=\sigma_{I}^{2}(X ; Y)$ and $\sigma_{I}^{2}(X ; Y, \lambda)=\delta^{\prime}(\lambda)$.
In parallel with Theorems 1, 2, and 4, we have the following result, which is referred to as the right NEP with respect to $I(X ; Y)$ and can be proved similarly.

Theorem 7 (Right NEP With Respect to $I(X ; Y)$ ). For any positive integer $n$,

$$
\begin{equation*}
\operatorname{Pr}\left\{\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{p\left(Y^{n}\right)}>I(X ; Y)+\delta\right\} \leq e^{-n r_{X ; Y}(\delta)} . \tag{4.28}
\end{equation*}
$$

Furthermore, under the assumptions (4.21) and (4.5), the following also hold:
(a) There exists a $\delta^{*}>0$ such that for any $\delta \in\left(0, \delta^{*}\right]$ and any positive integer $n$,

$$
\begin{equation*}
r_{X ; Y}(\delta)=\frac{1}{2 \sigma_{I}^{2}(X ; Y)} \delta^{2}+O\left(\delta^{3}\right) \tag{4.29}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\operatorname{Pr}\left\{\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{p\left(Y^{n}\right)}>I(X ; Y)+\delta\right\} \leq e^{-n\left(\frac{\delta^{2}}{2 \sigma_{I}^{2}(X ; Y)}+O\left(\delta^{3}\right)\right)} . \tag{4.30}
\end{equation*}
$$

(b) For any $\delta \in\left(0, \Delta^{*}(X ; Y)\right)$ and any positive integer $n$

$$
\begin{align*}
\underline{\xi}_{I}(X ; Y, \lambda, n) e^{-n r_{X ; Y}(\delta)} & \leq \operatorname{Pr}\left\{\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{p\left(Y^{n}\right)}>I(X ; Y)+\delta\right\} \\
& \leq \bar{\xi}_{I}(X ; Y, \lambda, n) e^{-n r_{X ; Y}(\delta)} \tag{4.31}
\end{align*}
$$

where $\lambda=r_{X ; Y}^{\prime}(\delta)>0$, and

$$
\begin{align*}
& \bar{\xi}_{I}(X ; Y, \lambda, n)=\frac{2 C M_{I}(X ; Y, \lambda)}{\sqrt{n} \sigma_{I}^{3}(X ; Y, \lambda)} \\
& \quad+e^{\frac{n \lambda^{2} \sigma_{I}^{2}(X ; Y, \lambda)}{2}}\left[Q\left(\sqrt{n} \lambda \sigma_{I}(X ; Y, \lambda)\right)-Q\left(\rho^{*}+\sqrt{n} \lambda \sigma_{I}(X ; Y, \lambda)\right)\right]  \tag{4.32}\\
& \underline{\xi}_{I}(X ; Y, \lambda, n)=e^{\frac{n \lambda^{2} \sigma_{I}^{2}(X ; Y, \lambda)}{2}} Q\left(\rho_{*}+\sqrt{n} \lambda \sigma_{I}(X ; Y, \lambda)\right) \tag{4.33}
\end{align*}
$$

with $Q\left(\rho^{*}\right)=\frac{C M_{I}(X ; Y, \lambda)}{\sqrt{n} \sigma_{I}^{3}(X ; Y, \lambda)}$ and $Q\left(\rho_{*}\right)=\frac{1}{2}-\frac{2 C M_{I}(X ; Y, \lambda)}{\sqrt{n} \sigma_{I}^{3}(X ; Y, \lambda)}$.
(c) For any $\delta \leq c \sqrt{\frac{\ln n}{n}}$, where $c<\sigma_{I}(X ; Y)$ is a constant,

$$
\begin{align*}
Q\left(\frac{\delta \sqrt{n}}{\sigma_{I}(X ; Y)}\right)-\frac{C M_{I}(X ; Y)}{\sqrt{n} \sigma_{I}^{3}(X ; Y)} & \leq \operatorname{Pr}\left\{\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{p\left(Y^{n}\right)}>I(X ; Y)+\delta\right\} \\
& \leq Q\left(\frac{\delta \sqrt{n}}{\sigma_{I}(X ; Y)}\right)+\frac{C M_{I}(X ; Y)}{\sqrt{n} \sigma_{I}^{3}(X ; Y)} \tag{4.34}
\end{align*}
$$

Remarks similar to those (Remark 1 and 2) following Theorem 2 can be drawn here concerning Theorems 6 and 7 .

## B. NEP With Respect to Relative Entropy

The IID source pair $(X, Y)=\left\{\left(X_{i}, Y_{i}\right)\right\}_{i=1}^{\infty}$ considered so far is arbitrary. Let us now focus on the case in which the source $X$ is discrete, but $Y$ could be either discrete or continuous. Let $\mathcal{P}$ denote the set of all probability distributions over the source alphabet $\mathcal{X}$. For any $t \in \mathcal{P}$, let

$$
\begin{gather*}
q_{t}(y) \triangleq \sum_{x \in \mathcal{X}} t(x) p(y \mid x)  \tag{4.35}\\
q_{t}\left(y^{n}\right) \triangleq \prod_{i=1}^{n} q_{t}\left(y_{i}\right)  \tag{4.36}\\
D(t, x) \triangleq \int p(y \mid x) \ln \frac{p(y \mid x)}{q_{t}(y)} d y \tag{4.37}
\end{gather*}
$$

and

$$
\begin{equation*}
I(t ; P) \triangleq \sum_{x \in \mathcal{X}} t(x) \int p(y \mid x) \ln \frac{p(y \mid x)}{q_{t}(y)} d y \tag{4.38}
\end{equation*}
$$

where $y^{n}=y_{1} y_{2} \cdots y_{n}$, and $P=\{p(y \mid x)\}$ represents a channel with $p(y \mid x)$ as its transitional pmf or pdf (as the case may be). Clearly, $D(t, x)$ is the divergence or relative entropy between $p(y \mid x)$ and $q_{t}(y)$; and $I(t ; P)$ is the mutual information between the input and output of the channel $P$ when the input is distributed according to $t$. To be specific, we denote the pmf of each $X_{i}$ by $p_{X}$. Without loss of generality, we assume that $p_{X}(x)>0$ for any $x \in \mathcal{X}$. Since

$$
\iint p(x, y)\left[\frac{p(y \mid x)}{p(y)}\right]^{-\lambda} d x d y=\sum_{a \in \mathcal{X}} p_{X}(a) \int p(y \mid a)\left[\frac{\sum_{b \in \mathcal{X}} p_{X}(b) p(y \mid b)}{p(y \mid a)}\right]^{\lambda} d y
$$

it is not hard to see that for any $\lambda>0$,

$$
\iint p(x, y)\left[\frac{p(y \mid x)}{p(y)}\right]^{-\lambda} d x d y<\infty
$$

if and only if

$$
\int p(y \mid a)\left[\frac{\sum_{b \in \mathcal{X}} p(y \mid b)}{p(y \mid a)}\right]^{\lambda} d y<\infty
$$

for any $a \in \mathcal{X}$. Therefore, $\lambda_{-}^{*}(X ; Y)$ defined in (4.1) is also equal to

$$
\sup \left\{\lambda \geq 0: \int p(y \mid a)\left[\frac{p(y \mid a)}{q_{t}(y)}\right]^{-\lambda} d y<\infty, a \in \mathcal{X}\right\}
$$

for any $t \in \mathcal{P}$ with $t(a)>0$ for any $a \in \mathcal{X}$ (such $t \in \mathcal{P}$ will be said to have full support).

Define for any $t \in \mathcal{P}$ with full support and any $\delta \geq 0$

$$
\begin{equation*}
r_{-}(t, \delta) \triangleq \sup _{\lambda \geq 0}\left[\lambda(\delta-I(t ; P))-\sum_{x \in \mathcal{X}} t(x) \ln \int p(y \mid x)\left[\frac{p(y \mid x)}{q_{t}(y)}\right]^{-\lambda} d y\right] \tag{4.39}
\end{equation*}
$$

and for any $\lambda \in\left[0, \lambda_{-}^{*}(X ; Y)\right)$ and any $t \in \mathcal{P}$ with full support

$$
\begin{gather*}
f_{-\lambda}(y \mid x) \triangleq \frac{\left[\frac{p(y \mid x)}{q_{t}(y)}\right]^{-\lambda}}{\int p(v \mid x)\left[\frac{p(v \mid x)}{q_{t}(v)}\right]^{-\lambda} d v}  \tag{4.40}\\
D(t, x, \lambda) \triangleq \int p(y \mid x) f_{-\lambda}(y \mid x)\left[\ln \frac{p(y \mid x)}{q_{t}(y)}\right] d y  \tag{4.41}\\
\delta_{-}(t, \lambda) \triangleq \sum_{x \in \mathcal{X}} t(x) \int p(y \mid x) f_{-\lambda}(y \mid x)\left[-\ln \frac{p(y \mid x)}{q_{t}(y)}\right] d y+I(t ; P) . \tag{4.42}
\end{gather*}
$$

It is not hard to verify that

$$
\delta_{-}(t, 0)=0
$$

and

$$
\begin{aligned}
\frac{\partial \delta_{-}(t, \lambda)}{\partial \lambda}= & \sum_{x \in \mathcal{X}} t(x)\left[\int p(y \mid x) f_{-\lambda}(y \mid x)\left[-\ln \frac{p(y \mid x)}{q_{t}(y)}\right]^{2} d y\right. \\
& \left.-\left(\int p(y \mid x) f_{-\lambda}(y \mid x)\left[-\ln \frac{p(y \mid x)}{q_{t}(y)}\right] d y\right)^{2}\right] \\
= & \sum_{x \in \mathcal{X}} t(x)\left[\int p(y \mid x) f_{-\lambda}(y \mid x)\left[\ln \frac{p(y \mid x)}{q_{t}(y)}\right]^{2} d y-D^{2}(t, x, \lambda)\right] \\
> & 0
\end{aligned}
$$

where the last inequality is due to (4.5). Therefore, $\delta_{-}(t, \lambda)$ as a function of $\lambda$ is strictly increasing over $\lambda \in\left[0, \lambda_{-}^{*}(X ; Y)\right)$. Let

$$
\Delta_{-}^{*}(t) \triangleq \lim _{\lambda \uparrow \lambda_{-}^{*}(X ; Y)} \delta_{-}(t, \lambda)
$$

By an argument similar to that in the proof of Theorem 1, it can be shown that $r_{-}(t, \delta)$ is strictly increasing, convex and continuously differentiable up to at least the third order inclusive over $\delta \in\left[0, \Delta_{-}^{*}(t)\right)$, and furthermore $r_{-}(t, \delta)$ has the following parametric expression

$$
\begin{equation*}
r_{-}\left(t, \delta_{-}(t, \lambda)\right)=\lambda\left(\delta_{-}(t, \lambda)-I(t ; P)\right)-\sum_{x \in \mathcal{X}} t(x) \ln \int p(y \mid x)\left[\frac{p(y \mid x)}{q_{t}(y)}\right]^{-\lambda} d y \tag{4.43}
\end{equation*}
$$

with

$$
\lambda=\frac{\partial r_{-}(t, \delta)}{\partial \delta}
$$

satisfying

$$
\delta_{-}(t, \lambda)=\delta
$$

Further define for any $\lambda \in\left[0, \lambda_{-}^{*}(X ; Y)\right)$

$$
\begin{equation*}
\sigma_{D,-}^{2}(t ; P, \lambda) \triangleq \sum_{x \in \mathcal{X}} t(x)\left[\int p(y \mid x) f_{-\lambda}(y \mid x)\left|\ln \frac{p(y \mid x)}{q_{t}(y)}-D(t, x, \lambda)\right|^{2} d y\right] \tag{4.44}
\end{equation*}
$$

and

$$
\begin{equation*}
M_{D,-}(t ; P, \lambda) \triangleq \sum_{x \in \mathcal{X}} t(x)\left[\int p(y \mid x) f_{-\lambda}(y \mid x)\left|\ln \frac{p(y \mid x)}{q_{t}(y)}-D(t, x, \lambda)\right|^{3} d y\right] \tag{4.45}
\end{equation*}
$$

Write $\sigma_{D,-}^{2}(t ; P, 0)$ simply as $\sigma_{D}^{2}(t ; P), M_{D,-}(t ; P, 0)$ as $M_{D}(t ; P), \sigma_{D}^{2}\left(p_{X} ; P\right)$ as $\sigma_{D}^{2}(X ; Y)$, and $M_{D}\left(p_{X} ; P\right)$ as $M_{D}(X ; Y)$. It is not hard to see that

$$
\begin{aligned}
& \sigma_{D}^{2}(t ; P)=\sum_{x \in \mathcal{X}} t(x)\left[\int p(y \mid x)\left|\ln \frac{p(y \mid x)}{q_{t}(y)}\right|^{2} d y-\left(\int p(y \mid x) \ln \frac{p(y \mid x)}{q_{t}(y)} d y\right)^{2}\right] \\
& \sigma_{D}^{2}(X ; Y)=\sum_{x \in \mathcal{X}} p(x)\left[\int p(y \mid x)\left|\ln \frac{p(y \mid x)}{p(y)}\right|^{2} d y-\left(\int p(y \mid x) \ln \frac{p(y \mid x)}{p(y)} d y\right)^{2}\right] \\
& M_{D}(t ; P) \triangleq \sum_{x \in \mathcal{X}} t(x)\left[\int p(y \mid x)\left|\ln \frac{p(y \mid x)}{q_{t}(y)}-\left(\int p(v \mid x) \ln \frac{p(v \mid x)}{q_{t}(v)} d v\right)\right|^{3} d y\right] \\
& M_{D}(X ; Y) \triangleq \sum_{x \in \mathcal{X}} p(x)\left[\int p(y \mid x)\left|\ln \frac{p(y \mid x)}{p(y)}-\left(\int p(v \mid x) \ln \frac{p(v \mid x)}{p(v)} d v\right)\right|^{3} d y\right]
\end{aligned}
$$

and

$$
\sigma_{D,-}^{2}(t ; P, \lambda)=\frac{\partial \delta_{-}(t, \lambda)}{\partial \lambda}
$$

For obvious reasons, we will refer to $\sigma_{D}^{2}(t ; P)\left(\sigma_{D}^{2}(X ; Y)\right.$, respectively) as the conditional divergence (or relative entropy) variance of $P$ given $t$ ( $Y$ given $X$, respectively).

In parallel with Theorems 3, 5, and 6, we have the following result, which is referred to as the left NEP with respect to relative entropy.

Theorem 8 (Left NEP With Respect to Relative Entropy). For any sequence $x^{n}=x_{1} \cdots x_{n}$ from $\mathcal{X}$, let $t \in \mathcal{P}$ be the type of $x^{n}$, i.e., nt $(a), a \in \mathcal{X}$, is the number of times the symbol $a$ appears in $x^{n}$. Assume that thas full support. Then

$$
\begin{equation*}
\operatorname{Pr}\left\{\left.\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)} \leq I(t ; P)-\delta \right\rvert\, X^{n}=x^{n}\right\} \leq e^{-n r_{-}(t, \delta)} . \tag{4.46}
\end{equation*}
$$

Furthermore, under the assumptions (4.2) and (4.5), the following also hold:
(a) There exists a $\delta^{*}>0$ such that for any $\delta \in\left(0, \delta^{*}\right]$

$$
\begin{equation*}
r_{-}(t, \delta)=\frac{1}{2 \sigma_{D}^{2}(t ; P)} \delta^{2}+O\left(\delta^{3}\right) \tag{4.47}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\operatorname{Pr}\left\{\left.\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)} \leq I(t ; P)-\delta \right\rvert\, X^{n}=x^{n}\right\} \leq e^{-n\left(\frac{\delta^{2}}{2 \sigma_{D}^{2}(t ; P)}+O\left(\delta^{3}\right)\right)} \tag{4.48}
\end{equation*}
$$

(b) For any $\delta \in\left(0, \Delta_{-}^{*}(X ; Y)\right)$

$$
\begin{align*}
\underline{\xi}_{D,-}(t ; P, \lambda, n) e^{-n r_{-}(t, \delta)} & \leq \operatorname{Pr}\left\{\left.\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)} \leq I(t ; P)-\delta \right\rvert\, X^{n}=x^{n}\right\} \\
& \leq \bar{\xi}_{D,-}(t ; P, \lambda, n) e^{-n r_{-}(t, \delta)} \tag{4.49}
\end{align*}
$$

where $\lambda=\frac{\partial r_{-}(t, \delta)}{\partial \delta}>0$, and

$$
\begin{align*}
& \bar{\xi}_{D,-}(t ; P, \lambda, n)=\frac{2 C M_{D,-}(t ; P, \lambda)}{\sqrt{n} \sigma_{D,-}^{3}(t ; P, \lambda)} \\
& \quad+e^{\frac{n \lambda^{2} \sigma_{D,-}^{2}(t ; P, \lambda)}{2}}\left[Q\left(\sqrt{n} \lambda \sigma_{D,-}(t ; P, \lambda)\right)-Q\left(\rho^{*}+\sqrt{n} \lambda \sigma_{D,-}(t ; P, \lambda)\right)\right]  \tag{4.50}\\
& \underline{\xi}_{D,-}(t ; P, \lambda, n)=e^{\frac{n \lambda^{2} \sigma_{D,-}^{2}(t ; P, \lambda)}{2}} Q\left(\rho_{*}+\sqrt{n} \lambda \sigma_{D,-}(t ; P, \lambda)\right) \tag{4.51}
\end{align*}
$$

with $Q\left(\rho^{*}\right)=\frac{C M_{D,-}(t ; P, \lambda)}{\sqrt{n} \sigma_{D,-}^{3}(t ; P, \lambda)}$ and $Q\left(\rho_{*}\right)=\frac{1}{2}-\frac{2 C M_{D,-}(t ; P, \lambda)}{\sqrt{n} \sigma_{D,-}^{3}(t ; P, \lambda)}$.
(c) For any $\delta \leq c \sqrt{\frac{\ln n}{n}}$, where $c<\sigma_{D}(t ; P)$ is a constant,

$$
\begin{align*}
Q\left(\frac{\delta \sqrt{n}}{\sigma_{D}(t ; P)}\right)-\frac{C M_{D}(t ; P)}{\sqrt{n} \sigma_{D}^{3}(t ; P)} & \leq \operatorname{Pr}\left\{\left.\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)} \leq I(t ; P)-\delta \right\rvert\, X^{n}=x^{n}\right\} \\
& \leq Q\left(\frac{\delta \sqrt{n}}{\sigma_{D}(t ; P)}\right)+\frac{C M_{D}(t ; P)}{\sqrt{n} \sigma_{D}^{3}(t ; P)} \tag{4.52}
\end{align*}
$$

Proof of Theorem 8. The inequality (4.46) comes from the Chernoff bound. To see this is indeed the case, note that

$$
\begin{align*}
\operatorname{Pr} & \left\{\left.\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)} \leq I(t ; P)-\delta \right\rvert\, X^{n}=x^{n}\right\} \\
& \leq \inf _{\lambda \geq 0} \frac{\mathbf{E}\left[\left.\left(\frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)}\right)^{-\lambda} \right\rvert\, X^{n}=x^{n}\right]}{e^{n \lambda(\delta-I(t ; P))}} \\
& =\inf _{\lambda \geq 0} \frac{\prod_{a \in \mathcal{X}}\left[\int p(y \mid a)\left(\frac{p(y \mid a)}{q_{t}(y)}\right)^{-\lambda} d y\right]^{n t(a)}}{e^{n \lambda(\delta-I(t ; P))}} \\
& =\inf _{\lambda \geq 0} \exp \left\{-n\left[\lambda(\delta-I(t ; P))-\sum_{a \in \mathcal{X}} t(a) \ln \int p(y \mid a)\left(\frac{p(y \mid a)}{q_{t}(y)}\right)^{-\lambda} d y\right]\right\} \\
& =e^{-n r_{-}(t, \delta)} \tag{4.53}
\end{align*}
$$

which completes the proof of (4.46).
The equation 4.47) follows from the Taylor expansion of $r_{-}(t, \delta)$ at $\delta=0$ and the fact that

$$
\frac{\partial^{2} r_{-}(t, \delta)}{\partial \delta^{2}}=\frac{1}{\sigma_{D}^{2}(t ; P)}
$$

What remains is to prove (4.49) and (4.52). To this end, let

$$
f_{-\lambda}\left(y^{n} \mid x^{n}\right)=\prod_{i=1}^{n} f_{-\lambda}\left(y_{i} \mid x_{i}\right)
$$

With $\lambda=\frac{\partial r_{-}(t, \delta)}{\partial \delta}$, it follows from (4.43) that

$$
r_{-}(t, \delta)=\lambda(\delta-I(t ; P))-\sum_{x \in \mathcal{X}} t(x) \ln \int p(y \mid x)\left[\frac{p(y \mid x)}{q_{t}(y)}\right]^{-\lambda} d y
$$

Then we have

$$
\begin{aligned}
\operatorname{Pr} & \left\{\left.\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)} \leq I(t ; P)-\delta \right\rvert\, X^{n}=x^{n}\right\} \\
& =\int_{\frac{1}{n} \ln \frac{p\left(y^{n} \mid x^{n}\right)}{q_{t}\left(y^{n}\right)} \leq I(t ; P)-\delta} p\left(y^{n} \mid x^{n}\right) d y^{n} \\
& =\int_{\frac{1}{n} \ln \frac{p\left(y^{n} \mid x^{n}\right)}{q_{t}\left(y^{n}\right)} \leq I(t ; P)-\delta} f_{-\lambda}^{-1}\left(y^{n} \mid x^{n}\right) f_{-\lambda}\left(y^{n} \mid x^{n}\right) p\left(y^{n} \mid x^{n}\right) d y^{n} \\
& =\int_{\frac{1}{n} \ln \frac{p\left(y^{n} \mid x^{n}\right)}{q_{t}\left(y^{n}\right)} \leq I(t ; P)-\delta} e^{\lambda \ln \frac{p\left(y^{n} \mid x^{n}\right)}{q_{t}\left(y^{n}\right)}+n \sum_{a \in \mathcal{X}} t(a) \ln \int p(v \mid a)\left(\frac{p(v \mid a)}{q_{t}(v)}\right)^{-\lambda} d v} f_{-\lambda}\left(y^{n} \mid x^{n}\right) p\left(y^{n} \mid x^{n}\right) d y^{n}
\end{aligned}
$$

$$
\begin{align*}
& =\int_{\frac{1}{n} \ln \frac{p\left(y^{n} \mid x^{n}\right)}{q_{t}\left(y^{n}\right)} \leq I(t ; P)-\delta} e^{\lambda \ln \frac{p\left(y^{n} \mid x^{n}\right)}{q_{t}\left(y^{n}\right)}+n \lambda(\delta-I(t ; P))-n r_{-}(t, \delta)} f_{-\lambda}\left(y^{n} \mid x^{n}\right) p\left(y^{n} \mid x^{n}\right) d y^{n} \\
& =e^{-n r_{-}(t, \delta)} \int_{\ln \frac{p\left(y^{n} \mid x^{n}\right)}{q_{t}\left(y^{n}\right)}-n(I(t ; P)-\delta) \leq 0} e^{\lambda\left[\ln \frac{p\left(y^{n} \mid x^{n}\right)}{q_{t}\left(y^{n}\right)}-n(I(t ; P)-\delta)\right]} f_{-\lambda}\left(y^{n} \mid x^{n}\right) p\left(y^{n} \mid x^{n}\right) d y^{n} \\
& =e^{-n r_{-}(t, \delta)} \int_{-0}^{\frac{l^{n} \frac{p\left(y^{n} \mid x^{n}\right)}{\left.q_{t} y^{n}\right)}-n(I(t ; P)-\delta)}{\sqrt{n} \sigma_{D,-}(t ; P, \lambda)}}=\rho \\
& =e^{-n r_{-}(t, \delta)} \int_{-\infty}^{0} e^{\lambda \sqrt{n} \sigma_{D,-}(t ; P, \lambda) \rho} f_{-\lambda}\left(y^{n} \mid x^{n}\right) p\left(y^{n} \mid x^{n}\right) d y^{n} \\
& =e^{-n r_{-}(t, \delta)}\left[F_{x^{n}(0)-(t ; P, \lambda) \rho} d F_{x^{n}}(\rho)\right.  \tag{4.54}\\
& \left.F_{-\infty}^{0} \lambda \sqrt{n} \sigma_{D,-}(t ; P, \lambda) e^{\lambda \sqrt{n} \sigma_{D,-}(t ; P, \lambda) \rho} F_{x^{n}}(\rho) d \rho\right] .
\end{align*}
$$

where

$$
F_{x^{n}}(\rho)=\operatorname{Pr}\left\{\frac{\ln \frac{p\left(Z^{n}| |^{n}\right)}{q_{t}\left(Z^{n}\right)}-n(I(t ; P)-\delta)}{\sqrt{n} \sigma_{D,-}(t ; P, \lambda)} \leq \rho\right\}
$$

and $Z_{i}$ takes values over the alphabet of $Y$ according to the pmf or pdf (as the case may be) $f_{-\lambda}\left(z \mid x_{i}\right) p\left(z \mid x_{i}\right)$. It is easy to verify that

$$
\mathbf{E}\left[\ln \frac{p\left(Z_{i} \mid x_{i}\right)}{q_{t}\left(Z_{i}\right)}\right]=D\left(t, x_{i}, \lambda\right)
$$

and

$$
\begin{aligned}
\sum_{i=1}^{n} \mathbf{E}\left[\ln \frac{p\left(Z_{i} \mid x_{i}\right)}{q_{t}\left(Z_{i}\right)}\right] & =\sum_{i=1}^{n} D\left(t, x_{i}, \lambda\right) \\
& =n \sum_{x \in \mathcal{X}} t(x) D(t, x, \lambda) \\
& =n(I(t ; P)-\delta)
\end{aligned}
$$

which further implies that

$$
F_{x^{n}}(\rho)=\operatorname{Pr}\left\{\frac{\sum_{i=1}^{n}\left[\ln \frac{p\left(Z_{i} \mid x_{i}\right)}{q_{t}\left(Z_{i}\right)}-D\left(t, x_{i}, \lambda\right)\right]}{\sqrt{n} \sigma_{D,-}(t ; P, \lambda)} \leq \rho\right\}
$$

Applying Lemma 1 to the independent sequence

$$
\left\{\ln \frac{p\left(Z_{i} \mid x_{i}\right)}{q_{t}\left(Z_{i}\right)}-D\left(t, x_{i}, \lambda\right)\right\}_{i=1}^{n}
$$

the argument similar to that in the proof of Theorem 2 can then be used to establish (4.49).

Finally, consider another sequence of independent random variables $W_{1}, W_{2}, \cdots, W_{n}$, where $W_{i}$ takes values over the alphabet of $Y$ according to the pmf or pdf (as the case may be) $p\left(w \mid x_{i}\right)$. Applying Lemma 1 directly to

$$
\left\{\ln \frac{p\left(W_{i} \mid x_{i}\right)}{q_{t}\left(W_{i}\right)}-D\left(t, x_{i}\right)\right\}_{i=1}^{n}
$$

we then get (4.52). This completes the proof of Theorem 8 .
The conditional probability that given $X^{n}=x^{n}, \frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)}$ is away from $I(t ; P)$ to the right can be bounded similarly. For completeness, we state these bounds below without proof. Define for any $t \in \mathcal{P}$ with full support and any $\delta \geq 0$

$$
\begin{equation*}
r(t, \delta) \triangleq \sup _{\lambda \geq 0}\left[\lambda(I(t ; P)+\delta)-\sum_{x \in \mathcal{X}} t(x) \ln \int p(y \mid x)\left[\frac{p(y \mid x)}{q_{t}(y)}\right]^{\lambda} d y\right] \tag{4.55}
\end{equation*}
$$

and for any $\lambda \in\left[0, \lambda^{*}(X ; Y)\right)$ and any $t \in \mathcal{P}$ with full support

$$
\begin{gather*}
f_{\lambda}(y \mid x) \triangleq \frac{\left[\frac{p(y \mid x)}{q_{t}(y)}\right]^{\lambda}}{\int p(v \mid x)\left[\frac{p(v \mid x)}{q_{t}(v)}\right]^{\lambda} d v}  \tag{4.56}\\
D_{+}(t, x, \lambda) \triangleq \int p(y \mid x) f_{\lambda}(y \mid x)\left[\ln \frac{p(y \mid x)}{q_{t}(y)}\right] d y  \tag{4.57}\\
\delta(t, \lambda) \triangleq \sum_{x \in \mathcal{X}} t(x) \int p(y \mid x) f_{\lambda}(y \mid x)\left[\ln \frac{p(y \mid x)}{q_{t}(y)}\right] d y-I(t ; P) . \tag{4.58}
\end{gather*}
$$

Then under the condition (4.5), $\delta(t, \lambda)$ as a function of $\lambda$ is strictly increasing over $\lambda \in$ $\left[0, \lambda^{*}(X ; Y)\right)$ with $\delta(t, 0)=0$. Let

$$
\Delta^{*}(t) \triangleq \lim _{\lambda \uparrow \lambda^{*}(X ; Y)} \delta(t, \lambda)
$$

By an argument similar to that in the proof of Theorem 1, it can be shown that $r(t, \delta)$ is strictly increasing, convex and continuously differentiable up to at least the third order over $\delta \in\left[0, \Delta^{*}(t)\right)$, and furthermore $r(t, \delta)$ has the following parametric expression

$$
\begin{equation*}
r(t, \delta(t, \lambda))=\lambda(I(t ; P)+\delta(t, \lambda))-\sum_{x \in \mathcal{X}} t(x) \ln \int p(y \mid x)\left[\frac{p(y \mid x)}{q_{t}(y)}\right]^{\lambda} d y \tag{4.59}
\end{equation*}
$$

with

$$
\lambda=\frac{\partial r(t, \delta)}{\partial \delta}
$$

satisfying

$$
\delta(t, \lambda)=\delta
$$

Further define for any $\lambda \in\left[0, \lambda^{*}(X ; Y)\right)$

$$
\begin{equation*}
\sigma_{D}^{2}(t ; P, \lambda) \triangleq \sum_{x \in \mathcal{X}} t(x)\left[\int p(y \mid x) f_{\lambda}(y \mid x)\left|\ln \frac{p(y \mid x)}{q_{t}(y)}-D_{+}(t, x, \lambda)\right|^{2} d y\right] \tag{4.60}
\end{equation*}
$$

and

$$
\begin{equation*}
M_{D}(t ; P, \lambda) \triangleq \sum_{x \in \mathcal{X}} t(x)\left[\int p(y \mid x) f_{\lambda}(y \mid x)\left|\ln \frac{p(y \mid x)}{q_{t}(y)}-D_{+}(t, x, \lambda)\right|^{3} d y\right] . \tag{4.61}
\end{equation*}
$$

Then the following result can be proved similarly, which is referred to as the right NEP with respect to relative entropy.

Theorem 9 (Right NEP With Respect to Relative Entropy). For any sequence $x^{n}=x_{1} \cdots x_{n}$ from $\mathcal{X}$, let $t \in \mathcal{P}$ be the type of $x^{n}$, i.e., $n t(a), a \in \mathcal{X}$, is the number of times the symbol a appears in $x^{n}$. Assume that $t$ has full support. Then

$$
\begin{equation*}
\operatorname{Pr}\left\{\left.\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)}>I(t ; P)+\delta \right\rvert\, X^{n}=x^{n}\right\} \leq e^{-n r(t, \delta)} \tag{4.62}
\end{equation*}
$$

Furthermore, under the assumptions (4.21) and (4.5), the following also hold:
(a) There exists a $\delta^{*}>0$ such that for any $\delta \in\left(0, \delta^{*}\right]$

$$
\begin{equation*}
r(t, \delta)=\frac{1}{2 \sigma_{D}^{2}(t ; P)} \delta^{2}+O\left(\delta^{3}\right) \tag{4.63}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\operatorname{Pr}\left\{\left.\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)}>I(t ; P)+\delta \right\rvert\, X^{n}=x^{n}\right\} \leq e^{-n\left(\frac{\delta^{2}}{2 \sigma_{D}^{2}(t ; P)}+O\left(\delta^{3}\right)\right)} \tag{4.64}
\end{equation*}
$$

(b) For any $\delta \in\left(0, \Delta^{*}(X ; Y)\right)$

$$
\begin{align*}
\underline{\xi}_{D}(t ; P, \lambda, n) e^{-n r(t, \delta)} & \leq \operatorname{Pr}\left\{\left.\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)}>I(t ; P)+\delta \right\rvert\, X^{n}=x^{n}\right\} \\
& \leq \bar{\xi}_{D}(t ; P, \lambda, n) e^{-n r(t, \delta)} \tag{4.65}
\end{align*}
$$

where $\lambda=\frac{\partial r(t, \delta)}{\partial \delta}>0$, and

$$
\begin{align*}
& \bar{\xi}_{D}(t ; P, \lambda, n)=\frac{2 C M_{D}(t ; P, \lambda)}{\sqrt{n} \sigma_{D}^{3}(t ; P, \lambda)} \\
& \quad+e^{\frac{n \lambda^{2} \sigma_{D}^{2}(t ; P, \lambda)}{2}}\left[Q\left(\sqrt{n} \lambda \sigma_{D}(t ; P, \lambda)\right)-Q\left(\rho^{*}+\sqrt{n} \lambda \sigma_{D}(t ; P, \lambda)\right)\right]  \tag{4.66}\\
& \underline{\xi}_{D}(t ; P, \lambda, n)=e^{\frac{n \lambda^{2} \sigma_{D}^{2}(t ; P, \lambda)}{2}} Q\left(\rho_{*}+\sqrt{n} \lambda \sigma_{D}(t ; P, \lambda)\right) \tag{4.67}
\end{align*}
$$

with $Q\left(\rho^{*}\right)=\frac{C M_{D}(t ; P, \lambda)}{\sqrt{n} \sigma_{D}^{3}(t ; P, \lambda)}$ and $Q\left(\rho_{*}\right)=\frac{1}{2}-\frac{2 C M_{D}(t ; P, \lambda)}{\sqrt{n} \sigma_{D}^{3}(t ; P, \lambda)}$.
(c) For any $\delta \leq c \sqrt{\frac{\ln n}{n}}$, where $c<\sigma_{D}(t ; P)$ is a constant,

$$
\begin{align*}
Q\left(\frac{\delta \sqrt{n}}{\sigma_{D}(t ; P)}\right)-\frac{C M_{D}(t ; P)}{\sqrt{n} \sigma_{D}^{3}(t ; P)} & \leq \operatorname{Pr}\left\{\left.\frac{1}{n} \ln \frac{p\left(Y^{n} \mid X^{n}\right)}{q_{t}\left(Y^{n}\right)}>I(t ; P)+\delta \right\rvert\, X^{n}=x^{n}\right\} \\
& \leq Q\left(\frac{\delta \sqrt{n}}{\sigma_{D}(t ; P)}\right)+\frac{C M_{D}(t ; P)}{\sqrt{n} \sigma_{D}^{3}(t ; P)} . \tag{4.68}
\end{align*}
$$

Remarks similar to those (Remark 1 and 2) following Theorem 2 can be drawn here concerning Theorems 8 and 9 . Theorem 8 will be used in [3] to establish a non-asymptotic coding theorem for Shannon random codes.

## V. NEP Application to Fixed Rate Source Coding

Assume that the source alphabet $\mathcal{X}$ is finite. In this section, we make use of the NEP with respect to $H(X)$ to establish a non-asymptotic fixed rate source coding theorem, which reveals, for any finite block length $n$, a complete picture about the tradeoff between the minimum rate of fixed rate coding of $X_{1} \cdots X_{n}$ and error probability when the error probability is a constant, or goes to 0 with block length $n$ at a sub-polynomial $n^{-\alpha}, 0<\alpha<1$, polynomial $n^{-\alpha}, \alpha \geq 1$, or sub-exponential $e^{-n^{\alpha}}, 0<\alpha<1$, speed. We begin with the definition of fixed rate source code.

Definition 1. Given a source from alphabet $\mathcal{X}$, a fixed rate source code with coding length $n$ is defined as a mapping $i: S_{n} \rightarrow\left\{1,2, \ldots,\left|S_{n}\right|\right\}$, where $S_{n}$ is a subset of $\mathcal{X}^{n}$. The performance of the code is measured by the rate $R_{n}=\frac{1}{n} \ln \left|S_{n}\right|$ (in nats) and error probability $\operatorname{Pr}\left\{X^{n} \notin S_{n}\right\}$.

As can be seen from the definition, the design of a fixed rate source code is equivalent to picking a subset of $\mathcal{X}^{n}$. Given the source statistics $p(x)$, one can easily show that the optimal way to pick $S_{n}$ is to order $x^{n}$ in the non-increasing order of $p\left(x^{n}\right)$, and include those $x^{n}$ with rank less than or equal to $\left|S_{n}\right|$. Then we have the following non-asymptotic fixed rate source coding theorem.

Theorem 10. Let $R_{n}\left(\epsilon_{n}\right)$ denote the minimum rate (in nats) of fixed rate coding of $X_{1} X_{2} \cdots X_{n}$ subject to the error probability not larger than $\epsilon_{n}$. Under the assumptions (2.2) and (2.5), for any $n$ and $\epsilon_{n}>0$,

$$
\begin{equation*}
\bar{\delta} \geq R_{n}\left(\epsilon_{n}\right)-H(X) \geq \underline{\delta}-r_{X}(\underline{\delta})+\frac{-d+\ln \left[\frac{1}{2}-Q\left(\frac{d}{\sqrt{n} \sigma_{H}(X, \lambda)}\right)-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right]}{n} \tag{5.1}
\end{equation*}
$$

for any constant $d$ satisfying $\frac{1}{2}-Q\left(\frac{d}{\sqrt{n} \sigma_{H}(X, \lambda)}\right)-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}>0$, where $\bar{\delta}$ is the solution to the equation

$$
\begin{equation*}
\epsilon_{n}=\bar{\xi}_{H}\left(X, r_{X}^{\prime}(\delta), n\right) e^{-n r_{X}(\delta)} \tag{5.2}
\end{equation*}
$$

$\underline{\delta}$ is the solution to the equation

$$
\begin{equation*}
\left(1+e^{-n}\right) \epsilon_{n}=\underline{\xi}_{H}\left(X, r_{X}^{\prime}(\delta), n\right) e^{-n r_{X}(\delta)} \tag{5.3}
\end{equation*}
$$

and $\lambda=r_{X}^{\prime}(\underline{\delta})$. In particular, the following hold, depending on whether $\epsilon_{n}$ is a constant, or how fast $\epsilon_{n}$ goes to 0 .
(a) When $\epsilon_{n}$ decreases exponentially with respect to $n$,

$$
\begin{align*}
r_{X}^{(i n v)}\left(-\frac{\ln \epsilon_{n}}{n}-\frac{\ln n}{2 n}\right)+O\left(n^{-1}\right) & \geq R_{n}\left(\epsilon_{n}\right)-H(X) \\
& \geq r_{X}^{(\text {inv })}\left(-\frac{\ln \epsilon_{n}}{n}-\frac{\ln n}{2 n}\right)+\frac{\ln \epsilon_{n}}{n}-O\left(n^{-1}\right) \tag{5.4}
\end{align*}
$$

where $r_{X}^{(\text {inv })}(\cdot)$ is the inverse function of $r_{X}(\cdot)$.
(b) When $\epsilon_{n}=n^{-\frac{\alpha}{2}} e^{-n^{\alpha}}$ for $\alpha \in(0,1)$,

$$
\begin{align*}
\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}+O\left(n^{-\frac{1+\alpha}{2}}\right) & \geq R_{n}\left(\epsilon_{n}\right)-H(X) \\
& \geq \sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}-O\left(n^{-\frac{1+\alpha}{2}}\right) \tag{5.5}
\end{align*}
$$

for $\alpha \in\left(0, \frac{1}{3}\right)$, and

$$
\begin{align*}
\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}+O\left(n^{-(1-\alpha)}\right) & \geq R_{n}\left(\epsilon_{n}\right)-H(X) \\
& \geq \sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}-O\left(n^{-(1-\alpha)}\right) \tag{5.6}
\end{align*}
$$

for $\alpha \in\left[\frac{1}{3}, 1\right)$.
(c) When $\epsilon_{n}=\frac{n^{-\alpha}}{\sqrt{\ln n}}$ for $\alpha>0$,

$$
\begin{align*}
\sigma_{H}(X) \sqrt{\frac{2 \alpha \ln n}{n}}+O\left(\sqrt{\frac{1}{n \ln n}}\right) & \geq R_{n}\left(\epsilon_{n}\right)-H(X) \\
& \geq \sigma_{H}(X) \sqrt{\frac{2 \alpha \ln n}{n}}-O\left(\sqrt{\frac{1}{n \ln n}}\right) \tag{5.7}
\end{align*}
$$

(d) When $\epsilon_{n}=\epsilon$ remains a constant,

$$
\begin{align*}
\frac{\sigma_{H}}{\sqrt{n}} Q^{-1}\left(\epsilon-\frac{C M_{H}(X)}{\sqrt{n} \sigma_{H}^{3}(X)}\right) & =\frac{\sigma_{H}}{\sqrt{n}} Q^{-1}(\epsilon)+O\left(\frac{1}{n}\right) \\
& \geq R_{n}\left(\epsilon_{n}\right)-H(X) \\
& \geq \frac{\sigma_{H}}{\sqrt{n}} Q^{-1}(\epsilon)-O\left(\frac{\ln n}{n}\right) . \tag{5.8}
\end{align*}
$$

where $Q^{-1}(\cdot)$ is the inverse function of $Q(\cdot)$.
Proof of Theorem 10: Define

$$
S_{n}(\delta) \triangleq\left\{x^{n}:-\frac{1}{n} \ln p\left(x^{n}\right) \leq H(X)+\delta\right\}
$$

and

$$
\epsilon_{n}(\delta)=\operatorname{Pr}\left\{X^{n} \notin S_{n}(\delta)\right\}
$$

Clearly $\epsilon_{n}(\delta)$ is a non-increasing function of $\delta$. Now let $\bar{\delta}$ and $\underline{\delta}$ satisfy that

$$
\begin{equation*}
\epsilon_{n}(\bar{\delta}) \leq \epsilon_{n}<\epsilon_{n}(\underline{\delta}) . \tag{5.9}
\end{equation*}
$$

According to the discussion on optimal fixed-rate source codes,

$$
\begin{equation*}
\frac{1}{n} \ln S_{n}(\underline{\delta})<R_{n}\left(\epsilon_{n}\right) \leq \frac{1}{n} \ln S_{n}(\bar{\delta}) . \tag{5.10}
\end{equation*}
$$

Observe that

$$
\begin{aligned}
\left|S_{n}(\bar{\delta})\right| e^{-n(H(X)+\bar{\delta})} & \leq \sum_{x^{n} \in S_{n}(\bar{\delta})} p\left(x^{n}\right) \\
& \leq \sum_{x^{n} \in \mathcal{X}^{n}} p\left(x^{n}\right) \\
& \leq 1
\end{aligned}
$$

which implies that

$$
\begin{equation*}
R_{n}\left(\epsilon_{n}\right) \leq \frac{1}{n} \ln \left|S_{n}(\bar{\delta})\right| \leq H(X)+\bar{\delta} \tag{5.11}
\end{equation*}
$$

Towards the lower bound on $R_{n}\left(\epsilon_{n}\right)$, further define

$$
S_{n}(\underline{\delta}, d) \triangleq\left\{x^{n}: H(X)+\underline{\delta}-\frac{d}{n} \leq-\frac{1}{n} \ln p\left(x^{n}\right) \leq H(X)+\underline{\delta}\right\}
$$

for some constant $d>0$. Then we have

$$
\begin{aligned}
\left|S_{n}(\underline{\delta}, d)\right| e^{-n\left(H(X)+\underline{\delta}-\frac{d}{n}\right)} & \geq \sum_{x^{n} \in S_{n}(\underline{\delta}, d)} p\left(x^{n}\right) \\
& =\sum_{x^{n} \in S_{n}(\underline{\delta}, d)} f_{\lambda}^{-1}(x) f_{\lambda}\left(x^{n}\right) p\left(x^{n}\right) \\
& =\sum_{x^{n} \in S_{n}(\underline{\delta}, d)} e^{-n\left[-\frac{1}{n} \lambda \ln p\left(x^{n}\right)-\ln \sum_{u \in \mathcal{X}} p^{-\lambda+1}(u)\right]} f_{\lambda}\left(x^{n}\right) p\left(x^{n}\right) \\
& \geq e^{-n r_{X}(\underline{\delta})} \sum_{x^{n} \in S_{n}(\underline{\delta}, d)} f_{\lambda}\left(x^{n}\right) p\left(x^{n}\right) \\
& =e^{-n r_{X}(\underline{\delta})} \operatorname{Pr}\left\{Z^{n} \in S_{n}(\underline{\delta}, d)\right\} \\
& =e^{-n r_{X}(\underline{\delta})} \operatorname{Pr}\left\{-\frac{d}{n} \leq \frac{1}{n} \sum_{i=1}^{n}-\ln p\left(Z_{i}\right)-(H(X)+\underline{\delta}) \leq 0\right\} \\
& \geq e^{-n r_{X}(\underline{\delta})}\left[\frac{1}{2}-Q\left(\frac{d}{\sqrt{n} \sigma_{H}(X, \lambda)}\right)-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right]
\end{aligned}
$$

where $\lambda=r_{X}^{\prime}(\underline{\delta}),\left\{Z_{i}\right\}_{i=1}^{n}$ are IID random variables with common $\operatorname{pmf} f_{\lambda}(z) p(z)$, and the last inequality is due to the direct application of Lemma 1 (Berry-Esseen Central Limit Theorem) to $\left\{-\ln p\left(Z_{i}\right)-(H(X)+\underline{\delta})\right\}_{i=1}^{n}$. And therefore

$$
\begin{align*}
R_{n}\left(\epsilon_{n}\right) \geq & \frac{1}{n} \ln \left|S_{n}(\underline{\delta})\right| \\
\geq & \frac{1}{n} \ln \left|S_{n}(\underline{\delta}, d)\right| \\
\geq & H(X)+\underline{\delta}-\frac{d}{n}-r_{X}(\underline{\delta}) \\
& +\frac{1}{n} \ln \left[\frac{1}{2}-Q\left(\frac{d}{\sqrt{n} \sigma_{H}(X, \lambda)}\right)-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}\right] . \tag{5.12}
\end{align*}
$$

Note that $\frac{1}{2}-Q\left(\frac{d}{\sqrt{n} \sigma_{H}(X, \lambda)}\right)-\frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}=\Theta\left(\frac{1}{\sqrt{n}}\right)$ for constant $d>0$. Then (5.1) is proved by showing $\bar{\delta}$ and $\underline{\delta}$ calculated according to (5.2) and (5.3) indeed satisfy (5.9), where we invoke Theorem 2, i.e.

$$
\begin{aligned}
\epsilon_{n}(\bar{\delta}) & =\operatorname{Pr}\left\{X^{n} \notin S_{n}(\bar{\delta})\right\} \\
& \leq \bar{\xi}_{H}\left(X, r_{X}^{\prime}(\bar{\delta}), n\right) e^{-n r_{X}(\bar{\delta})} \\
& =\epsilon_{n}
\end{aligned}
$$

while

$$
\begin{aligned}
\epsilon_{n}(\underline{\delta}) & =\operatorname{Pr}\left\{X^{n} \notin S_{n}(\underline{\delta})\right\} \\
& \geq \underline{\xi}_{H}\left(X, r_{X}^{\prime}(\underline{\delta}), n\right) e^{-n r_{X}(\underline{\delta})} \\
& >\epsilon_{n} .
\end{aligned}
$$

Let us now look at special cases.
(a) When $\epsilon_{n}$ decreases exponentially with respect to $n$, i.e. $\frac{1}{n} \ln \epsilon_{n} \rightarrow c$ as $n \rightarrow+\infty$ for some constant $c<0$, we have

$$
\begin{equation*}
\frac{\ln \epsilon_{n}}{n}=\frac{\ln \bar{\xi}_{H}\left(X, r_{X}^{\prime}(\bar{\delta}), n\right)}{n}-r_{X}(\bar{\delta}) . \tag{5.13}
\end{equation*}
$$

Note that

$$
\bar{\xi}_{H}(X, \lambda, n) \geq \frac{2 C M_{H}(X, \lambda)}{\sqrt{n} \sigma_{H}^{3}(X, \lambda)}=\Omega\left(\frac{1}{\sqrt{n}}\right) .
$$

Taking $n \rightarrow+\infty$ in (5.13), it can be seen that $r_{X}(\bar{\delta}) \rightarrow-c$. And therefore, $\bar{\xi}_{H}\left(X, r_{X}^{\prime}(\bar{\delta}), n\right)=$ $\Theta\left(\frac{1}{\sqrt{n}}\right)$, which further implies that

$$
\begin{align*}
\bar{\delta} & =r_{X}^{(i n v)}\left(-\frac{\ln \epsilon_{n}}{n}+\frac{\ln \bar{\xi}_{H}\left(X, r_{X}^{\prime}(\bar{\delta}), n\right)}{n}\right) \\
& =r_{X}^{(i n v)}\left(-\frac{\ln \epsilon_{n}}{n}-\frac{\ln n}{2 n}+O\left(n^{-1}\right)\right) \\
& =r_{X}^{(i n v)}\left(-\frac{\ln \epsilon_{n}}{n}-\frac{\ln n}{2 n}\right)+O\left(n^{-1}\right) \tag{5.14}
\end{align*}
$$

On the other hand,

$$
\begin{equation*}
\frac{\ln \epsilon_{n}}{n}+\frac{\ln \left(1+e^{-n}\right)}{n}=\frac{\ln \underline{\xi}_{H}\left(X, r_{X}^{\prime}(\underline{\delta}), n\right)}{n}-r_{X}(\underline{\delta}) . \tag{5.15}
\end{equation*}
$$

and by the same argument, $r_{X}(\underline{\delta}) \rightarrow-c$ as $n \rightarrow+\infty$. Consequently, $\underline{\xi}_{H}\left(X, r_{X}^{\prime}(\underline{\delta}), n\right)=$ $\Theta\left(\frac{1}{\sqrt{n}}\right)$, which further implies

$$
\begin{equation*}
\frac{\ln \epsilon_{n}}{n}=-r_{X}(\underline{\delta})-\frac{\ln n}{2 n}+O\left(n^{-1}\right) \tag{5.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\underline{\delta}=r_{X}^{(\text {inv })}\left(-\frac{\ln \epsilon_{n}}{n}-\frac{\ln n}{2 n}\right)-O\left(n^{-1}\right) \tag{5.17}
\end{equation*}
$$

Combining (5.1) with (5.14), (5.16) and (5.17) yields,

$$
\begin{align*}
r_{X}^{(\text {inv })}\left(-\frac{\ln \epsilon_{n}}{n}-\frac{\ln n}{2 n}\right)+O\left(n^{-1}\right) & \geq R_{n}\left(\epsilon_{n}\right)-H(X) \\
& \geq \underline{\delta}-r_{X}(\underline{\delta})-\frac{\ln n}{2 n}-O\left(n^{-1}\right) \\
& =r_{X}^{(\text {inv })}\left(-\frac{\ln \epsilon_{n}}{n}-\frac{\ln n}{2 n}\right)+\frac{\ln \epsilon_{n}}{n}-O\left(n^{-1}\right) \tag{5.18}
\end{align*}
$$

This completes the proof of (5.4).
(b) First of all, let us consider the case when $\alpha \in\left(0, \frac{1}{3}\right)$. Towards proving (5.5), let us show that $\bar{\delta}=\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}+\eta n^{-\frac{1+\alpha}{2}}$ for some properly chosen constant $\eta$ will guarantee

$$
\begin{equation*}
\epsilon_{n}(\bar{\delta}) \leq n^{-\frac{\alpha}{2}} e^{-n^{\alpha}} \tag{5.19}
\end{equation*}
$$

By Theorem 2 and Remark 1 ,

$$
\epsilon_{n}(\bar{\delta}) \leq \bar{\xi}_{H}\left(X, r_{X}^{\prime}(\bar{\delta}), n\right) e^{-n r_{X}(\bar{\delta})}
$$

while

$$
\begin{aligned}
\bar{\xi}_{H} & \left(X, r_{X}^{\prime}(\bar{\delta}), n\right) \\
& =\bar{\xi}_{H}\left(X, r_{X}^{\prime}\left(\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}+\eta n^{-\frac{1+\alpha}{2}}\right), n\right) \\
& =\Theta\left(\frac{1}{\sqrt{n} r_{X}^{\prime}\left(\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}+\eta n^{-\frac{1+\alpha}{2}}\right)}\right) \\
& =\Theta\left(n^{-\frac{\alpha}{2}}\right) \leq \eta_{1} n^{-\frac{\alpha}{2}}
\end{aligned}
$$

for some constant $\eta_{1}>0$, and

$$
\begin{aligned}
& e^{-n r_{X}(\bar{\delta})} \\
& \quad=\exp \left\{-n r_{X}\left(\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}+\eta n^{-\frac{1+\alpha}{2}}\right)\right\} \\
& \quad=\exp \left\{-n\left[\frac{1}{2 \sigma_{H}^{2}(X)}\left(\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}+\eta n^{-\frac{1+\alpha}{2}}\right)^{2}+O\left(n^{-\frac{3(1-\alpha)}{2}}\right)\right]\right\} \\
& \quad=\exp \left\{-n^{\alpha}-\frac{\sqrt{2} \eta}{\sigma_{H}(X)}-O\left(n^{-\alpha}+n^{-\frac{1-3 \alpha}{2}}\right)\right\} \\
& \quad=\exp \left\{-n^{\alpha}-\frac{\sqrt{2} \eta}{\sigma_{H}(X)}-o(1)\right\}
\end{aligned}
$$

since $\alpha \in\left(0, \frac{1}{3}\right)$. Now it is trivial to see that we can select a constant $\eta$ such that

$$
\eta_{1} e^{-\frac{\sqrt{2} \eta}{\sigma_{H}(X)}-o(1)} \leq 1
$$

which will make (5.19) satisfied, and consequently

$$
\begin{aligned}
\bar{\delta} & =\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}+\eta n^{-\frac{1+\alpha}{2}} \\
& =\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}+O\left(n^{-\frac{1+\alpha}{2}}\right) \\
& \geq R_{n}\left(\epsilon_{n}\right)-H(X) .
\end{aligned}
$$

In the similar manner, we can show that by making $\underline{\delta}=\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}-\eta^{\prime} n^{-\frac{1+\alpha}{2}}$ for another constant $\eta^{\prime}>0$,

$$
\epsilon_{n}(\underline{\delta})>\epsilon_{n} .
$$

Consequently,

$$
\begin{aligned}
R_{n}\left(\epsilon_{n}\right)-H(X) & \geq \underline{\delta}-r_{X}(\underline{\delta})-\frac{\ln n}{2 n}-O\left(n^{-1}\right) \\
& =\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}-\eta^{\prime} n^{-\frac{1+\alpha}{2}}-O\left(n^{-(1-\alpha)}\right) \\
& =\sqrt{2} \sigma_{H}(X) n^{-\frac{1-\alpha}{2}}-O\left(n^{-\frac{1+\alpha}{2}}\right)
\end{aligned}
$$

for $\alpha \in\left(0, \frac{1}{3}\right)$. The proof of (5.6) for the case $\alpha \in\left[\frac{1}{3}, 1\right)$ is essentially the same, and therefore omitted.
(c) Following the same spirit of the proof for part (b), one can verify that constants $\eta$ and $\eta^{\prime}$ can be chosen respectively such that

$$
\left[\left.\epsilon_{n}(\bar{\delta})\right|_{\bar{\delta}=\sigma_{H}} \sqrt{\frac{2 \alpha \ln n}{n}}+\eta \sqrt{\frac{1}{n \ln n}}\right] \leq \frac{n^{-\alpha}}{\sqrt{\ln n}}
$$

and

$$
\left[\left.\epsilon_{n}(\underline{\delta})\right|_{\underline{\delta}=\sigma_{H}} \sqrt{\frac{2 \alpha \ln n}{n}}-\eta^{\prime} \sqrt{\frac{1}{n \ln n}}\right]>\frac{n^{-\alpha}}{\sqrt{\ln n}}
$$

which, together with (5.1), proves (5.7).
(d) It can be readily seen that by Theorem 2 (b), $\bar{\delta}=\frac{\sigma_{H}(X)}{\sqrt{n}} Q^{-1}\left(\epsilon-\frac{C M_{H}(X)}{\sqrt{n} \sigma_{H}^{3}(X)}\right)$ is the right choice to guarantee

$$
\epsilon_{n}(\bar{\delta}) \leq \epsilon
$$

while $\underline{\delta}=\frac{\sigma_{H}(X}{\sqrt{n}} Q^{-1}\left(\epsilon+\frac{2 C M_{H}(X)}{\sqrt{n} \sigma_{H}^{3}(X)}\right)$ will make

$$
\epsilon_{n}(\underline{\delta})>\epsilon
$$

satisfied. (5.8) then follows immediately from (5.1) and the choices of $\bar{\delta}$ and $\underline{\delta}$. This completes the proof of Theorem 10 .

Remark 3. To show Theorem 10 provides a non-trivial bound, we claim that

$$
\delta>r_{X}(\delta)
$$

for $0<\delta<\ln |\mathcal{X}|-H(X)$. Indeed, recall the definition of $\delta(\lambda)$ and

$$
0 \leq r_{X}(\delta(1))=H(X)+\delta(1)-\ln |\mathcal{X}|
$$

which implies that $\delta(1) \geq \ln |\mathcal{X}|-H(X)$ or $r_{X}^{\prime}(\delta)<1$ for $0<\delta<\ln |\mathcal{X}|-H(X)$. The claim then follows immediately from the fact that $r_{X}(0)=0$.

Remark 4. In Part (d) of Theorem 10, we can see that if $\epsilon_{n}=\epsilon>0.5$ is selected, then $R_{n}\left(\epsilon_{n}\right)$ could be strictly less than $H(X)$ for finite block length $n$ ! This means that if the error probability is allowed to be slightly larger than 0.5 , the rate of source code can be even less than the entropy rate. For an IID binary source with $p=\operatorname{Pr}\left\{X_{1}=1\right\}=0.12$, Figure 3 shows the tradeoff between the error probability and block length when the code rate is $0.21 \%$ below the entropy rate, where in Figure 3, both the entropy rate and code rate are expressed in terms of bits. As can be seen from Figure 3, at the block length 1000, the error probability is around 0.65 , and the code rate is $0.21 \%$ below the entropy rate. Similar phenomenon can be seen for channel coding shown in [3].

Remark 5. Related to Part (d) of Theorem 10 is the second order source coding analysis in [5] with a fixed error probability $0<\epsilon<1$. Both results are concerned with the scenario where the rate is around the entropy rate in the order of $\frac{1}{\sqrt{n}}$ and the error probability is a constant. However, the work in [5] is asymptotic. On the other hand, Theorem 10 ((5.1) and Part (d)) is non-asymptotic and valid for any block length $n$. It reveals a complete picture about the tradeoff between the rate and error probability when the error probability is constant, or approaches 0 with block length $n$ at an exponential (Part (a)), a sub-exponential (Part (b)), a polynomial (Part (c) with $\alpha \geq 1$ ), or a sub-polynomial (Part (c) with $0<\alpha<1$ ) speed.

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Fig. 3. Tradeoff between the error probability and block length when the rate is below the entropy rate with $p=0.12$
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