# Info-Clustering: An Efficient Algorithm by Network Information Flow 

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#### Abstract

Motivated by the fact that entities in a social network or biological system often interact by exchanging information, we propose an efficient info-clustering algorithm that can group entities into communities using a parametric max-flow algorithm. This is a meaningful special case of the info-clustering paradigm where the dependency structure is graphical and can be learned readily from data.


## I. Introduction

Info-clustering was proposed in [1] as an application of network information theory to the problem of clustering in machine learning. It regards each object as a piece of information, namely a random variable, and groups random variables with sufficiently large amount of mutual information together. Clustering is often an important first step in studying a large biological system such as the human connectome and genome. It can also identify communities in a social network so that resources can be allocated efficiently based on the communities discovered. Since entities in a social or biological system often possess information and interact with each other by the transmission of information, clustering them by their mutual information intuitively gives meaningful results.

Using the multivariate mutual information (MMI) in [2] as the similarity measure, info-clustering defines a hierarchy of clusters of possibly different sizes at different levels of mutual information. The clustering solution is intimately related to the principal sequence of partitions (PSP) [3] of a submodular function, namely, that of the entropy function of the set of random variables to be clustered. From this, it follows that the clustering solution is unique and solvable using a polynomial number of oracle calls to evaluate the entropy function. However, in general, learning the entropy function of a finite set $V$ of random variables from data takes
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exponential time in the size of $V$ [4]. The computation of the PSP [3, 5], [1, Algorithm 3] without any approximation also takes $\Omega\left(|V|^{2}\right)$ calls to a submodular function minimization (SFM) algorithm, which in turn makes $\Omega\left(|V|^{5}\right)$ oracle calls to evaluate the submodular function. Hence, the practicality of the general info-clustering algorithm is limited by both the sample complexity and computational complexity.

Fortunately, info-clustering reduces to faster algorithms under special statistical models that are also easier (compared to a general model) to learn from data. For instance, under the Markov tree model, info-clustering reduces to an edgefiltering procedure that runs in $O\left(|V|^{2}\right)$ time [6]. Furthermore, this procedure coincides with an existing functional genomic clustering method by mutual information relevance networks (MIRN) [7]. While rediscovering a simple clustering algorithm under the Markov tree simplification, the info-clustering paradigm provides a theoretical justification of the MIRN algorithm and helps discover how the algorithm may fail when the Markov tree assumption does not hold [1, Example 6].

In this work, we propose an efficient info-clustering algorithm under a different graphical model called the pairwise independent network (PIN) [8, 9]. Using the idea of the matroidal network link model in [10], the MMI has a concrete operational meaning as the maximum network broadcast throughput [11]. The info-clustering solution therefore identifies clusters with large intra-cluster communication rates, which naturally maps to communities of closely related entities in a social network. Learning the PIN model simplifies to learning the weights of $O\left(|V|^{2}\right)$ edges in a graph on $V$. In a social network, the weight of each edge can simply be the amount/rate of communication between the edge's incident nodes.

As shown in [1, Proposition 9], the info-clustering solution for the PIN model can be obtained from the PSP of the cut function of a weighted graph. It is well-known that faster SFM algorithms are possible for the cut function using min-cut or max-flow algorithms (e.g., see [12-14]). An algorithm was given in [15] that computes the PSP efficiently by reducing the problem to a parametric max-flow problem, where the capacities of the edges are certain monotonic functions of a parameter. The reduction is carefully done such that the parametric max-flow algorithm in [16] can compute the PSP in $O\left(|V|^{3} \sqrt{|E|)}\right.$ time, where $E$ is the set of edges with nonzero weight. We will adapt this algorithm to compute the infoclustering solution and modify it to improve the performance further.


Fig. 1: Identifying the clusters of the PIN in (2.1a) by brute-force search over subsets satisfying the threshold constraint (2.9).

## II. PRELIMINARIES ON INFO-CLUSTERING

## A. Formulation

Let $V$ be the finite index set of the objects we want to cluster. Without loss of generality, we assume

$$
V=[|V|]:=\{1, \ldots,|V|\} \quad \text { and } \quad|V|>1
$$

The idea of info-clustering is to treat each object $i \in V$ as a random variable $Z_{i}$ (denoted in san serif font) taking values from a finite set $Z_{i}$ (denoted in the usual math font), and cluster the objects according to their mutual information. $P_{Z_{V}}$ denotes the distribution of the entire vector of random variables

$$
\mathrm{Z}_{V}:=\left(\mathrm{Z}_{i} \mid i \in V\right)
$$

We will illustrate the idea of info-clustering via a simple example shown in Fig. 1a, where $V=[3]=\{1,2,3\}$ and the corresponding random variables are defined as

$$
\left.\begin{array}{l}
\mathrm{Z}_{1}:=\left(\mathrm{X}_{\mathrm{a}}, \quad \mathrm{X}_{\mathrm{c}}\right) \\
\mathrm{Z}_{2}:=\left(\mathrm{X}_{\mathrm{a}}, \mathrm{X}_{\mathrm{b}}\right.
\end{array}\right), \begin{aligned}
& \left.\mathrm{X}_{\mathrm{b}}, \mathrm{X}_{\mathrm{c}}\right), \tag{2.1a}
\end{aligned}
$$

with $X_{a}, X_{b}$ and $X_{c}$ being independent uniformly random variables with entropies

$$
\begin{align*}
& H\left(\mathrm{X}_{\mathrm{a}}\right)=H\left(\mathrm{X}_{\mathrm{b}}\right)=1 \\
& H\left(\mathrm{X}_{\mathrm{c}}\right)=5 \tag{2.1b}
\end{align*}
$$

This is a PIN (Definition 2.2) with correlation represented by the weighted triangle $G$ shown in Fig. 1a characterized by the weight function $c$ where

$$
\begin{align*}
& c(\{1,2\})=c(\{2,3\}):=H\left(\mathrm{X}_{\mathrm{a}}\right)=H\left(\mathrm{X}_{\mathrm{b}}\right)=1 \\
& c(\{1,3\}):=H\left(\mathrm{X}_{\mathrm{c}}\right)=5 \tag{2.1c}
\end{align*}
$$

The vertex set is $V:=[3]$ and the edge set is

$$
\begin{equation*}
\mathcal{E}:=\operatorname{supp}(c)=\{\{1,2\},\{2,3\},\{1,3\}\} . \tag{2.1d}
\end{equation*}
$$

Note that, for ease of comparison, this is the same graph used as an example in [15] to illustrate the algorithm. A formal definition of the (hyper-)graphical source model is as follows:

Definition 2.1 (Definition 2.4 of [17]) $Z_{V}$ is a hypergraphical source w.r.t. a hypergraph $(V, E, \xi)$ with edge functions $\xi: E \rightarrow 2^{V} \backslash\{\emptyset\}$ iff, for some independent (hyper)edge variables $\mathrm{X}_{e}$ for $e \in E$ with $H\left(\mathrm{X}_{e}\right)>0$,

$$
\begin{equation*}
\mathrm{Z}_{i}:=\left(\mathrm{X}_{e} \mid e \in E, i \in \xi(e)\right), \text { for } i \in V \tag{2.2}
\end{equation*}
$$

The weight function $c: 2^{V} \backslash\{\emptyset\} \rightarrow \mathbb{R}$ of a hypergraphical source is defined as

$$
\begin{align*}
c(B) & :=H\left(\mathrm{X}_{e} \mid e \in E, \xi(e)=B\right) \text { with support }  \tag{2.3a}\\
\operatorname{supp}(c) & :=\left\{B \in 2^{V} \backslash\{\emptyset\} \mid c(B)>0\right\} \tag{2.3b}
\end{align*}
$$

The PIN model [9] is an example, where the corresponding hypergraph is a graph.

Definition 2.2 ([9]) $\mathrm{Z}_{V}$ is a pairwise independent network (PIN) iff it is hypergraphical w.r.t. a graph $(V, E, \xi)$ with edge function $\xi: E \rightarrow V^{2} \backslash\{(i, i) \mid i \in V\}$ (i.e., no self loops). $\square$

The mutual information among multiple random variables is measured by the multivariate mutual information (MMI) defined in [2] as

$$
\begin{align*}
I\left(\mathrm{Z}_{V}\right) & :=\min _{\mathcal{P} \in \Pi^{\prime}(V)} I_{\mathcal{P}}\left(\mathrm{Z}_{V}\right), \text { with }  \tag{2.4a}\\
I_{\mathcal{P}}\left(\mathrm{Z}_{V}\right) & :=\frac{1}{|\mathcal{P}|-1}[\underbrace{\sum_{C \in \mathcal{P}} H\left(\mathrm{Z}_{C}\right)-H\left(\mathrm{Z}_{V}\right)}_{=D\left(P_{\mathrm{Z}_{V}} \| \Pi_{C \in \mathcal{P}} P_{\mathrm{Z}_{C}}\right)}] \tag{2.4b}
\end{align*}
$$

and $\Pi^{\prime}(V)$ being the set of partitions of $V$ into at least 2 nonempty disjoint subsets of $V$. We may also write $I_{\mathcal{P}}\left(\mathrm{Z}_{V}\right)$ more explicitly as

$$
\begin{equation*}
I_{\mathcal{P}}\left(\mathrm{Z}_{V}\right)=I\left(\mathrm{Z}_{C_{1}} \wedge \cdots \wedge \mathrm{Z}_{C_{k}}\right) \tag{2.5}
\end{equation*}
$$

for $\mathcal{P}=\left\{C_{1}, \ldots, C_{k}\right\}$. Note that Shannon's mutual information $I\left(\mathrm{Z}_{1} \wedge \mathrm{Z}_{2}\right)$ is the special case when $\mathcal{P}$ is a bipartition. It is sometimes convenient to expand $I_{\mathcal{P}}\left(Z_{V}\right)$ using Shannon's mutual information [2, (5.18)] as follows:

$$
\begin{align*}
I_{\mathcal{P}}\left(\mathrm{Z}_{V}\right) & =I\left(\mathrm{Z}_{C_{1}} \wedge \cdots \wedge C_{k}\right) \\
& =\frac{1}{k-1} \sum_{i=1}^{k-1} I\left(\mathrm{Z}_{C_{i}} \wedge \mathrm{Z}_{\mathrm{U}_{j=i+1}^{k} C_{j}}\right) \tag{2.6}
\end{align*}
$$

For the example with the random vector defined in (2.1a),

$$
\begin{align*}
I\left(\mathrm{Z}_{\{v, w\}}\right) & =I\left(\mathrm{Z}_{v} \wedge \mathrm{Z}_{w}\right)=c(\{v, w\}) \\
& = \begin{cases}1, & \{v, w\} \in\{\{1,2\},\{2,3\}\} \\
5, & \{v, w\}=\{1,3\}\end{cases} \tag{2.7a}
\end{align*}
$$

which reduces to Shanon's mutual information, and

$$
\begin{gather*}
I\left(\mathrm{Z}_{[3]}\right)=\min \left\{I_{\{\{2,3\},\{1\}\}}\left(\mathrm{Z}_{[3]}\right),\right. \\
I_{\{\{1,3\},\{2\}\}}\left(\mathrm{Z}_{[3]}\right), \\
I_{\{\{1,2\},\{3\}\}}\left(\mathrm{Z}_{[3]}\right), \\
\left.I_{\{\{1\},\{2\},\{3\}\}}\left(\mathrm{Z}_{[3]}\right)\right\} \\
=\min \left\{I\left(\mathrm{Z}_{2}, \mathrm{Z}_{3} \wedge \mathrm{Z}_{1}\right),\right.  \tag{2.7b}\\
I\left(\mathrm{Z}_{1}, \mathrm{Z}_{3} \wedge \mathrm{Z}_{2}\right), \\
I\left(\mathrm{Z}_{1}, \mathrm{Z}_{2} \wedge \mathrm{Z}_{3}\right), \\
\left.\frac{I\left(\mathrm{Z}_{1}, \mathrm{Z}_{3} \wedge \mathrm{Z}_{2}\right)+I\left(\mathrm{Z}_{1} \wedge \mathrm{Z}_{3}\right)}{2}\right\} \\
=\min \left\{6,6,2, \frac{2+5}{2}\right\}=2,
\end{gather*}
$$

where we have applied (2.6) to calculate $I_{\{\{1\},\{2\},\{3\}\}}\left(\mathrm{Z}_{[3]}\right)$ for the partition into singletons as the average of the value $I\left(\mathrm{Z}_{1}, \mathrm{Z}_{3} \wedge \mathrm{Z}_{2}\right)$ of the cut that separates node 2 from nodes 1 and 3 , and the value $I\left(\mathrm{Z}_{1} \wedge \mathrm{Z}_{3}\right)$ of the cut that further separates node 1 from node 3 .

Note that the sequence of two cuts effectively partitions the vertex set into singletons. From this expansion, it is clear that the partition into singletons cannot be optimal in this case, since the mutual information between nodes 1 and 3 is very large. Indeed, the optimal partition turns out to be a clustering of the random variable into correlated groups. In general, the set of optimal partitions to (2.4a), denoted as $\Pi^{*}\left(Z_{V}\right)$, form a semi-lattice w.r.t. the partial order that $\mathcal{P} \preceq \mathcal{P}^{\prime}$ for the partitions $\mathcal{P}$ and $\mathcal{P}^{\prime}$ when

$$
\begin{equation*}
\forall C \in \mathcal{P}, \exists C^{\prime} \in \mathcal{P}^{\prime}: C \subseteq C^{\prime} \tag{2.8}
\end{equation*}
$$

$\mathcal{P} \prec \mathcal{P}^{\prime}$ denotes the strict inequality. There is a unique finest/minimum partition $\mathcal{P}^{*}\left(Z_{V}\right)$, referred to as the fundamental partition for $\mathrm{Z}_{V}$ [2, Theorem 5.2]. For a threshold $\gamma \in \mathbb{R}$, the set of clusters is defined as [1, Definition 1]

$$
\begin{equation*}
\mathcal{C}_{\gamma}\left(\mathrm{Z}_{V}\right):=\operatorname{maximal}\left\{B \subseteq V| | B \mid>1, I\left(\mathrm{Z}_{B}\right)>\gamma\right\} \tag{2.9}
\end{equation*}
$$

where maximal $\mathcal{F}$ is used to denote the inclusion-wise maximal elements of $\mathcal{F}$, i.e.,

$$
\begin{equation*}
\operatorname{maximal} \mathcal{F}:=\left\{B \in \mathcal{F} \mid \nexists B^{\prime} \supsetneq B, B^{\prime} \in \mathcal{F}\right\} \tag{2.10}
\end{equation*}
$$

Proposition 2.1 ([1, Theorem 5]) $\mathcal{C}_{\gamma}\left(\mathrm{Z}_{V}\right)=\mathcal{P}^{*}\left(\mathrm{Z}_{V}\right) \backslash\{i \mid$ $i \in V\}$ with $\gamma=I\left(Z_{V}\right)$, i.e., the non-singleton subsets in the fundamental partition are the maximal subsets (clusters) with MMI larger than that of the entire set.

For the example, applying the definition (2.9) of clusters with the MMI calculated in (2.7), the clustering solution is

$$
\mathcal{C}_{\gamma}\left(\mathrm{Z}_{[3]}\right)= \begin{cases}\{[3]\} & \gamma \in(-\infty, 2)  \tag{2.11}\\ \{1,3\} & \gamma \in[2,5) \\ \emptyset & \gamma \in[5, \infty)\end{cases}
$$

The info-clustering solution above consists of two clusters shown in Fig. 1c for different intervals of the threshold $\gamma$. For $\gamma \geq 2$, the subset $\{1,3\}$ is the only feasible solution that satisfies the threshold constraint in (2.9). For $\gamma \leq 2$, the entire set [3] also satisfies the threshold constraint and is maximal. Recall from (2.7b) that there is a unique optimal partition for $I\left(\mathrm{Z}_{\{1,2,3\}}\right)$, which is therefore the finest optimal partition

$$
\begin{equation*}
\mathcal{P}^{*}\left(\mathrm{Z}_{[3]}\right)=\underbrace{\{\{1,3\}}_{\mathcal{C}_{2}\left(\mathrm{Z}_{[3]}\right)},\{2\}\} . \tag{2.12}
\end{equation*}
$$

As expected from Proposition 2.1, the non-singleton element $\{1,3\}$ is the only possible subset with MMI larger than 2 .

It turns out that the computation of the MMI, fundamental partition, and the entire info-clustering solution can be done in strongly polynomial time from the principal sequence of partition (PSP) of the entropy function

$$
\begin{equation*}
h: B \subseteq V \mapsto H\left(\mathrm{Z}_{B}\right) \tag{2.13}
\end{equation*}
$$

The PSP is a more general mathematical structure [3] in combinatorial optimization defined for a submodular function. More precisely, a reveal-valued set function $g: 2^{V} \rightarrow \mathbb{R}$ is said to be submodular iff

$$
\begin{equation*}
g\left(B_{1}\right)+g\left(B_{2}\right) \geq g\left(B_{1} \cap B_{2}\right)+g\left(B_{1} \cup B_{2}\right) \tag{2.14}
\end{equation*}
$$

for all $B_{1}, B_{2} \subseteq V$. The entropy function, in particular, is a submodular function [18]. The PSP of the submodular function $g$ is the characterization of the solutions to the following for all $\gamma \in \mathbb{R}$ :

$$
\begin{equation*}
\hat{g}(V):=\min _{\mathcal{P} \in \Pi(V)} g_{\gamma}[\mathcal{P}] \tag{2.15a}
\end{equation*}
$$

referred to as the Dilworth truncation [19], where $\Pi(V)$ is the partition of $V$ into one or more non-empty disjoint subsets,

$$
\begin{align*}
g_{\gamma}[\mathcal{P}] & :=\sum_{C \in \mathcal{P}} g_{\gamma}(C)  \tag{2.15b}\\
g_{\gamma}(C) & :=g(C)-\gamma \tag{2.15c}
\end{align*}
$$

(n.b., $\Pi^{\prime}(V)$ in (2.4a) is $\Pi(V)$ but without the trivial partition $\{V\}$, i.e., $\Pi^{\prime}(V)=\Pi(V) \backslash\{\{V\}\}$.) For every $\gamma$, submodularity of $g$ implies that there exists a unique finest/minimum (w.r.t. the partial order (2.8)) optimal partition to (2.15a), denoted as $\mathcal{P}^{*}(\gamma)$. It can be characterized as

$$
\begin{equation*}
\mathcal{P}^{*}(\gamma)=\mathcal{P}_{\ell} \quad \forall \gamma \in\left[\gamma_{\ell}, \gamma_{\ell+1}\right), \ell \in\{0, \ldots, N\} \tag{2.16a}
\end{equation*}
$$

for some integer $N>0$, a sequence of critical values of $\gamma$

$$
\begin{equation*}
-\infty<\gamma_{1}<\cdots<\gamma_{N}<\infty \tag{2.16b}
\end{equation*}
$$

with $\gamma_{0}:=-\infty$ and $\gamma_{N+1}:=\infty$ for convenience, and a sequence of successively finer partitions

$$
\begin{equation*}
\mathcal{P}_{0}=V \succ \mathcal{P}_{1} \succ \cdots \succ \mathcal{P}_{N}=\{\{i\} \mid i \in V\} \tag{2.16c}
\end{equation*}
$$

The sequence of partitions (together with the corresponding critical values) is referred to as the PSP of $g$. The PSP of the entropy function (2.13) characterizes the info-clustering solution as follows:

Proposition 2.2 ([1, Corollary 2]) For a finite set $V$ with size $|V|>1$ and a random vector $Z_{V}$,

$$
\begin{gather*}
\mathcal{C}_{\gamma}\left(\mathrm{Z}_{V}\right)=\left[\min \left\{\mathcal{P} \in \Pi(V) \mid h_{\gamma}[\mathcal{P}]=\hat{h}_{\gamma}(V)\right\}\right]  \tag{2.17}\\
\backslash\{\{i\} \mid i \in V\}
\end{gather*}
$$

namely, the non-singleton elements of the finest optimal partition to the Dilworth truncation (2.15a).

## III. InFORMATION FLOW INTERPRETATION

For PIN, the MMI can be interpreted as the maximum broadcast throughput of a network [10, 11], and hence infoclustering reduces to clustering by network information flow. When applied to clustering social network, it can identify communities naturally based on the amount of information flow.

More precisely, treating each edge as an undirected communication link with capacity $c$, at most a total of 1 bit can be communicated between node 1 and 2 , and between node 2 and 3 ; and at most a total of 5 bits can be communicated between node 1 and 3 . It can be seen that, for every pair of distinct nodes $v, w \in V$, the broadcast throughput between $v$ and $w$ is given by the MMI in (2.7a). Fig. 1b illustrates how 2 bits of information can be broadcast in the entire network, achieving the MMI of the entire set of random variables in (2.7b). With the interpretation of the MMI as information flow, a cluster at threshold $\gamma$ is therefore a maximal subnetwork with broadcast throughput larger than $\gamma$. For instance, the cluster $\{1,3\} \in \mathcal{C}_{2}\left(\mathrm{Z}_{\{1,2,3\}}\right)$ is the only subset of nodes on which the induced subnetwork has a throughput exceeding 2 .

Specializing to the (hyper-)graphical model, it was shown in [1, Proposition 8] that the clustering solutions can be obtained directly as the non-singleton subsets from the PSP of the incut function. More precisely, from the weighted graph $G$ with vertex set $V$ and capacity function $c$, define for every pair $(v, w)$ of vertices $v, w \in V$

$$
c(v, w):= \begin{cases}c(\{v, w\}), & v<w  \tag{3.1}\\ 0, & \text { otherwise }\end{cases}
$$

This defines the capacity function of a weighted digraph $D$ with vertex set $V$ and edge set $E$ which can be defined as the set of $\operatorname{arcs}(v, w)$ with positive capacity $c(v, w)>0$. For example, Fig. 2a is the weighted digraph obtained by orienting the weighted graph in Fig. 1a according to (3.1), i.e., by directing an edge from the incident node with a smaller label to the other incident node with a larger one.

For convenience, we also write for arbitrary subsets $B_{1}, B_{2} \subseteq V$

$$
\begin{align*}
c\left(B_{1}, B_{2}\right) & :=\sum_{v \in B_{1}} c\left(v, B_{2}\right) & \text { where }  \tag{3.2a}\\
c\left(v, B_{2}\right) & :=\sum_{w \in B_{2}} c(v, w) & \text { for } v \in V, \text { and }  \tag{3.2b}\\
c\left(B_{1}, w\right) & :=\sum_{w \in B_{1}} c(v, w) & \text { for } w \in V \tag{3.2c}
\end{align*}
$$

The incut function of the weighted digraph is defined as

$$
\begin{equation*}
g(B):=c(V \backslash B, B) \tag{3.3}
\end{equation*}
$$

The incut function for the digraph in Fig. 2a is shown in Fig. 2b and calculated below:

$$
\begin{align*}
g(\{2\}) & =c(\{1,3\},\{2\})=c(1,2)+c(3,2)=1 \\
g(\{3\}) & =c(\{1,2\},\{3\})=c(1,3)+c(2,3)=6 \\
g(\{2,3\}) & =c(\{1\},\{2,3\})=c(1,2)+c(1,3)=6  \tag{3.4}\\
g(\{1,3\}) & =c(\{2\},\{1,3\})=c(2,1)+c(2,3)=1 \\
g(\{1\}) & =g(\{1,2\})=g(\{1,2,3\})=0 .
\end{align*}
$$

To compute the PSP of $g$, we first evaluate (2.15b) for different partitions as follows:

$$
\begin{align*}
g_{\gamma}[\{\{1,2,3\}\}] & =g_{\gamma}(\{1,2,3\}) \\
& =g([3])-\gamma \\
& =-\gamma \\
g_{\gamma}[\{1,3\},\{2\}] & =g_{\gamma}(\{1,3\})+g_{\gamma}(\{2\})  \tag{3.5a}\\
& =g(\{1,3\})+g(\{2\})-2 \gamma \\
& =2-2 \gamma \\
g_{\gamma}[\{1\},\{2\},\{3\}] & =g_{\gamma}(\{1\})+g_{\gamma}(\{2\})+g_{\gamma}(\{3\}) \\
& =7-3 \gamma
\end{align*}
$$

Similarly,

$$
\begin{align*}
& g_{\gamma}[\{1,2\},\{3\}]=6-2 \gamma>g_{\gamma}[\{1,3\},\{2\}]  \tag{3.5b}\\
& g_{\gamma}[\{1\},\{2,3\}]=6-2 \gamma>g_{\gamma}[\{1,3\},\{2\}]
\end{align*}
$$

Fig. 2c plots the Dilworth truncation $\hat{g}_{\gamma}(V)$ in (2.15) against $\gamma$ as the minimum of $g_{\gamma}[\mathcal{P}]$ over all partitions $\mathcal{P} \in \Pi(V)$. It can be seen that for a given $\mathcal{P}, g_{\gamma}[\mathcal{P}]$ is linear with integer slope $-|\mathcal{P}| \in\{-|V|, \ldots, 1\}$ and so $\hat{g}_{\gamma}(V)$ is piecewise linear consisting, in this example, of $|V|=3$ line segments (highlighted in blue) and $|V|-1=2$ break points (highlighted in red). (In general, the number of line segments is at most $|V|$.) The finest optimal partition for each value of $\gamma$ is

$$
\mathcal{P}^{*}(\gamma)= \begin{cases}\underbrace{\{\{1,2,3\}\}}_{\mathcal{P}_{0}}, & \gamma \in(-\infty, 2]  \tag{3.6}\\ \underbrace{\{\{1,3\},\{2\}\}}_{\mathcal{P}_{1}}, & \gamma \in[\underbrace{2}_{\gamma_{1}}, 5) \\ \underbrace{\{\{1\},\{2\},\{3\}\}}_{\mathcal{P}_{2}}, & \gamma \in[\underbrace{5}_{\gamma_{2}}, \infty)\end{cases}
$$

with the PSP and the corresponding critical values annotated above and in the Fig. 2c.

## IV. CLUSTERING USING PARAMETRIC MAX-FLOW

By [1, Proposition 9], the PSP of the incut function $B \mapsto c(V \backslash B, B)$ of the digraph $D$ coincides with the PSP of the cut function (divided by 2) of the corresponding undirected graph $G$, which was shown in [15] to be computable by running a parametric max-flow algorithm $O(|V|)$ times. The parametric max-flow algorithm was introduced by [16], which runs in $O\left(|V|^{2} \sqrt{|\mathcal{E}|}\right)$ times using the well-known


Fig. 2: Computing the clusters of the PIN model (2.1a) as the non-singleton elements of the PSP (2.16).
push-relable/preflow algorithm [20,21] implemented with the highest-level selection rule [22]. Hence, the info-clustering algorithm solution for the PIN model can be obtained in $O\left(|V|^{3} \sqrt{|\mathcal{E}|}\right)$ time.

In this section, we will adapt and improve the algorithm in [15] to compute the desired PSP for the info-clustering solution. The algorithm will be illustrated using the same example as in the last section, which is chosen to be the same example as in [15] for ease of comparison.

We first give a procedure in Algorithm 1 for computing the minimum minimizer $\mathcal{P}^{*}(\gamma)$ to (2.15) for all $\gamma \in \mathbb{R}$ and any submodular function $g$, assuming a parametric submodular function minimizer. This procedure can be specialized further to the PIN model where $g$ is chosen to be the incut function (3.3), so that the parametric max-flow algorithm can be applied instead.

Consider the example with $g$ defined in (3.4) and illustrated in Fig. 2b, and with $g_{\gamma}$ defined in (2.15c). When $j=2$, Line 1 initializes $x_{\gamma, 1}$ as

$$
\begin{equation*}
x_{\gamma, 1}=g(\{1\})-\gamma=-\gamma \tag{4.2}
\end{equation*}
$$

Then, (4.1) becomes

$$
\begin{aligned}
\min \left\{g_{\gamma}(\{1,2\})-x_{\gamma, 1}, g_{\gamma}(\{2\})\right\} & =\min \{0,1-\gamma\} \\
& =\left\{\begin{array}{ll}
1-\gamma, & \gamma<1 \\
0 & \gamma \geq 1,
\end{array} \quad \mathcal{P}^{*}(\gamma)\right. \text { to }
\end{aligned}
$$

which is a piecewise linear function plotted in Fig. 3a. The minimum minimizer is therefore given by

$$
B^{*}(\gamma)= \begin{cases}\underbrace{\{1,2\}}_{B_{0}}, & \gamma<1  \tag{4.3}\\ \underbrace{\{2\}}_{B_{1}}, & \gamma \geq \underbrace{1}_{\gamma_{1}^{\prime}}\end{cases}
$$

```
Algorithm 1: Computing the PSP as a parametric SFM.
    Input: Submodular function \(g: 2^{V} \mapsto \mathbb{R}\) defined on a
            finite ground set \(V\).
    Output: The function \(\mathcal{P}^{*}(\gamma)\) of \(\gamma \in \mathbb{R}\) defined in (2.16a).
    \(\mathcal{P}^{*} \leftarrow\{\{1\}\}, B^{*}(\gamma) \leftarrow\{1\}, x_{\gamma, 1} \leftarrow g_{\gamma}(\{1\}), \mu_{i} \leftarrow-\infty\)
    for all \(i \in V\);
    for \(j=2\) to \(|V|\) do
        set \(B^{*}(\gamma)\) as the (inclusion-wise) minimum
        minimizer to
                        \(\min _{B \subseteq[j]: j \in B} g_{\gamma}(B)-x_{\gamma}(B \backslash\{j\})\),
        where \(x_{\gamma}(C):=\sum_{i \in C} x_{\gamma, i}\) for convenience;
        remove every \(C\) that intersects \(B^{*}(\gamma)\) from \(\mathcal{P}^{*}(\gamma)\);
        add \(B^{*}(\gamma)\) to \(\mathcal{P}^{*}(\gamma)\);
        if \(j<|V|\) then
            for \(i=1\) to \(j\) do
                \(\mu_{i} \leftarrow \max \left\{\mu_{i}, \min \left\{\gamma \mid i \notin B^{*}(\gamma)\right\}\right\} ;\)
                \(x_{\gamma, i} \leftarrow g_{\max \left\{\gamma, \mu_{i}\right\}}(\{i\}) ;\)
            end
        end
    end
```

$$
\mathcal{P}^{*}(\gamma)= \begin{cases}\{\underbrace{\{1,2\}}_{B_{0}}\}, & \gamma<\underbrace{1}_{\gamma_{1}^{\prime}}  \tag{4.4}\\ \{\{1\}, \underbrace{\{2\}}_{B_{1}}\}, & \gamma \geq \underbrace{1}_{\gamma_{2}^{\prime}}\end{cases}
$$

With $\mathcal{P}^{*}(\gamma)$ initialized in Line 1 to $\{\{1\}\}$, Line $4-5$ update Next, with $\mu_{1}$ and $\mu_{2}$ initialized to $-\infty$ in Line 1 , the

$$
\min _{B \subseteq\{1,2\}: 2 \in B} g_{\gamma}(B)-x_{\gamma}(B \backslash\{2\})
$$

(a) For the loop with $j=2$.

$$
\min _{B \subseteq\{1,2,3\}: 3 \in B} g_{\gamma}(B)-x_{\gamma}(B \backslash\{3\})
$$


(b) For the loop with $j=3$.

Fig. 3: Illustration of the computation of PSP in Algorithm 1.
subsequent steps following Line 5 give

$$
\begin{align*}
\mu_{1} & =\max \left\{-\infty, \gamma_{1}^{\prime}\right\}=1  \tag{4.5a}\\
x_{\gamma, 1} & =g_{\max \{\gamma, 1\}}(\{1\})= \begin{cases}-1, & \gamma<1 \\
-\gamma, & \gamma \geq 1 .\end{cases}  \tag{4.5b}\\
\mu_{2} & =\max \{-\infty,-\infty\}=-\infty  \tag{4.5c}\\
x_{\gamma, 2} & =g_{\max \{\gamma,-\infty\}}(\{2\})=1-\gamma . \tag{4.5d}
\end{align*}
$$

Similarly, when $j=3$, the function of the minimum in (4.1) is plotted in Fig. 3b. It follows that the minimum minimizer is

$$
B^{*}(\gamma)= \begin{cases}\underbrace{\{1,2,3\}}_{B_{0}}, & \gamma \in(-\infty, 2]  \tag{4.6}\\ \underbrace{\{1,3\}}_{B_{1}}, & \gamma \in[\underbrace{2}_{\gamma_{1}^{\prime}}, 5) \\ \underbrace{\{3\}}_{B_{2}}, & \gamma \in[\underbrace{5}_{\gamma_{2}^{\prime}}, \infty)\end{cases}
$$

With $\mathcal{P}^{*}(\gamma)$ given by (4.4) and illustrated in Fig. 3a, Lines 4-

5 update $\mathcal{P}^{*}(\gamma)$ to the desired solution (3.6) characterized by the PSP.

The procedure is said to be parametric since the solution $\mathcal{P}^{*}$ is computed for all possible values of $\gamma \in \mathbb{R}$ rather than a particular value. To realize such a procedure, the characterization (2.16a) of $\mathcal{P}^{*}(\gamma)$ through the PSP in (2.16c) and the corresponding critical values of $\gamma$ in (2.16b) is computed instead.

The minimization in (4.1) is a submodular function minimization (SFM) (over a lattice family). In contrast with [15], we perform (4.1) on a growing set $[j]=\{1, \ldots, j\}$ instead of the entire set $V$ in every loop. The idea follows from the algorithm for computing Dilworth truncation such as the one given in [19]. In contrast with [19], however, we follow [15] to consider the update rule in Lines 8-9, which guarantees $x_{\gamma, i}$ to be piecewise linear with at most one break point, possibly at $\gamma=\mu_{i}$ if $\mu_{i}$ is finite. When $i=j$, the step in Line 8 gives $\mu_{j}=-\infty$ as $B^{*}(\gamma)$ always contain $j$ (see (4.1)), and so $x_{\gamma, j}=g_{\gamma}(\{j\})$, which is linear without any breakpoint. As pointed out in [15], the update rule is particularly useful for the the parametric procedure since the complexity often grows with the number of break points.

Specializing to PIN models where $g$ is the incut function defined in (3.3), the parametric SFM in (4.1) can be solved as a parametric min-cut problem as shown in Algorithm 2.

```
Algorithm 2: Computing the parametric SFM in (4.1) as
a parametric min-cut.
    Input: A weighted digraph \(D\) on vertex set \(V\) with
            capacity function \(c: V^{2} \rightarrow \mathbb{R}\) satisfying (3.1).
    Output: The minimum minimizer to (4.1) with \(g\) defined
                as the incut function in (3.3).
    1 define a weighted digraph \(D_{j}(\gamma)\) with vertex set
    \(U \leftarrow\{s, 1, \ldots, j\}\) (where \(s\) is a new node outside \([j]\) )
    and capacity function \(c_{\gamma}: U^{2} \rightarrow \mathbb{R}\) initialized to \(\mathbf{0}\);
    for \(v=1\) to \(j-1\) do
        \(c_{\gamma}(s, v) \leftarrow \max \left\{0,-x_{\gamma, v}\right\} ;\)
        \(c_{\gamma}(v, j) \leftarrow \max \left\{0, x_{\gamma, v}\right\}+c(v, j)\);
        \(c_{\gamma}(v, w) \leftarrow c(v, w)\) for all \(w \in[j-1] \backslash[v] ;\)
    end
    7 compute the minimum minimizer \(B^{*}(\gamma)\) to
\[
\begin{equation*}
\min _{T \subseteq U \backslash\{s\}: j \in T} c_{\gamma}(U \backslash T, T) \tag{4.7}
\end{equation*}
\]
which is the minimum \(s-j\) cut value of \(D_{j}(\gamma)\).
```

For the current example, the digraph $D_{j}(\gamma)$ with $j=2$ is shown in Fig. 4. The vertex set is $U=\{s, 1,2\}$. The for-loop sets $v=1$ and initializes the capacity $c(s, 1)=\left|-x_{\gamma, 1}\right|^{+}$and $c(1,2)=\left|x_{\gamma, 1}\right|^{+}+1$, where

$$
\begin{equation*}
|x|^{+}:=\max \{0, x\} \quad \text { for } x \in \mathbb{R} \tag{4.8}
\end{equation*}
$$

Evaluating the capacities with the initial value of $x_{\gamma, 1}$ in (4.2) gives the non-decreasing and non-increasing piecewise linear functions $c(s, 1)$ and $c(1,2)$ respectively shown in the figure.


Fig. 4: Illustration of the parametric min-cut problem in Algorithm 2 for the for-loop with $j=2$.


Fig. 5: Illustration of the parametric min-cut problem in Algorithm 2 for the for-loop with $j=3$.

Similarly, the digraph $D_{j}(\gamma)$ with $j=3$ is shown in Fig. 5, with the vertex set now being $U=\{s, 1,2,3\}$ instead, and $x_{\gamma, 1}$ and $x_{\gamma, 2}$ updated to the functions in (4.5).

The weighted digraph $D_{j}(\gamma)$ can be viewed as a result of processing the weighted digraph $D$ as follow:

1. Augment the digraph $D$ with two new nodes $s$ and $t$ (both outside $V$ ) such that
a. the capacity from $j$ to $t$ is set to infinity;
b. for $v$ from 1 to $j-1$, if $x_{\gamma, v}<0$, add an arc with capacity $-x_{\gamma, v}$ from $s$ to $v$, else add an arc with capacity $x_{\gamma, v}$ from $v$ to $t$.
2. Remove all outgoing arcs from node $j$ and contract node $t$ to node $j$.
3. Remove all incoming arcs of nodes $j+1, \ldots,|V|$ and contract the nodes to node $s$. (4.7).
This procedure is illustrated for $D_{2}(\gamma)$ and $D_{3}(\gamma)$ in Fig. 4 b and Fig. 5b respectively, where the red dotted arcs are removed, and the nodes circled together by blue lines are contracted.

Step 2 implies the formula in Line 4 directly. Step 3 gives

$$
c_{\gamma}(s, v)=\max \left\{0,-x_{\gamma, v}\right\}+\sum_{u \in V \backslash[j]} c(u, v)
$$

but this reduces to the formula in Line 3 because the last summation is zero by the assumption (3.1) that all the arcs point from a node with a smaller label to a node with a larger label. For the same reason, in Line 5, we do not need to set
$c_{\gamma}(v, w)=c(v, w)$ for $w \in[v-1]$ as they are zero by default. In contrast with [15], the additional node contraction and edge removal in Steps 2 and 3 above reduce the number of vertices from $|V|$ to $|U|=j$ and therefore the complexity in solving (4.7).

For each $j \in V$, (4.7) can be solved by the parametric maxflow algorithm in [16] in $O\left(j^{3} \sqrt{|E|}\right)$ time by invoking $O(j)$ times the preflow algorithm [20,21] implemented with highest level selection rule [22], which in turn runs in $O\left(j^{2} \sqrt{|E|}\right)$ times. The procedure is described in Algorithm 3, which returns the characterization of the solution $B^{*}(\gamma)$ to (4.7) as

$$
\begin{equation*}
B^{*}(\gamma)=B_{\ell} \quad \text { for } \gamma \in\left[\gamma_{\ell}^{\prime}, \gamma_{\ell+1}^{\prime}\right), \ell \in\left\{0, \ldots, N^{\prime}\right\} \tag{4.9}
\end{equation*}
$$

for some integer $N^{\prime}>0$, where $\gamma_{0}^{\prime}:=-\infty, \gamma_{N^{\prime}+1}^{\prime}:=+\infty$ and $B_{0}:=[j]$. Note also that $B^{*}(\gamma)=\{j\}$ for sufficiently large $\gamma$ and so $B_{N^{\prime}+1}=\{j\}$.

To solve (4.7) for any fixed $j$, we assume the following subroutine

$$
\begin{equation*}
\left[f^{*}, T^{*}\right]=\operatorname{MaxFlow}(\bar{c}, \bar{U}, \bar{s}, \bar{t}, \bar{f}) \tag{4.11}
\end{equation*}
$$

which takes as arguments the capacity function $\bar{c}$ (fixed and not parametric), the vertex set $\bar{U}$ on which $\bar{c}$ is defined, the source node $\bar{s} \in \bar{U}$, the sink node $\bar{t} \in \bar{U}$ and a valid preflow $\bar{f}$ associated with the weighted digraph $D_{j}(\gamma)$ defined by the previous arguments. It returns the maximum $\bar{s}-\bar{t}$ flow $f^{*}$ and

```
Algorithm 3: Solving the parametric min-cut problem in
(4.7) using the parametric max-flow algorithm [16].
    Input: The weighted digraph \(D_{j}(\gamma)\) on vertex set \(U\) with
        capacity function \(c_{\gamma}\) created in Algorithm 2.
    Output: A list L containing \(\left(\gamma_{\ell}^{\prime}, B_{\ell}\right)\) for \(\ell \in\left[N^{\prime}\right]\) that
            characterizes (4.7) as in (4.9).
    create empty lists L and PL ;
    \(\gamma^{+} \leftarrow \max _{v \in[j-1]}\{c([v-1], v)+c(v,[j] \backslash[v])\} ;\)
    \(\gamma^{-} \leftarrow \min \left\{\max \left\{\mu_{v}, c(v, j)\right\} \mid v \in[j-1]\right\}\);
    if \(\gamma^{-}=\gamma^{+}\)then
        \(L \leftarrow\left(\gamma^{-},\{j\}\right)\) and return \(L ;\)
    end
    set \(f\) as the zero flow \(\mathbf{0}\) from \(s\) to \(j\);
    \(\left[f^{*}, T^{*}\right] \leftarrow \operatorname{MaxFlow}\left(c_{\gamma^{-}}, U, s, j, f\right)\);
    add \(\left(\gamma^{-}, \gamma^{+}, f^{*},\{s\},\{j\}\right)\) to PL ;
    while PL is not empty do
        withdraw any element \(\left(\gamma^{-}, \gamma^{+}, f, S, T\right)\) from L ;
        compute \(\bar{\gamma} \in\left[\gamma^{-}, \gamma^{+}\right]\)as the solution to
\[
\begin{equation*}
c_{\gamma}(S, U \backslash S)=c_{\gamma}(U \backslash T, T) \tag{4.10}
\end{equation*}
\]
define a weighted digraph \(\bar{D}_{j}\) with vertex set \(\bar{U} \leftarrow([j] \backslash(S \cup T)) \cup\{s, j\}\) and capacity function \(\bar{c}: \bar{U}^{2} \rightarrow \mathbb{R}\) initialized to \(\mathbf{0}\);
for \(v\) in \(\bar{U} \backslash\{s, j\}\) do
\(\bar{c}(s, v) \leftarrow \sum_{u \in S} c_{\bar{\gamma}}(u, v) ;\)
\(\bar{c}(v, j) \leftarrow \sum_{w \in T} c_{\bar{\gamma}}(v, w) ;\)
\(\bar{c}(v, w) \leftarrow c(v, w)\) for all \(w \in \bar{U} \backslash\{s, j, v\} ;\)
end
for \(v\) in \(\bar{U} \backslash\{j\}\) do
\(\bar{f}(v, j) \leftarrow \sum_{w \in T} \min \{f(v, w), \bar{c}(v, j)\}\) and
\(\bar{f}(j, v) \leftarrow-\bar{f}(v, j)\) to ensure anti-symmetry;
\(\bar{f}(v, w) \leftarrow f(v, w)\) for all \(w \in \bar{U} \backslash\{j, v\} ;\) end
\(\left[f^{*}, T^{*}\right] \leftarrow \operatorname{MaxFlow}(\bar{c}, \bar{U}, s, j, f)\); if \(T^{*}=\{j\}\) then
add \(\left(\gamma^{-}, T\right)\) to \(L\);
end
add \(\left(\gamma^{-}, \bar{\gamma}, f, S, T \cup T^{*}\right)\) and
\(\left(\bar{\gamma}, \gamma^{+}, f^{*}, S \cup\left(U \backslash T^{*}\right), T\right)\) to PL ;
end
```

the inclusion-wise minimum set $T^{*}$ that solves

$$
\begin{equation*}
\min _{T \subseteq \bar{U} \backslash\{\bar{s}\}: \bar{t} \in T} c(\bar{U} \backslash T, T) \tag{4.12}
\end{equation*}
$$

and is referred to as the minimum $\bar{s}-\bar{t}$ cut.
Roughly speaking, $\gamma^{+}$in Line 2 is the value of $\gamma$ at which (4.7) (i.e., $B^{*}(\gamma)$ ) is constant for $\gamma \geq \gamma^{+}$. Similarly, $\gamma^{-}$in Line 3 is the value of $\gamma$ at which (4.7) is constant for $\gamma \leq \gamma^{-}$. When $\gamma^{-}=\gamma^{+}$, there is only one critical value $\gamma_{1}^{\prime}$ of $\gamma$ where $B^{*}(\gamma)$ changes from $B_{0}=[j]$ to $B_{1}=\{j\}$.

To illustrate the above, consider $j=2$, i.e., with $D_{2}(\gamma)$ shown in Fig. 4a as the input to Algorithm 3. Then, Lines 2-

3 give

$$
\begin{aligned}
& \gamma^{+}=c(1,2)=1 \\
& \gamma^{-}=\max \left\{\mu_{1}, c(1,2)\right\}=1
\end{aligned}
$$

where the last equality is because $\mu_{1}$ is initialized to be $-\infty$ by Line 1 of Algorithm 1. Since $\gamma^{-}=\gamma^{+}$in this case, the algorithm returns at Line 5 the list

$$
\mathrm{L}=[(\underbrace{1}_{\gamma_{1}^{\prime}}, \underbrace{\{1\}}_{B_{1}})] .
$$

This gives the desired $B^{*}(\gamma)$ in (4.3). Fig. 4c shows the digraph $D_{2}(\gamma)$ at $\gamma=1$. It can be seen that both $\{1,2\}$ and $\{2\}$ are solutions to the minimization in (4.7).
If $\gamma^{-} \neq \gamma^{+}$(or more specifically $\gamma^{-}<\gamma^{+}$), then the interval $\left(\gamma^{-}, \gamma^{+}\right)$must contain other critical values of $\gamma$ where $B^{*}(\gamma)$ changes. The critical values are then computed iteratively by the preflow algorithm MaxFlow (4.11) (Lines 8 and 23) applied on the digraph $\bar{D}_{j}$ with capacities derived from those of $D_{j}(\gamma)$ (Lines 15-17), and with $\gamma$ evaluated at some value $\bar{\gamma} \in\left(\gamma^{-}, \gamma^{+}\right)$satisfying (4.10). This either resolves $B^{*}(\gamma)$ for the entire interval (in which case the solution is updated in Line 25) or reduces the problem to two smaller subproblems for later processing (i.e., with the original interval $\left(\gamma^{-}, \gamma^{+}\right)$replaced by the two smaller intervals $\left(\gamma^{-}, \bar{\gamma}\right)$ and ( $\bar{\gamma}, \gamma^{+}$) in Line 27).

To illustrate the procedure above, consider $j=3$, i.e., with $D_{3}(\gamma)$ shown in Fig. 5a as the input to Algorithm 3. Then, Lines 2-3 give

$$
\begin{aligned}
\gamma^{+} & =\max \{c(1,2)+c(1,3), c(1,2)+c(2,3)\} \\
& =\max \{1+5,1+1\}=6 \\
\gamma^{-} & =\min \left\{\max \left\{\mu_{1}, c(1,3)\right\}, \max \left\{\mu_{2}, c(2,3)\right\}\right\} \\
& =\min \{\max \{1,5\}, \max \{-\infty, 1\}\}=1
\end{aligned}
$$

where we used the values $\mu_{1}=1$ and $\mu_{2}=-\infty$ by (4.5). Since $\gamma^{-}<\gamma^{+}$in this case, Line 5 is skipped. Line 8 invokes the preflow algorithm for the graph $D_{3}\left(\gamma^{-}\right)=D_{3}(1)$ shown in Fig. 5c. The min-cut is $T^{*}=[3]=U \backslash\{s\}$ (where $U=$ $\{s, 1,2,3\}$ is the vertex set of $D_{3}$ ) by the construction of $\gamma^{-}$. The max-flow is

$$
\begin{align*}
& f^{*}(s, 1)=f^{*}(1,3)=1 \\
& f^{*}(1, s)=f^{*}(3,1)=-1 \tag{4.13}
\end{align*}
$$

and 0 otherwise. Note that the second line of equations ensures the anti-symmetry property of a flow function, i.e.,

$$
\begin{equation*}
f^{*}(w, v)=-f^{*}(v, w) \tag{4.14}
\end{equation*}
$$

for all pairs of distinct nodes $v$ and $w$. The flow along each arc is indicated in Fig. 5c by the parentheses next to the corresponding capacity of the arc.

The tuple $\left(\gamma^{-}, \gamma^{+}, f^{*},\{s\},\{j\}\right)$ is then added to PL in Line 9 and then retrieved (and deleted from PL) subsequently inside the while-loop (Line 11). With $\gamma^{-}=1, \gamma^{+}=6$, $S=\{s\}$ and $T=\{j\}$ in Line 11, the l.h.s. of (4.10) is


Fig. 6: Illustration of the parametric max-flow algorithm in Algorithm 3.
given as (see Fig. 5a)

$$
\begin{aligned}
c_{\gamma}(S, U \backslash S) & =c_{\gamma}(s,\{1,2,3\}) \\
& =\left\{\begin{array}{ll}
1, & \gamma<1 \\
\gamma, & \gamma \geq 1
\end{array}+ \begin{cases}0, & \gamma<1 \\
\gamma-1, & \gamma \geq 1\end{cases} \right. \\
& = \begin{cases}1, & \gamma<1 \\
2 \gamma-1, & \gamma \geq 1\end{cases}
\end{aligned}
$$

and r.h.s. of (4.10) is given as

$$
\begin{aligned}
c_{\gamma}(U \backslash T, T) & =c_{\gamma}(\{s, 1,2\}, 3) \\
& =5+ \begin{cases}2-\gamma, & \gamma<1 \\
1, & \gamma \geq 1\end{cases} \\
& = \begin{cases}7-\gamma, & \gamma<1 \\
6, & \gamma \geq 1\end{cases}
\end{aligned}
$$

$\bar{\gamma}$ is computed as the solution to (4.10), namely $\bar{\gamma}=3.5$. In general, such a value must exist and is unique because $U \backslash S$ and $T$ are optimal solutions to (4.7) at $\gamma^{-}$and $\gamma^{+}$ respectively. The computation is in $O(j)$ time since both sides of the equations are piecewise linear with at most $O(j)$ break points.

The new weighted digraph $\bar{D}_{j}$ with the capacity function $\bar{c}$ assigned in the first for-loop (Lines 15-17) can be obtained from $D_{j}(\gamma)$ by

1) setting $\gamma=\bar{\gamma}$, contracting $S$ to the source node $s$,
2) contracting $T$ to the sink node $j$, and then
3) removing the incoming arcs to $s$ and outgoing arcs from $j$.
The second for-loop turns $f$ to a valid preflow $\bar{f}$ of $\bar{D}_{j}$.
Recall that for the current example, $\bar{\gamma}=3.5$ in the last execution of the algorithm. Fig. 6a shows two digraphs, where the top one is the digraph $D_{3}(\gamma)$ at $\gamma=\bar{\gamma}=3.5$ and the bottom one is the new weighted digraph $\bar{D}_{3}$. The sets $S, T$ and the flow $f$ indicated on the top digraph $D_{3}(3.5)$ satisfy (4.10), while $\bar{D}_{3}$ is annotated with the max-flow $f^{*}$ and min-cut $T^{*}$ computed by Line 23 . Note that, since $T^{*}=\{1,3\} \neq\{j\}$,

Line 25 will be skipped. Instead, Line 27 adds the following two tuples to the list PL, which becomes

$$
\begin{equation*}
\mathrm{PL}=\left[\left(1,3.5, f^{*},\{s\},\{1,3\}\right),(3.5,6, f,\{s, 2\},\{3\})\right] . \tag{4.15}
\end{equation*}
$$

Repeating the while-loop with the first element retrieved from PL, it can be shown that (4.10) is solved by the value $\bar{\gamma}=2$. Similar to Fig. 6a, Fig. 6b shows the digraph $D_{3}(2)$ at the top and $\bar{D}_{3}$ at the bottom. It can be verified that $S$ and $T$ satisfies (4.10) for the top graph and $T^{*}$ is the min-cut in the bottom graph. Since $T^{*}=\{3\}$ in this case, a new element $(2,\{1,3\})$ is added to $L$ in Line 25.

Finally, repeating the while-loop again with the last element retrived from PL (4.15), it can be shown that $\bar{\gamma}=5$. Fig. 6c again gives $D_{3}(5)$ and the min-cut $T^{*}=\{3\}$, in which case a new element $(5,\{3\})$ is added to $L$ again in Line 25 . Since PL is not empty, the algorithm terminates with

$$
\mathrm{L}=[(\underbrace{2}_{\gamma_{1}^{\prime}}, \underbrace{\{1,3\}}_{B_{1}}),(\underbrace{5}_{\gamma_{2}^{\prime}}, \underbrace{\{3\}}_{B_{2}})] .
$$

This gives the desired $B^{*}(\gamma)$ in (4.6) that yields the desired PSP in (3.6), and therefore the info-clustering solution in (2.11) for the PIN model (2.1a).

## V. Conclusion

We have adapted the parametric max-flow algorithm of computing the PSP to an info-clustering algorithm that clusters a graphical network based on the information flow over its edges. The overall running time is $O\left(|V|^{3} \sqrt{|E|}\right)$, where $|V|$ is the size of the network and $|E|$ is the number of edges or communication link. The algorithm simplifies the general info-clustering algorithm by a few orders of magnitude, and is applicable to systems, such as the social networks, where similarity can be measured by mutual information.

To implement the algorithm in a large-scale social network, the preflow algorithm may be made distributive and adaptive: Servers may be deployed in different parts of the network to measure and store the information exchange rates of different
pair of nodes. The push and relabel operations in the preflow algorithm can be done locally by the servers first and then communicated to other servers when necessary. The preflow of the network may be stored in conjunction with the clustering solution, so that the clusters can be updated incrementally over time based on the changes of information exchange rates. The allocation of the servers and other resources may also be adapted to the clustering solution. For instance, as intra-cluster communication is more frequent than inter-cluster communication, the nodes in a cluster with larger mutual information may be assigned to the same server so that changes in the network can be updated more frequently without much communication overhead among the servers.

## REFERENCES

[1] C. Chan, A. Al-Bashabsheh, Q. Zhou, T. Kaced, and T. Liu, "Infoclustering: A mathematical theory for data clustering," IEEE Transactions on Molecular, Biological and Multi-Scale Communications, vol. 2, no. 1, pp. 64-91, June 2016.
[2] C. Chan, A. Al-Bashabsheh, J. Ebrahimi, T. Kaced, and T. Liu, "Multivariate mutual information inspired by secret-key agreement," Proceedings of the IEEE, vol. 103, no. 10, pp. 1883-1913, Oct 2015.
[3] H. Narayanan, "The principal lattice of partitions of a submodular function," Linear Algebra and its Applications, vol. 144, no. 0, pp. 179 - 216, 1990.
[4] Y. Wu and P. Yang, "Minimax rates of entropy estimation on large alphabets via best polynomial approximation," IEEE Trans. Inf. Theory, vol. 62, no. 6, pp. 3702-3720, 2016.
[5] K. Nagano, Y. Kawahara, and S. Iwata, "Minimum average cost clustering." in NIPS, J. D. Lafferty, C. K. I. Williams, J. Shawe-Taylor, R. S. Zemel, and A. Culotta, Eds. Curran Associates, Inc., 2010, pp. 1759-1767.
[6] C. Chan and T. Liu, "Clustering of random variables by multivariate mutual information on Chow-Liu tree approximations," in Fifty-Third Annual Allerton Conference on Communication, Control, and Computing, Allerton Retreat Center, Monticello, Illinois, Sep. 2015.
[7] A. J. Butte and I. S. Kohane, "Mutual information relevance networks: functional genomic clustering using pairwise entropy measurements," in Pac Symp Biocomput, vol. 5, 2000, pp. 418-429.
[8] S. Nitinawarat, C. Ye, A. Barg, P. Narayan, and A. Reznik, "Secret key generation for a pairwise independent network model," IEEE Trans. Inf. Theory, vol. 56, no. 12, pp. 6482-6489, Dec 2010.
[9] S. Nitinawarat and P. Narayan, "Perfect omniscience, perfect secrecy, and steiner tree packing," IEEE Trans. Inf. Theory, vol. 56, no. 12, pp. 6490-6500, Dec. 2010.
[10] C. Chan, "Matroidal undirected network," in Information Theory Proceedings (ISIT), 2012 IEEE International Symposium on, July 2012, pp. 1498-1502.
[11] , "The hidden flow of information," in Proc. IEEE Int. Symp. on Inf. Theory, St. Petersburg, Russia, Jul. 2011.
[12] S. Fujishige and S. Iwata, "Minimizing a submodular function arising from a concave function," Discrete Applied Mathematics, vol. 92, no. 2, pp. 211-215, 1999.
[13] M. Queyranne, "Minimizing symmetric submodular functions," Mathematical Programming, vol. 82, no. 1-2, pp. 3-12, 1998.
[14] S. Jegelka, H. Lin, and J. A. Bilmes, "On fast approximate submodular minimization," in Advances in Neural Information Processing Systems, 2011, pp. 460-468.
[15] V. Kolmogorov, "A faster algorithm for computing the principal sequence of partitions of a graph," Algorithmica, vol. 56, no. 4, pp. 394412, 2010.
[16] G. Gallo, M. D. Grigoriadis, and R. E. Tarjan, "A fast parametric maximum flow algorithm and applications," SIAM Journal on Computing, vol. 18, no. 1, pp. 30-55, 1989.
[17] C. Chan and L. Zheng, "Mutual dependence for secret key agreement," in Proceedings of 44th Annual Conference on Information Sciences and Systems, 2010.
[18] S. Fujishige, "Polymatroidal dependence structure of a set of random variables," Information and Control, vol. 39, no. 1, pp. 55-72, 1978.
[19] A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency. Springer, 2002.
[20] A. V. Goldberg, "Efficient graph algorithms for sequential and parallel computers," Ph.D. dissertation, Massachusetts Institute of Technology, Dept. of Electrical Engineering and Computer Science, 1987.
[21] A. V. Goldberg and R. E. Tarjan, "A new approach to the maximumflow problem," Journal of the ACM (JACM), vol. 35, no. 4, pp. 921-940, 1988.
[22] B. V. Cherkassky and A. V. Goldberg, "On implementing the pushrelabel method for the maximum flow problem," Algorithmica, vol. 19, no. 4, pp. 390-410, 1997.

