A Novel Index Coding Scheme and its Application to Coded Caching

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Abstract—This paper proposes a novel achievable scheme for the index problem and applies it to the caching problem. Index coding and caching are noiseless broadcast channel problems where receivers have message side information. In the index coding problem the side information sets are fixed, while in the caching problem the side information sets correspond the cache contents, which are under the control of the system designer. The proposed index coding scheme, based on distributed source coding and non-unique decoding, is shown to strictly enlarge the rate region achievable by composite coding. The novel index coding scheme applied to the caching problem is then shown to match an outer bound (previously proposed by the authors and also based on known results for the index coding problem) under the assumption of uncoded cache placement/prefetching.

I. INTRODUCTION

The index coding problem, originally proposed by Birk and Kol in [1], is a distributed source coding problem with side information that has received considerable attention over the past decade. In a general multicast index coding problem, a server/sender wishes to communicate N' independent messages to K' users through an error-free link. Each client/receiver knows a subset of the N' messages and demands a subset of the unknown messages. The server broadcasts packets such that each client can recover the desired messages. The objective is to determine the largest message rate region for a fixed assignment of side information sets. If each client demands a single district message, we have a so-called multiple unicast index coding problem.

For the general index coding problem, an outer bound based on the polymatroidal properties of the entropy function [2] was originally proposed in [3, Theorem 3.1] and later extended in [4, Theorem 1]. A looser version of [4, Theorem 1] but easier to evaluate for the multiple unicast index coding problem was given in [4, Corollary 1], which we shall refer to as acyclic outer bound in the following. Several inner bounds are known for the index coding problem. A scheme based on rank minimization of certain matrices was proposed in [5], and interference alignment based schemes were proposed in [6], [7]. Since a multiple unicast index coding problem can be represented as a directed graph, schemes leveraging graph proprieties such as clique-cover, partial clique-cover, local clique cover, partial local clique covering were proposed in [5], [8]–[10], respectively. Random coding schemes have also been

studied. The schemes proposed in [11]-[14] are based on the Heegard-Berger [15] idea of source compression with different receiver side information sets. By using Slepian-Wolf coding [16], the authors in [4] proposed a scheme known as composite coding, which is optimal for the multiple unicast index coding problem with up to five messages.

The index coding problem has connection to the coded caching problem as originally formulated by Maddah-Ali and Niensen in [17], [18], where a server with a library of N files is connected via a shared error-free link to K users. Each user has a local cache of size M < N files to store information. There are two phases in the caching problem. In the placement phase (during network peak-off traffic times) users store parts of the files within their cache without knowledge of later demands. When each user directly copies some bits of the files in his cache, the placement phase is said to be uncoded; otherwise it is *coded*. If central coordination (among users) during the placement phase is possible, the caching system is said to be *centralized*; otherwise it is *decentralized*. In the delivery phase (during network peak traffic times) each user demands a specific file and, based on the users' demands and cache contents, the server broadcasts packets so that each user can recover the demanded file. The objective is to design a two-phase scheme so that the number of transmitted packets in the delivery phase is minimized for the worst-case demands, referred to as worst-case load, or just load for simplicity.

The connection between caching and index coding is as follows [17]. After the users' demands are revealed in a caching scheme with uncoded cache placement, the delivery phase is equivalent to a general index coding problem. Even if the capacity region of the general index coding problem is not known, available inner and outer bounds can be used to bound the worst-case load in the caching problem. To the best of our knowledge, the first outer bound on the worstcase load under the constraint of uncoded cache placement for centralized caching systems was derived in [19], [20] and for decentralized caching systems in [21]. To this end, we used the acyclic index coding outer bound in [4, Corollary 1] and leveraged the intrinsic symmetries of the caching problem to derive an outer bound that not only outperforms cut-set-based bounds (which are valid for coded cache placement too) but shows the optimality of the Maddah-Ali and Niensen's original

schemes in [17], [18] for systems with more files than users and under the constraint of uncoded cache placement.

Our outer bound in [19], [20] has been recently shown to be tight for caching systems with more users than files as well in [22]. The key observation is that certain packets sent in the Maddah-Ali and Niensen's original scheme in [17] are linear combinations of other packets and thus need not be sent. In [22] matching inner and outer bounds for systems with uniform demands were given.

Contributions: This work is motivated by the observation that the coding inner bound is not optimal when applied to the caching problem with uncoded placement. We first propose an inner bound for the index coding problem based on Han's coding scheme [23], Slepian-Wolf coding [16], and non-unique decoding [24]. This inner bound is proved to strictly improve on composite coding by way of an example. We then apply the novel inner bound to the caching problem with uncoded cache placement and show that it matches our worst-case load outer bound in [19], [20], thus providing an alternate 'source coding with side information' proof to some results in [22]. Compared to the achievable scheme in [22], which is a clever analysis of the linear code originally proposed by Maddah-Ali and Niensen in [17], our inner bound has the following pleasing features: (i) it applies to the general index coding problem, (ii) it is not restricted to linear codes, and (iii) it can be easily extended to index coding problems over noisy broadcast channels.

Paper Outline: The rest of the paper is organized as follows. Section II presents the system models for the index coding and the caching problems, and formally connects them. Section III proves the main result of this paper. Section IV concludes the paper.

Notation: Calligraphic symbols denotes sets. $|\cdot|$ is the cardinality of a set. We denote $[1 : K] := \{1, 2, ..., K\}$ and $\mathcal{A} \setminus \mathcal{B} := \{x \in \mathcal{A} | x \notin \mathcal{B}\}$. \oplus represents the bit-wise XOR operation (zeros may need to be appended to make the vectors have the same length).

II. SYSTEM MODELS AND RELATED RESULTS

In this section, we start by describing the caching problem and the index coding problem, and we finish by discussing their relationship. By way of an example, we show the need to improve the composite coding inner bound for the index coding problem before inner bounds from index coding can be applied to the caching problem.

A. Caching Problem

In the coded caching problem a central server is equipped with N independent files of B bits. Files are denoted by F_1, \ldots, F_N . The server is connected to K users through an error-free broadcast link.

In the placement phase, user $i \in [1 : K]$ stores information about the N files in his cache of size MB bits without knowledge of users' demands. Here $M \in [0, N]$. The cache content for user $i \in [1 : K]$ is denoted by Z_i ; we let $\mathbf{Z} := (Z_1, \ldots, Z_K)$. Centralized systems allow for coordination among users in the placement phase, while decentralized systems do not. So in decentralized systems the caching functions are random and independent functions.

In the delivery phase, each user demands one file and the demand vector $\mathbf{d} := (d_1, \ldots, d_K)$ is revealed to everyone, where $F_{d_i}, d_i \in [1 : N]$, is the file demanded by user $i \in [1 : K]$. Given (\mathbf{Z}, \mathbf{d}) , the server broadcasts a message $X_{\mathbf{Z}, \mathbf{d}}$ of length $B R(\mathbf{d}, M)$ bits. It is required that user $i \in [1 : K]$ recovers his desired file from the broadcast message and his local cache content with arbitrary high probability as $B \to \infty$.

The objective is to minimize the worst-case load

$$R_t^*(M) := \min\max_{\mathbf{d}} R(\mathbf{d}, M), \tag{1}$$

where t = c if the placement phase is centralized, and t = d the placement phase is decentralized. Note that $R_t^*(M)$ represents the number of transmissions needed to deliver one file to each user.

We briefly revise the details of the scheme originally proposed by Maddah-Ali and Niensen in [17], [18] next.

1) Centralized Caching Systems (cMAN) [17]: Let the cache size be $M = t\frac{N}{K}$, for some positive integer $t \in [0:K]$, and R[t] be the corresponding worst-case load. The worst-case load R(M) for other values of M is obtained as the lower convex envelope of the set of points $(t\frac{N}{K}, R[t])$ for $t \in [0:K]$.

In the placement phase, each file is split into $\binom{K}{t}$ nonoverlapping sub-files of equal size. The sub-files of F_i are denoted by $F_{i,W}$ for $W \subseteq [1 : K]$ where |W| = t. User $k \in [1 : K]$ fills his cache as

$$Z_k = \left(F_{i,\mathcal{W}} : k \in \mathcal{W} \subseteq [1:K], |\mathcal{W}| = t, \ i \in [1:N]\right).$$
(2)

In the delivery phase, the server transmits

$$X_{\mathbf{Z},\mathbf{d}} = \left(\bigoplus_{s \in \mathcal{S}} F_{d_s,\mathcal{S} \setminus \{s\}} : \mathcal{S} \subseteq [1:K], |\mathcal{S}| = t+1 \right), \quad (3)$$

which requires broadcasting at a rate

$$R_{\text{cMAN}}[t] := \frac{\binom{K}{t+1}}{\binom{K}{t}}.$$
(4)

Let $\mathcal{N}(\mathbf{d})$ be set of distinct demanded files in the demand vector **d**. In [22] it was shown that among all the $\binom{K}{t+1}$ linear combinations in (3), $\binom{K-|\mathcal{N}(\mathbf{d})|}{t+1}$ of them can be obtained by linear combinations of the remaining ones and thus need not be transmitted. Hence, the worst-case load is attained for $|\mathcal{N}(\mathbf{d})| = \min(K, N)$, which requires a broadcast rate of [22]

$$R_{\text{c,uncoded placement}}[t] := \frac{\binom{K}{t+1} - \binom{K-\min(K,N)}{t+1}}{\binom{K}{t}}.$$
 (5)

The worst-case load in (5) coincides with the outer bound under the constraint of uncoded cache placement in [19], [20] and it is thus optimal. 2) Decentralized Caching Systems (dMAN) [18]: In decentralized systems, user coordination during the placement phase is not possible, so each user stores a subset of $\frac{M}{N}B$ bits of each file, chosen uniformly and independently at random. Given the cache content of all the users, the bits of the files can be grouped into sub-files $F_{i,W}$, where $F_{i,W}$ is the set of bits of file $i \in [1 : N]$ that are only known by the users in $W \subseteq [1 : K]$. By the Law of Large Numbers, the size of the sub-files converges in probability to

$$\frac{|F_{i,\mathcal{W}}|}{B} \xrightarrow{p} \left(\frac{M}{N}\right)^{|\mathcal{W}|} \left(1 - \frac{M}{N}\right)^{K-|\mathcal{W}|} \text{ when } B \to \infty.$$
 (6)

In the delivery phase, for each $t \in [0 : K - 1]$, all the $\binom{K}{t+1}$ sub-files $F_{i,\mathcal{W}}$ with $|\mathcal{W}| = t$ and $i \in [1 : N]$ are gathered together; since they all have approximately the same length that only depends on how many users have stored them in their cache (given by (6)), the server uses the cMAN scheme for $M = t\frac{N}{K}$ to deliver them. Thus, the worst-case load of the dMAN scheme is

$$R_{\text{dMAN}}(M) := \sum_{t \in [0:K-1]} {\binom{K}{t+1} \left(\frac{M}{N}\right)^t \left(1 - \frac{M}{N}\right)^{K-t}} \\ = \frac{1 - \frac{M}{N}}{\frac{M}{N}} \left[1 - \left(1 - \frac{M}{N}\right)^K\right].$$
(7)

The optimal load for decentralized caching systems with uncoded cache placement can be achieved following the dMAN original idea without the redundant transmissions in the underlying cMAN scheme, which leads to [22]

$$R_{d,\text{uncoded placement}}(M) := \frac{1 - \frac{M}{N}}{\frac{M}{N}} \left[1 - \left(1 - \frac{M}{N} \right)^{\min(K,N)} \right].$$
(8)

The worst-case load in (8) coincides with the outer bound under the constraint of uncoded cache placement in [21] and it is thus optimal.

Before we connect the caching problem with uncoded cache placement to the index coding problem, we need to introduce the index coding problem formally.

B. Index Coding Problem

In the index coding problem a central server with N'independent messages is connected to K' users. Each user $j \in [1 : K']$ demands a set of messages indexed by $\mathcal{D}_j \subseteq [1 : N']$ and knows a set of messages indexed by $\mathcal{A}_j \subseteq [1 : N']$. In order to avoid trivial problems, it is assumed that $\mathcal{D}_j \neq \emptyset$, $\mathcal{A}_j \neq [1 : N']$, and $\mathcal{D}_j \cap \mathcal{A}_j = \emptyset$. The server is connected to the users through a noiseless channel with alphabet \mathcal{X} . Without loss of generality we can take \mathcal{X} to be GF(2) [4]. A $(2^{nR_1}, \ldots, 2^{nR_{N'}}, n, \epsilon_n)$ -code for this index coding problem is defined as follows.

Each message M_i , for $i \in [1 : N']$, is uniformly distributed in $[1 : 2^{nR_i}]$ where n is the block-length, $R_i \ge 0$ is the transmission rate in bits per channel use. In order to satisfy users' demands, the server broadcasts $X^n = \operatorname{enc}(M_1, \ldots, M_{N'}) \in$ \mathcal{X}^n where enc is the encoding function. Each user $j \in [1:K']$ estimates the messages indexed by \mathcal{D}_j by the decoding function dec_j $(X^n, (M_i: i \in \mathcal{A}_j))$. The probability of error is

$$\epsilon_n := \max_{j \in [1:K']} \Pr\left[\mathsf{dec}_j \left(X^n, (M_i : i \in \mathcal{A}_j) \right) \neq (M_i : i \in \mathcal{D}_j) \right]$$

A rate vector $(R_1, \ldots, R_{N'})$ is said to be achievable if there exists a family of $(2^{nR_1}, \ldots, 2^{nR_{N'}}, n, \epsilon_n)$ -codes with $\lim_{n\to\infty} \epsilon_n = 0.$

For later use, we close this subsection with a description of the composite coding inner bound, which was proposed for the multiple unicast index coding problem in [4]. We trivially extended it here to the general index coding problem.

Composite Coding Inner Bound: Composite coding is a two-stage scheme based on binning and non-unique decoding. In the first encoding stage, for each $\mathcal{J} \subseteq [1 : N']$, the messages $(M_i : i \in \mathcal{J})$ are encoded into the 'composite index' $W_{\mathcal{J}} \in [1 : 2^{nS_{\mathcal{J}}}]$ based on random binning at some rate $S_{\mathcal{J}} \geq 0$. By convention $S_{\emptyset} = 0$. In the second encoding stage, the collection of all composite indices $(W_{\mathcal{J}} : \mathcal{J} \subseteq [1 : N'])$ is mapped into a length-*n* sequence $X^n \in \mathcal{X}^n$. In the first decoding stage, every user recovers all composite indices by making use of the available side information. In the second decoding stage, user $j \in [1 : K']$ chooses a set \mathcal{K}_j such that $\mathcal{D}_j \subseteq \mathcal{K}_j \subseteq [1 : N'] \setminus \mathcal{A}_j$ and simultaneously decodes all messages $(M_i : i \in \mathcal{K}_j)$, based on the recovered $(W_{\mathcal{J}} : \mathcal{J} \subseteq \mathcal{K}_j \cup \mathcal{A}_j)$. The achievable rate region with composite coding is as follows.

Theorem 1 (Composite Coding Inner Bound, generalization of [4] to allow for multicast messages). A non-negative rate tuple $\mathbf{R} := (R_1, \ldots, R_{N'})$ is achievable for the index coding problem $((\mathcal{A}_j, \mathcal{D}_j) : j \in [1 : K'])$ with $N' = |\bigcup_{j \in [1:K']} \mathcal{A}_j \cup \mathcal{D}_j|$ if

$$\mathbf{R} \in \bigcap_{j \in [1:K']} \bigcup_{\mathcal{K}_j: \mathcal{D}_j \subseteq \mathcal{K}_j \subseteq [1:N'] \setminus \mathcal{A}_j} \mathscr{R}_{cc}(\mathcal{K}_j | \mathcal{A}_j, \mathcal{D}_j), \quad (9a)$$

$$\mathscr{R}_{cc}(\mathcal{K}|\mathcal{A},\mathcal{D}) := \bigcap_{\mathcal{J}:\mathcal{J}\subseteq\mathcal{K}} \left\{ \sum_{i\in\mathcal{J}} R_i < v_{\mathcal{J}} \right\},\tag{9b}$$

where in (9b) $v_{\mathcal{J}}$ is defined as

$$v_{\mathcal{J}} := \sum_{\mathcal{P}: \mathcal{P} \subseteq \mathcal{A} \cup \mathcal{K}, \mathcal{P} \cap \mathcal{J} \neq \emptyset} S_{\mathcal{P}}, \tag{9c}$$

and where in (9c) the non-negative quantities $(S_{\mathcal{J}} : \mathcal{J} \subseteq [1 : N])$ must satisfy

$$\sum_{\mathcal{J}:\mathcal{J}\in[1:N'],\mathcal{J}\notin\mathcal{A}_{j}}S_{\mathcal{J}}\leq \log_{2}(|\mathcal{X}|), \quad \forall j\in[1:K'].$$
(9d)

Note that the constrain in (9d) is from the first decoding stage and the region $\mathscr{R}_{cc}(\mathcal{K}_j|\mathcal{A}_j, \mathcal{D}_j)$ in (9a) is from the second decoding stage at receiver $j \in [1:K']$.

C. Connecting Caching to Index Coding

Under the constraint of uncoded cache placement, when the cache contents and the demands are fixed, the delivery phase of the caching problem is equivalent to the following index coding problem. For each $i \in \mathcal{N}(\mathbf{d})$ and for each $\mathcal{W} \subseteq [1:K]$, the sub-file $F_{i,\mathcal{W}}$ (containing the bits of file F_i within the cache of the users indexed by \mathcal{W}) is an independent message in the index coding problem with user set [1:K] Hence, by using the notation introduced in Sections II-B and II-A, K' = K and $N' = |\mathcal{N}(\mathbf{d})|(2^K - 1)$. For each user $k \in [1:K]$ in this general index coding problem, the desired message and side information sets are

$$\mathcal{D}_k = \left(F_{d_k, \mathcal{W}} : \mathcal{W} \subseteq [1:K], k \notin \mathcal{W} \right), \tag{10}$$

$$\mathcal{A}_{k} = \left(F_{i,\mathcal{W}}: \mathcal{W} \subseteq [1:K], i \in \mathcal{N}(\mathbf{d}), k \in \mathcal{W}\right).$$
(11)

In [19], [20], we proposed an outer bound on the worstcase load in centralized caching systems under the constraint of uncoded cache placement by exploiting the acyclic index coding outer bound in [4, Corollary 1]. For a demand vector d, we considered all possible multiple unicast index coding problems with $|\mathcal{N}(\mathbf{d})|$ users. By summing together the resulting bounds and by taking the worst-case demand vector d, we showed that (5) is a lower bound to the worst-case load under uncoded cache placement for centralized systems [19], [20]. We followed a similar approach for decentralized caching systems in [21].

When we attempted to match the worst-case load lower bounds in (5) and (8) with an achievable load from the composite coding inner bound for index coding in Theorem 1 we failed¹. The following example shows that composite coding is insufficient for the index coding problem. This was already pointed out in [4]. The example we give next will be used later on to show that our proposed index coding inner bound is strictly better than composite coding.

Example 1. Consider a multiple unicast index coding problem with K = 6 equal rate messages and with

$$\begin{aligned} \mathcal{D}_1 &= \{1\}, & \mathcal{A}_1 &= \{3,4\}, \\ \mathcal{D}_2 &= \{2\}, & \mathcal{A}_2 &= \{4,5\}, \\ \mathcal{D}_3 &= \{3\}, & \mathcal{A}_3 &= \{5,6\}, \\ \mathcal{D}_4 &= \{4\}, & \mathcal{A}_4 &= \{2,3,6\} \\ \mathcal{D}_5 &= \{5\}, & \mathcal{A}_5 &= \{1,4,6\} \\ \mathcal{D}_6 &= \{6\}, & \mathcal{A}_6 &= \{1,2\}. \end{aligned}$$

Composite Coding Inner Bound. In [25, Example 1] the authors showed that the largest symmetric rate with the composite coding inner bound in Theorem 1 for this problem is $R_{\text{sym,cc}} = 0.2963 \cdot \log_2(|\mathcal{X}|)$. It the same paper, the authors proposed an extension of the composite coding idea (see [25, Section III.B]) and showed that this extended scheme for this problem gives $R_{\text{sym,enhanced cc}} = 0.2987 \cdot \log_2(|\mathcal{X}|)$.

Converse. Give message F_5 as additional side information to receiver 1 so that the new side information set satisfied $\{3,4,5\} \subset A_2$. With this receiver 1, in addition to message 1, can decode message 2 and then message 6. Thus

$$3R_{\text{sym}} \le \lim_{n \to \infty} \frac{1}{n} H(X^n) \le \log_2(|\mathcal{X}|).$$
(12)

Next we show that $R_{\text{sym}} = 1/3 \cdot \log_2(|\mathcal{X}|)$ is tight. This shows the strict sub-optimality of composite coding and its extension.

Achievability. Take the messages to be binary digits. All users can be satisfied by the transmission of the three coded bits $X = (F_1 \oplus F_3 \oplus F_4, F_2 \oplus F_4 \oplus F_5, F_1 \oplus F_2 \oplus F_6)$. Receivers 1, 2 and 6 can 'read off' the desired message bit from one of the transmitted bits after subtracting the known bits. Receiver 3 first sums the three transmitted bits and then recovers F_3 thanks to its side information; receivers 4 and 5 proceed similarly. This shows that one bit per user can be delivered in one channel use, where one channel use corresponds to three bits. Therefore, $R_{\text{sym}} = 1/3 \cdot \log_2(|\mathcal{X}|)$ is achievable and is optimal.

Given that composite coding is insufficient, in the rest of the paper we derive a novel index coding achievable scheme, which we shall prove to strictly improve on composite coding and to be sufficient for caching.

III. NOVEL INDEX CODING SCHEME AND ITS APPLICATION TO THE CACHING PROBLEM

A. Novel Index Coding Scheme

In this section, we first introduce a novel achievable scheme for index coding and then prove that it strictly outperforms composite coding by continuing Example 1. Intuitively, the improvements in our scheme come from:

- For each subset J ⊆ [1 : K'] in the composite coding scheme, the composite index W_J is determined by the messages indexed by J. Thus, composite indices are correlated among themselves. We leverage this correlation to lower the required rate in the first decoding stage.
- In the composite coding scheme, decoder j ∈ [1 : K'] wants to recover uniquely the messages in K_j, and for that he only uses the composite indices (W_J : J ⊆ K_j ∪ Aj). In our proposed scheme, every user uses all the composite messages (X_J : J ⊆ [1 : N']) to uniquely recover the desired messages in D_j and non-uniquely those in K_j\D_j, while the remaining messages are treated as noise.

Theorem 2 (Novel Achievable Scheme for Index Coding). A non-negative rate tuple $\mathbf{R} := (R_1, \ldots, R_{N'})$ is achievable for the index coding problem $((\mathcal{A}_j, \mathcal{D}_j) : j \in [1 : K'])$ with $N' = |\bigcup_{j \in [1:K']} \mathcal{A}_j \cup \mathcal{D}_j|$ if

$$\mathbf{R} \in \bigcap_{j \in [1:K']} \bigcup_{\mathcal{K}_j: \mathcal{D}_j \subseteq \mathcal{K}_j \subseteq [1:N'] \setminus \mathcal{A}_j} \mathscr{R}(\mathcal{K}_j | \mathcal{A}_j, \mathcal{D}_j), \quad (13a)$$

$$\mathscr{R}(\mathcal{K}|\mathcal{A},\mathcal{D}) := \bigcap_{\mathcal{J}:\mathcal{J}\subseteq\mathcal{K},\mathcal{D}\cap\mathcal{J}\neq\emptyset} \left\{ \sum_{i\in\mathcal{J}} R_i < \kappa_{\mathcal{J}} \right\}, \quad (13b)$$

¹ The reason why we do not consider the other index coding achievable schemes we mentioned in the Introduction is because they do not provide easily computable rate expressions for the general index coding problem, or because they were designed for the case of two messages only.

where in (13b) $\kappa_{\mathcal{J}}$ is defined as

$$\kappa_{\mathcal{J}} := I\Big(\big(U_i : i \in \mathcal{J}\big) ; \ \big(X_{\mathcal{P}} : \mathcal{P} \subseteq [1 : N']\big) \\ \Big|\big(U_i : i \in \mathcal{A}_j \cup \mathcal{K}_j \setminus \mathcal{J}\big)\Big),$$
(13c)

for some independent auxiliary random variables $(U_i : i \in$ [1:N'] and some functions $(f_{\mathcal{P}}: \mathcal{P} \subseteq [1:N'])$, such that $X_{\mathcal{P}} = f_{\mathcal{P}}((U_i : i \in \mathcal{P}))$ and satisfying for all $j \in [1 : K']$ $H\Big(\big(X_{\mathcal{P}}:\mathcal{P}\subseteq[1:N']\big)\big|\big(U_i:i\in\mathcal{A}_j\big)\Big)\leq\log_2(|\mathcal{X}|).$ (13d)

Proof: Intuitively, the proof is as follows.

Encoding. Each message $M_i, i \in [1 : N']$, is encoded into a codeword U_i^n generated in an i.i.d. fashion according to some distribution p_{U_i} . Then the collection $(U_i^n : i \in \mathcal{P})$ is mapped into a 'composite index' $X_{\mathcal{P}}^n \in [1:2^{nS_{\mathcal{P}}}]$, for all $\mathcal{P} \subseteq [1 : N']$, by using the function $f_{\mathcal{P}}$ component-wise. Each receiver observes the 'channel input' $X^n := bin(X^n_{\mathcal{P}})$ $\mathcal{P} \subseteq [1 : N'] \in \mathcal{X}^n$, where bin is the bin index of the collection $(X_{\mathcal{P}}^n:\mathcal{P}\subseteq [1:N'])$. Binning is done uniformly and independently.

Decoding. Receiver $j \in [1:K']$, given the side information \mathcal{A}_i , can recover the 'channel input' X^n if the condition in (13d) is satisfied, i.e., only the 'composite indices' that are not fully determined by the side information must be recovered. Finally, receiver $j \in [1:K']$ chooses a set $\mathcal{K}_j \in [1:N']$ such that it includes all the desired messages but none of the side information messages (that is, $\mathcal{D}_j \subseteq \mathcal{K}_j \subseteq [1:N'] \setminus \mathcal{A}_j$); he then simultaneously decodes all messages $(M_i : i \in \mathcal{K}_j)$, but uniquely only the messages in \mathcal{D}_i . For this equivalent multiple access channel with user set \mathcal{K}_i , the achievable region is in the form of (13a) where the messages indexed by \mathcal{J} can be reliably decoded, given that those in the side information or already decoded are known (that is, given the messages indexed by $A_j \cup K_j \setminus J$, if the condition in (13c) is satisfied.

This concludes the proof.

Corollary 1. The composite coding region in Theorem 1 is a special case of our Theorem 2.

Proof: In general, for a set $\mathcal{B} \subseteq [1 : N']$ and for the auxiliary random variables as defined in Theorem 2, we have

$$H\Big(\big(X_{\mathcal{P}}: \mathcal{P} \subseteq [1:N']\big)\Big|\big(U_{i}: i \in \mathcal{B}\big)\Big)$$

$$\leq H\Big(\big(X_{\mathcal{P}}: \mathcal{P} \subseteq [1:N'], \mathcal{P} \not\subseteq \mathcal{B}\big)\Big)$$

$$\leq \sum_{\mathcal{P}: \mathcal{P} \subseteq [1:N'], \mathcal{P} \not\subseteq \mathcal{B}} H\big(X_{\mathcal{P}}\big)$$

$$\leq \sum_{\mathcal{P}: \mathcal{P} \subseteq [1:N'], \mathcal{P} \not\subseteq \mathcal{B}} S_{\mathcal{P}}, \text{ where } \log_{2}(|\mathcal{X}_{\mathcal{P}}|) = S_{\mathcal{P}}. \quad (14)$$

In the following, we choose $(U_i : i \in [1 : N'])$ and $(X_{\mathcal{P}} : \mathcal{P} \subseteq [1 : N'])$ such that all the inequality leading to (14) holds with equality for any $\mathcal{B} \subseteq [1 : N']$, that is, we construct random variables $(X_{\mathcal{P}} : \mathcal{P} \subseteq [1 : N'])$ that are independent and uniformly distributed, where the alphabet of $X_{\mathcal{P}}$ has support of size $|\mathcal{X}_{\mathcal{P}}| = 2^{S_{\mathcal{P}}}$. With this choice of auxiliary random variables we show that Theorem 2 reduces to Theorem 1.

Assume that $S_{\mathcal{P}} \log_2(|\mathcal{X}|)$ is an integer for all $\mathcal{P} \subseteq [1:N']$. Let U_i , for $i \in [1 : N']$, be an equally likely binary vector of length L_i . Let $X_{\mathcal{P}}$ be a binary vector of length $S_{\mathcal{P}} \log_2(|\mathcal{X}|)$ obtained as a linear code for the collection of bits in $(U_i, i \in$ \mathcal{P}). If $L_i \geq \sum_{\mathcal{P} \subseteq [1:N']: i \in \mathcal{P}} S_{\mathcal{P}} \log_2(|\mathcal{X}|)$ for all $i \in [1:N']$, then all the linear combinations that determine $X_{\mathcal{P}}$ can be chosen to be independent and therefore all the inequalities leading to (14) holds with such choice of auxiliary random variables. As a result, we have that the bound in (13d) reduces to the one in (9d) by using (14) with $\mathcal{B} = \mathcal{A}_i$, and that the bound in (13c) reduces to the one in (9c) by using (14) twice, once with $\mathcal{B} = \mathcal{A} \cup \mathcal{K} \setminus \mathcal{J}$ and once with $\mathcal{B} = \mathcal{A} \cup \mathcal{K}$, which is so because

$$\begin{aligned} \kappa_{\mathcal{J}} &= \sum_{\mathcal{P}: \mathcal{P} \subseteq [1:N']: \mathcal{P} \not\subseteq (\mathcal{A} \cup \mathcal{K} \setminus \mathcal{J})} S_{\mathcal{P}} - \sum_{\mathcal{P}: \mathcal{P} \subseteq [1:N']: \mathcal{P} \not\subseteq (\mathcal{A} \cup \mathcal{K})} S_{\mathcal{P}} \\ &= \sum_{\mathcal{P}: \mathcal{P} \subseteq \mathcal{A} \cup \mathcal{K}: \mathcal{P} \cap \mathcal{J} \neq \emptyset} S_{\mathcal{P}}. \end{aligned}$$

This concludes the proof.

Example 2. We continue Example 1. Let each file be an independent bit, $\mathcal{K}_j = \mathcal{D}_j$ for $j \in [1:6]$, and

$$U_{1} = F_{1}, U_{2} = F_{2}, \cdots, U_{6} = F_{6},$$

for all $\mathcal{P} \subseteq [1:6]$ set $X_{\mathcal{P}} = 0$ except
 $X_{\{1,3,4\}} = U_{1} \oplus U_{3} \oplus U_{4},$
 $X_{\{2,4,5\}} = U_{2} \oplus U_{4} \oplus U_{5},$
 $X_{\{1,2,6\}} = U_{1} \oplus U_{2} \oplus U_{6},$
 $X = (X_{\{1,3,4\}}, X_{\{2,4,5\}}, X_{\{1,2,6\}}).$

Here $\mathcal{X} = GF(2^3)$ so one channel use corresponds to three bits. From (13c), we have that for example the rate of user 5 is bounded by

$$\begin{aligned} R_{\text{sym}} &\leq I(U_5; U_1 \oplus U_3 \oplus U_4, U_2 \oplus U_4 \oplus U_5, U_1 \oplus U_2 \oplus U_6 | U_1, U_4, U_6) \\ &= I(U_5; U_3, U_2 \oplus U_5, U_2) = I(U_5; U_2, U_3, U_5) \\ &= I(U_5; U_5) = H(U_5) = 1/3 \cdot \log_2(|\mathcal{X}|), \end{aligned}$$

and similarly for all the other users. As a result, $R_{\rm sym} = 1/3$. $\log_2(|\mathcal{X}|)$ is achievable by the proposed scheme and coincides with the outer bound. \square

B. Application to the Caching Problem

We are now ready to show that Theorem 2 can be used to determine the optimal load in caching problems under the constraint of uncoded cache placement.

Theorem 3. For a caching system under the constraint of uncoded cache placement, Theorem 2 achieves the worst-case loads in (5) and (8) for centralized and decentralized caching systems, respectively.

Proof: We only do the proof for centralized caching systems under the constraint of uncoded cache placement as the one for decentralized systems follows similarly.

We use the same placement phase as cMAN for $M = t\frac{N}{K}$, for $t \in [0 : K]$, so that the delivery phase is equivalent to an index coding problem with K users in which each subfile $F_{i,W}$, for $i \in \mathcal{N}(\mathbf{d})$, $\mathcal{W} \subseteq [1 : K]$ and $|\mathcal{W}| = t$, is an independent message, and where the desired message and side information sets are given by (10) and (11), respectively. Note that the message rates in this equivalent index coding problem are identical by construction and the number of messages for the worst case-load is $N' = \min(N, K) {K \choose t}$.

In Theorem 2, following in Example 2, we let $\mathcal{K}_j = \mathcal{D}_j$ for $j \in [1:K]$, we represent $F_{i,\mathcal{W}}$ as a binary vector for length k and we let the corresponding random variable U to be equal to the message. We also let $X_{\mathcal{P}}$ to be non zero only for the linear combinations of messages sent by the scheme in [22]. With this we have $R_{\text{sym}} = H(U) = k$ and $\log_2(|\mathcal{X}|) = H(X) = k \left(\binom{K}{t+1} - \binom{K-\min(N,K)}{t+1} \right)$, so the symmetric rate is

$$R_{\text{sym}} = \frac{1}{\binom{K}{t+1} - \binom{K - |\mathcal{N}(\mathbf{d})|}{t+1}} \log_2(|\mathcal{X}|)$$

Each receiver in the original caching problem is interested in recovering $\binom{K}{t}$ messages, or one file of $k\binom{K}{t}$ bits, thus the 'sum-rate rate' delivered to each user is

$$R_{\text{sum-rate}} = \frac{\binom{K}{t}}{\binom{K}{t+1} - \binom{K - |\mathcal{N}(\mathbf{d})|}{t+1}} \log_2(|\mathcal{X}|) \quad \left[\frac{\text{bits}}{\text{ch.use}}\right]$$

The load in the caching problem is the number of transmissions (channel uses) needed to deliver one file to each user, thus the inverse of $R_{\text{sum-rate}}$ for $|\mathcal{X}| = 2$ indeed corresponds to the load in (5).

IV. CONCLUSION

In this paper, we investigated the index coding problem and its application to the caching problem with uncoded placement. We proposed a novel index coding inner bound based on distributed source coding that provably strictly improves on composite coding. The novel index coding scheme was then shown to be sufficient to match a known outer bound on the optimal worst-case load in caching systems under the constraint of uncoded cache placement.

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