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Rajeev Verma and Domitilla Del Vecchio

***Abstract*— We consider the problem of safety control in Hidden Mode Hybrid Systems (HMHS) that arises in the development of a semi-autonomous cooperative active safety system for collision avoidance at an intersection. We utilize the approach of constructing a new hybrid automaton whose discrete state is an estimate of the HMHS mode. A dynamic feedback map can then be designed that guarantees safety on the basis of the current mode estimate and the concept of the capture set. In this work, we relax the conditions for the termination of the algorithm that computes the capture set by constructing an abstraction of the new hybrid automaton. We present a relation to compute the capture set for the abstraction and show that this capture set is equal to the one for the new hybrid automaton.**

I. INTRODUCTION

The continuous advances of embedded computing and communication technologies are pushing most engineering systems toward increased levels of autonomy. One such example is vehicles that can drive autonomously or semi-autonomously interacting with drivers and other human-driven vehicles. These technologies fall under Intelligent Transportation System initiatives of government and industry consortia. The availability of vehicle-to-vehicle (V2V) and vehicle-to-infrastructure (V2I) wireless communication will bring these technologies one step closer to reality in the near future. For example, collision avoidance among multiple cars merging on an intersection is studied in [10]. The scheme employs a centralized control scheme that resides on the intersection and acts as a scheduler that assigns a safe time slot to each car for crossing the intersection.

While newer vehicles will be equipped with wireless radio to communicate and cooperate with other vehicles and the infrastructure, there will still be vehicles that will not be able to communicate. The control algorithms developed for guaranteeing safety must be able to operate in this *partially autonomous* real world scenario as long as road-side infrastructure (e.g., cameras, radar, and magnetic-induction loops) is employed to measure the approximate position of the non-communicating vehicles. This approach can be elegantly formulated as a safety control problem for hidden mode hybrid systems [20,21].

The safety control problem for hybrid systems has been extensively considered in the literature when both the continuous and discrete state are available for measurement [14,16,18,19]. These measurements are required to compute a safe control input. In [1,15], hybrid systems whose continuous dynamics is linear time-invariant and discrete state switching is due to transition guards are considered. An over approximation of the reachable set is computed using simulation techniques over bounded time in [15] and by using zonotopes in [1]. In [17], a hybrid system is considered whose discrete state can switch due to discrete control, discrete disturbance and discrete human input. Hybrid reachability results are then utilized to create an invariance-preserving discrete event system abstraction of the so called hybrid human-automation system. The knowledge of discrete input and perfect state information is assumed.

A number of works have addressed the control problem for special classes of hybrid systems with imperfect state information [5,6,24]. In [24], a controller that relies on a state estimator is proposed for finite state systems. The results are then extended to control a class of rectangular hybrid automata with imperfect state information, which can be abstracted by a finite state system. In [5–7, 11], computationally efficient state estimation and control algorithms were proposed for special classes of hybrid system with order-preserving dynamics. The problem of safety control for hidden mode hybrid systems has been addressed in [20,21]. A perfect state information control problem is obtained by constructing a new hybrid automaton, whose discrete state is an estimate of the HMHS mode and is known. This problem is solved by computing the capture set and the least restrictive control map for the new hybrid automaton. Sufficient conditions for the termination of the algorithm that computes the capture set are provided in [20,21]. It has also been shown that the solved perfect state information control problem is equivalent to the original problem with imperfect state information under suitable assumptions. The main contribution of this paper is to show that in the case where the termination conditions for the algorithm that computes the capture set are not satisfied, an abstraction of the new hybrid automaton can be constructed for which the algorithm is guaranteed to terminate and such that the fixed point gives the capture set for the new hybrid automaton.

This paper is organized as follows. We recall some results from [20,21] in Section II, the construction of the abstraction is shown in Section III and Section IV presents an application example.

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II. Safety control problem for hidden mode hybrid systems

In this section, we summarize the results on safety control of HMHS from [20,21]. We first present the general hybrid automaton model.

Definition 1. A *Hybrid Automaton with Uncontrolled Mode Transitions* H is a tuple $H = (Q, X, U, D, \Sigma, R, f)$, in which Q is the set of modes; X is the continuous state space; U is the continuous set of control inputs; D is the continuous set of disturbance inputs; Σ is the set of disturbance events that trigger transitions among modes; $\epsilon \in \Sigma$ is the silent event, which correspond to no transition occurring; $R : Q \times \Sigma \to Q$ is the mode update map and $f : X \times Q \times U \times D \to X$ is the vector field, which is allowed to be piecewise continuous with its arguments.

For a hybrid automaton H, a hybrid time trajectory [16] is denoted by $\mathcal{T} = \bigcup_{i=0}^{N}[\tau_i, \tau_i']]$ with $\sigma(\tau_i') \in \Sigma/\epsilon$ and $\sigma(t) = \epsilon$ for $t \in [\tau_i, \tau_i')$ for all i such that $\tau_i < \tau_i'$. Since the last interval may be open or closed (if $N < \infty$), a ")]" parenthesis is used. We thus represent H by $q(\tau_{i+1}) = R(q(\tau_i'), \sigma(\tau_i'))$, $\sigma(\tau_i') \in \Sigma/\epsilon$ and $\dot{x}(t) = f(x(t), q(t), u(t), d(t)), d(t) \in D, \sigma(t) = \epsilon$, where τ_i for $i \in \{0, ..., N\}$ are the times at which a discrete transition takes place and satisfy $\tau_i \leq \tau_i' = \tau_{i+1}$, the value of q after the i^{th} transition is denoted by $q(\tau_{i+1})$, $q(t) := q(\sup_{\tau_i \leq t} \tau_i)$ for $t \in \mathcal{T}$ and $\sigma(t) = \epsilon$, $x(0) = x_0 \in X$, and $q(\tau_0) = q_0 \in Q$. The initial state x_0 is known (the case where x_0 is subject to uncertainty is considered in [11]). We assume without loss of generality that $\tau_0 = 0$. The continuous state remains the same after the discrete transition, i.e., $x(\tau_{i+1}) = x(\tau_i')$ for all i. For input signals $\mathbf{u} : \mathcal{T} \to U$, $\mathbf{d} : \mathcal{T} \to D$, $\boldsymbol{\sigma} : \mathcal{T} \to \Sigma$, we denote the continuous trajectory of the system by $x(t) = \phi_x(t, (q_0, x_0), \mathbf{u}, \mathbf{d}, \boldsymbol{\sigma})$ for all $t \geq 0$, in which $x(0) = \phi_x(0, (q_0, x_0), \mathbf{u}, \mathbf{d}, \boldsymbol{\sigma}) := x_0$ and the discrete state trajectory by $\phi_q(t, q_0, \boldsymbol{\sigma}) = q(t)$ with $q(0) = \phi_q(0, q_0, \boldsymbol{\sigma})$. The set of reachable modes from any initial set of modes $\bar{q} \subset Q$ is denoted $\text{Rch}(\bar{q}) := \bigcup_{q_0 \in \bar{q}} \bigcup_{t \geq 0} \bigcup_{\sigma} \phi_q(t, q_0, \boldsymbol{\sigma})$.

Definition 2. A *Hidden Mode Hybrid System* (HMHS) is a hybrid automaton with uncontrolled mode transitions in which the discrete state $q(t)$ is not measured and q_0 is only known to belong to a set $\bar{q}_0 \subseteq Q$.

Thus, the mode of a *HMHS* is not known, the only measured state is $x(t)$ and its evolution is driven by hidden mode transitions. In the remainder of the paper, we denote a HMHS by H. Let $Bad \subseteq X$ be a bad set of states. The control task is to keep the continuous state $x(t)$ outside Bad for all time using all the available information. The available information at any time is the initial mode uncertainty, denoted $\bar{q}_0 \subseteq Q$, the measured signals $x(t)$ and the control signal $u(t)$.

Definition 3. A *discrete state estimate* is a time-dependent set, denoted $\hat{q}(t) \in \hat{Q} \subseteq 2^Q$, with the properties that (i) $q(t) \in \hat{q}(t)$ for all $t \geq 0$; (ii) For $t_2 \geq t_1$, we have that $\hat{q}(t_2) \subseteq \text{Rch}(\hat{q}(t_1))$.

Define the new hybrid automaton $\hat{H} = (\hat{Q}, X, U, D, Y, \hat{R}, f)$, in which $\hat{Q} \subseteq 2^Q$ is a new set of discrete states, Y is a set of discrete events, $\epsilon \in Y$ is the silent event, $\hat{R} : \hat{Q} \times Y \to \hat{Q}$ is a discrete state transition map. Let $\hat{\mathcal{T}} = \bigcup_{i=0}^{\hat{N}}[\hat{\tau}_i, \hat{\tau}_i']]$ be a hybrid time trajectory such that $\hat{\tau}_0 = \tau_0$, $y(\hat{\tau}_i') \in Y/\epsilon$ and $y(t) = \epsilon$ for $t \in [\hat{\tau}_i, \hat{\tau}_i')$ for all i such that $\hat{\tau}_i < \hat{\tau}_i'$. We represent $\hat{H}$ by $\hat{q}(\hat{\tau}_{i+1}) = \hat{R}(\hat{q}(\hat{\tau}_i'), y(\hat{\tau}_i'))$, $y(\hat{\tau}_i') \in Y/\epsilon$ and $\dot{\hat{x}}(t) \in f(\hat{x}(t), \hat{q}(t), u(t), d(t)), d(t) \in D, y(t) = \epsilon$, where we have defined $\hat{q}(t) := \hat{q}(\sup_{\hat{\tau}_i \leq t} \hat{\tau}_i)$ for all $t \in \hat{\mathcal{T}}$. Let the map $\hat{R}$ be such that $\hat{q}(t)$ is a discrete state estimate, $\hat{x}(0) = x_0$ and $\hat{q}(\hat{\tau}_0) = \bar{q}_0$. Then, we refer to system $\hat{H}$ as *an estimator*. This in turn implies that (a) $\hat{R}(\hat{q}, y) \subseteq \text{Rch}(\hat{q})$ for all $y \in Y$ and $\hat{q} \in \hat{Q}$ and that (b) $\hat{\tau}_0' = \hat{\tau}_0 = 0$ and $y(\hat{\tau}_0')$ is such that $\hat{R}(\hat{q}(\hat{\tau}_0'), y(\hat{\tau}_0')) := \text{Rch}(\hat{q}(\hat{\tau}_0)) = \text{Rch}(\bar{q}_0)$. The discrete input $y(t)$ derives information from the measured continuous state signal about the values of $\dot{x}(\tau)$ for $\tau < t$ and utilizes this information to determine the current set of modes compatible with such a derivative (see [3,8,9] for more information on mode estimators).

Since for system $\hat{H}$, the state $\hat{q}(t)$ and $\hat{x}(t) = x(t)$ is measured, a safety control problem now becomes a problem with perfect state information. Specifically, given a feedback map $u(t) = \hat{\pi}(\hat{q}(t), \hat{x}(t))$ for system $\hat{H}$, we denote the *closed loop system* by $\hat{H}^{\hat{\pi}}$. The flow of $\hat{H}^{\hat{\pi}}$ is denoted by $\hat{\phi}^{\hat{\pi}}(t, (\bar{q}_0, x_0), \mathbf{d}, \mathbf{y})$ and the continuous flow by $\phi_{\hat{x}}^{\hat{\pi}}(t, (\bar{q}_0, x_0), \mathbf{d}, \mathbf{y})$. Also, a feedback map that guarantees safety for $\hat{H}$ also guarantees safety for H as the set of trajectories of $\hat{H}$ contain also those of H. For more details on the relations between the solutions to the imperfect and the perfect information control problem, the reader is referred to [23]. The capture set for system $\hat{H}$ is given by $\hat{C} := \bigcup_{\hat{q} \in \hat{Q}} \left(\hat{q} \times \hat{C}_{\hat{q}}\right)$, in which $\hat{C}_{\hat{q}} := \{x_0 \in X \mid \forall \hat{\pi}, \exists \mathbf{d}, \mathbf{y}, t \geq 0 \textit{ s.t.} \text{ some } \phi_{\hat{x}}^{\hat{\pi}}(t, (\hat{q}, x_0), \mathbf{d}, \mathbf{y}) \in Bad\}$ is called the mode-dependent capture set. It represents the set of all continuous states that are taken to Bad for all feedback maps when the initial mode estimate is equal to $\hat{q}$.

Problem 1. (Control Problem with Perfect State Information) Determine the set $\hat{C}$ and a feedback map $\hat{\pi}$ that keeps any initial condition $(\bar{q}_0, x_0) \notin \hat{C}$ outside $\hat{C}$.

The solution to Problem 1 can be obtained by leveraging results available for control of hybrid automata with perfect state information [20,21]. For this purpose, for any $\hat{q} \in \hat{Q}$ and $S \subseteq X$ define the operator Pre as $\text{Pre}(\hat{q}, S) := \{x \in X \mid \forall \hat{\pi}, \exists \mathbf{d}, t \geq 0 \text{ s.t. some } \phi_{\hat{x}}^{\hat{\pi}}(t, (\hat{q}, x), \mathbf{d}, \epsilon) \in S\}$. The set $\text{Pre}(\hat{q}, S)$ is the set of all continuous states that are taken to S for all feedback maps when the mode estimate is kept constant to $\hat{q}$.

A. Computation of the capture set

An algorithmic procedure is defined in [20,21] for obtaining set $\hat{C}_{\hat{q}}$. We recall this procedure here. We use for all $\hat{q} \in \hat{Q}$ the notation $\hat{R}(\hat{q}, Y) := \{\hat{q}' \in \hat{R}(\hat{q}, y) \mid y \in Y\}$, in which we set $\hat{R}(\hat{q}, y) := \emptyset$ if $\hat{R}(\hat{q}, y)$ is not defined for some $y \in Y$.

Definition 4. A set $\hat{W} \subseteq \hat{Q} \times X$ is termed a *controlled invariant set* for $\hat{H}$ if there is a feedback map $\hat{\pi}$ such that for all $(\bar{q}_0, x_0) \in \hat{W}$, we have that all flows $\hat{\phi}^{\hat{\pi}}(t, (\bar{q}_0, x_0), \mathbf{d}, \mathbf{y}) \in \hat{W}$ for all t, $\mathbf{d}$, and $\mathbf{y}$. A set $\hat{W} \subseteq \hat{Q} \times X$ is *the maximal controlled invariant set* for $\hat{H}$ provided it is a controlled invariant set for $\hat{H}$ and any other controlled invariant set for $\hat{H}$ is a subset of $\hat{W}$.

The next result (Proposition 1 of [20]) states that the complement of the capture set is the maximal controlled invariant set for $\hat{H}$.

Proposition 1. *The set $\hat{W} := (\hat{Q} \times X)/\hat{C}$ is the maximal controlled invariant set for $\hat{H}$ contained in $(\hat{Q} \times X)/(\hat{Q} \times Bad)$.*

Let $\hat{Q} = \{\hat{q}_1, ..., \hat{q}_M\}$ with $\hat{q}_i \in 2^Q$ for $i \in \{1, \dots, M\}$, $S_i \in 2^X$ for $i \in \{1, \dots, M\}$, and define $S := (S_1, \dots, S_M) \subseteq (2^X)^M$. We define the map $G : (2^X)^M \to (2^X)^M$ as

$$G(S) := \begin{bmatrix} \mathrm{Pre}\left(\hat{q}_1, \bigcup_{\{j|\hat{q}_j \in \hat{R}(\hat{q}_1, Y)\}} S_j \cup Bad\right) \\ \vdots \\ \mathrm{Pre}\left(\hat{q}_M, \bigcup_{\{j|\hat{q}_j \in \hat{R}(\hat{q}_M, Y)\}} S_j \cup Bad\right) \end{bmatrix}.$$

Algorithm 1. $S^0 := (S_1^0,\ S_2^0, \dots, S_M^0) := (\emptyset, \dots,\ \emptyset)$,
$S^1 = G(S^0)$
while $S^{k-1} \neq S^k$
$\quad S^{k+1} = G(S^k)$
end.

If Algorithm 1 terminates, that is, if there is a K^* such that $S^{K^*} = (S_1^{K^*}, ..., S_M^{K^*}) = (S_1^{K^*+1}, ..., S_M^{K^*+1}) = S^{K^*+1}$, we denote the fixed point by S^*. It can be shown that if Algorithm 1 terminates, the fixed point S^* is such that $S^* = (\hat{C}_{\hat{q}_1}, ..., \hat{C}_{\hat{q}_M})$ (see Theorem 1 of [20]).

B. The control map

To determine the set of feedback maps that keep the complement of $\hat{C}$ invariant, we employ notions from viability theory [2].

Definition 5. A set-valued map $F : X \to 2^X$ is said to be *piecewise Lipschitz continuous* on X if it is Lipschitz continuous on a finite number of sets $X_i \subset X$ for $i = 1, ..., N$ that cover X, that is, $\bigcup_{i=1}^N X_i = X$, and $X_i \cap X_j = \emptyset$ for $i \neq j$.

Let X be a normed space and let $S \subset X$ be nonempty. The *contingent cone* to S at $x \in S$ is the set given by $T_S(x) := \{v \in S \mid \liminf_{h \to 0+} \frac{d_S(x+h\,v)}{h} = 0\}$, in which $d_S(y)$ denotes the distance of y from set S, that is, $d_S(y) := \inf_{z \in K} \|y - z\|$. The next result (Proposition 6 of [20]) extends conditions for set invariance as found in [2] to the case of piece-wise Lipschitz continuous set-valued maps. This extension is required in our case because the vector field f is allowed to be piece-wise continuous.

Proposition 2. *Let $F : X \to 2^X$ be a set-valued Marchaud map. Assume that F is piecewise Lipschitz continuous on X. A closed set $S \subseteq X$ is invariant under F if and only if $F(x) \subseteq T_S(x)$ for all $x \in S$.*

For simplifying notation, for each mode $\hat{q} \in \hat{Q}$ define the set-valued map $\bar{f} : X \times \hat{Q} \times U \to 2^X$ as $\bar{f}(\hat{x}, \hat{q}, u) = \{\hat{f}(\hat{x}, \hat{q}, u, d), d \in D\}$ for all $(\hat{x}, \hat{q}, u) \in X \times \hat{Q} \times U$. Define $L_{\hat{q}} := X \backslash \hat{C}_{\hat{q}}$ for all $\hat{q} \in \hat{Q}$ and consider the set-valued map defined as

$$\Pi(\hat{q}, \hat{x}) := \{u \in U \mid \bar{f}(\hat{x}, \hat{q}, u) \subset T_{L_{\hat{q}}}(\hat{x})\}. \qquad (1)$$

The following theorem (Theorem 3 of [20]) states that a control map can be selected that makes the complement of the capture set controlled invariant.

Theorem 1. *Assume that $\hat{\pi} : \hat{Q} \times X \to U$ is such that for all $\hat{q} \in \hat{Q}$ the set-valued map $F(\hat{x}, \hat{q}) := \bar{f}(\hat{x}, \hat{q}, \hat{\pi}(\hat{x}, \hat{q}))$ is Marchaud and piecewise Lipschitz continuous on X. Then, the set $(\hat{Q} \times X) \backslash \hat{C}$ is invariant for $\hat{H}^{\hat{\pi}}$ if and only if $\hat{\pi}(\hat{q}, \hat{x}) \in \Pi(\hat{q}, \hat{x})$.*

III. Termination of Algorithm 1

For the termination of Algorithm 1, sufficient conditions on $\hat{H}$ are provided in [20]. For the systems that do not satisfy these conditions, we show that one can construct an abstraction of $\hat{H}$ for which Algorithm 1 always terminates and such that the fixed point gives the mode-dependent capture sets of $\hat{H}$. In order to proceed, we introduce the notion of kernel sets for $\hat{H}$.

Definition 6. (Kernel set) The *kernel set* corresponding to a mode $\hat{q}^* \in \hat{Q}$ is defined as $ker(\hat{q}^*) := \{\hat{q} \in \hat{Q} \mid \hat{q} \in \hat{\mathrm{Rch}}(\hat{q}^*)$ and $\hat{q}^* \in \hat{\mathrm{Rch}}(\hat{q})\}$.

The kernel set for a mode $\hat{q}^*$ is thus the set of all modes that can be reached from $\hat{q}^*$ and from which $\hat{q}^*$ can be reached. One can verify that for all pairs of modes $\hat{q}_i, \hat{q}_j \in \hat{Q}$, we have that $\hat{q}_i \in \hat{\mathrm{Rch}}(\hat{q}_j)$ and $\hat{q}_j \in \hat{\mathrm{Rch}}(\hat{q}_i)$ if and only if $ker(\hat{q}_i) = ker(\hat{q}_j)$. The next result shows that any two modes of $\hat{H}$ in the same kernel set have the same mode-dependent capture set and hence the same set of safe feedback maps.

Proposition 3. *For every kernel set $ker \subseteq \hat{Q}$ and for any two modes $\hat{q}, \hat{q}' \in ker$, we have that $\hat{C}_{\hat{q}} = \hat{C}_{\hat{q}'}$ and hence that $\Pi(\hat{q}, x) = \Pi(\hat{q}', x)$.*

Proof. Since $\hat{q}, \hat{q}' \in ker$, we have that $\hat{q}' \in \hat{\mathrm{Rch}}(\hat{q})$ and that $\hat{q} \in \hat{\mathrm{Rch}}(\hat{q}')$. By Proposition 4 of [20], the first inclusion implies that $\hat{C}_{\hat{q}'} \subseteq \hat{C}_{\hat{q}}$, while the second inclusion implies that $\hat{C}_{\hat{q}} \subseteq \hat{C}_{\hat{q}'}$. Hence, we must have that $\hat{C}_{\hat{q}} = \hat{C}_{\hat{q}'}$. By equation (1), this in turn implies also that $\Pi(\hat{q}, x) = \Pi(\hat{q}', x)$. ◻

Let $\mathcal{K} := \{ker(\hat{q}_1), \ldots, ker(\hat{q}_M)\}$. Let there be p distinct elements in $\mathcal{K}$ denoted $ker_1, \ldots, ker_p$. Note that $ker_i \cap ker_j = \emptyset$, for $i \neq j$. If each of the kernel sets is just one element in $\hat{Q}$, it means that there are no discrete transitions possible in $\hat{R}$ that bring a discrete state $\hat{q}$ back to itself. That is, there is no loop in any of the trajectories of $\hat{q}$. In this case, one can verify that Algorithm 1 terminates in a finite number of steps (see [20]). If there are loops, then the existence of a maximal element in each kernel set guarantees termination, as has been shown in Theorem 2 of [20]. However, when not all kernel sets have a maximal element, this result does not hold. Hence, we propose a different approach based on constructing an abstraction of $\hat{H}$ that merges all the modes that belong to the same kernel set in a unique new mode.

Definition 7. Given hybrid system $\hat{H} = (\hat{Q}, X, U, D, Y, \hat{R}, \hat{f})$, the abstraction $\tilde{H} = (\tilde{Q}, X, U, D, \tilde{Y}, \tilde{R}, \tilde{f})$ is a hybrid system with uncontrolled mode transitions such that

(i) $\tilde{Q} = \{\tilde{q}_1, ..., \tilde{q}_p\}$, $\tilde{Y}$ is such that $\epsilon \in \tilde{Y}$ and $\tilde{R}(\tilde{q}, \epsilon) = \tilde{q}$ for all $\tilde{q} \in \tilde{Q}$;

(ii) for all $i, j \in \{1, ..., p\}$ there is $\tilde{y} \in \tilde{Y}$ such that $\tilde{q}_i = \tilde{R}(\tilde{q}_j, \tilde{y})$ if and only if there are $\hat{q}' \in ker_i$, $\hat{q} \in ker_j$, and $y \in Y$ such that $\hat{q}' = \hat{R}(\hat{q}, y)$;

(iii) for all $i \in \{1, ..., p\}$, $x \in X$, $d \in D$, and $v \in U$, we have that $\tilde{f}(x, \tilde{q}_i, v, d) := \bigcup_{\hat{q} \in ker_i} \hat{f}(x, \hat{q}, v, d)$.

Since $p \leq m$, the number of discrete states in system $\tilde{H}$ is always less than or equal to that of system $\hat{H}$. For a feedback map $\tilde{\pi} : \tilde{Q} \times X \to U$, initial states $x_0 \in X$ and $\tilde{q}_0 \in \tilde{Q}$, and signals $\tilde{\mathbf{y}}$, $\mathbf{d}$, we denote the flows of the closed loop system $\tilde{H}^{\tilde{\pi}}$ by $\phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}})$ and $\phi_{\tilde{x}}^{\tilde{\pi}}(t, (\tilde{q}_0, x_0), \mathbf{d}, \tilde{\mathbf{y}})$, in which $\tilde{x}(t) := \phi_{\tilde{x}}^{\tilde{\pi}}(t, (\tilde{q}_0, x_0), \mathbf{d}, \tilde{\mathbf{y}})$ satisfies $\dot{\tilde{x}}(t) \in \tilde{f}(\tilde{x}(t), \phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}}), \tilde{\pi}(\phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}}), \tilde{x}), d(t))$. We also denote by $\tilde{C}_{\tilde{q}_i}$ for $i \in \{1, ..., p\}$ the mode-dependent capture sets of $\tilde{H}$. For any $\tilde{q} \in \tilde{Q}$, we define $ker(\tilde{q}) := ker_i$ provided $\tilde{q} = \tilde{q}_i$. Also, for all $\tilde{q} \in \tilde{Q}$, we denote the set of reachable modes from $\tilde{q}$ as $\tilde{\text{Rch}}(\tilde{q}) := \bigcup_{t \geq 0} \bigcup_{\tilde{\mathbf{y}}} \phi_{\tilde{q}}(t, \tilde{q}, \tilde{\mathbf{y}})$. In the sequel, we denote $\tilde{R}(\tilde{q}, \tilde{Y}) := \bigcup_{\tilde{y} \in \tilde{Y}} \tilde{R}(\tilde{q}, \tilde{y})$, in which we set $\tilde{R}(\tilde{q}, \tilde{y}) := \tilde{q}$ if $\tilde{R}(\tilde{q}, \tilde{y})$ is not defined for some $\tilde{y} \in \tilde{Y}$. The following proposition is a direct consequence of the fact that all kernel sets of $\tilde{H}$ are singletons and there is no loop in any of the trajectories of $\tilde{q}$.

Proposition 4. *Algorithm 1 terminates for system $\tilde{H}$.*

The next result shows that any piece-wise continuous signal, which is continuous from the right and contained in $ker(\phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}}))$ is a possible discrete flow of $\hat{H}$ for suitable $\mathbf{y}$ starting from some $\hat{q}_0 \in ker(\tilde{q}_0)$.

Proposition 5. *For any piece-wise continuous signal $\boldsymbol{\alpha}$ that is continuous from the right and such that $\alpha(t) \in ker(\phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}}))$, there are $\hat{q}_0 \in ker(\tilde{q}_0)$ and $\mathbf{y}$ such that $\alpha(t) = \phi_{\hat{q}}(t, \hat{q}_0, \mathbf{y})$ for all t.*

Proof. Since $\alpha(t) \in ker(\phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}}))$ for all t, there are times $t_0, ..., t_N \leq t$ and a sequence $j_0, ..., j_N \in \{1, ..., p\}$ such that $\alpha(t) \in ker_{j_i}$ for all $t \in [t_i, t_{i+1})$. Since any mode in ker_{j_i} can transit to any other mode in ker_{j_i} instantaneously under the discrete transitions of $\hat{H}$, we have that there are $\hat{q}_{o,i} \in ker_{j_i}$ and $\mathbf{y}_i$ such that $\alpha(t) = \phi_{\hat{q}}(t - t_i, \hat{q}_{o,i}, \mathbf{y}_i)$ for all $t \in [t_i, t_{i+1})$. Also, for any two modes $\alpha_i \in ker_{j_i}$ and $\alpha_{i+1} \in ker_{j_{i+1}}$ we have that $\alpha_{i+1} \in \hat{\text{Rch}}(\alpha_i)$. Hence, let $\alpha_i^- := \lim_{t \to t_{i+1}^-} \phi_{\hat{q}}(t - t_i, \hat{q}_{o,i}, \mathbf{y}_i)$ and $\alpha_i^+ := \lim_{t \to t_{i+1}^+} \phi_{\hat{q}}(t - t_{i+1}, \hat{q}_{o,i+1}, \mathbf{y}_{i+1})$. Then, since multiple transitions are possible in $\hat{H}$ at the same time, there is a signal $\mathbf{y}_{i,i+1}$ such that $\alpha_i^+ = \phi_{\hat{q}}(0, \alpha_i^-, \mathbf{y}_{i,i+1})$. Hence, there is a signal $\mathbf{y}$ such that $\alpha(t) = \phi_{\hat{q}}(t, \hat{q}_{o,0}, \mathbf{y})$ for all t. □

Theorem 2. *For all kernel sets ker_i with $i \in \{1, ..., p\}$ and for all $\hat{q} \in ker_i$, we have that $\hat{C}_{\hat{q}} = \tilde{C}_{\tilde{q}_i}$.*

Proof. Let $\hat{q} \in ker_i$. We first show that $\hat{C}_{\hat{q}} \subseteq \tilde{C}_{\tilde{q}_i}$. Let $x_0 \in \hat{C}_{\hat{q}}$, then for all $\hat{\pi} : \hat{Q} \times X \to U$, there are $\mathbf{y}$, $\mathbf{d}$, and $t > 0$ such that $\phi_{\hat{x}}^{\hat{\pi}}(t, (\hat{q}, x), \mathbf{d}, \mathbf{y}) \in Bad$. This is in particular true for all those feedback maps $\hat{\pi}$ such that $\hat{\pi}(\hat{q}, x) = \hat{\pi}(\hat{q}', x)$ whenever $\hat{q}, \hat{q}' \in ker_j$ for some $j \in \{1, ..., p\}$. Hence, we also have that for all $\tilde{\pi} : \tilde{Q} \times X \to U$, there are $\mathbf{y}$, $\mathbf{d}$, and $t > 0$ such that $\hat{x}(t) := \phi_{\hat{x}}^{\tilde{\pi}}(t, (\hat{q}, x), \mathbf{d}, \mathbf{y}) \in Bad$, in which $\dot{\hat{x}} \in \hat{f}(\hat{x}(t), \phi_{\hat{q}}(t, \hat{q}, \mathbf{y}), \tilde{\pi}(\alpha(t), x(t)), d(t))$ with $\alpha(t) := \tilde{q}_j$ if $\phi_{\hat{q}}(t, \hat{q}, \mathbf{y}) \in ker_j$. Such a signal $\hat{x}(t)$ also satisfies $\dot{\hat{x}} \in \tilde{f}(\hat{x}(t), \alpha(t), \tilde{\pi}(\alpha(t), x(t)), d(t))$ by the definition of $\tilde{f}$. By the definition of $\tilde{R}$, there is $\tilde{\mathbf{y}}$ such that $\alpha(t) = \phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}})$ for all t. Hence, $\hat{x}(t)$ is also a continuous flow of $\tilde{H}$ starting at $(\tilde{q}_i, x_0)$ and therefore $x_0 \in \tilde{C}_{\tilde{q}_i}$.

We now show that $\tilde{C}_{\tilde{q}_i} \subseteq \hat{C}_{\hat{q}}$. If $x_0 \in \tilde{C}_{\tilde{q}_i}$, then for all feedback maps $\tilde{\pi} : \tilde{Q} \times X \to U$, there are $\tilde{\mathbf{y}}$, $\mathbf{d}$, and $t > 0$ such that $\tilde{x}(t) := \phi_{\tilde{x}}^{\tilde{\pi}}(t, (\tilde{q}_i, x_0), \tilde{\mathbf{y}}, \mathbf{d}) \in Bad$. Here, we have that $\tilde{x}(t)$ satisfies $\dot{\tilde{x}}(t) \in \tilde{f}(\tilde{x}(t), \phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}}), \tilde{\pi}(\phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}}), \tilde{x}), d(t))$, which is equivalent (by the definition of $\tilde{f}$) to $\dot{\tilde{x}}(t) \in \hat{f}(\tilde{x}(t), ker(\phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}})), \tilde{\pi}(\phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}}), \tilde{x}), d(t))$, which is equivalent to $\dot{\tilde{x}}(t) = \hat{f}(\tilde{x}(t), \alpha(t), \tilde{\pi}(\phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}}), \tilde{x}), d(t))$ for piece-wise continuous signal $\boldsymbol{\alpha}$ (continuous from the right) such that $\alpha(t) \in ker(\phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}}))$. By Proposition 5, any such $\alpha(t)$ is such that there are $\mathbf{y}$ and $\hat{q}_0 \in ker(\tilde{q}_i)$ such that $\alpha(t) = \phi_{\hat{q}}(t, \hat{q}_0, \mathbf{y})$ for all t, that is, it is a discrete flow of system $\hat{H}$. Hence, for all $\pi' : \hat{Q} \times X \to U$ with $\hat{\pi}'(\hat{q}, x) = \hat{\pi}'(\hat{q}', x)$ for all $\hat{q}, \hat{q}' \in ker_j$ for all j, there are $\mathbf{y}$, $\mathbf{d}$, $\hat{q}_0 \in ker_i$, such that $\phi_{\hat{x}}^{\hat{\pi}'}(t, (\hat{q}_0, x_0), \mathbf{y}, \mathbf{d}) \in Bad$. By Proposition 3, this implies that for all $\pi : \hat{Q} \times X \to U$ there are $\mathbf{y}$, $\mathbf{d}$, $\hat{q}_0 \in ker_i$, such that $\phi_{\hat{x}}^{\hat{\pi}}(t, (\hat{q}_0, x_0), \mathbf{y}, \mathbf{d}) \in Bad$. Hence, $x_0 \in \hat{C}_{\hat{q}_0}$. □

The above theorem can be utilized to compute the capture set of $\hat{H}$ by constructing the abstraction $\tilde{H}$ and applying Algorithm 1 to it, which is guaranteed to terminate for $\tilde{H}$. The next two technical propositions provide a characterization of the Pre operator computed for system $\tilde{H}$ and the relationship between $\tilde{R}$ and R. Specifically, denote the predecessor operator for system $\tilde{H}$ for some $S \subseteq X$ as

$\text{Pre}^a(\tilde{q}, S) := \{x_0 \in X \mid \forall\ \tilde{\pi}\ \exists\ t, \mathbf{d},\ \text{s.t. } \phi_{\tilde{x}}^{\tilde{\pi}}(t, (\tilde{q}, x_0), \mathbf{d}, \epsilon) \in S\}$.

Proposition 6. *For all $\tilde{q} \in \tilde{Q}$ and $S \subseteq X$, we have that $Pre^a(\tilde{q}, S) = Pre(\bigvee ker(\tilde{q}), S)$.*

Proof. From the definition of $\text{Pre}^a(\tilde{q}, S)$, we have that $x_0 \in \text{Pre}^a(\tilde{q}, S)$ if and only if for all $\tilde{\pi}$, there are $t, \mathbf{d}$ such that $\tilde{x}(t) = \phi_{\tilde{x}}^{\tilde{\pi}}(t, (\tilde{q}, x_0), \mathbf{d}, \epsilon) \in S$, in which $\dot{\tilde{x}}(t) \in \tilde{f}(\tilde{x}(t), \tilde{q}, \tilde{\pi}(\tilde{x}(t)), d(t))$, which, by the definition of $\tilde{f}$ and of $\hat{f}$ is equivalent to $\dot{\tilde{x}}(t) \in f(\tilde{x}(t), \bigcup_{\hat{q} \in ker(\tilde{q})} \bigcup_{q \in \hat{q}} q, \tilde{\pi}(\tilde{x}(t)), d(t)) = f(\tilde{x}(t), \bigvee ker(\tilde{q}), \tilde{\pi}(\tilde{x}(t)), d(t))$. Hence, by the definition of Pre, we have that $x_0 \in \text{Pre}^a(\tilde{q}, S)$ if and only if $x_0 \in \text{Pre}(\bigvee ker(\tilde{q}), S)$. □

Proposition 7. *Let $\tilde{q}_{j_1}, \tilde{q}_{j_0} \in \tilde{Q}$. If $\tilde{q}_{j_1} \in \tilde{R}(\tilde{q}_{j_0}, \tilde{Y})$ then $\bigvee ker(\tilde{q}_{j_1}) \subseteq Rch(\bigvee ker(\tilde{q}_{j_0}))$.*

Proof. If $\tilde{q}_{j_1} \in \tilde{R}(\tilde{q}_{j_0}, \tilde{Y})$, then by the definition of $\tilde{R}$ there are $\hat{q} \in ker(\tilde{q}_{j_0})$ and $\hat{q}' \in ker(\tilde{q}_{j_1})$ such that $\hat{q}' = \hat{R}(\hat{q}, y)$ for some $y \in Y$. By the definition of a kernel set, this also implies that for all $\hat{q} \in ker(\tilde{q}_{j_0})$ and $\hat{q}' \in ker(\tilde{q}_{j_1})$, there is a sequence of events $y_1, ..., y_k$ and of modes $\hat{q}_{j_0}, ..., \hat{q}_{j_k} \in \hat{Q}$ such that $\hat{q}_{j_0} = \hat{q}$, $\hat{q}_{j_k} = \hat{q}'$ and $\hat{q}_{j_{i+1}} = \hat{R}(\hat{q}_{j_i}, y_{i+1})$ for $i \in \{0, ..., k-1\}$. Since $\hat{R}(\hat{q}, y) \subseteq \text{Rch}(\hat{q})$ for all $y \in Y$ and $\hat{q} \in \hat{Q}$, this in turn implies that $\hat{q}_{j_{i+1}} \subseteq \text{Rch}(\hat{q}_{j_i})$ for $i \in \{0, ..., k-1\}$. This leads to $\hat{q}' \subseteq \text{Rch}(\hat{q})$ for all $\hat{q} \in ker(\tilde{q}_{j_0})$ and $\hat{q}' \in ker(\tilde{q}_{j_1})$. This also implies that $\hat{q}' \subseteq \text{Rch}(\bigvee ker(\tilde{q}_{j_0}))$ and hence (since this holds for all $\hat{q}' \in ker(\tilde{q}_{j_1})$) to $\bigvee ker(\tilde{q}_{j_1}) \subseteq \text{Rch}(\bigvee ker(\tilde{q}_{j_0}))$. □

Lemma 1. *For all $\bar{q} \in \hat{Q}$, we have that $\hat{C}_{\bar{q}} = Pre(Rch(\bar{q}), Bad)$.*

Proof. First, we show that $\hat{C}_{\bar{q}} \subseteq \text{Pre}(\text{Rch}(\bar{q}), Bad)$. Since Algorithm 1 terminates in a finite number n of steps for $\tilde{H}$, we have that $\tilde{C}_{\tilde{q}} = \text{Pre}^a\big(\tilde{q}, \bigcup_{\tilde{q}_{j_1} \in \tilde{R}(\tilde{q}, \tilde{Y})} \text{Pre}^a\big(\tilde{q}_{j_1}, \bigcup_{\tilde{q}_{j_2} \in \tilde{R}(\tilde{q}_{j_1}, \tilde{Y})} \text{Pre}^a\big(\tilde{q}_{j_2}, ..., \bigcup_{\tilde{q}_{j_{n-1}} \in \tilde{R}(\tilde{q}_{j_{n-2}}, \tilde{Y})} \text{Pre}^a(\tilde{q}_{j_{n-1}}, Bad)...\big)\big)\big)$. By Proposition 6, we also have that $\tilde{C}_{\tilde{q}} = \text{Pre}\big(\bigvee ker(\tilde{q}), \bigcup_{\tilde{q}_{j_1} \in \tilde{R}(\tilde{q}, \tilde{Y})} \text{Pre}\big(\bigvee ker(\tilde{q}_{j_1}), \bigcup_{\tilde{q}_{j_2} \in \tilde{R}(\tilde{q}_{j_1}, \tilde{Y})} \text{Pre}\big(\bigvee ker(\tilde{q}_{j_2}), ... \bigcup_{\tilde{q}_{j_{n-1}} \in \tilde{R}(\tilde{q}_{j_{n-2}}, \tilde{Y})} \text{Pre}(\bigvee ker(\tilde{q}_{j_{n-1}}), Bad)...\big)\big)\big)$. By Proposition 7, we have that $\bigvee ker(\tilde{q}_{j_1}) \subseteq \text{Rch}(\bigvee ker(\tilde{q}))$ and that $\bigvee ker(\tilde{q}_{j_{i+1}}) \subseteq \text{Rch}(\bigvee ker(\tilde{q}_{j_i}))$ for $i < n$. Since the Pre operator and Rch preserve the inclusion relation in the first argument, these imply that $\tilde{C}_{\tilde{q}} \subseteq \text{Pre}(\text{Rch}(\bigvee ker(\tilde{q})), Bad)$. Since for all $\bar{q}_1, \bar{q}_2 \in ker(\tilde{q})$ we have that $\text{Rch}(\bar{q}_1) = \text{Rch}(\bar{q}_2)$, we also have that $\text{Rch}(\bar{q}) = \text{Rch}(\bigvee ker(\tilde{q}))$ for all $\bar{q} \in ker(\tilde{q})$. Hence, $\hat{C}_{\bar{q}} \subseteq \text{Pre}(\text{Rch}(\bar{q}), Bad)$ for all $\bar{q} \in ker(\tilde{q})$. This along with Theorem 2 finally imply that for all $\bar{q} \in ker(\tilde{q})$ we have $\hat{C}_{\bar{q}} \subseteq \text{Pre}(\text{Rch}(\bar{q}), Bad)$.

To show that $\hat{C}_{\bar{q}} \supseteq \text{Pre}(\text{Rch}(\bar{q}), Bad)$, we employ the properties of the Pre operator and Proposition 4 of [20]. By such a proposition, by the fact that (since $\hat{H}$ is derived from H) for all $\bar{q} \in \hat{Q}$ there is $y \in Y$ such that $\hat{R}(\bar{q}, y) = \text{Rch}(\bar{q})$, and by property (iii) of Proposition 2 from [20], it follows that $\hat{C}_{\bar{q}} \supseteq \text{Pre}(\bar{q}, \hat{C}_{\text{Rch}(\bar{q})})$. In turn we have that $\hat{C}_{\text{Rch}(\bar{q})} \supseteq \text{Pre}(\text{Rch}(\bar{q}), Bad)$ by Proposition 4 of [20] and property (iii) of Proposition 2 from [20]. Hence, we have that $\hat{C}_{\bar{q}} \supseteq \text{Pre}(\bar{q}, \text{Pre}(\text{Rch}(\bar{q}), Bad))$, which by property (i) of Proposition 2 from [20] leads to $\hat{C}_{\bar{q}} \supseteq \text{Pre}(\text{Rch}(\bar{q}), Bad)$. □

This result shows that the mode-dependent capture set $\hat{C}_{\bar{q}}$ can be computed by computing the Pre operator only once as opposed to being determined through a (finite, by Theorem 2 and Proposition 4) iteration of Pre operator computations (as was performed in [20,21]). To illustrate this point, consider as an example a tuple $(\hat{R}, \hat{Q}, Y)$ with $\hat{Q} = \{\hat{q}_1, \hat{q}_2\}$, $Y = \{\epsilon, y\}$, $\hat{R}(\hat{q}_1, y) = \hat{q}_2$ and $\hat{R}(\hat{q}_2, y) = \hat{q}_1$ with $\hat{q}_1 \not\subseteq \hat{q}_2$. Since there is a loop between $\hat{q}_1$ and $\hat{q}_2$ and the kernel set does not contain a maximal element, Theorem 2 of [20] cannot guarantee the termination of Algorithm 1. However, the results presented in this paper show that the desired capture set can be obtained by utilizing Lemma 1, that is, $\hat{C}_{\hat{q}_1} = \text{Pre}(\text{Rch}(\hat{q}_1), Bad)$ and $\hat{C}_{\hat{q}_2} = \text{Pre}(\text{Rch}(\hat{q}_2), Bad)$, in which $\text{Rch}(\hat{q}_2) = \text{Rch}(\hat{q}_1) = \{\hat{q}_1, \hat{q}_2\}$. The computation of such a Pre can be efficiently performed if the continuous dynamics for $q \in \hat{q}_1 \cup \hat{q}_2$ has suitable order preserving properties [23]. We show an application example in the next section.

IV. Application scenario

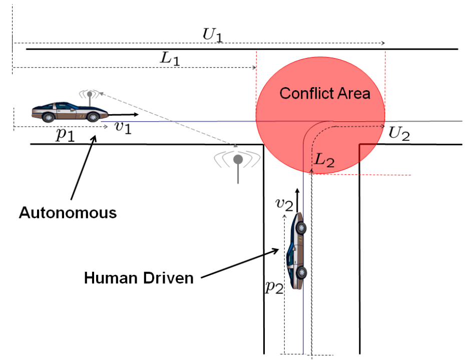

Fig. 1. **Two-vehicle Conflict Scenario**. Vehicle 1 (autonomous) is equipped with a cooperative active safety system and communicates with the infrastructure via wireless. Vehicle 2 (human-driven) is not equipped and does not communicate with the infrastructure. A collision occurs when more than one vehicle occupies the conflict area at one time.

Referring to Figure 1, vehicle 1 is autonomous and communicates with the infrastructure, while vehicle 2 is human-driven and does not communicate its intent to the infrastructure nor to the other vehicle. We assume that the infrastructure measures the position and speed of vehicle 2 through road-side sensors such as cameras and magnetic-induction loops and that it transmits this information to the on-board controller of vehicle 1. Vehicle 1 has to use this information to avoid a collision.We assume that the

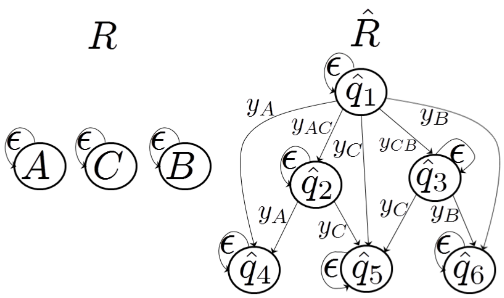

Fig. 2. Map R and map $\hat{R}$.

human driver decides to either accelerate (A), coast (C) or brake (B) the vehicle when he/she is near the intersection. The intersection system is a hybrid automaton with uncontrolled mode transitions H, in which $Q = \{A, C, B\}$; $X = \mathbb{R}^4$ and $x \in X$ is such that $x = (p_1, v_1, p_2, v_2)$, where p_i is the longitudinal displacement along the path and v_i is the longitudinal speed of the i^{th} vehicle, with $i \in \{1, 2\}$; $U = [u_L, u_H] \subset \mathbb{R}$ represents the maximum braking and throttle control input; $D = [-\bar{d}, \bar{d}] \subset \mathbb{R}$; $\Sigma = \{\epsilon\}$ as there is no transition allowed between the modes; $R : Q \times \Sigma \to Q$ is the mode update map, and $f : X \times Q \times U \times D \to X$ is the vector field, which is piecewise continuous and is given by $f(x, q, u, d) = (f_1(p_1, v_1, u), f_2(p_2, v_2, q, d))$ in which

$$f_1(p_1, v_1, u) = \left(\begin{array}{c} v_1 \\ \left\{ \begin{array}{ll} 0 & \text{if } (v_1 = v_{min} \text{ and } \alpha_1 < 0) \\ & \text{or } (v_1 = v_{max} \text{ and } \alpha_1 > 0) \\ \alpha_1 & \text{otherwise} \end{array} \right. \end{array} \right),$$

$$f_2(p_2, v_2, q, d) = \left(\begin{array}{c} v_2 \\ \left\{ \begin{array}{ll} 0 & \text{if } (v_2 = v_{min} \text{ and } \alpha_2 < 0) \\ & \text{or } (v_1 = v_{max} \text{ and } \alpha_2 > 0) \\ \alpha_2 & \text{otherwise} \end{array} \right. \end{array} \right),$$

with $\alpha_1 = au + b - cv_1^2$; $\alpha_2 = \beta_q + d$; $b < 0$ represents the static friction term; $c > 0$ with the cv_1^2 term modeling air drag (see [13,22] for more details on the model); $q \in \{A, C, B\}$; $d \in [-\bar{d}, \bar{d}]$ and $\bar{d} > 0$. The value of β_q corresponds to the nominal dynamics of mode q and thus $\beta_A > 0, \beta_C = 0$ and $\beta_B < 0$. The disturbance d models the error with respect to the nominal mode. There is a lower non-negative speed limit, v_{min}, implying that vehicles cannot go in reverse and guaranteeing liveness of the system. Similarly there is an upper speed limit, denoted v_{max}. The assumption that the driver cannot change his mind once he selects a mode is a fair assumption near an intersection. In [12], the authors study drivers who either accelerate or brake while approaching a traffic light. Driver behavior that allows switching from acceleration to coasting to braking is considered in [23]. Referring to Figure 1, the set of bad states for system H models collision configurations and it is given by $Bad := \{(p_1, v_1, p_2, v_2) \in \mathbb{R}^4 \mid (p_1, p_2) \in [L_1, U_1] \times [L_2, U_2]\}$.

The system $\hat{H} = (\hat{Q}, X, U, D, Y, \hat{I}nv, \hat{R}, f)$, in which $\hat{Q} = \{\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_4, \hat{q}_5, \hat{q}_6\}$ with $\hat{q}_1 = \{A, C, B\}$, $\hat{q}_2 = \{A, C\}$, $\hat{q}_3 = \{C, B\}$, $\hat{q}_4 = \{A\}$, $\hat{q}_5 = \{C\}$, $\hat{q}_6 = \{B\}$, and $\hat{q}(0) = \hat{q}_1$, is uniquely defined once the set Y and map $\hat{R}$ are defined. We define $Y = \{y_{AC}, y_{CB}, y_A, y_C, y_B, \epsilon\}$. Let us consider the following estimate $\hat{\beta}(t) = \frac{1}{t}\int_{t-T}^{t} \dot{v}_2(\tau) d\tau,\ \ t \geq T,$ [1] where $T > 0$ is a time window ($\hat{\beta}(t)$ is the average acceleration over time window of length T). If the mode is q, then we have that $|\hat{\beta}(t) - \beta_q| \leq \bar{d}$. Thus, for $t > T$, define $y(t) = y_A$ if $|\hat{\beta}(t) - \beta_C| > \bar{d}$ and $|\hat{\beta}(t) - \beta_B| > \bar{d}$; $y(t) = y_C$ if $|\hat{\beta}(t) - \beta_A| > \bar{d}$ and $|\hat{\beta}(t) - \beta_B| > \bar{d}$; $y(t) = y_B$ if $|\hat{\beta}(t) - \beta_A| > \bar{d}$ and $|\hat{\beta}(t) - \beta_C| > \bar{d}$; $y(t) = y_{AC}$ if $|\hat{\beta}(t) - \beta_C| \leq \bar{d}$, $|\hat{\beta}(t) - \beta_A| \leq \bar{d}$ and $|\hat{\beta}(t) - \beta_B| > \bar{d}$; $y(t) = y_{CB}$ if $|\hat{\beta}(t) - \beta_B| \leq \bar{d}$, $|\hat{\beta}(t) - \beta_C| \leq \bar{d}$ and $|\hat{\beta}(t) - \beta_A| > \bar{d}$; and $y(t) = \epsilon$ otherwise. The resulting $\hat{R}$ is shown in Figure 2. For system $\hat{H}$, we have from Lemma 1 that $\hat{C}_{\hat{q}_i} = \text{Pre}(\hat{q}_i, Bad)$ for $i \in \{1, 2, 3, 4, 5, 6\}$. Since a mode switch is not allowed, identifying the mode reduces the size of the capture set.

The sets $\text{Pre}(\hat{q}_i, Bad)$ can be easily calculated with a linear complexity discrete time algorithm, as in the i^{th} mode the dynamics are given by the parallel composition of two order-preserving systems and Bad is an interval [11]. In particular, these sets are given as $\text{Pre}(\hat{q}, Bad) = \text{Pre}(\hat{q}, Bad)_L \cap \text{Pre}(\hat{q}, Bad)_H$, in which $\text{Pre}(\hat{q}, Bad)_L = \{x \in X \mid \exists\, t,\ \mathbf{d}\ \ s.t.\ \text{some}\ \phi_{\hat{x}}(t, (x, \hat{q}), \mathbf{d}, u_L, \epsilon) \in Bad\}$ and $\text{Pre}(\hat{q}, Bad)_H = \{x \in X \mid \exists\ t,\ \mathbf{d}\ \ s.t.\ \text{some}\ \phi_{\hat{x}}(t, (x, \hat{q}), \mathbf{d}, u_H, \epsilon) \in Bad\}$ (see [7,11] for more details on these computational techniques). The map $\hat{\pi}(\hat{q}, x)$ for every mode estimate $\hat{q}$ is active only when x is on the boundary of $\hat{C}_{\hat{q}}$ and in such a case it makes the continuous state slide on the boundary of $\hat{C}_{\hat{q}}$ [7,11]. A feedback map $\hat{\pi}(\hat{q}, x)$, that satisfies Theorem 1 is given by

$$\hat{\pi}(\hat{q}, x) := \left\{ \begin{array}{ll} u_L & if\ \ x \in \text{Pre}(\hat{q}, Bad)_H \wedge x \in \partial\text{Pre}(\hat{q}, Bad)_L \\ u_H & if\ \ x \in \text{Pre}(\hat{q}, Bad)_L \wedge x \in \partial\text{Pre}(\hat{q}, Bad)_H \\ u_L & if\ \ x \in \partial\text{Pre}(\hat{q}, Bad)_L \wedge x \in \partial\text{Pre}(\hat{q}, Bad)_H \\ * & \text{otherwise.} \end{array} \right.$$

Simulation results are shown in Figure 3.

V. Conclusion

In this paper, we considered the problem of safety control of hidden mode hybrid systems. In particular, we solve the problem by utilizing an existing approach from [20,21] to construct a new hybrid automaton (an estimator) whose discrete state is an estimate of the hidden mode. The main contribution of this work is in showing that the algorithm that computes the capture set is guaranteed to terminate under substantially less restrictive conditions than those considered in [20,21]. Moreover, we provide a simple formula for the computation of the capture set. Independently of the number of discrete states in the estimator, the capture set for each discrete state is efficiently calculable for systems whose continuous

[1] Note that in practice, we will not require measurement of acceleration as we will consider discrete time models where derivative is replaced by time anticipation.

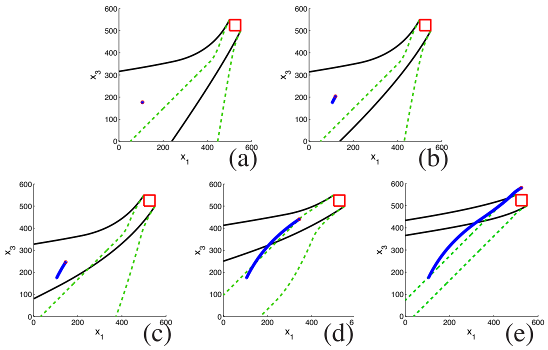

Fig. 3. In each of the plots (a)–(e), the red box represents $[L_1, U_1] \times [L_2, U_2]$. We plot the slice of $\hat{C}_{\hat{q}}$ in the (x_1, x_3) position plane corresponding to the current speed (x_2, x_4). In the (x_1, x_3) plane and for the current speed values (x_2, x_4), the black solid lines delimit the set $\text{Pre}(\hat{q}, Bad)_H$, the green dashed lines delimit the set $\text{Pre}(\hat{q}, Bad)_L$ and the intersection of these two sets is the current mode dependent capture set $\hat{C}_{\hat{q}}$. The red circle denotes the current position x_1, x_3, while the blue trace represents the projection in the position plane of the continuous trajectory of H. Plot (a) shows the initial configuration in the position plane. Here, the current mode estimate is $\hat{q} = \{A, C, B\}$. Plot (b) shows the mode estimate switching to $\hat{q} = \{C, B\}$ and the corresponding capture set shrinking. Plot (c) shows the time at which the mode estimate becomes $\hat{q} = \{B\}$, so that the current mode is locked and the capture set shrinks further. Plot (d) shows when the continuous state hits the boundary of the current mode-dependent capture set thus resulting in the application of a safe control.

dynamics have suitable order-preserving properties [23]. We introduce an example of a semi-autonomous cooperative active safety system that belongs to this class and present simulation results for collision avoidance between a human-driven and an autonomous vehicle merging at an intersection. In future work, we intend to consider situations with more than two vehicles merging on an intersection, in which some of the vehicles are human-driven and some are autonomous. The approach presented in this paper cannot be directly extended to the multiple vehicle scenario due to the bad set not being convex. Alternative approaches are being investigated, including discrete abstraction techniques exploiting the fact that the vehicles dynamics are differentially flat and order preserving [4].

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