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String Stability of Heterogeneous Platoons with Non-connected Automated Vehicles

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***Abstract*— It is expected that automated vehicles will gradually penetrate on public roads, resulting in mixed traffic in the next decades. This can impact traffic flow operations, especially the roadway capacity and flow stability. It is of paramount importance to understand and predict the implications of automated driving systems on traffic flow at the early design phase to avoid disruptive impacts on traffic. String stability properties of automated vehicle platoons are a fundamental block to understand their traffic flow stability impact. Previous reports on string stability analysis focussed on homogeneous vehicle strings and simplify the time delays in vehicle systems. This work propose an analytical approach to determine string stability conditions for non-connected vehicle platoons with heterogeneous parameters. To this end, a third-order linear vehicle dynamics model is used in the control design and Laplace transform of the spacing and speed error dynamics in time domain to frequency domain enables the determination of sufficient string stability criteria of heterogeneous vehicle strings. The analytical string stability conditions give new insights into the relationship between the string stability properties of vehicle strings in relation to the system properties of time delays and controller design parameters of feedback gains and desired time gap. Analytical results are verified via systematic simulation of both homogeneous and heterogeneous strings. Simulations demonstrate the predictive power of the analytical string stability conditions.**

I. Introduction

Automated vehicles have attracted considerable attention from the public since they may completely change the way we operate our vehicles today and consequently may have great implications for the traffic operations. It is therefore important to design such systems in a scrutinized manner to ensure benefits to the traffic systems. Automated vehicles can be classified as non-connected/autonomous and connected/cooperative vehicle systems. Non-connected automated vehicles rely solely on on-board sensors [1], [2], [3], while connected automated vehicles exchange (state and control) information with each other via Vehicle-to-Vehicle (V2V) communication or with road infrastructure via Vehicle-to-Infrastructure (V2I) communication to improve situation awareness and/or to manoeuvre together under a common goal [4], [5], [6], [7], [8].

Adaptive Cruise Control (ACC) is one of the earliest automated vehicle systems. The most widely used ACC controller is a linear state feedback controller, where the vehicle acceleration is proportional to the deviation of gap from a desired gap, e.g. constant time gap policy (CTG), and the speed error, i.e. the relative speed with respect to the preceding vehicle [9].

One of the problems with autonomous ACC is the string instability, i.e. tracking errors in one vehicle can be amplified when propagating in a string of vehicles. The major influencing factors for vehicular string stability properties are the system properties, notably time delays of the vehicle dynamic system, and control (design) parameters, e.g. the desired gap and feedback gains. Two types of system delays can be distinguished, being sensor delay and actuator lag [10], [11]. Sensor delay is caused by the process of sensing and filtering, due to the discrete sampling of on-board sensors, the lag due to the radar or lidar filter, and the bandwidth of low pass filters used for other sensors such as wheel speed sensors [10]. The actuator lag lies in the lower level of the vehicle control system when executing the desired acceleration command from the upper level ACC controller, due to the time delay in the generation of traction/brake wheel torques due to the lag of the powertrain actuator or the lag of brake actuator when braking [10], [12].

Much work on control design and stability analysis of vehicle platoons does not explicitly address both time delays [13]. Omitting the combination of sensor delay and actuator lag in the control loop may result in over-optimistic evaluations of the controller performance and the corresponding impacts on traffic flow [11]. Few work has addressed both sensor delay and actuator lag, but is restricted to the linear feedback control law with the CTG policy [10]. String stability conditions of autonomous vehicle platoons employing general nonlinear gap policies with both sensor delay and actuator lag remain largely unresolved.

The aforementioned problems lead to the main objective of this contribution: to develop a methodology for string stability analysis for the stability of a general longitudinal control system for non-connected automated vehicle systems with heterogeneous parameters. Application of the proposed approach will yield new insights into the impacts of the system properties of time delays and controller design parameters of feedback gains and desired time gap on resulting string stability properties. To this end, we generalize previous work by deriving the sufficient conditions of the string stability of a general heterogeneous platoon with mixed vehicle classes and different parameter settings. To this end, a third order linearised vehicle dynamics model is used in the control design and Laplace transform of the speed and gap error dynamics in time domain to frequency domain enables the determination of sufficient string stability criteria of a

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heterogeneous vehicle string. We verify the analytical results with simulation of vehicle strings subject to exogenous disturbance.

The remainder of the paper is organized as follows: Section II introduces the model for longitudinal vehicle dynamics. Section III presents the string stability conditions for homogeneous vehicle strings with two delays and a generalised gap policy, followed by extension to string stability conditions for heterogeneous strings in Section IV. We summarise the study in Section V.

II. Vehicle longitudinal dynamics model

We first introduce the linear vehicle model used for controller design that resembles driveline dynamics and then add delay in the system.

Let x_i denote the longitudinal position of vehicle i in a platoon, which increases in the forward driving direction, then its longitudinal dynamics can be expressed as a third-order equation as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_i \\ \dot{x}_i \\ \ddot{x}_i \end{pmatrix} = \begin{pmatrix} \dot{x}_i \\ \ddot{x}_i \\ \frac{u_i - \ddot{x}_i}{\tau_i} \end{pmatrix} \tag{1}$$

Here we introduce the control input u_i, which can be interpreted as the *desired acceleration* of the controlled vehicle i. τ_i represents the driveline dynamics involved in the lower-level vehicle control. It is caused by the lag of the powertrain actuator or the lag of brake actuator when braking [9], [10], [12], [14] and implies that t*he commanded acceleration u_i cannot be realized instantaneously* only after a retarded time τ_i. The linear model enables us analyse the properties of the feedback ACC/CACC control law without consideration of the lower-level complexity.

For car-following control of vehicle i, it is convenient to define the system state such that it couples the dynamics of the predecessor $i-1$. Hence, we define the system state (vector) $\mathbf{X}$ as: $\mathbf{X} = (s_i, v_{i-1}, v_i, a_i)$, where $s_i = x_{i-1} - x_i - l_i$ denotes the gap or spacing to the preceding vehicle $i-1$, l_i denotes the vehicle length. For state-feedback control laws, the control input u_i can be expressed as an explicit functional of the state vector: $u = f(\mathbf{X})$. The state variables can be measured/estimated from on-board sensors of vehicle i but are subject to a time delay ξ_i. Hence when including feedback delay, the system dynamics can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t) = \begin{pmatrix} \Delta v_i(t) \\ a_{i-1}(t) \\ a_i(t) \\ \frac{u_i(t) - a_i(t)}{\tau_i} \end{pmatrix} = \begin{pmatrix} \Delta v_i(t) \\ a_{i-1}(t) \\ a_i(t) \\ \frac{f(\mathbf{X}_i(t-\xi_i)) - a_i(t)}{\tau_i} \end{pmatrix} \tag{2}$$

where $\Delta v_i := v_{i-1} - v_i$ denotes the relative speed with respect to the preceding vehicle.

III. String stability of homogeneous platoon

In this section, we derive string stability criteria for a homogeneous platoon with sensor delay and actuator lag.

A. General control law and equilibria

Due to low signal-to-noise ratio of the vehicle acceleration measurement for on-board sensors, it is not desirable to include acceleration variable a_i in the feedback control law. Hence a general formulation of a nonlinear control law in state feedback form can be written as:

$$u_i(t) = f(s_i(t-\xi_i), v_i(t-\xi_i), v_{i-1}(t-\xi_i)) \tag{3}$$

Inserting the feedback law (3) into the system dynamics equation (2) we get the compact description of the closed-loop system dynamics as:

$$\dot{a}_i(t) = \frac{f(s(t-\xi_i), v_i(t-\xi_i), v_{i-1}(t-\xi_i)) - a_i(t)}{\tau_i} \tag{4}$$

For car-following control, a desired gap $s_{d,i}$ is usually defined under some desired gap/spacing policy. The general (nonlinear) equilibrium speed-gap policy can be written as a function of vehicle speed:

$$s_{d,i} = \mathfrak{g}(v_i) \tag{5}$$

The main control objective for the controller for vehicle i is to track the predecessor with the desired gap with zero relative speed. Hence, at equilibria, all vehicles in a string travel at the same speed v_e and at their desired gaps $s_{e,i} = \mathfrak{g}(v_e)$. The desired and actual accelerations are thus equal to zero, $u_{e,i} = a_{e,i} = 0$.

B. Definition of string stability of homogeneous platoon

Before we detail the analytical stability analysis approach, we first define the string stability we use throughout this paper.

Definition 3.1: Let y denote the signal of interest, e.g. disturbance in speed or gap, and let Γ denote the frequency response function between the scalar output y_{i-1} of the preceding vehicle $i-1$ and the scalar output y_i of the follower i, i.e. $\Gamma(z) \equiv \frac{Y_i(z)}{Y_{i-1}(z)}$. A string of length m is said to be string stable if the condition

$$\sup|\Gamma_i(z)| = \sup_{\omega>0}|\Gamma_i(j\omega)| \le 1 \text{ , } 2 \le i \le m \tag{6}$$

holds for all frequencies of $\omega > 0$.
z denotes the complex variable of frequency. Definition 3.1 states that the disturbance of the signal of interest will not be amplified when propagating along the considered string.

Both speed and gap disturbances can be used as signals in determining string stability condition. However, we remark that the string stability conditions derived from speed disturbance and gap disturbance are only the same in homogeneous platoons, not in heterogeneous platoons as we will show in Section 4.

C. Speed error transfer function

To get the speed error transfer function, we first approximate the general control law (3) around equilibria:

$$\begin{aligned} &f(s_i(t-\xi_i), v_i(t-\xi_i), v_{i-1}(t-\xi_i)) \\ \approx\ & f_{s_i}\tilde{s}(t-\xi_i) + f_{v_i}\tilde{v}_i(t-\xi_i) + f_{v_{i-1}}\tilde{v}_{i-1}(t-\xi_i) \end{aligned} \tag{7}$$

where $\tilde{s}_i = s_i - s_e$, $\tilde{v}_i = v_i - v_e$ and $\tilde{v}_{i-1} = v_{i-1} - v_e$ denote the small perturbations in equilibrium gap and equilibrium speeds respectively and $f_{s_i} = \frac{\partial f}{\partial s_i}|_{v_e}$, $f_{v_i} = \frac{\partial f}{\partial v_i}|_{v_e}$, $f_{v_{i-1}} = \frac{\partial f}{\partial v_{i-1}}|_{v_e}$ are evaluated at equilibria. Inserting (7) into (4) and differentiating both sides of (4) and rearranging the equation we get:

$$\begin{aligned} \tau_i \ddot{a}(t) + f_{s_i}\tilde{v}_i(t-\xi_i) - f_{v_i}a_i(t-\xi_i) + \dot{a}_i(t) \\ = f_{s_i}\tilde{v}_{i-1}(t-\xi_i) + f_{v_{i-1}}a_{i-1}(t-\xi_i) \end{aligned} \tag{8}$$

Assuming zero initial conditions for speeds and accelerations, applying Laplace transform and rearranging the resulting equation gives the speed error transfer function in frequency domain as:

$$G_i(z) = \frac{\tilde{V}_i(z)}{\tilde{V}_{i-1}(z)} = \frac{(f_{v_{i-1}}z + f_{s_i})e^{-\xi_i z}}{\tau_i z^3 + z^2 - f_{v_i}e^{-\xi_i z}z + f_{s_i}e^{-\xi_i z}} \tag{9}$$

Equation (9) represents the speed disturbance propagation from the immediate predecessor to the follower.

D. Gap error transfer function

Given a general nonlinear gap policy of Eq. (5), the gap error is defined as:

$$e_{s,i} = s_i - \mathfrak{g}_i(v_i) \tag{10}$$

Differentiating the above equation gives

$$\dot{e}_{s,i} = v_{i-1} - v_i - z\dot{\mathfrak{g}}_i(v_i) = v_{i-1} - v_i - z\mathfrak{g}_i'(v_i)a_i \tag{11}$$

Note that the derivative term $\mathfrak{g}_i'(v_i) = \frac{d\mathfrak{g}_i}{dv_i}$ has physical meanings, i.e. it represents the *desired time gap* that the subject vehicle aims to maintain at equilibrium conditions and is a non-decreasing function of vehicle speed.

Performing Laplace transform and assuming zero initial conditions, we get:

$$zE_{s,i}(z) = V_{i-1}(z) - V_i(z) - zV_i(z)\mathfrak{g}_i'(v_i) \tag{12}$$

Similarly for vehicle $i-1$, we have:

$$zE_{s,i-1}(z) = V_{i-2}(z) - V_{i-1}(z) - zV_{i-1}(z)\mathfrak{g}_{i-1}'(v_{i-1}) \tag{13}$$

Using the relation of $G_i(z) = \frac{\tilde{V}_i(z)}{\tilde{V}_{i-1}(z)}$ and $G_{i-1}(z) = \frac{\tilde{V}_{i-1}(z)}{\tilde{V}_{i-2}(z)}$, we arrive at the following gap error transfer function:

$$H_i(z) = \frac{E_{s,i}}{E_{s,i-1}} = G_i(z)\mathcal{G}_i(z) \tag{14a}$$

$$\mathcal{G}_i(z) = \frac{1/G_i(z) - 1 - z\mathfrak{g}_i'(v_i)}{1/G_{i-1}(z) - 1 - z\mathfrak{g}_{i-1}'(v_{i-1})} \tag{14b}$$

Equation (14) represents the gap disturbance propagation from the immediate predecessor to the follower and has the speed disturbance propagation function as one of its components.

E. String stability criteria

The disturbance propagation functions derived so far, we can specify Condition 3.1 as:

Condition 3.2: String stability of a homogeneous vehicle platoon is guaranteed if:

$$|H_i(j\omega)| < 1 \text{ for } \forall\omega > 0 \tag{15}$$

Condition 3.3: One sufficient condition for string stability of a general vehicle platoon is:

$$|G_i(j\omega)| < 1 \text{ and } |\mathcal{G}_i(j\omega)| < 1 \text{ for } \forall\omega > 0 \tag{16}$$

Remark 3.4: For homogeneous vehicle strings, where $G_i(z) = G_{i-1}(z)$ and $\mathfrak{g}_i'(v_i) = \mathfrak{g}_{i-1}'(v_{i-1})$, we have $\mathcal{G}_i(j\omega) = 1$ and thus:

$$H_i(z) = G_i(z) = \frac{(f_{v_{i-1}}z + f_{s_i})e^{-\xi_i z}}{\tau_i z^3 + z^2 - f_{v_i}e^{-\xi_i z}z + f_{s_i}e^{-\xi_i z}} \tag{17}$$

Remark 3.4 states that for homogeneous vehicle platoons, the gap error propagation is equivalent to the speed error propagation in frequency domain.

For notation simplicity, we drop the index i and use $f_{v_p} = f_{v_{i-1}}$ to represent the derivative of the predecessor in the following Theorem:

Theorem 3.5: For any given frequency $\omega > 0$, the upper bound of the magnitude of the gap error transfer function is:

$$\sup|H(j\omega)| = \sup|G(j\omega)| = \frac{f_{v_p}^2\omega^2 + f_s^2}{\tau^2\omega^6 + (1 + 2f_v\tau + 2f_s\tau\xi + 2f_v\xi)\omega^4 + (f_v^2 - 2f_s)\omega^2 + f_s^2} \tag{18}$$

Proof: See Appendix.

Corollary 5.1: Sufficient string stability condition 1: The first sufficient condition for string stability of homogeneous strings with two delays derived from Theorem 3.5 is:

$$A_2 = -2f_s + f_v^2 - f_{v_p}^2 > 0 \tag{19a}$$

$$A_4 = 1 + 2f_v\tau + 2f_s\tau\xi + 2f_v\xi > 0 \tag{19b}$$

Proof: Proof of Corollary 5.1 is straight forward. Notice that $A_6 = \tau^2 > 0$, string stability requires $\sup|H(jw)| < 1$, which is unconditionally satisfied if $A_4 = 1 + 2f_v\tau + 2f_s\tau\xi + 2f_v\xi > 0$ and $-2f_s + f_v^2 > f_{v_p}^2$ (or equivalently $A_2 = -2f_s + f_v^2 - f_{v_p}^2 > 0$)

Conditions (19a,19b) give the relations between model/design parameters (embedded in the partial derivatives of the acceleration model/controller) and the two delays, which can be used for control parametrisation, c.f. Section III-F. The two conditions imply that the string stability condition for systems with delay are stricter than those without delay.

Corollary 5.2: Sufficient string stability condition 2: a second sufficient condition for string stability derived from 3.5 is:

$$A_4 < 0 \text{ and } A_2 > \frac{A_4^2}{4A_6} \tag{20}$$

Proof: If $A_4 < 0$, $|H(z)| < 1$ can still be respected if

$$\tau^2\omega^6 + (1 + 2f_v\tau + 2f_s\tau\xi + 2f_v\xi)\omega^4 + (-2f_s + f_v^2)\omega^2 > f_{v_p}^2\omega^2 \tag{21}$$

Equivalently, we can solve the following polynomial inequality:

$$\min_{\omega>0} Y(\omega) = A_6\omega^4 + A_4\omega^2 + A_2 > 0 \text{ , } \forall\omega > 0 \tag{22}$$

with $A_6 = \tau^2 > 0$. One can find the roots of the polynomial with $\frac{\partial Y(\omega)}{\partial \omega} = 0$ and after removing the negative and zero roots and inserting ω^* to Y we arrive at:

$$\min_{\omega>0} Y(\omega) = A_6 \left(\sqrt{\frac{-A_4}{2A_6}}\right)^4 + A_4 \left(\sqrt{\frac{-A_4}{2A_6}}\right)^2 + A_2 > 0 \quad (23)$$

which will be respected if $A_2 > \frac{A_4^2}{4A_6}$.

In the remainder of this section, we will demonstrate the applicability of the analytical conditions for parameters design of ACC systems.

F. Worked example: Linear ACC controller with CTG policy

Consider the widely used linear ACC controller with the so-called constant time gap policy [1] as:

$$u_i = k_v(v_{i-1} - v_i) + k_s(s_i - v_i * t_d - s_0) \quad (24)$$

with sensor delay ξ_i and actuator lag τ_i. k_v and k_s are feedback gains on speed error and gap error respectively and t_d is the constant desired time gap. s_0 denotes the gap at standstill conditions.

The derivatives of the control law are given as:

$$f_s = k_s,\ f_{v_p} = k_v,\ f_v = -k_v - k_s * t_d \quad (25)$$

The gap/speed error transfer function 9 is specified as

$$H(z) = \frac{(k_v z + k_s)e^{-\xi_i z}}{\tau_i z^3 + z^2 + (k_v + t_d k_s) z e^{-\xi_i z} + k_s e^{-\xi_i z}} \quad (26)$$

Note that $k_s > 0$, $k_v > 0$ and given the reasonable assumption that the desired time gap is larger than sensor delay or actuator lag, i.e. $t_d > \xi$ and $t_d > \tau$, applying Corollary 5.1 gives the first sufficient condition for string stability as:

$$k_s t_d^2 + 2k_v t_d - 2 > 0 \quad (27a)$$

$$1 - 2k_v(\tau + \xi) > 0 \quad (27b)$$

Applying Corollary 5.2 gives the second sufficient condition for string stability as:

$$k_s > \frac{t_d - 2(\tau + \xi)}{2t_d(t_d(\tau + \xi) - \tau\xi)} \quad (28a)$$

$$k_s^2 t_d^2 + 2k_s k_v t_d - 2k_s > \frac{(1 - 2(k_v + k_s t_d)(\tau + \xi) + 2k_s \tau \xi)^2}{\tau^2} \quad (28b)$$

We can plot the stability conditions in a two-dimensional plane of control and system parameters in Fig.s 1-2, which gives four regions:

- *Type I stability*, stable due to satisfying string stability condition 27;
- *Type II stability*, stable due to satisfying string stability condition 28;
- *Type I instability*, unstable due to violating $A_2 > 0$;
- *Type II instability*, unstable due to violating $A_2 > \frac{A_4^2}{4A_6}$ while $A_4 < 0$.

In the sequel, we present some interesting insights from the stability diagrams 1-2.

1) Influence of control parameters: feedback gains and desired time gap: Insights into the influence of control parameters on the string stability of a homogeneous platoon can be gained via examining the stability diagram of Fig. 1 with $\xi = 0.2$ s, $\tau = 0.2$ s.

- Influence of feedback gain on gap error k_s: when k_s is small, increasing k_s has the stabilising effect. However, when it is large, increasing it may destabilise the string.
- Similar trend is observed for the feedback gain on speed error k_v.
- At small time gaps, e.g. $t_d = 1.0$ s, the feedback gain on speed error needs to be set high while the feedback gain on gap error needs to be set low to guarantee Type I string stability.
- When t_d is below some threshold, e.g. $t_d < 1.6$ s, increasing t_d enlarges the area of stability I and II regions. When t_d increases further, the stability regions does not increase significantly, but shift towards the origin (0,0).

2) Influence of system properties: sensor delay and actuator lag: It can be shown that without any delay, only $A_2 > 2$ is needed to guarantee string stability and the larger A_2 , the smaller the magnitude of the transfer function (18). In this case, increasing k_v and k_s will certainly increase string stability. However, this is not the case when considering delays. The stabilising effect of increasing feedback gains of k_v and k_s only works in certain regions, but not in all regions, cf. Fig. 1. This implies that delays may have negative effects on string stability. Below we analytically impart the intuition of the influence of two delays on string stability. We show in Eq. (52) in Appendix that $f_v + f_s\tau < 0$. The same procedure can be used to derive the relation between f_s and ξ as:

$$f_v + f_s\xi < 0 \quad (29)$$

Using these two relations, Condition 19b can be rewritten as:

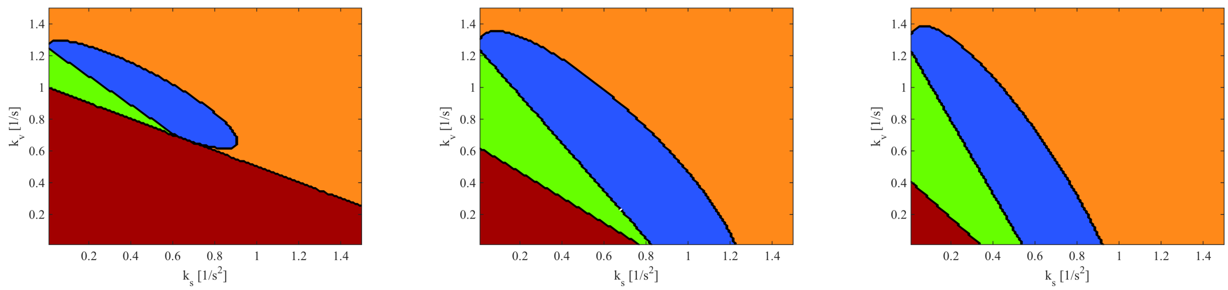
$$1 + 2(f_v + f_s\tau)\xi + 2f_v\tau > 0 \quad (30a)$$

$$\text{or} \quad 1 + 2(f_v + f_s\xi)\tau + 2f_v\xi > 0 \quad (30b)$$

Note that $f_v < 0$, hence increasing the value of τ and ξ tends to violate Condition 19b, thus deteriorates string stability. The relative change is the same for τ and ξ in Eq. (30a) since it is symmetric in both parameters.

Nevertheless, increasing τ also increase the coefficient of the higher order term A_6 in (18), which has a stabilising effect. Thus the destabilisation effect is less serious for increasing τ than the same magnitude of increase in ξ [10]. When the frequency of the disturbance is high enough, e.g. higher than the system and control parameters (which are often less than 2), it even circumvents the destabilising effect of τ and becomes the major determinant of the resulting string stability.

Fig. 2 depicts the supreme of the magnitude of the gap error transfer function (18) at different frequencies in the two dimension-plane of τ and ξ using Eq. 18. As we can see from the figure:



(a) $t_d = 1.0$ s (b) $t_d = 1.6$ s (c) $t_d = 2.4$ s

Fig. 1. Stability diagram in $k_s - k_v$ plane with two delays $\xi = 0.2$ s and $\tau = 0.2$ s. Green region depicts Type I stability; blue region depicts Type II stability; red region depicts Type I instability; orange region depicts Type II instability.

- At low disturbance frequencies (e.g., $\omega <= 0.1$ rad/s), increasing delay or actuator lag tends to increase the magnitude of the gap error transfer function and consequently destabilises vehicle strings.
- At high frequencies (e.g., $\omega >> 1$), the higher order term $\tau^2\omega^6$ will dominate the magnitude of the transfer function and increasing actuator lag tends to stabilise the string while increasing sensor delay always tends to destabilise vehicle strings.
- At mediate frequencies, e.g. $w = 0.8$ rad/s, increasing sensor delay will destabilise the string. Increasing actuator lag will first deteriorate string stability performance, but after a certain level, it will tend to stabilise the string. This clearly shows the trade off of the destabilising effect of the 4th order term and stabilising effect of the 6th order term with the increase of τ.

G. *Verification by simulation*

The stability diagram shown so far gives an intuitive means to design control parameters for string stability. To test the analytical results, we simulate a platoon of vehicles under an exogenous leader. We impose step deceleration disturbance followed by acceleration disturbance of the leader (see leader acceleration profile in Fig. 3(d)) and a platoon of 5 followers employing the linear ACC control law with the linear system dynamics (2) are simulated. We test different k_s and k_v combinations at the desired time gap of $t_d = 1.2$ s to generate stable string (Fig. 3(a)), unstable string with Type I instability (Fig. 3(b)) and unstable string with Type II instability (Fig. 3(c)) as predicted by Corollary 5.1 and 5.2. The vehicle accelerations of the vehicles for the three platoons are shown in Fig. 3(d), 3(e), 3(f) respectively.

The acceleration plots verify the analytical predictions.

IV. String stability condition of heterogeneous platoon

The previous section lays the foundation for string stability analysis by deriving conditions for homogeneous strings. This section goes a step further to string stability condition of heterogeneous platoon with different control parameters and system properties.

A. *Strict and head-to-tail string stability: definition*

For a vehicle string with heterogeneous system and control parameters, string stability can be examined by the transfer function of the gap error from the first follower to the last follower. For a heterogeneous platoon of length k, we can distinguish between string and weak string stability concepts. For strict string stability, we require the magnitude of the gap error transfer function in each predecessor-follower pair in the platoon less than unity:

$$|H(z)| = \left|\frac{E_{s,i}(z)}{E_{s,i-1}(z)}\right| < 1, \forall i \leq k \tag{31}$$

Derivation of strict string stability condition is straightforward in the sense that one can apply Corollary 5.1 and 5.2 to each predecessor-follower pair. Condition (31) is rather strict whereas in a heterogeneous string setting, disturbances amplified by a string-unstable predecessor-follower pair can be damped out by an upstream string-stable predecessor-follower pair. Hence, a weak string stability concept is defined as:

$$|\bar{H}_{1-k}(z)| = \left|\frac{E_{s,k}(z)}{E_{s,1}(z)}\right| < 1 \tag{32}$$

The weak string stability is also referred to as head-to-tail string stability [5]. In the sequel, We employ this definition to derive the analytical string stability criterion for heterogeneous platoons with length k.

B. *Conditions for strict and head-to-tail string stability*

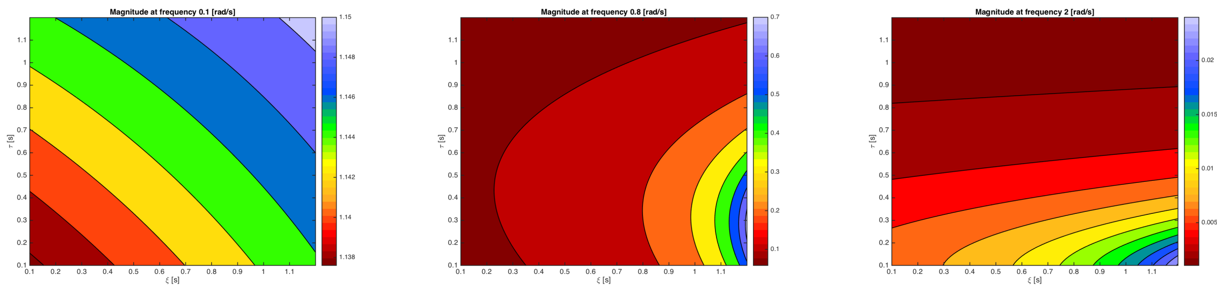
Given a general nonlinear gap policy of Eq. (5), the gap error function of the 1st and the kth vehicle is:

$$e_{s,1} = x_0 - x_1 - l_1 - \mathfrak{g}_1(v_1) \tag{33a}$$
$$e_{s,k} = x_{k-1} - x_k - l_k - \mathfrak{g}_k(v_k) \tag{33b}$$

where l_k is the length of vehicle k. Here we use $\mathfrak{g}_k$ to differentiate the gap policies of vehicles in the heterogeneous string, whereas in literature, most of them refer to a homogeneous gap policy.

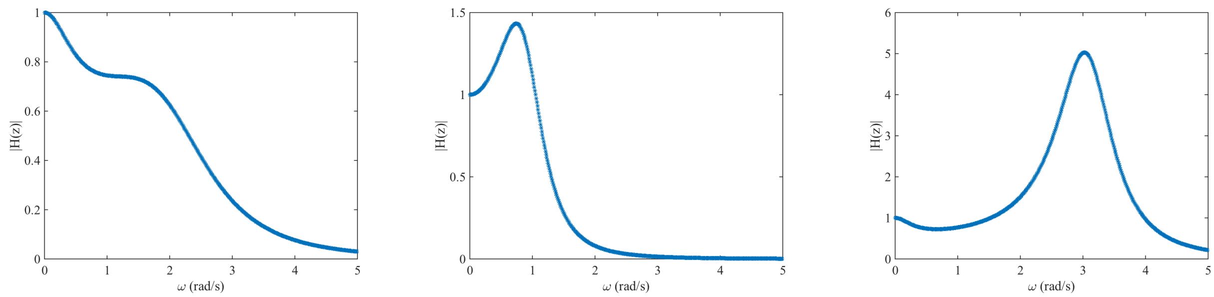
Differentiating the above equation arrives at:

$$\begin{aligned}\dot{e}_{s,1} &= v_0 - v_1 - \mathfrak{g}_1'(v_1)a_1 \\ \dot{e}_{s,k} &= v_{k-1} - v_k - \mathfrak{g}_k'(v_k)a_k\end{aligned} \tag{34}$$

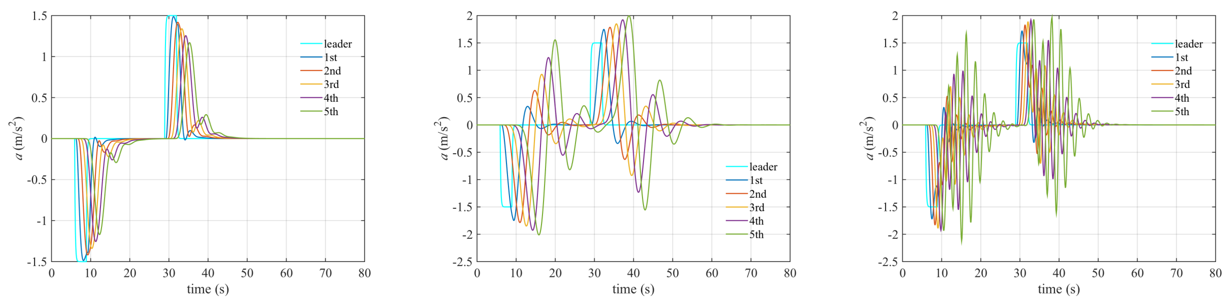


(a) Magnitude at $\omega = 0.1$ rad/s (b) Magnitude at $\omega = 0.8$ rad/s (c) Magnitude at $\omega = 2$ rad/s

Fig. 2. Magnitude contour plot in $\xi - \tau$ plane at different frequencies with $k_s = 0.1$, $k_v = 0.15$, $t_d = 1.5$



(a) $\sup|H(z)|$ under string stability with $k_s = 0.6$, $k_v = 0.8$, and $t_d = 1.2$ (b) $\sup|H(z)|$ under Type I instability with $k_s = 0.6$, $k_v = 0.2$, and $t_d = 1.2$ (c) $\sup|H(z)|$ under Type II instability with $k_s = 0.6$, $k_v = 1.5$, and $t_d = 1.2$



(d) Acceleration profiles under string stability (e) Acceleration profiles under Type I instability (f) Acceleratio profiles under Type II instability

Fig. 3. Magnitude as a function of frequency under different parameter settings

Performing Laplace transform to the above equation and using the relation of $G_i(z) = \frac{V_i(z)}{V_{i-1}(z)}$ we can get:

$$H_i(z) = \frac{E_{s,i}}{E_{s,i-1}} = G_i(z)\frac{1/G_i(z) - 1 - z\mathfrak{g}_i'(v_i)}{1/G_{i-1}(z) - 1 - z\mathfrak{g}_{i-1}'(v_{i-1})} \quad (35)$$

The head-to-tail gap error propagation function $\bar{H}_{1-k}(z)$ is:

$$\begin{aligned}\bar{H}_{1-k}(z) &= \frac{E_{s,k}}{E_{s,1}} = \frac{E_{s,k}}{E_{s,k-1}} \cdot \frac{E_{s,k-1}}{E_{s,k-2}} \cdot \ldots \cdot \frac{E_{s,2}}{E_{s,1}} \\ &= \frac{1/G_k(z) - 1 - z\mathfrak{g}_k'(v_k)}{1/G_0(z) - 1 - z\mathfrak{g}_0'(v_0)} \prod_{j=1}^{k} G_j(z) \end{aligned} \quad (36)$$

where the speed error transfer function $G_i(z)$ is defined in Eq. (9).

Strict string stability is guaranteed if:

$$\left|\frac{1/G_i(z) - 1 - z\mathfrak{g}_i'(v_i)}{1/G_{i-1}(z) - 1 - z\mathfrak{g}_{i-1}'(v_{i-1})}\right| |G_i(z)| < 1 \quad (37)$$

$\forall \omega > 0$ and $\forall i \in [1, k]$.

Head-to-tail string stability is guaranteed if:

$$\left|\frac{1/G_k(j\omega) - 1 - j\omega\mathfrak{g}_k'(v_k)}{1/G_0(\omega) - 1 - j\omega\mathfrak{g}_0'(v_0)}\right| \prod_{j=1}^{k} |G_j(j\omega)| < 1 \quad (38)$$

$\forall \omega > 0$.

Since we have derived the upper bound of the speed error transfer function $G(z)$ in (18), this gives the following *sufficient condition for strict string stability* as:

Condition 4.1: Strict string stability for heterogeneous vehicle string is guaranteed if:

$$\left|\frac{1/G_i(j\omega) - 1 - z\mathfrak{g}_i'(v_i)}{1/G_{i-1}(j\omega) - 1 - z\mathfrak{g}_{i-1}'(v_{i-1})}\right| < 1 \quad (39a)$$

$$|G_i(j\omega)| < 1 \quad (39b)$$

for $1 \leq i \leq k$ with G_i specified as Eq. (18).

Similarly, we can get the following *sufficient condition for head-to-tail string stability* as:

Condition 4.2: Head-to-tail string stability is guaranteed if:

$$\left|\frac{1/G_k(j\omega) - 1 - z\mathfrak{g}'_k(v_k)}{1/G_0(j\omega) - 1 - z\mathfrak{g}'_0(v_0)}\right| < 1 \quad (40a)$$

$$|G_i(j\omega)| < 1 \quad (40b)$$

for $1 \leq i \leq k$ with G_i specified as Eq. (18).

If the model parameters are known, Eqs. (39a,40a) can be numerically calculated using Padé approximation.

C. Verification by simulation

To verify the sufficient conditions for strict and head-to-tail string stability, we simulate three heterogeneous platoon of four vehicles with three ACC followers employing different control parameters following an exogenous leader under periodic speed oscillations. We choose the parameters such that the three platoons exhibit strict string stability (Fig. 4(a)), head-to-tail string stability (Fig. 4(b)) and string instability (Fig. 4(c)) according to analytical prediction respectively.

Fig. 4(a) shows the strict string stable case, as the magnitudes of the gap error transfer functions for all followers are less than unity (see Fig. 4(a)). The simulation confirms the analytical prediction, as we can see clearly from Fig. 4(d) that the disturbance in gap error is attenuated monotonically when propagating along the platoon.

Fig. 4(b) shows the not strict but head-to-tail string stable case, as the first and the third followers are string stabilisers while the second follower is a destabiliser (see Fig. 4(b)). As we can see from this figure, homogeneous strings consisting of only vehicle 1 or 3 are string stable, while homogeneous strings consisting of only vehicle 2 is string unstable. This violates the strict string stability condition 4.1. Nevertheless, the upper bound of the gap error transfer function for the whole platoon (head-to-tail) is less than unity, thus Condition 4.2 is respected and the mixed string is head-to-tail string stable. Fig. 4(e) verifies the analytical prediction, i.e. follower 2 amplifies the disturbance as a reaction to the behaviour of follower 1. But follower 3 is able to damp out the disturbance amplification. As a result, the gap error of follower 3 is not amplified compared to follower 1.

Fig. 4 shows the string unstable case. depicts the analytical prediction of the disturbance transfer function for the three followers. As we can see from the analytical prediction in Fig. 4(c), homogeneous strings for each follower is string unstable. Consequently, strict and head-to-tail string stability conditions are violated. This prediction is verified by simulations in Fig. 4(f). These figures demonstrate that the disturbance in the leader speed profile is amplified through the platoon.

V. Concluding remarks

We developed a methodology to analyse string stability properties for heterogeneous vehicles employing a general class of vehicle following control systems. The frequency domain analysis of gap error transfer function enables the determination of sufficient string stability criteria of heterogeneous vehicle strings. The analytical string stability conditions give new insights into the relationship between the string stability properties of vehicle strings in relation to the system properties of time delays and controller design parameters of feedback gains and desired time gap. Analytical results are verified via systematic simulation of both homogeneous and heterogeneous strings. Simulations demonstrate the predictive power of the analytical string stability conditions.

Future work is directed to apply the method in designing communication topologies for cooperative vehicle systems under V2V communication and adaptively changing the control parameters of the controlled vehicles using V2I communications.

Appendix: Proof of Theorem 3.5

For homogeneous strings with delays, the speed transfer function 9 can rewritten as:

$$\begin{aligned} G(z) &= \frac{(f_{v_p}z + f_s)e^{-\xi z}}{\tau z^3 + z^2 - f_v e^{-\xi z} z + f_s e^{-\xi z}} \cdot \frac{e^{\xi z}}{e^{\xi z}} \\ &= \frac{(f_{v_p}z + f_s)}{\tau z^3 e^{\xi z} + z^2 e^{\xi z} - f_v z + f_s} = \frac{UP}{LP} \end{aligned} \quad (41)$$

Inserting $z = j\omega$ into the upper part of the aforementioned equation and take the magnitude:

$$UP = j\omega f_{v_p} + f_s \quad (42)$$

$$|UP| = \omega^2 f_{v_p}^2 + f_s^2 \quad (43)$$

Inserting $z = j\omega$ into the lower part of the aforementioned equation and note:

$$e^{\xi z} = e^{\xi j\omega} = \cos(\xi\omega) + j\sin(\xi\omega) \quad (44)$$

$$LP = \underbrace{\tau z^3 e^{\xi z}}_{(1)} + \underbrace{z^2 e^{\xi z}}_{(2)} \underbrace{- f_v z}_{(3)} + \underbrace{f_s}_{(4)} \quad (45)$$

Inserting Eq. (44) gives:

$$\begin{aligned} LP = &\underbrace{\omega^2(\tau\omega\sin(\xi\omega) - \cos(\xi\omega)) + f_s}_{(5)} \\ &\underbrace{- j\left(\omega^2(\tau\omega\cos(\xi\omega) + \sin(\xi\omega)) + f_v\omega\right)}_{(6)} \end{aligned} \quad (46)$$

Thus the magnitude of the lower part $|LP| = (5)^2 + (6)^2$, with

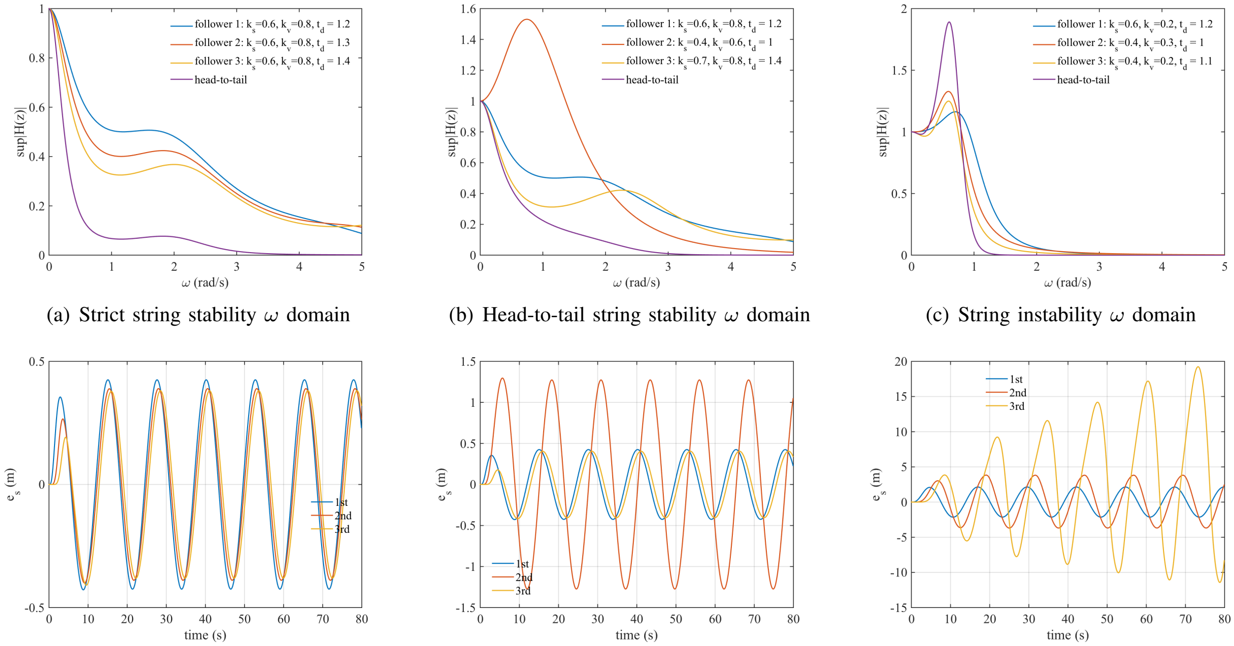
$$\begin{aligned} &(5)^2 = \sin^2(\xi\omega)\omega^6 - 2\tau\sin(\xi\omega)\cos(\xi\omega)\omega^5 \\ &+ \cos^2(\xi\omega)\omega^4 + 2f_s\tau\sin(\xi\omega)\omega^3 - 2f_s\cos(\xi\omega)\omega^2 + f_s^2 \end{aligned} \quad (47)$$

$$\begin{aligned} &(6)^2 = \tau^2\cos^2(\xi\omega)\omega^6 + 2\tau\sin(\xi\omega)\cos(\xi\omega)\omega^5 \\ &+ \sin^2(\xi\omega)\omega^4 + f_v^2\omega^2 + 2f_v\tau\cos(\xi\omega)\omega^4 + 2f_v\sin(\xi\omega)\omega^3 \end{aligned} \quad (48)$$

$$\begin{aligned} &(5)^2 + (6)^2 = \tau^2\omega^6 + (1 + 2f_v\tau\cos(\xi\omega))\omega^4 \\ &+ 2(f_s\tau + f_v)\sin(\xi\omega)\omega^3 + (-2f_s\cos(\xi\omega) + f_v^2)\omega^2 + f_s^2 \end{aligned} \quad (49)$$

String stability requires $\left|\frac{UP}{LP}\right| \leq 1$, which is equivalent to:

$$\|LP\|_2 \geq \|UP\|_2 \quad (50)$$



(a) Strict string stability ω domain

(b) Head-to-tail string stability ω domain

(c) String instability ω domain

(d) Strict string stability t domain

(e) Head-to-tail string stability t domain

(f) String instability t domain

Fig. 4. Strict string stability, head-to-tail string stability and string instability of heterogeneous platoon

with

$$|LP| = \tau^2\omega^6 + (1 + 2f_v\tau\cos(\xi\omega))\omega^4$$
$$+2(f_s\tau + f_v)\sin(\xi\omega)\omega^3 + (-2f_s\cos(\xi\omega) + f_v^2)\omega^2 + f_s^2 \quad (51)$$

We can prove [14] that if *the desired time gap* $\mathfrak{g}'(v)$ *is larger than the actuator delay*, then:

$$f_v + f_s\tau < f_v + f_s\mathfrak{g}'(v) \leq 0 \quad (52)$$

Note that $\sin(\xi\omega) \leq \xi\omega$ for $\xi \geq 0$ and $\omega > 0$, $\cos(\xi\omega) \leq 1$, $f_v < 0$ and $f_v + f_s\tau < 0$, $|LP|$ becomes:

$$|LP| \geq$$
$$\tau^2\omega^6 + (1 + 2f_v\tau)\omega^4 + 2(f_s\tau + f_v)\xi\omega\omega^3 + (-2f_s + f_v^2)\omega^2 + f_s^2$$
$$= \tau^2\omega^6 + (1 + 2f_v\tau + 2f_s\tau\xi + 2f_v\xi)\omega^4 + (-2f_s + f_v^2)\omega^2 + f_s^2$$

thus condition $\left|\frac{UP}{LP}\right| \leq 1$ simplifies as:

$$\frac{f_{v_p}^2\omega^2 + f_s^2}{\tau^2\omega^6 + (1 + 2f_v\tau + 2f_s\tau\xi + 2f_v\xi)\omega^4 + (-2f_s + f_v^2)\omega^2 + f_s^2} \leq 1$$

Q.E.D.

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