Joint Route Guidance and Demand Management Strategy in the Presence of Uncertain Demand

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***Abstract*— This work proposes a novel formulation for the joint route guidance and demand management problem, taking into account the uncertainty in traffic demand. Previous attempts to address this problem aimed to minimize the total time spent by all vehicles in the network by determining the optimal routes and departure times, assuming perfect knowledge of traffic demand. In contrast to prior approaches, this work introduces a more realistic model that incorporates uncertain demand. Doing so results to a stochastic model predictive control model with nonconvex nonlinear constraints. To address the stochastic nature of the problem, a scenario-based formulation is introduced, which uses a Gaussian Processes framework to generate multiple scenarios. In addition, an efficient solution methodology over the scenario-based formulation is proposed that relaxes the original nonlinear problem into a linear problem, significantly enhancing its computational tractability. Moreover, in the proposed solution methodology a quadratic reformulation is derived that ensures feasibility over the original problem space. Simulation results demonstrate the superiority of the proposed scenario-based methodology over the worst-case and simplistic averaging approaches.**

I. Introduction

Recent advances in Information and Communication Technologies (ICT) have enabled the development of a plethora of traffic management schemes that can address the problem of traffic congestion [1]. Among these, the primary strategies are route guidance and perimeter control approaches [2]. Route guidance provides drivers with optimized route instructions, helping them avoid congested parts of the network and reducing travel times [1]. Perimeter control regulates the transfer flows between neighboring regions to prevent congestion from spilling over into adjacent areas [3]. Although these strategies are highly popular in the literature, their effectiveness is limited, as they can only reduce but not entirely avoid congestion [4].

An alternative approach to manage traffic is through demand management strategies, which aim to regulate the inflow rate of vehicles into the network [5]. These strategies are mainly economic policies that influence drivers' decisions on choosing alternative departure times or modes of transport [5]. Their effectiveness is based on the fact that these solutions aim to enhance the social optimum by redistributing traffic across time and space. By doing so, the emergence of congestion is avoided, and travel times are sustained around those achieved under free-flow speed conditions [6].

A novel formulation that jointly integrated route guidance with demand management has been proposed earlier by the authors in [7]. This approach computes and manages vehicle routes (i.e., route guidance) and departure times (i.e., demand management) based on macroscopic traffic dynamics. The strong benefit of using macroscopic traffic dynamics is that they rely solely on the three fundamental traffic parameters of speed, flow, and density, as defined by the concept of the Macroscopic Fundamental Diagram (MFD) [8]. Evidently, many previous works have been proposed in this context, such as multi-regional route guidance, which guides vehicular flows to follow multi-regional routes with the aim of reducing the experienced travel times [2]. Similarly, the work in [9] leverages macroscopic traffic dynamics to develop a multi-regional demand management method that regulates vehicles' departure times to minimize their total travel time in the network. However, despite their effectiveness, a major drawback of the aforementioned approaches is that they do not consider the effects of stochastic uncertainty and fluctuations in traffic dynamics and demand. These stochastic effects have shown to significantly impact their performance, making them impractical for their real-life application [10].

Several recent works have focused on addressing modeling uncertainties in traffic dynamics. For instance, the work in [11] provides a detailed case study that presents a methodological framework for incorporating stochastic effects into traffic modeling. Likewise, the work in [12] extends the MFD model to include uncertainty and proposes a Stochastic Model Predictive Control (SMPC) method [13] for traffic management. The latter approach achieves higher trip completion rates compared to the deterministic case, highlighting the significance of considering uncertainty in traffic management strategies.

In this work, we propose a SMPC formulation [14] to address the challenges of joint route guidance and

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demand management, considering the uncertainty associated with vehicle departure times, which represent demand. We build upon our earlier work in [7], but we no longer assume that the exact demand is known. Instead, we consider demands that are uncertain characterized by known mean and variance. To mitigate the inherent stochastic nature of the SMPC formulation, we introduce a Scenario-Based Model Predictive Control (SBMPC) formulation. This formulation utilizes the Gaussian Processes (GP) [15] to generate a diverse set of scenarios, incorporating the mean and variance of the demands. To generate these scenarios, we rely on historical data of traffic demands for each region in the considered urban network. Under this setting, demands are treated as unknown random functions of time, modeled as multivariable Normal distributions [15]. To learn the underlying demand function, we employ Gaussian Processes Regression (GPR), which allows us to predict a set of future scenarios for a specific time period on a given day [15]. By generating multiple scenarios using GPR, this work is able to capture a wide range of possible outcomes, providing an accurate representation of the uncertain demand.

Moreover, to efficiently solve the resulting nonlinear SBMPC formulation, we propose an efficient algorithmic methodology. This methodology relaxes the original nonlinear problem into a linearized model, significantly enhancing its computational tractability. Additionally, we develop a quadratic optimization procedure to ensure the feasibility of the solution derived for the linear model, thereby overcoming the potential issue of infeasibility in the original nonlinear program. The main contributions of this work can be summarized as follows:

- We propose a mathematical formulation for the joint route guidance and demand management problem, considering demand uncertainties. The resulting formulation is a stochastic, nonconvex, nonlinear program that effectively manages the admission of vehicles entering the network and the transfer of flows between regions. The objective is to minimize the *expected* total time spent by all vehicles in the network. To deal with the challenges arising from the stochastic formulation, we also propose a scenario-based strategy that approximates the SMPC problem with a deterministic formulation.
- We develop a linear approximation approach that provides fast and high-quality solutions for the SBMPC formulation. Recognizing that these solutions might not always be feasible, we further develop a quadratic optimization procedure to obtain feasible solutions.

Section II presents the demand and traffic flow model in subsections II-A and II-B, respectively. Thereafter, Section III provides the mathematical formulation of the SMPC and SBMPC problems in subsections III-C and III-D, respectively. Next, Section IV develops a practical and efficient upper-bound algorithmic methodology. It begins by approximating the scenario-based formulation with a linear program and then uses the obtained results to create high-quality feasible control decisions. Subsequently, Section V evaluates the proposed solutions, and finally, Section VI wraps up this work and explores potential avenues for future research.

II. Traffic flow model dynamics

A. Demand Model

Consider an urban area that is divided into a set of homogeneous regions, with each region $r \in \mathcal{R}$ where, $\mathcal{R} = \{1, \ldots, R\}$. Let $\mathcal{O}, \mathcal{D} \subseteq \mathcal{R}$ represent the set of regions considered as origins and destinations, respectively. Let the time be discretized into time steps of duration T_s, where at each discrete time step, $\tau \in \mathcal{T}$, where $\mathcal{T} = \{1, 2, \ldots, T\}$, a certain number of vehicles intend to enter the network, from an origin region $o \in \mathcal{O}$ towards a destination region $d \in \mathcal{D}$. This quantity is termed as the *instantaneous external demand* and is denoted by variable, $d_{od}(\tau)$, (veh). An assumption of this work is that the instantaneous external demand is considered as an unknown time-dependent function, with its value being revealed at the current time step. Moreover, let the variables $\tilde{d}_{od}(\tau)$ (veh) and $D_{od}(\tau)$ (veh) denote the *admitted external demand* and the *cumulative external demand*, respectively. Specifically, the admitted external demand refers to the actual number of vehicles that are permitted to start their trips at $\tau \in \mathcal{T}$. Moreover, the cumulative external demand represents the demand that remains outside the network at each time step τ. Essentially, the admitted external demand represents the portion of the cumulative external demand which is admitted to the network, expressed as follows:

$$\tilde{d}_{od}(\tau) = \tilde{u}_{od}(\tau) D_{od}(\tau), \quad \tau = 1, 2, \ldots, \tag{1}$$

where, the variable $\tilde{u}_{od}(\tau) \in [0, 1]$ represents the control variable which denotes the portion of demand allowed to enter the network. The dynamics of the demand at each time step τ are defined as:

$$D_{od}(\tau+1) = D_{od}(\tau) - \tilde{d}_{od}(\tau) + d_{od}(\tau+1), \quad \tau = 1, 2, \ldots, \tag{2}$$

where, $D_{od}(1) = d_{od}(1)$.

B. Traffic Flow Model

This work assumes that the dynamics of each region are captured by macroscopic traffic dynamics, using a generalized MFD shape [8]. The key macroscopic parameters of each region $r \in \mathcal{R}$ include the *jam density*, ρ_r^J, the *critical density*, ρ_r^C, the *free-flow speed*, u_r^f, and the *capacity*, q_r^C. Each time step τ is assumed to have a duration of T_s (s), while parameters L_r, λ_r and l_r represent the total length of region r, the average trip length within region r, and the ratio between L_r and λ_r (i..e., $l_r = L_r/\lambda_r$), respectively. Furthermore, the variable $q_r(\rho_r(\tau))$ (veh/h) denotes the *intended outflow* of region r at τ that is the prospective traffic discharge from region r at timestep τ. This is a function of density, $\rho_r(\tau)$ (veh/km), and is calculated as the product of density and speed, $v_r(\rho_r(\tau))$ (km/h), such that:

$$q_r(\rho_r(\tau)) = l_r \rho_r(\tau) v_r(\rho_r(\tau)). \tag{3}$$

In other words, the intended outflow characterizes the total flow that region r could potentially transit either, as completed trips in region r, or to its neighboring regions, assuming that there are no constraints on inter-region boundary capacity. In this work the intended outflow is modeled via a uni-modal nonlinear MFD function (e.g., third order polynomial), $f_r(\rho_r(\tau))$, which can be mathematically expressed as:

$$q_r(\rho_r(\tau)) = l_r f_r(\rho_r(\tau)). \tag{4}$$

Further, we represent all neighboring regions of a region $r \in \mathcal{R}$, that are directly accessible from it by the set $\mathcal{J}_r^- \subseteq \mathcal{R}$. Additionally, we define the set $\mathcal{J}_r^+ = \mathcal{J}_r^- \cup r$ as follows:

$$\mathcal{J}_r = \begin{cases} \mathcal{J}_r^+, & \text{if } r \in \mathcal{D}, \\ \mathcal{J}_r^-, & \text{otherwise.} \end{cases} \tag{5}$$

Furthermore, to account for the portion of traffic destined to different regions, we introduce variables $\rho_{rd}(\tau)$ which denote the density in region $r \in \mathcal{R}$ destined for region $d \in \mathcal{D}$ associated as follows:

$$\rho_r(\tau) = \sum_{d \in \mathcal{D}} \rho_{rd}(\tau). \tag{6}$$

Likewise, the variables $q_{rd}(\tau)$ and $q_{rjd}(\tau)$ express the *intended transfer flow* from $r \in \mathcal{R}$ to $d \in \mathcal{D}$ and from $r \in \mathcal{R}$ to $d \in \mathcal{D}$, through neighbouring region $j \in \mathcal{J}_r$ at time step τ, respectively, defined as:

$$q_{rjd}(\tau) = u_{rjd}(\tau) q_{rd}(\tau), \tag{7}$$

$$q_{rd}(\tau) = l_r v_r(\tau) \rho_{rd}(\tau), \tag{8}$$

$$q_{rd}(\tau) = \sum_{j \in \mathcal{J}_r} q_{rjd}(\tau), \tag{9}$$

$$q_r(\tau) = \sum_{d \in \mathcal{D}} q_{rd}(\tau), \tag{10}$$

$$q_r(\tau) = \sum_{d \in \mathcal{D}} \sum_{j \in \mathcal{J}_r} q_{rjd}(\tau), \tag{11}$$

where $u_{rjd}(\tau), \in [0,1]$, denotes the control variable that defines the ratio of vehicles that transit from r to d through a neighbouring region j, in such a way that:

$$\sum_{j \in \mathcal{J}_r^-} u_{rjd}(\tau) = 1, \; r \in \mathcal{R}, \; d \in \mathcal{D} \tag{12}$$

The equality condition in Eq. (12) ensures that the intended flow destined outside region r will be transferred potentially to its neighboring regions j. Note that, the flow of vehicles that arrive at their destination $d \in \mathcal{D}$ and exit the network at time-step τ is determined by $q_{ddd}(\tau)$, i.e. variable $q_{rjd}(\tau)$ when $\{r = j = d\}$ while it always holds that $u_{ddd}(\tau) = 1$.

Moreover, the maximum flow that can be exchanged between neighbouring regions $r \in \mathcal{R}$ and $j \in \mathcal{J}_r^-$ is restricted by the flow/storage capacity of its neighbouring region with variable $C_{rj}(\rho_j(\tau))$, denoting the inter-boundary capacity from region r to region j, mathematically expressed as follows:

$$C_{rj}(\rho_j(\tau)) = \begin{cases} C_{rj}^{\text{MAX}}, & \text{if } \rho_j(\tau) \leq \beta_{rj} \rho_j^J, \\ \frac{C_{rj}^{\text{MAX}}}{1-\beta_{rj}} \left(1 - \frac{\rho_j(\tau)}{\rho_j^J}\right), & \text{otherwise,} \end{cases} \tag{13}$$

where C_{rj}^{MAX} is the maximum inter-boundary capacity and $\beta_{rj}\rho_j^J$ is the point where the inter-boundary capacity starts to decrease with $0 < \beta_{rj} < 1$. Subsequently, the *actual transfer flow* from $r \in \mathcal{R}$ to $j \in \mathcal{J}_r$, denoted by variable $\tilde{q}_{rjd}(\tau)$, is defined as:

$$\tilde{q}_{rjd}(\tau) = \min \left(q_{rjd}(\tau), C_{rj}(\rho_j(\tau)) \frac{q_{rjd}(\tau)}{\sum_{y \in D} q_{rjy}(\tau)} \right). \tag{14}$$

Considering all above, the traffic dynamics of density of region $r \in \mathcal{R}$ towards region $d \in \mathcal{D}$ can be defined as:

$$\rho_{rd}(\tau+1) = \rho_{rd}(\tau) + \frac{1}{L_r}\tilde{d}_{rd}(\tau) + \frac{T_s}{L_r}\Bigg(\sum_{j \in \mathcal{J}_r^-} \tilde{q}_{jrd}(\tau) - \sum_{j \in \mathcal{J}_r} \tilde{q}_{rjd}(\tau) \Bigg). \tag{15}$$

III. Problem Formulation

A. Demand Uncertainty

In our previous work [7], the joint Route Guidance and Demand Management Problem is addressed using a Model Predictive Control (MPC) framework. The aforementioned work however, assumed perfect prior knowledge of the *instantaneous demand*, $d_{od}(\tau)$. In practice however, acquiring exact knowledge of future instantaneous demand can be challenging, if not impossible, due to the daily and hourly fluctuations in traffic demand. Hence in this work we relax the assumption that instantaneous traffic demand is known in advance and consider that it only becomes known at the current time step. Without loss of generality, in this work the instantaneous traffic demand for each origin-destination pair is an unknown time-dependent function, with know mean and variance, that follows a normal distribution as expressed below:

$$d_{od}(\tau) \sim \mathcal{N}(\mu_{od}(\tau), \sigma_{od}^2(\tau)), \; \tau = 1, 2, 3, \ldots, \tag{16}$$

with mean $\mu_{od}(\tau)$ and variance $\sigma_{od}^2(\tau)$ for each time step τ, respectively.

B. Objective Function

Given the uncertain nature of instantaneous demand, our primary objective is to minimize the *Total Time Spent* (TTS) for all vehicles. The TTS is the cumulative sum of the *Total Travel Time* (TTT) and the *Total Waiting Time* (TWT) for all vehicles. It's crucial to note that our demand management strategy might require some flows to delay their departure. Consequently, the TWT is computed as the difference between the time when vehicles are to depart and the actual time when they are permitted to do so.

To define the objective function, we introduce variables $S_a(\tau)$ that represent the cumulative number of vehicles that wait to enter the road network and $S_b(\tau)$ to denote the cumulative number of vehicles that arrive at their destination at time step τ, mathematically expressed as

$$S_a(\tau+1) = S_a(\tau) + \sum_{o\in\mathcal{O}}\sum_{d\in\mathcal{D}} d_{od}(\tau),\ \tau = 1, 2, \dots, \tag{17}$$

$$S_b(\tau+1) = S_b(\tau) + T_s \sum_{d\in\mathcal{D}} q_{ddd}(\tau),\ \tau = 1, 2, \dots, \tag{18}$$

with $S_a(1) = 0$ and $S_b(1) = 0$. Then, the TTS is equal with the summation of the difference $S_a(\tau)$ and $S_b(\tau)$ over all time-steps such that the objective function , J_{TTS} (veh·h), mathematically described as,

$$J_{TTS} = T_s \sum_{\tau} \big(S_a(\tau) - S_b(\tau)\big). \tag{19}$$

C. Stochastic Model Predictive Control Approach

Extending the classical MPC receding horizon approach of [7], a finite horizon SMPC problem is proposed. Similar to MPC, the SMPC assumes discrete time steps, each of duration T_s, wherein a different control action can be employed at each time step. Within this framework, each SMPC problem includes a control horizon and a prediction horizon, each covering N^C and N^P time steps, respectively. Hence, every new SMPC problem is solved at intervals of $N^C \le N^P$ time-steps. At time step $t = N^C(p-1)$, the initial state which includes the measured current states $\bar{\rho}_r(t)$, $\bar{\rho}_{rd}(t)$, the cumulative external demands $\bar{D}_{od}(t)$, the current realization of instantaneous demand $\bar{d}_{od}(t)$ and the distributions of future demands $\mathcal{N}(\mu_{od}(\tau), \sigma^2_{od}(\tau)),\ \tau = \{t, t+1, \dots, T\}$ is used as input to the traffic controller on each iteration to solve the next instance of SMPC, for the next prediction horizon $\mathcal{T}_p = \{N^C(p-1)+1, \dots, N^C(p-1)+N^P\}$, with p denoting the p-th instance. Note that at t the instantaneous demand reveals its true value. The control decisions derived from the SMPC problem are the ratio of vehicles that transfer from r toward d through the immediate neighboring region j, $u_{rjd}(\tau)$ (route guidance), and the portion of cumulative external demand that is admitted to enter r and destined d, $\tilde{u}_{od}(\tau)$ (demand management). With each new computation of the MPC problem, the derived decisions are used as the control input for traffic management, with the procedure repeated every N^C time-steps. Considering all the above, the p-th SMPC problem can be formulated using the following mathematical program:

$$(\mathrm{P}_1)\ \min_{u_{rjd}(\tau),\tilde{u}_{rd}(\tau)} T_s\, \mathrm{E}\left[\sum_{\tau\in\mathcal{T}_p}\Big(S_a(\tau) - S_b(\tau)\Big) \mid \text{Initial State}\right] \tag{20a}$$

$$\begin{aligned}
\text{s.t.}\quad & \textbf{Dynamics: } (1)-(2), (4)-(15), (17)-(18),\\
& \tilde{d}_{od}(\tau) \le D^{MAX}_{od},\ \forall\tau\in\mathcal{T}_p, \forall o\in\mathcal{O}, \forall d\in\mathcal{D}, && (20b)\\
& \tilde{d}_{od}(\tau) \le D_{od}(\tau),\ \forall\tau\in\mathcal{T}_p, \forall o\in\mathcal{O}, \forall d\in\mathcal{D}, && (20c)\\
& 0 \le \rho_r(\tau) \le \rho^J_r, \forall\tau\in\mathcal{T}_p, \forall r\in\mathcal{R}, && (20d)\\
& \tilde{u}_{od}(\tau) \in [0,1], \forall\tau\in\mathcal{T}_p, \forall o\in\mathcal{O}, \forall d\in\mathcal{D}, && (20e)\\
& u_{rjd}(\tau) \in [0,1],\ \forall\tau\in\mathcal{T}_p,\ \forall r\in\mathcal{R},\ \forall j\in\mathcal{J},\ \forall d\in\mathcal{D}, && (20f)\\
& u_{ddd}(\tau) = 1, \forall\tau\in\mathcal{T}_p,\ \forall d\in\mathcal{D}, && (20g)\\
& \textbf{Initialization: } \rho_r(t) = \bar{\rho}_r(t),\ \rho_{rd}(t) = \bar{\rho}_{rd}(t),\ D_{rd}(t) = \bar{D}_{rd}(t),\ d_{od}(t) = \bar{d}_{od}(t),\\
& \forall r\in\mathcal{R}, \forall d\in\mathcal{D},\ t = N^C(p-1), && (20h)\\
& \textbf{Variables: } S_a(\tau), S_b(\tau), \rho_{rd}(\tau), \tilde{d}_{od}(\tau), D_{rd}(\tau),\\
& \rho_r(\tau), q_r(\tau), v_r(\tau), q_{rd}(\tau), q_{rjd}(\tau), \tilde{q}_{rjd}(\tau), \tilde{u}_{od}(\tau),\\
& u_{rjd}(\tau), \forall r\in\mathcal{R}, \forall j\in\mathcal{J}, \forall d\in\mathcal{D}, \forall\tau\in\mathcal{T}_p.
\end{aligned}$$

Problem P_1 aims to minimize the **expected value** of the *Total Time Spent* of all vehicles within the considered prediction horizon, conditioned on the initial state as defined in constraint (20h). Constraints (1)-(2), (4)-(15) and (17)-(18) model the demand and traffic flow model dynamics. Furthermore, (20b) and (20c) impose the physical limits on the external demand inflows ensuring that it is always smaller than the D^{MAX}_{od} and the $D_{od}(\tau)$, (20d) makes sure that the density of each region is within the physical limits. Constraints (20e) - (20g) restrict both control variables within their limits and finally, constraint (20h) defines the initial state of the network.

D. Scenario-Based Approach

Problem P_1 is a challenging optimization problem due to its stochastic nature. The inherent complexity arises from the requirement to consider all potential realizations of the unknown demand. However, such solutions are hard, if not impossible, as the SMPC is an infinite-dimensional optimization problem. A practical solution to this problem is to approximate the uncertain demand function by multiple possible scenarios [16].

1) ***Scenario Generation:*** This work employs GP to generate multiple scenarios based on historical data of the demand function [15]. GP is a multivariate Gaussian function, i.e., $g(\tau)$, characterized by its mean, $m(\tau)$, and covariance[1], $k(\tau, \tau')$, defined as follows:

$$g_(\tau) \sim \mathcal{GP}\big(m(\tau), k(\tau, \tau')\big). \tag{21}$$

To efficiently sample multiple scenarios using GP, this work considers the instantaneous demand as an unknown function of time, given by

$$y^n_{od}(\tau) = g_{od}(\tau) + \epsilon_{od},\ \tau = 1, 2, \dots, \tag{22}$$

[1]In this work, the covariance function is assumed to be the squared exponential radial basis function [15].

where $y_{od}^n(\tau)$ represents the n-th measured output of demand at time step τ, $g_{od}(\tau)$ is the unknown function of demand and $\epsilon_{od} \sim \mathcal{N}(o, \sigma^2_{\epsilon_{od}})$ denotes white measurement noise. To estimate $g_{od}(\tau)$, we employ the Gaussian Processes Regression (GPR) procedure as detailed in [15]. GPR procedure considers a given set of n demand measurements and learns the mean $\bar{g}_*$ and covariance $cov(g_*)$ of the Gaussian Process [15]. Therefore, we can generate M unique demand scenarios for each origin-destination pair by simply sampling the derived GP as follows:

$$\check{d}_{od}^m(\tau) \sim \mathcal{GP}\big(\bar{g}_*, cov(g_*)\big) \tag{23}$$

where $\check{d}_{od}^m(\tau)$ denotes the m-th sampled scenario for a specific origin-destination pair, with $m \in \mathcal{M}$ and $\mathcal{M} = \{1, \ldots, M\}$.

2) ***Scenario-Based Formulation****:* The proposed Scenario-Based Model Predictive Control (SBMPC) formulation aims to approximate the SMPC formulation of P_1 into a deterministic counterpart. This process employs the proposed GPR method to generate M unique demand scenarios for each origin-destination pair, i.e., $\check{d}_{od}^m(\tau)$. Within this framework, the SBMPC formulation considers distinct demand and traffic dynamics for each scenario assuming that the control variables are shared among them. Hence, control variables $\tilde{u}_{rd}(\tau)$ and $u_{rjd}(\tau)$, correspond to all scenarios as they are shared among them. On the other hand, each scenario $m \in \mathcal{M}$ is governed by its own demand and traffic dynamics, as specified by Eqs. (1)-(18), but for the sake of brevity, we will not reintroduce the dynamics for each scenario in this discussion. Hereafter, the superscript m will be used to denote variables specific to the m-th scenario. For instance, $\rho_r^m(\tau)$ refers to the density of region $r \in \mathcal{R}$ at time step $\tau \in T$, reflecting the demand of scenario $m \in \mathcal{M}$.

The proposed SBMPC formulation operates similarly to the SMPC, with a new SBMPC problem solved every $N^C \leq N^P$ time steps. At time step $t = N^C(p-1)$, the current measured states $\bar{\rho}_r(t)$, $\bar{\rho}_{rd}(t)$, along with the cumulative external demand $\bar{D}_{od}(t)$, the current instantaneous demand $\bar{d}_{od}(t)$ and the instantaneous demand for each scenario $\check{d}_{od}^m(\tau)$, $\forall \tau \in \mathcal{T}_p$ and $\forall m \in \mathcal{M}$, are fed into the traffic controller. These inputs are used to solve the p-th SBMPC optimization problem, for the time horizon $\mathcal{T}_p = \{N^C(p-1)+1, \ldots, N^C(p-1)+N^P\}$. Just as in the SMPC formulation, at time step t the instantaneous demand reveal its true value with the control decisions involving the variables $u_{rjd}(\tau)$ and $\tilde{u}_{od}(\tau)$. Once an MPC problem has been solved, the resulting decisions are utilized as the control input for the traffic network. This process is then repeated every N^C time-steps. Considering all the above, the pth SBMPC problem can be mathematically formulated as follows:

$$(\text{P}_2) \min_{u_{rjd}(\tau), \tilde{u}_{rd}(\tau)} \frac{T_s}{M} \sum_{m\in\mathcal{M}} \sum_{\tau\in\mathcal{T}_p} \Big(S_a^m(\tau) - S_b^m(\tau)\Big) \tag{24a}$$

$$\begin{aligned}
\text{s.t.}\quad & \textbf{Dynamics: } (1)-(2), (4)-(15), (17)-(18),\\
& \tilde{d}_{od}^m(\tau) \leq D_{od}^{MAX},\ \forall\tau \in \mathcal{T}_p, \forall o \in \mathcal{O}, \forall d \in \mathcal{D},\\
& \qquad \forall m \in \mathcal{M}, && (24\text{b})\\
& \tilde{d}_{od}^m(\tau) \leq D_{od}^m(\tau),\ \forall\tau \in \mathcal{T}_p, \forall o \in \mathcal{O}, \forall d \in \mathcal{D},\\
& \qquad \forall m \in \mathcal{M}, && (24\text{c})\\
& 0 \leq \rho_r^m(\tau) \leq \rho_r^J, \forall\tau \in \mathcal{T}_p, \forall r \in \mathcal{R},\ \forall m \in \mathcal{M}, && (24\text{d})\\
& \tilde{u}_{od}(\tau) \in [0,1],\ \forall\tau \in \mathcal{T}_p, \forall o \in \mathcal{O}, \forall d \in \mathcal{D}, && (24\text{e})\\
& u_{rjd}(\tau) \in [0,1],\ \forall\tau \in \mathcal{T}_p,\ \forall r \in \mathcal{R},\ \forall j \in \mathcal{J},\ \forall d \in \mathcal{D}, && (24\text{f})\\
& u_{ddd}(\tau) = 1, \forall\tau \in \mathcal{T}_p,\ \forall d \in \mathcal{D}, && (24\text{g})\\
& \textbf{Initialization: } \rho_r^m(t) = \bar{\rho}_r(t),\ \rho_{rd}^m(t) =\\
& \bar{\rho}_{rd}(t),\ D_{rd}^m(t) = \bar{D}_{rd}(t),\ \check{d}_{od}^m(t) = \bar{d}_{od}(t),\\
& \forall r \in \mathcal{R}, \forall d \in \mathcal{D}, \forall m \in \mathcal{M},\ t = N^C(p-1), && (24\text{h})\\
& \textbf{Variables: } u_{rjd}(\tau), \tilde{u}_{od}(\tau), \tilde{q}_{rjd}^m(\tau), q_{rd}^m(\tau), q_{rjd}^m(\tau),\\
& \rho_r^m(\tau), q_r^m(\tau), v_r^m(\tau), \rho_{rd}^m(\tau), S_a^m(\tau), S_b^m(\tau), \tilde{d}_{od}^m(\tau),\\
& D_{rd}^m(\tau), \forall r \in \mathcal{R}, \forall j \in \mathcal{J}, \forall d \in \mathcal{D}, \forall m \in \mathcal{M}, \forall\tau \in \mathcal{T}_p.
\end{aligned}$$

Problem P_2 is designed to minimize the average *Total Time Spent* cost for all vehicles across all scenarios within the given prediction horizon, related to the initial state given in Constraint (24h). Within this formulation, constraints (1)-(2), (4)-(15) and (17)-(18) characterize the demand and traffic flow dynamics for all considered scenarios. Drawing parallels to SMPC, constraints (24b) and (24c) set physical limits on the external demand, while constraint (24d) makes sure that the density of each region is between the physical limits across all scenarios. Constraints (24e) - (24g) set the physical boundaries for the shared control variables. Moreover, constraint (24e) denotes the initial state of the variables, with all scenarios beginning from the same values. Despite its deterministic nature, Problem (P_2) remain challenging as it is a nonconvex and nonlinear mathematical program (NLP). This complexity arises due to the presence of the bilinear term in Eq. (1), the nonlinear unimodal MFD function in Eq. (4), the bilinear terms in Eqs. (7) and (8), and the nonlinear functions in Eqs. (13) and (14).

IV. Solution Approach

As aforementioned, despite the deterministic nature of the SBMPC formulation, it remains a nonlinear and nonconvex problem. As such, it can not be efficiently addressed by standard nonlinear solvers. Hence, in this section we introduce an effective solution capable of generating high-quality and accurate results. The proposed methodology approximates all nonlinearities within the SBMPC using linear constraints, aligning with the approach presented in [7]. Similar to the method presented in [7], our proposed solution approximates all nonlinearities involved in the SBMPC using linear constraints. The approach involves two key steps. First, acknowledging that the shared control variables introduce significant nonlinearities, instead of solving a single MPC instance which simultaneously considers all scenarios, we address the approximate linear problem for each scenario individually. Subsequently, the second step employs a quadratic optimization process to pinpoint optimal shared control variables across all scenarios.

A. Linear Approximate SBMPC Formulation

As noted earlier, instead of determining shared control variables across all scenarios simultaneously, we opt to solve each scenario independently. This approach allows us to omit Eqs. (1) and (7) from our proposed formulation. Specifically, by excluding these equations, the variables $d^m_{od}(\tau)$ and $q^m_{rjd}(\tau)$ are free to adopt the value that best optimizes the objective function within each individual scenario.

The nonlinear unimodal MFD function in Eq. (4), is approximated by a triangular MFD $h_r(\rho^m_r(\tau))$ which consists of two linear segments which are intersect at the critical density ρ^C_r. These segments correspond to intervals $\rho^m_r(\tau) \in [0, \rho^C_r]$ and $\rho^m_r(\tau) \in (\rho^C_r, \infty)$ and are identified using an optimal least-squares fitting procedure [17]. The fitted triangular function takes the form:

$$h_r(\rho^m_r(\tau)) = \begin{cases} \alpha_{r_1}\rho^m_r(\tau), & 0 \le \rho^m_r(\tau) \le \rho^C_r \\ \alpha_{r_2}\rho^m_r(\tau) + \alpha_{r_3}, & \rho^C_r < \rho^m_r(\tau) \le -\alpha_{r_1}/\alpha_{r_2}, \end{cases} \tag{25}$$

where $\alpha_{r_1}, \alpha_{r_2}$ and α_{r_3} are fitted parameters with α_{r_1} and α_{r_2} denoting the approximated *free-flow* and *backward congestion propagation* speed, respectively. Despite being composed of two linear segments, the triangular MFD remains nonlinear. To relax the nonlinearity, Eq. (25) is linearized by replacing the equality "=" with the inequality sign "≤". This results in,

$$q^m_r(\tau) \le \alpha_{r_1}\rho^m_r(\tau), \tag{26}$$
$$q^m_r(\tau) \le \alpha_{r_2}\rho^m_r(\tau) + \alpha_{r_3}. \tag{27}$$

Furthermore, constraints (8) is relaxed into:

$$q^m_{rd}(\tau) \le \alpha_{r_1}\rho^m_{rd}(\tau) l_r, \tag{28}$$

since for the triangular MFD it is always true that $v_r(\tau) \le \alpha_{r_1}$, where $\alpha_{r_1} = v^f_r$. Finally, constraints (13) and (14) are handled together. Constraint (14) is the minimum of two functions, it can be linearized by replacing the equality "=" with the inequality sign "≤" as follows,

$$\tilde{q}^m_{rjd}(\tau) \le q^m_{rjd}(\tau), \tag{29}$$

$$\tilde{q}^m_{rjd}(\tau) \le C^m_{rj}(\rho_j(\tau)) \frac{q^m_{rjd}(\tau)}{\sum_{y \in \mathcal{D}} q^m_{rjy}(\tau)}. \tag{30}$$

Next, the constraint (30) is further relaxed by taking the sum over all $\tilde{q}^m_{rjd}(\tau)$ for $d \in \mathcal{D}$ as follows:

$$\sum_{d \in \mathcal{D}} \tilde{q}^m_{rjd}(\tau) \le C^m_{rj}(\rho_j(\tau)), \tag{31}$$

which is a relaxed version of (30). Considering that constraint (31) is still nonlinear, we combine constraints (29) and (30), to reduce constraint (31) into:

$$\sum_{d \in \mathcal{D}} \tilde{q}^m_{rjd}(\tau) \le C^{MAX}_{rj}, \tag{32}$$

$$\sum_{d \in \mathcal{D}} \tilde{q}^m_{rjd}(\tau) \le \frac{C^{MAX}_{rj}}{1-\beta_{rj}} \left(1 - \frac{\rho^m_j(\tau)}{\rho^J_j}\right). \tag{33}$$

Hence, the nonlinear constraints of (13) and (14) are relaxed into the linear constraints (29), (32) and (33). Given all the approximations outlined above, each individual scenario is linearized by the following MPC formulation:

$$(\mathrm{P}_3) \quad \min_{\tilde{q}_{rjd}(\tau), \tilde{d}_{rd}(\tau)} T_s \sum_{\tau \in \mathcal{T}_p} (S^m_a(\tau) - S^m_b(\tau)) \tag{34a}$$

$$\begin{aligned}
\text{s.t.} \quad & \textbf{Dynamics: } (2), (5)-(6), (9)-(11), (15), \\
& (17)-(18), (26)-(28), (29), (32)-(33), \\
& \tilde{d}^m_{od}(\tau) \le D^{MAX}_{od}, \ \forall \tau \in \mathcal{T}_p, \forall o \in \mathcal{O}, \forall d \in \mathcal{D}, && (34\text{b}) \\
& \tilde{d}^m_{od}(\tau) \le D^m_{od}(\tau), \ \forall \tau \in \mathcal{T}_p, \forall o \in \mathcal{O}, \forall d \in \mathcal{D}, && (34\text{c}) \\
& 0 \le \rho^m_r(\tau) \le \rho^J_r, \forall \tau \in \mathcal{T}_p, \forall r \in \mathcal{R}, && (34\text{d}) \\
& \textbf{Variables: } D^m_{rd}(\tau), \tilde{q}^m_{rjd}(\tau), q^m_{rd}(\tau), q^m_{rjd}(\tau), \rho^m_r(\tau), \\
& q^m_r(\tau), v^m_r(\tau), \rho^m_{rd}(\tau), S^m_a(\tau), S^m_b(\tau), \tilde{d}^m_{od}(\tau), \\
& \forall r \in \mathcal{R}, \forall j \in \mathcal{J}, \forall d \in \mathcal{D}, \forall \tau \in \mathcal{T}_p.
\end{aligned}$$

Formulation (34) is a Linear Program (LP) that delivers an approximate solution for each individual scenario of the original nonlinear Problem P_2. This formulation is applied to each Model Predictive Control (MPC) instance, taking into account the time horizon $\mathcal{T}_p = N^C(p-1)+1, \dots, N^C(p-1)+N^P$. The proposed formulation takes as input each individual control scenario $d^m_{od}(\tau), \forall \tau \in \mathcal{T}_p$ and in return provides the values of $\tilde{q}_{rjd}(\tau)$ and $\tilde{d}_{od}(\tau) \, \forall \tau \in \mathcal{T}_p$. However, the main objective is not to solve each scenario independently, but to find the optimal control decisions for the variables $u_{rjd}(\tau)$ and $\tilde{u}_{od}(\tau)$, across all scenarios. This is achieved through the following quadratic optimization procedure:

$$(\mathrm{P}_4) \quad \min_{u_{rjd}(\tau), \tilde{u}_{rd}(\tau)} \sum_{m \in \mathcal{M}} \Bigg(\sum_{\tau \in \mathcal{T}_p} \big(\tilde{d}^m_{od}(\tau) - \tilde{u}_{rd}(\tau) D_{od}(\tau)\big)^2 + \big(\tilde{q}_{rjd} - u_{rjd}(\tau) q_{rd}(\tau)\big)^2 \Bigg) \tag{35a}$$

$$\begin{aligned}
\text{s.t.} \quad & \tilde{u}_{od}(\tau) \in [0,1], \forall \tau \in \mathcal{T}_p, \forall o \in \mathcal{O}, \forall d \in \mathcal{D}, && (35\text{b}) \\
& u_{rjd}(\tau) \in [0,1], \forall \tau \in \mathcal{T}_p, \forall r \in \mathcal{R}, \forall j \in \mathcal{J}, \forall d \in \mathcal{D}, && (35\text{c}) \\
& \sum_{j \in \mathcal{J}^-_r} u_{rjd}(\tau) = 1, \forall \tau \in \mathcal{T}_p, \forall r \in \mathcal{R}, \forall d \in \mathcal{D}, && (35\text{d})
\end{aligned}$$

Algorithm 1 Linear Approximation of SBMPC

1: Input: ***External demand Scenarios:*** $d_{od}^{m}(\tau)$, $o \in \mathcal{O}$, $d \in \mathcal{D}$, $\tau \in \mathcal{T}, m \in \mathcal{M}$.
Traffic network parameters: $h_r(\rho)$, v_r^f, ρ_r^C, ρ_r^J, C_{rj}^{MAX}, β_{rj}, $j \in \mathcal{J}_r$, $r \in \mathcal{R}$.
2: *Initialization:* Get the initial traffic state of the network and set $p = 1$.
for $N^C(p-1) \leq T$ **do**
  3: *Initiated the MPC state according to previous traffic state.*
  for $m \in \mathcal{M}$ **do**
    4: Solve P_3 for $t = N^C(p-1)$ and initial state.
  end for
  5: Considering the solutions of all $m \in \mathcal{M}$ of P_3, solve P_4 and derive $u_{rjd}(\tau)$ and $\tilde{u}_{od}(\tau), \forall \tau \in [t, t+N^c]$.
  6: $p = p + 1$ and updated the traffic state.
end for
7: Output: $u_{rjd}(\tau)$ and $\tilde{u}_{od}(\tau)$, $\forall\ r \in \mathcal{R}$, $j \in \mathcal{J}_r$, $d \in \mathcal{D}$, $\tau \in T$.

$$u_{ddd}(\tau) = 1, \forall \tau \in \mathcal{T}_p, \forall d \in \mathcal{D}, \tag{35e}$$

Variables: $u_{rjd}(\tau), \tilde{u}_{od}(\tau)$

Problem P_4 is a Quadratic Program (QP) that uses the solution from Problem P_2 (i.e.,$\tilde{q}_{rjd}(\tau)$ and $\tilde{d}_{od}(\tau)\, \forall \tau \in \mathcal{T}_p$) as input. Its aim is to find the optimal control decisions for the variables $u_{rjd}(\tau)$ and $\tilde{u}_{od}(\tau)$ that are shared across all scenarios. Therefore, the objective of Problem P_4 is to find the best control variables that ensure that the nonlinear Eqs. (7) and (1) hold across the majority of constraints, assuming that control variables are shared across all scenarios. In simpler terms, the goal of Problem P_4 is to minimize the summed square differences between $\tilde{d}_{od}^{m}(\tau)$ and $\tilde{u}_{rd}(\tau)D_{od}(\tau)$, as well as between $\tilde{q}_{rjd}$ and $u_{rjd}(\tau)q_{rd}(\tau)$. These sums are calculated for all time steps within the considered time horizon and across all scenarios.

The SBMPC approach is efficiently solved through the algorithmic procedure detailed in Algorithm 1. Initially, Algorithm 1 takes the generated external demand scenarios and the traffic-related parameters for each region as input, with traffic dynamics within each region defined according to a triangular-shaped MFD. Following this, the algorithm initializes the traffic state based on the current traffic conditions and launches an iterative procedure. This process aims to determine the control variables of $u_{rjd}(\tau)$ and $\tilde{u}_{od}(\tau)$ within the considered time horizon $\mathcal{T}$. This procedure is repeated every N^C steps. During each iteration, we solve Problem P_3 separately for each demand scenario. Following this, the results of all scenarios are used to infer the control decisions for the forthcoming control horizon, N^C, applying the mathematical formulation of P_4. This procedure continues until the end of the time horizon.

		Demand Level (veh/h)			
		2000	3500	4500	5500
TTS (min)	SBA	2.93	3.98	5.38	6.60
	Average	3.72	4.58	5.80	6.89
	Worst	4.21	5.02	6.28	7.11
	Known	**2.93**	**3.94**	**5.38**	**6.54**

TABLE I: Performance evaluation of different solution approaches for varying demand levels.

V. Simulation Results

The effectiveness of our proposed scenario-based solution methodology is assessed using a 16-region network topology. This topology includes four regions designated as origins and another four regions as destinations, mirroring the network structure used in [7]. Traffic dynamics in each region are modeled using a third-order polynomial function, given by $q_r(\tau) = a_{r1}\rho_r^3(\tau) + a_{r2}\rho_r^2(\tau) + a_{r3}\rho_r(\tau)$, with parameters: $a_{r1} = 8/3675$, $a_{r2} = -1192/2205$, $a_{r3} = 14768/441$, $\rho_r^C = 43$ veh/km, $\rho_r^J = 118$ veh/km, $L_r = 1$ km, $l_r = 0.3$ and $q_r^C = 1850/3$ veh/h, $\forall r \in \mathcal{R}$, $C_{rj}^{\mathrm{MAX}} = 2000$ veh/h and $\beta_{rj} = 0.25$, $\forall r \in \mathcal{R}$, $\forall j \in \mathcal{J}_r$. In the evaluation, four scenarios with different demand levels of 2000, 3500, 4500, and 5500 veh/h are assessed. The scenario with 2000 veh/h represents the lowest demand, while the 5500 veh/h scenario signifies the peak demand. In this framework, we examine the performance of the following solution approaches:

- **SBA**: This refers to the solution attained through the proposed Scenario-Based Approximation (SBA) solution methodology presented in Algorithm 1 In total thirty one scenarios are generated where the first thirty are used as input to Algorithm 1 and the latter is used as the ground truth.
- **Known**: This signifies the solution to problem P_3, assuming perfect knowledge of the demand which is given by solving a single scenario.
- **Average**: This denotes the solution to problem P_3 assuming the average demand given by a singular scenario. The average demand can be derived using the proposed Gaussian Processes Regression (GPR) method.
- **Worst**: This represents the solution for the worst-case scenario. This can be obtained by solving P_3, assuming that the control variables $\tilde{d}_{od}(\tau)$ and $\tilde{q}_{rjd}(\tau)$ are uniform across all scenarios.

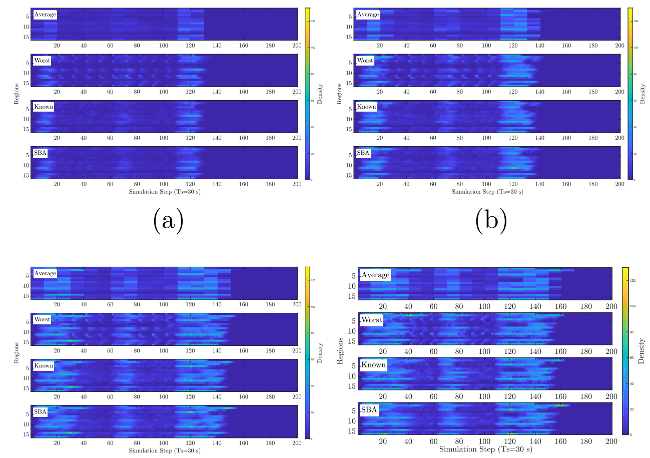
Note that all of these approaches are MPC strategies, which assume a triangular-shaped MFD function. All MPC parameters are set to $T_s = 30$ s, $N^C = 10$, and $N^P = 25$, respectively. Finally, by applying a least-square fit to the provided third-order polynomial MFD the resulting triangular MFD parameters are: $\delta_{r1} = 4243/297$, $\delta_{r2} = -4257/519$, and $\delta_{r3} = 55091/57$.

		Demand Level (veh/h)			
		2000	3500	4500	5000
Gap (%)	SBA	0.0	1.0	0.0	1.0
	Average	26.9	16.24	7.8	5.3
	Worst	43.6	27.4	16.7	8.7

TABLE II: Optimality Gap for varying demand levels.

Table I presents the results of all considered approaches in terms of the TTS metric, as defined by Eq: (19). From the table, it becomes evident that both the average and worst-case approaches lead to increased travel times, particularly in high-demand scenarios. In contrast, the SBA strategy performs better by providing the shortest TTS for all vehicles, highlighting its superior performance in every situation. The SBA strategy only has a small increase in travel times compared to the scenario with perfect demand knowledge. Additionally, Table II shows the "optimality gap" defined the difference between the TTS achieved when we know demand perfectly and the results from other methods. Interestingly, the table shows that the proposed SBA strategy has less than a 1% drop in performance compared to the scenario with perfect demand knowledge. When the demand is low, both the average and worst-case strategies show big differences when compared to the scenario where the demand is perfectly known. However, this is not the case in high-demand scenarios. In such situations, a lot of vehicles are suggested to leave late because the demand is so high. As a result, many vehicles are held outside the network over time. However, as time goes by, the feedback mechanism of MPC identifies the true value of demand enabling them to improve their quality. The data in both tables confirm that our proposed Scenario-Based Approximation (SBA) method performs better and is more reliable in various demand situations.

Figure 1 shows space-time density diagrams for all methods, covering the four demand scenarios. Figure 1 (a) relates to the scenario with the lowest demand, where all methods avoid major traffic congestion. Interestingly, the SBA and the case with known demand perform the best in this scenario, as there is a minor difference between the two. The same pattern is observed in scenarios with higher demand. While all strategies manage to avoid congestion in all the scenarios, the SBA approach is the only one that can keep the TTS metric close to the value seen with known demand. Therefore, the SBA strategy demonstrates its effectiveness in managing different traffic conditions and dealing with uncertainties in demand across all considered scenarios.



(a) (b)

(c) (e)

Fig. 1: Space-time density diagrams of densities of all investigated solution approaches across the four demand scenarios: (a) 2000, (b) 3500, (c) 4500 and (d) 5500 veh/h.

VI. Conclusions and Future Work

This work proposes a novel formulation for the joint route guidance and demand management problem, considering uncertainty in traffic demand, unlike previous approaches, which assume perfect knowledge of traffic demand. The proposed formulation results in a stochastic, nonconvex, nonlinear MPC program where a scenario-based MPC formulation is introduced to address the stochastic nature of the problem. The scenario-based MPC formulation is efficiently solved through a novel solution methodology, which relaxes the original nonlinear problem into a linear problem and employs a quadratic optimization procedure to ensure feasibility. Finally, simulation results demonstrate the superiority of the proposed scenario-based methodology over the worst case and the simplistic averaging approaches. Future research avenues will explore how to integrate modeling uncertainties and disturbances into the proposed methodology.

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