# Duality between Packings and Coverings of the Hamming Space

**Gérard** Cohen

Département Informatique Ecole Nationale Supérieure des Télécommunications 46 rue Barrault, 75634 Paris, FRANCE *cohen@enst.fr* 

#### **Alexander Vardy**

Department of Electrical and Computer Engineering Department of Computer Science and Engineering Department of Mathematics University of California San Diego 9500 Gilman Drive, La Jolla, CA 92093, U.S.A. vardy@kilimanjaro.ucsd.edu

November 4, 2018

Dedicated to the memory of Jack van Lint

#### Abstract

We investigate the packing and covering densities of linear and nonlinear binary codes, and establish a number of duality relationships between the packing and covering problems. Specifically, we prove that if almost all codes (in the class of linear or non-linear codes) are good packings, then only a vanishing fraction of codes are good coverings, and vice versa: if almost all codes are good coverings, then at most a vanishing fraction of codes are good packings. We also show that any *specific* maximal binary code is either a good packing or a good covering, in a certain well-defined sense.

Supported in part by the David and Lucile Packard Fellowship and by the National Science Foundation.

## 1. Introduction

Let  $\mathbb{F}_2^n$  be the vector space of all the binary *n*-tuples, endowed with the Hamming metric. Specifically, the *Hamming distance* d(x, y) between  $x, y \in \mathbb{F}_2^n$  is defined as the number of positions where x and y differ. A *binary code* of length n is a subset of  $\mathbb{F}_2^n$ , while a *binary linear code* of length n and dimension k is a k-dimensional subspace of  $\mathbb{F}_2^n$ . Since we are concerned *only* with binary codes in this paper, we henceforth omit the "binary" quantifier throughout. The *minimum distance* d of a code  $\mathbb{C} \subseteq \mathbb{F}_2^n$  is defined as the minimum Hamming distance between distinct elements of  $\mathbb{C}$ . The *covering radius* of  $\mathbb{C}$  is the smallest integer R such that for all  $x \in \mathbb{F}_2^n$ , there is a  $y \in \mathbb{C}$  with  $d(x, y) \leq R$ . For all other notation from coding theory, we refer the reader to the book of van Lint [8]. Van Lint [8, p.34] calls the covering radius the "counterpart of minimum distance." Indeed, the trade-off between the parameters  $|\mathbb{C}|, n, d$ , and R is one of the fundamental problems in coding theory.

Let  $\mathscr{C}(n, M)$  denote the set of all codes  $\mathbb{C} \subseteq \mathbb{F}_2^n$  with  $|\mathbb{C}| = M$ . Thus  $|\mathscr{C}(n, M)| = {\binom{2^n}{M}}$ . Similarly, let  $\mathscr{L}(n, k)$  denote the set of all linear codes of length *n* and dimension *k*. Thus the cardinality of  $\mathscr{L}(n, k)$  is given by  $|\mathscr{L}(n, k)| = \prod_{i=0}^{k-1} (2^n - 2^i)/(2^k - 2^i)$ . We will be interested in questions of the following kind. Given a property **P** which determines the packing or covering density of a code, what fraction of codes in  $\mathscr{C}(n, M)$  and/or  $\mathscr{L}(n, k)$  have this property? Moreover, how does this fraction behave as  $n \to \infty$ ? Our main results are curious duality relationships between such packing and covering problems. In particular, we show that:

- Any maximal code is good. That is, any specific maximal code  $\mathbb{C} \subseteq \mathbb{F}_2^n$  is either a good packing or a good covering, in a certain well-defined sense.
- **\*** If almost all codes in  $\mathscr{C}(n, M)$  are good coverings, then almost all codes in  $\mathscr{C}(n, M+1)$  are bad packings. Vice versa, if almost all codes in  $\mathscr{C}(n, M+1)$  are good packings, then almost all codes in  $\mathscr{C}(n, M)$  are bad coverings.
- **\*** The same is true for linear codes. That is, **\*** holds with  $\mathcal{C}(n, M)$  and  $\mathcal{C}(n, M+1)$  replaced by  $\mathcal{L}(n, k)$  and  $\mathcal{L}(n, k+1)$ , respectively.

The definition of what we mean by "good packing" and "good covering" is given in the next section. Precise statements of  $\diamondsuit$  and  $\divideontimes$ ,  $\divideontimes$  may be found in §3 and §4, respectively.

## **2.** Definitions

The *covering density* of a code  $\mathbb{C} \subseteq \mathbb{F}_2^n$  is defined in [3] as the sum of the volumes of spheres of covering radius *R* about the codewords of  $\mathbb{C}$  divided by the volume of the space:

$$\mu(\mathbb{C}) \stackrel{\text{def}}{=} \frac{\sum_{c \in \mathbb{C}} |B_R(c)|}{|\mathbb{F}_2^n|} = \frac{|\mathbb{C}| V(n, R)}{2^n}$$

where  $B_r(x) = \{y \in \mathbb{F}_2^n : d(x, y) \leq r\}$  is the sphere (ball) of radius *r* centered at  $x \in \mathbb{F}_2^n$ and  $V(n, r) = \sum_{i=0}^r {n \choose i}$  is the volume (cardinality) of  $B_r(x)$ . We find it extremely convenient to extend this definition of density to arbitrary radii as follows. **Definition 1.** Given a code  $\mathbb{C} \subseteq \mathbb{F}_2^n$  and a nonnegative integer  $r \leq n$ , the *r*-density of  $\mathbb{C}$  is defined as

$$\varphi_r(\mathbb{C}) \stackrel{\text{def}}{=} \frac{\sum_{c \in \mathbb{C}} |B_r(c)|}{|\mathbb{F}_2^n|} = \frac{|\mathbb{C}| V(n, r)}{2^n}$$
(1)

Many well-known bounds on the packing and covering density of codes can be concisely stated in terms of the *r*-density. For example, if *R*, *d*, and  $t = \lfloor (d-1)/2 \rfloor$  denote the covering radius, the minimum distance, and the packing radius, respectively, then

Sphere-packing bound: 
$$\varphi_t(\mathbb{C}) \leq 1$$
 for all  $\mathbb{C} \subseteq \mathbb{F}_2^n$  (2)

Sphere-covering bound: 
$$\varphi_R(\mathbb{C}) \ge 1$$
 for all  $\mathbb{C} \subseteq \mathbb{F}_2^n$  (3)

The classical Gilbert-Varshamov bound [8] asserts that for all *n* and  $d \le n$ , there exist codes in  $\mathscr{C}(n, M)$  whose minimum distance *d* satisfies  $M \ge 2^n/V(n, d-1)$ . Equivalently

*Gilbert-Varshamov bound:*  $\forall n, \forall d \leq n$ , there exist  $\mathbb{C} \subseteq \mathbb{F}_2^n$ , such that  $\varphi_{d-1}(\mathbb{C}) \geq 1$  (4)

Recently, this bound was improved upon by Jiang and Vardy [7] who proved that for all sufficiently large *n* and all<sup>\*</sup>  $d \leq 0.499n$ , there exist codes  $\mathbb{C} \subset \mathbb{F}_2^n$  with minimum distance *d* such that  $|\mathbb{C}| \geq cn 2^n/V(n, d-1)$ , where *c* is an absolute constant. Equivalently

$$\exists c > 0, \forall n \ge n_0, \forall d \le 0.499n$$
, there exist  $\mathbb{C} \subseteq \mathbb{F}_2^n$ , such that  $\varphi_{d-1}(\mathbb{C}) \ge cn$  (5)

The best known existence bounds for covering codes can be also expressed in terms of the *r*-density, except that one should set r = R rather than r = d - 1. Thus

$$\forall n, \forall R < n/2$$
, there exist linear  $\mathbb{C} \subseteq \mathbb{F}_2^n$ , such that  $\varphi_R(\mathbb{C}) \leq n^2$  (6)

$$\forall n, \forall R < n/2, \text{ there exist } \mathbb{C} \subseteq \mathbb{F}_2^n, \text{ such that } \varphi_R(\mathbb{C}) \leq (\ln 2)n$$
 (7)

where the first result is due to Cohen [4] while the second is due to Delsarte and Piret [5]. Motivated by (4) - (7), we introduce the following definition.

**Definition 2.** Let f(n) be a given function, and let  $\mathbb{C} \subseteq \mathbb{F}_2^n$  be a code with minimum distance *d* and covering radius *R*. We say that  $\mathbb{C}$  is an f(n)-good packing if  $\varphi_{d-1}(\mathbb{C}) \ge f(n)$ . We say that  $\mathbb{C}$  is an f(n)-good covering if  $\varphi_R(\mathbb{C}) \le f(n)$ .

## 3. Duality for a specific maximal code

A code  $\mathbb{C} \subseteq \mathbb{F}_2^n$  is said to be *maximal* if it is not possible to adjoin any point of  $\mathbb{F}_2^n$  to  $\mathbb{C}$  without decreasing its minimum distance. Equivalently, a code  $\mathbb{C}$  with minimum distance d

<sup>\*</sup>The condition  $d \leq 0.499n$  has been now improved to the more natural d < n/2 by Vu and Wu [9]. Vu and Wu [9] also show that a similar bound holds over any finite filed  $\mathbb{F}_q$  provided d < n(q-1)/q.

and covering radius *R* is maximal if and only if  $R \le d - 1$ . Our first result is an easy theorem, which says that *any* maximal code is either a good packing or a good covering.

**Theorem 1.** Let f(n) be an arbitrary function of n, and let  $\mathbb{C} \subseteq \mathbb{F}_2^n$  be a maximal code. Then  $\mathbb{C}$  is an f(n)-good packing or an f(n)-good covering (or both).

*Proof.* By definition,  $\mathbb{C}$  is not an f(n)-good packing if  $\varphi_{d-1}(\mathbb{C}) < f(n)$ . But this implies that  $\varphi_R(\mathbb{C}) \leq \varphi_{d-1}(\mathbb{C}) < f(n)$ , so  $\mathbb{C}$  is an f(n)-good covering.

For example, taking  $f(n) = \theta(n)$ , Theorem 1 implies that, up to a constant factor, any maximal code attains either the Jiang-Vardy bound (5) or the Delsarte-Piret bound (7).

## 4. Duality for almost all codes

We begin with three simple lemmas, which are needed to prove Theorems 5 and 6, our main results in this section. The following "supercode lemma" is well known.

**Lemma 2.** Given a code  $\mathbb{C}$ , let  $d(\mathbb{C})$  and  $R(\mathbb{C})$  denote its minimum distance and covering radius, respectively. If  $\mathbb{C}$  is a proper subcode of another code  $\mathbb{C}'$ , then  $R(\mathbb{C}) \ge d(\mathbb{C}')$ .

*Proof.* Since  $\mathbb{C} \subset \mathbb{C}'$ , there exists an  $x \in \mathbb{C}' \setminus \mathbb{C}$ . For any  $c \in \mathbb{C}$ , we have  $d(x, c) \ge d(\mathbb{C}')$ . Hence  $R(\mathbb{C}) \ge d(\mathbb{C}')$  by definition.

**Lemma 3.** Let  $S' \subseteq \mathscr{C}(n, M+1)$  be an arbitrary set of codes of length n and size M + 1, and let  $S = \{\mathbb{C} \in \mathscr{C}(n, M) : \mathbb{C} \subset \mathbb{C}' \text{ for some } \mathbb{C}' \in S'\}$ . Then the fraction of codes in S is greater or equal to the fraction of codes in S', namely

$$\frac{|\mathcal{S}|}{|\mathscr{C}(n,M)|} \geq \frac{|\mathcal{S}'|}{|\mathscr{C}(n,M+1)|}$$

*Proof.* Define a bipartite graph  $\mathcal{G}$  as follows. The left vertices, respectively the right vertices, of  $\mathcal{G}$  are all the codes in  $\mathscr{C}(n, M)$ , respectively all the codes in  $\mathscr{C}(n, M+1)$ , with  $\mathbb{C} \in \mathscr{C}(n, M)$  and  $\mathbb{C}' \in \mathscr{C}(n, M+1)$  connected by an edge iff  $\mathbb{C} \subset \mathbb{C}'$ . Then  $\mathcal{G}$  is bi-regular with left-degree  $2^n - M$  and right-degree M + 1. Hence the number of edges in  $\mathcal{G}$  is

$$|E(\mathcal{G})| = (M+1)|\mathscr{C}(n, M+1)| = (2^{n} - M)|\mathscr{C}(n, M)|$$
(8)

Now consider the subgraph  $\mathcal{H}$  induced in  $\mathcal{G}$  by the set  $\mathcal{S}'$ . Then the left vertices in  $\mathcal{H}$  are precisely the codes in  $\mathcal{S}$ , and every such vertex has degree at most  $2^n - M$ . The degree of every right vertex in  $\mathcal{H}$  is still M + 1. Thus, counting the number of edges in  $\mathcal{H}$ , we obtain

$$|E(\mathcal{H})| = (M+1)|\mathcal{S}'| \leq (2^n - M)|\mathcal{S}|$$
(9)

The lemma follows immediately from (8) and (9). Observe that the specific expressions for the left and right degrees of  $\mathcal{G}$  are, in fact, irrelevant for the proof.

**Lemma 4.** Let  $S' \subseteq \mathscr{L}(n, k+1)$  be an arbitrary set of linear codes of length n and dimension k + 1, and let  $S = \{\mathbb{C} \in \mathscr{L}(n, k) : \mathbb{C} \subset \mathbb{C}' \text{ for some } \mathbb{C}' \in S'\}$ . Then the fraction of codes in S is greater or equal to the fraction of codes in S', namely

$$\frac{|\mathcal{S}|}{|\mathscr{L}(n,k)|} \ge \frac{|\mathcal{S}'|}{|\mathscr{L}(n,k+1)|}$$

*Proof.* The argument is identical to the one given in the proof of Lemma 3, except that here we use the bipartite graph defined on  $\mathcal{L}(n,k) \cup \mathcal{L}(n,k+1)$ .

The next theorem establishes the duality between the fraction of good coverings in  $\mathscr{C}(n, M)$ and the fraction of good packings in  $\mathscr{C}(n, M+1)$ . In order to make its statement precise, we need to exclude the degenerate cases. Thus we henceforth assume that  $n \leq M \leq 2^n - 1$ .

**Theorem 5.** Let f(n) be an arbitrary function. Let  $\alpha \in [0, 1]$  denote the fraction of codes in  $\mathscr{C}(n, M)$  that are f(n)-good coverings, and let  $\beta \in [0, 1]$  denote the fraction of codes in  $\mathscr{C}(n, M+1)$  that are f(n)-good packings. Then  $\alpha + \beta \leq 1$ .

*Proof.* Let S' be the set of all codes in  $\mathscr{C}(n, M+1)$  that are f(n)-good packings. Thus  $|S'|/|\mathscr{C}(n, M+1)| = \beta$ . Further, let  $S = \{\mathbb{C} \in \mathscr{C}(n, M) : \mathbb{C} \subset \mathbb{C}' \text{ for some } \mathbb{C}' \in S'\}$  as in Lemma 3. We claim that none of the codes in S is an f(n)-good covering. Indeed, let  $\mathbb{C} \in S$ , and let  $\mathbb{C}' \in S'$  be a code such that  $\mathbb{C} \subset \mathbb{C}'$ . Set  $R = R(\mathbb{C})$  and  $d = d(\mathbb{C}')$ . Then

 $\varphi_R(\mathbb{C}) \ge \varphi_d(\mathbb{C})$  (by Lemma 2) (10)

$$> \varphi_{d-1}(\mathbb{C}')$$
 (trivial from (1) if  $M \ge n$ ) (11)

$$\geq f(n)$$
 ( $\mathbb{C}'$  is an  $f(n)$ -good packing) (12)

Thus  $\mathbb{C}$  is not an f(n)-good covering, as claimed. Hence  $1 - \alpha \ge |\mathcal{S}|/|\mathcal{C}(n, M)|$ . The theorem now follows immediately from Lemma 3.

For linear codes, exactly the same argument works, except that we need a factor of 2 in (11), since  $|\mathbb{C}'| = 2|\mathbb{C}|$  for any  $\mathbb{C} \in \mathcal{L}(n, k)$  and  $\mathbb{C}' \in \mathcal{L}(n, k+1)$ . For the functions f(n) of the kind one usually considers, such constant factors are not particularly significant.

**Theorem 6.** Let f(n) be an arbitrary function. Let  $\alpha \in [0, 1]$  denote the fraction of codes in  $\mathcal{L}(n, k)$  that are f(n)-good coverings, and let  $\beta \in [0, 1]$  denote the fraction of codes in  $\mathcal{L}(n, k+1)$  that are 2f(n)-good packings. Then  $\alpha + \beta \leq 1$ .

*Proof.* Follows from Lemmas 2 and 4 in the same way as Theorem 5 follows from Lemmas 2 and 3. Explicitly, (10)–(12) becomes  $\varphi_R(\mathbb{C}) \ge \varphi_d(\mathbb{C}) > \frac{1}{2} \varphi_{d-1}(\mathbb{C}') \ge f(n)$ .

## 5. Discussion

Clearly, Theorems 5 and 6 imply the statements  $\Re$  and  $\Re$ , respectively, made in §1. If  $\alpha$  tends to one as  $n \to \infty$ , then  $\beta$  tends to zero, and vice versa if  $\beta \to 1$  then  $\alpha \to 0$ .

It is well known [8] that almost all linear<sup>\*</sup> codes achieve the Gilbert-Varshamov bound (4). Hence an intriguing question is what fraction of codes in  $\mathcal{L}(n,k)$  achieve the stronger bound (5) of Jiang and Vardy [7]. Combining Theorem 6 with the results of Blinovskii [2] on random *covering* codes establishes the following theorem.

**Theorem 7.** Let *n* and  $k = \lambda n$  be positive integers, with  $0 < \lambda < 1$ . For any real  $\varepsilon > 0$ , let  $\beta_{\varepsilon}(n,k)$  denote the fraction of codes in  $\mathscr{L}(n,k)$  whose minimum distance *d* is such that  $\varphi_{d-1}(\mathbb{C}) \ge n^{1+\varepsilon}$ . Then  $\beta_{\varepsilon}(n,k)$  tends to zero as  $n \to \infty$ , for all  $\varepsilon$  and  $\lambda$ .

We omit the proof of Theorem 7, since Dumer [6] recently proved a stronger result. Specifically, Dumer [6] shows that the fraction of linear codes that are f(n)-good packings tends to zero as  $n \to \infty$  for *any* function f(n) such that  $\lim_{n\to\infty} f(n) = \infty$ . This implies that as  $n \to \infty$ , almost all linear codes satisfy  $\varphi_{d-1}(\mathbb{C}) = \theta(1)$ .

Acknowledgment. We are grateful to Alexander Barg and Ilya Dumer for helpful discussions. We are especially indebted to Ilya Dumer for sending us his proof in [6].

# References

- [1] A. BARG, Complexity issues in coding theory, Chapter 7 of *Handbook of Coding Theory*. V.S. Pless and W.C. Huffman (Eds.), North-Holland/Elsevier, 1998.
- [2] V.M. BLINOVSKII, Asymptotic Combinatorial Coding Theory. Boston: Kluwer, 1997.
- [3] G.D. COHEN, I. HONKALA, S. LITSYN, and A. LOBSTEIN, *Covering Codes*. Amsterdam: North-Holland/Elsevier, 1997.
- [4] G.D. COHEN, A nonconstructive upper bound on covering radius, *IEEE Trans. Inform. Theory*, 29 (1983), 352–353.
- [5] PH. DELSARTE and P. PIRET, Do most binary linear codes achieve the Goblick bound on the covering radius?, *IEEE Trans. Inform. Theory*, **32** (1986), 826–828.
- [6] I.I. DUMER, private communication, March 2005.
- [7] T. JIANG and A. VARDY, Asymptotic improvement of the Gilbert-Varshamov bound on the size of binary codes, *IEEE Trans. Inform. Theory*, **50** (2004), 1655–1664.
- [8] J.H. VAN LINT, Introduction to Coding Theory. New York: Springer-Verlag, 1982.
- [9] V. VU and L. WU, Improving the Gilbert-Varshamov bound for *q*-ary codes, *IEEE Trans. Inform. Theory*, submitted for publication, June 2004.

<sup>\*</sup>It is also known [1] that almost all nonlinear codes do not achieve the Gilbert-Varshamov bound.

# Duality between Packings and Coverings of the Hamming Space

## **Gérard** Cohen

Département Informatique Ecole Nationale Supérieure des Télécommunications 46 rue Barrault, 75634 Paris, FRANCE *cohen@enst.fr* 

### **Alexander Vardy**

Department of Electrical and Computer Engineering Department of Computer Science and Engineering Department of Mathematics University of California San Diego 9500 Gilman Drive, La Jolla, CA 92093, U.S.A. *vardy@kilimanjaro.ucsd.edu* 

### July 4, 2005

### Dedicated to the memory of Jack van Lint

#### Abstract

We investigate the packing and covering densities of linear and nonlinear binary codes, and establish a number of duality relationships between the packing and covering problems. Specifically, we prove that if almost all codes (in the class of linear or non-linear codes) are good packings, then only a vanishing fraction of codes are good coverings, and vice versa: if almost all codes are good coverings, then at most a vanishing fraction of codes are good packings. We also show that any *specific* maximal binary code is either a good packing or a good covering, in a certain well-defined sense.

Supported in part by the David and Lucile Packard Fellowship and by the National Science Foundation.

## 1. Introduction

Let  $\mathbb{F}_2^n$  be the vector space of all the binary *n*-tuples, endowed with the Hamming metric. Specifically, the *Hamming distance* d(x, y) between  $x, y \in \mathbb{F}_2^n$  is defined as the number of positions where x and y differ. A *binary code* of length n is a subset of  $\mathbb{F}_2^n$ , while a *binary linear code* of length n and dimension k is a k-dimensional subspace of  $\mathbb{F}_2^n$ . Since we are concerned *only* with binary codes in this paper, we henceforth omit the "binary" quantifier throughout. The *minimum distance* d of a code  $\mathbb{C} \subseteq \mathbb{F}_2^n$  is defined as the minimum Hamming distance between distinct elements of  $\mathbb{C}$ . The *covering radius* of  $\mathbb{C}$  is the smallest integer R such that for all  $x \in \mathbb{F}_2^n$ , there is a  $y \in \mathbb{C}$  with  $d(x, y) \leq R$ . For all other notation from coding theory, we refer the reader to the book of van Lint [8]. Van Lint [8, p.34] calls the covering radius the "counterpart of minimum distance." Indeed, the trade-off between the parameters  $|\mathbb{C}|, n, d$ , and R is one of the fundamental problems in coding theory.

Let  $\mathscr{C}(n, M)$  denote the set of all codes  $\mathbb{C} \subseteq \mathbb{F}_2^n$  with  $|\mathbb{C}| = M$ . Thus  $|\mathscr{C}(n, M)| = {\binom{2^n}{M}}$ . Similarly, let  $\mathscr{L}(n, k)$  denote the set of all linear codes of length *n* and dimension *k*. Thus the cardinality of  $\mathscr{L}(n, k)$  is given by  $|\mathscr{L}(n, k)| = \prod_{i=0}^{k-1} (2^n - 2^i)/(2^k - 2^i)$ . We will be interested in questions of the following kind. Given a property **P** which determines the packing or covering density of a code, what fraction of codes in  $\mathscr{C}(n, M)$  and/or  $\mathscr{L}(n, k)$  have this property? Moreover, how does this fraction behave as  $n \to \infty$ ? Our main results are curious duality relationships between such packing and covering problems. In particular, we show that:

- Any maximal code is good. That is, any specific maximal code  $\mathbb{C} \subseteq \mathbb{F}_2^n$  is either a good packing or a good covering, in a certain well-defined sense.
- **\*** If almost all codes in  $\mathscr{C}(n, M)$  are good coverings, then almost all codes in  $\mathscr{C}(n, M+1)$  are bad packings. Vice versa, if almost all codes in  $\mathscr{C}(n, M+1)$  are good packings, then almost all codes in  $\mathscr{C}(n, M)$  are bad coverings.
- **\*** The same is true for linear codes. That is, **\*** holds with  $\mathcal{C}(n, M)$  and  $\mathcal{C}(n, M+1)$  replaced by  $\mathcal{L}(n, k)$  and  $\mathcal{L}(n, k+1)$ , respectively.

The definition of what we mean by "good packing" and "good covering" is given in the next section. Precise statements of  $\diamondsuit$  and  $\divideontimes$ ,  $\divideontimes$  may be found in §3 and §4, respectively.

## **2.** Definitions

The *covering density* of a code  $\mathbb{C} \subseteq \mathbb{F}_2^n$  is defined in [3] as the sum of the volumes of spheres of covering radius *R* about the codewords of  $\mathbb{C}$  divided by the volume of the space:

$$\mu(\mathbb{C}) \stackrel{\text{def}}{=} \frac{\sum_{c \in \mathbb{C}} |B_R(c)|}{|\mathbb{F}_2^n|} = \frac{|\mathbb{C}| V(n, R)}{2^n}$$

where  $B_r(x) = \{y \in \mathbb{F}_2^n : d(x, y) \leq r\}$  is the sphere (ball) of radius *r* centered at  $x \in \mathbb{F}_2^n$ and  $V(n, r) = \sum_{i=0}^r {n \choose i}$  is the volume (cardinality) of  $B_r(x)$ . We find it extremely convenient to extend this definition of density to arbitrary radii as follows. **Definition 1.** Given a code  $\mathbb{C} \subseteq \mathbb{F}_2^n$  and a nonnegative integer  $r \leq n$ , the *r*-density of  $\mathbb{C}$  is defined as

$$\varphi_r(\mathbb{C}) \stackrel{\text{def}}{=} \frac{\sum_{c \in \mathbb{C}} |B_r(c)|}{|\mathbb{F}_2^n|} = \frac{|\mathbb{C}| V(n, r)}{2^n}$$
(1)

Many well-known bounds on the packing and covering density of codes can be concisely stated in terms of the *r*-density. For example, if *R*, *d*, and  $t = \lfloor (d-1)/2 \rfloor$  denote the covering radius, the minimum distance, and the packing radius, respectively, then

Sphere-packing bound: 
$$\varphi_t(\mathbb{C}) \leq 1$$
 for all  $\mathbb{C} \subseteq \mathbb{F}_2^n$  (2)

Sphere-covering bound: 
$$\varphi_R(\mathbb{C}) \ge 1$$
 for all  $\mathbb{C} \subseteq \mathbb{F}_2^n$  (3)

The classical Gilbert-Varshamov bound [8] asserts that for all *n* and  $d \leq n$ , there exist codes in  $\mathscr{C}(n, M)$  whose minimum distance *d* satisfies  $M \geq 2^n/V(n, d-1)$ . Equivalently

*Gilbert-Varshamov bound:* 
$$\forall n, \forall d \leq n$$
, there exist  $\mathbb{C} \subseteq \mathbb{F}_2^n$ , such that  $\varphi_{d-1}(\mathbb{C}) \geq 1$  (4)

Recently, this bound was improved upon by Jiang and Vardy [7] who proved that for all sufficiently large *n* and all\*  $d \leq 0.499n$ , there exist codes  $\mathbb{C} \subset \mathbb{F}_2^n$  with minimum distance *d* such that  $|\mathbb{C}| \geq cn 2^n/V(n, d-1)$ , where *c* is an absolute constant. Equivalently

$$\exists c > 0, \forall n \ge n_0, \forall d \le 0.499n$$
, there exist  $\mathbb{C} \subseteq \mathbb{F}_2^n$ , such that  $\varphi_{d-1}(\mathbb{C}) \ge cn$  (5)

The best known existence bounds for covering codes can be also expressed in terms of the *r*-density, except that one should set r = R rather than r = d - 1. Thus

$$\forall n, \forall R < n/2$$
, there exist linear  $\mathbb{C} \subseteq \mathbb{F}_2^n$ , such that  $\varphi_R(\mathbb{C}) \leq n^2$  (6)

$$\forall n, \forall R < n/2$$
, there exist  $\mathbb{C} \subseteq \mathbb{F}_2^n$ , such that  $\varphi_R(\mathbb{C}) \leq (\ln 2)n$  (7)

where the first result is due to Cohen [4] while the second is due to Delsarte and Piret [5]. Motivated by (4) - (7), we introduce the following definition.

**Definition 2.** Let f(n) be a given function, and let  $\mathbb{C} \subseteq \mathbb{F}_2^n$  be a code with minimum distance *d* and covering radius *R*. We say that  $\mathbb{C}$  is an f(n)-good packing if  $\varphi_{d-1}(\mathbb{C}) \ge f(n)$ . We say that  $\mathbb{C}$  is an f(n)-good covering if  $\varphi_R(\mathbb{C}) \le f(n)$ .

## 3. Duality for a specific maximal code

A code  $\mathbb{C} \subseteq \mathbb{F}_2^n$  is said to be *maximal* if it is not possible to adjoin any point of  $\mathbb{F}_2^n$  to  $\mathbb{C}$  without decreasing its minimum distance. Equivalently, a code  $\mathbb{C}$  with minimum distance d and covering radius R is maximal if and only if  $R \leq d - 1$ . Our first result is an easy theorem, which says that *any* maximal code is either a good packing or a good covering.

<sup>\*</sup>The condition  $d \leq 0.499n$  has been now improved to the more natural d < n/2 by Vu and Wu [9]. Vu and Wu [9] also show that a similar bound holds over any finite filed  $\mathbb{F}_q$  provided d < n(q-1)/q.

**Theorem 1.** Let f(n) be an arbitrary function of n, and let  $\mathbb{C} \subseteq \mathbb{F}_2^n$  be a maximal code. Then  $\mathbb{C}$  is an f(n)-good packing or an f(n)-good covering (or both).

*Proof.* By definition,  $\mathbb{C}$  is not an f(n)-good packing if  $\varphi_{d-1}(\mathbb{C}) < f(n)$ . But this implies that  $\varphi_R(\mathbb{C}) \leq \varphi_{d-1}(\mathbb{C}) < f(n)$ , so  $\mathbb{C}$  is an f(n)-good covering.

For example, taking  $f(n) = \theta(n)$ , Theorem 1 implies that, up to a constant factor, any maximal code attains either the Jiang-Vardy bound (5) or the Delsarte-Piret bound (7).

## 4. Duality for almost all codes

We begin with three simple lemmas, which are needed to prove Theorems 5 and 6, our main results in this section. The following "supercode lemma" is well known.

**Lemma 2.** Given a code  $\mathbb{C}$ , let  $d(\mathbb{C})$  and  $R(\mathbb{C})$  denote its minimum distance and covering radius, respectively. If  $\mathbb{C}$  is a proper subcode of another code  $\mathbb{C}'$ , then  $R(\mathbb{C}) \ge d(\mathbb{C}')$ .

*Proof.* Since  $\mathbb{C} \subset \mathbb{C}'$ , there exists an  $x \in \mathbb{C}' \setminus \mathbb{C}$ . For any  $c \in \mathbb{C}$ , we have  $d(x, c) \ge d(\mathbb{C}')$ . Hence  $R(\mathbb{C}) \ge d(\mathbb{C}')$  by definition.

**Lemma 3.** Let  $S' \subseteq \mathscr{C}(n, M+1)$  be an arbitrary set of codes of length n and size M + 1, and let  $S = \{\mathbb{C} \in \mathscr{C}(n, M) : \mathbb{C} \subset \mathbb{C}' \text{ for some } \mathbb{C}' \in S'\}$ . Then the fraction of codes in S is greater or equal to the fraction of codes in S', namely

$$\frac{|\mathcal{S}|}{|\mathscr{C}(n,M)|} \geq \frac{|\mathcal{S}'|}{|\mathscr{C}(n,M+1)|}$$

*Proof.* Define a bipartite graph  $\mathcal{G}$  as follows. The left vertices, respectively the right vertices, of  $\mathcal{G}$  are all the codes in  $\mathcal{C}(n, M)$ , respectively all the codes in  $\mathcal{C}(n, M+1)$ , with  $\mathbb{C} \in \mathcal{C}(n, M)$  and  $\mathbb{C}' \in \mathcal{C}(n, M+1)$  connected by an edge iff  $\mathbb{C} \subset \mathbb{C}'$ . Then  $\mathcal{G}$  is bi-regular with left-degree  $2^n - M$  and right-degree M + 1. Hence the number of edges in  $\mathcal{G}$  is

$$|E(\mathcal{G})| = (M+1)|\mathscr{C}(n,M+1)| = (2^n - M)|\mathscr{C}(n,M)|$$
(8)

Now consider the subgraph  $\mathcal{H}$  induced in  $\mathcal{G}$  by the set  $\mathcal{S}'$ . Then the left vertices in  $\mathcal{H}$  are precisely the codes in  $\mathcal{S}$ , and every such vertex has degree at most  $2^n - M$ . The degree of every right vertex in  $\mathcal{H}$  is still M + 1. Thus, counting the number of edges in  $\mathcal{H}$ , we obtain

$$|E(\mathcal{H})| = (M+1)|\mathcal{S}'| \leq (2^n - M)|\mathcal{S}|$$
(9)

The lemma follows immediately from (8) and (9). Observe that the specific expressions for the left and right degrees of  $\mathcal{G}$  are, in fact, irrelevant for the proof.

**Lemma 4.** Let  $S' \subseteq \mathscr{L}(n, k+1)$  be an arbitrary set of linear codes of length *n* and dimension k + 1, and let  $S = \{\mathbb{C} \in \mathscr{L}(n, k) : \mathbb{C} \subset \mathbb{C}' \text{ for some } \mathbb{C}' \in S'\}$ . Then the fraction of codes in *S* is greater or equal to the fraction of codes in *S'*, namely

$$\frac{|\mathcal{S}|}{|\mathcal{L}(n,k)|} \ge \frac{|\mathcal{S}'|}{|\mathcal{L}(n,k+1)|}$$

*Proof.* The argument is identical to the one given in the proof of Lemma 3, except that here we use the bipartite graph defined on  $\mathcal{L}(n,k) \cup \mathcal{L}(n,k+1)$ .

The next theorem establishes the duality between the fraction of good coverings in  $\mathscr{C}(n, M)$ and the fraction of good packings in  $\mathscr{C}(n, M+1)$ . In order to make its statement precise, we need to exclude the degenerate cases. Thus we henceforth assume that  $n \leq M \leq 2^n - 1$ .

**Theorem 5.** Let f(n) be an arbitrary function. Let  $\alpha \in [0, 1]$  denote the fraction of codes in  $\mathscr{C}(n, M)$  that are f(n)-good coverings, and let  $\beta \in [0, 1]$  denote the fraction of codes in  $\mathscr{C}(n, M+1)$  that are f(n)-good packings. Then  $\alpha + \beta \leq 1$ .

*Proof.* Let S' be the set of all codes in  $\mathscr{C}(n, M+1)$  that are f(n)-good packings. Thus  $|S'|/|\mathscr{C}(n, M+1)| = \beta$ . Further, let  $S = \{\mathbb{C} \in \mathscr{C}(n, M) : \mathbb{C} \subset \mathbb{C}' \text{ for some } \mathbb{C}' \in S'\}$  as in Lemma 3. We claim that none of the codes in S is an f(n)-good covering. Indeed, let  $\mathbb{C} \in S$ , and let  $\mathbb{C}' \in S'$  be a code such that  $\mathbb{C} \subset \mathbb{C}'$ . Set  $R = R(\mathbb{C})$  and  $d = d(\mathbb{C}')$ . Then

 $\varphi_R(\mathbb{C}) \ge \varphi_d(\mathbb{C})$  (by Lemma 2) (10)

$$> \varphi_{d-1}(\mathbb{C}')$$
 (trivial from (1) if  $M \ge n$ ) (11)

$$\geq f(n)$$
 ( $\mathbb{C}'$  is an  $f(n)$ -good packing) (12)

Thus  $\mathbb{C}$  is not an f(n)-good covering, as claimed. Hence  $1 - \alpha \ge |\mathcal{S}|/|\mathcal{C}(n, M)|$ . The theorem now follows immediately from Lemma 3.

For linear codes, exactly the same argument works, except that we need a factor of 2 in (11), since  $|\mathbb{C}'| = 2|\mathbb{C}|$  for any  $\mathbb{C} \in \mathcal{L}(n,k)$  and  $\mathbb{C}' \in \mathcal{L}(n,k+1)$ . For the functions f(n) of the kind one usually considers, such constant factors are not particularly significant.

**Theorem 6.** Let f(n) be an arbitrary function. Let  $\alpha \in [0, 1]$  denote the fraction of codes in  $\mathcal{L}(n, k)$  that are f(n)-good coverings, and let  $\beta \in [0, 1]$  denote the fraction of codes in  $\mathcal{L}(n, k+1)$  that are 2f(n)-good packings. Then  $\alpha + \beta \leq 1$ .

*Proof.* Follows from Lemmas 2 and 4 in the same way as Theorem 5 follows from Lemmas 2 and 3. Explicitly, (10)–(12) becomes  $\varphi_R(\mathbb{C}) \ge \varphi_d(\mathbb{C}) > \frac{1}{2} \varphi_{d-1}(\mathbb{C}') \ge f(n)$ .

## 5. Discussion

Clearly, Theorems 5 and 6 imply the statements  $\frac{3}{2}$  and  $\frac{3}{2}$ , respectively, made in §1. If  $\alpha$  tends to one as  $n \to \infty$ , then  $\beta$  tends to zero, and vice versa if  $\beta \to 1$  then  $\alpha \to 0$ .

It is well known [8] that almost all linear<sup>\*</sup> codes achieve the Gilbert-Varshamov bound (4). Hence an intriguing question is what fraction of codes in  $\mathcal{L}(n,k)$  achieve the stronger bound (5) of Jiang and Vardy [7]. Combining Theorem 6 with the results of Blinovskii [2] on random *covering* codes establishes the following theorem.

**Theorem 7.** Let *n* and  $k = \lambda n$  be positive integers, with  $0 < \lambda < 1$ . For any real  $\varepsilon > 0$ , let  $\beta_{\varepsilon}(n,k)$  denote the fraction of codes in  $\mathscr{L}(n,k)$  whose minimum distance *d* is such that  $\varphi_{d-1}(\mathbb{C}) \ge n^{1+\varepsilon}$ . Then  $\beta_{\varepsilon}(n,k)$  tends to zero as  $n \to \infty$ , for all  $\varepsilon$  and  $\lambda$ .

<sup>\*</sup>It is also known [1] that almost all nonlinear codes do not achieve the Gilbert-Varshamov bound.

We omit the proof of Theorem 7, since Dumer [6] recently proved a stronger result. Specifically, Dumer [6] shows that the fraction of linear codes that are f(n)-good packings tends to zero as  $n \to \infty$  for *any* function f(n) such that  $\lim_{n\to\infty} f(n) = \infty$ . This implies that as  $n \to \infty$ , almost all linear codes satisfy  $\varphi_{d-1}(\mathbb{C}) = \theta(1)$ .

Acknowledgment. We are grateful to Alexander Barg and Ilya Dumer for helpful discussions. We are especially indebted to Ilya Dumer for sending us his proof in [6].

# References

- A. BARG, Complexity issues in coding theory, Chapter 7 of Handbook of Coding Theory. V.S. Pless and W.C. Huffman (Eds.), North-Holland/Elsevier, 1998.
- [2] V.M. BLINOVSKII, Asymptotic Combinatorial Coding Theory. Boston: Kluwer, 1997.
- [3] G.D. COHEN, I. HONKALA, S. LITSYN, and A. LOBSTEIN, *Covering Codes*. Amsterdam: North-Holland/Elsevier, 1997.
- [4] G.D. COHEN, A nonconstructive upper bound on covering radius, *IEEE Trans. Inform. Theory*, **29** (1983), 352–353.
- [5] PH. DELSARTE and P. PIRET, Do most binary linear codes achieve the Goblick bound on the covering radius?, *IEEE Trans. Inform. Theory*, **32** (1986), 826–828.
- [6] I.I. DUMER, private communication, March 2005.
- [7] T. JIANG and A. VARDY, Asymptotic improvement of the Gilbert-Varshamov bound on the size of binary codes, *IEEE Trans. Inform. Theory*, **50** (2004), 1655–1664.
- [8] J.H. VAN LINT, Introduction to Coding Theory. New York: Springer-Verlag, 1982.
- [9] V. VU and L. WU, Improving the Gilbert-Varshamov bound for *q*-ary codes, *IEEE Trans. Inform. Theory*, submitted for publication, June 2004.