Additive extensions of a quantum channel

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Abstract—We study extensions of a quantum channel whose one-way capacities are described by a single-letter formula. This provides a simple technique for generating powerful upper bounds on the capacities of a general quantum channel. We apply this technique to two qubit channels of particular interest—the depolarizing channel and the channel with independent phase and amplitude noise. Our study of the latter demonstrates that the key rate of BB84 with one-way post-processing and quantum bit error rate q cannot exceed H(1/2-2q(1-q))-H(2q(1-q)).

I. INTRODUCTION

Perhaps the central problem of information theory is finding the rate at which information can be transmitted through a noisy channel. Indeed, Shannon created the field with his 1948 paper [1] showing that the capacity of a noisy channel is equal to the maximum mutual information over all input distributions to *a single use* of the channel, even though the encoding needs, in general, to use an asymptotically large number of channel uses.

However, it has long been known that the apparent quantum generalization of the mutual information, namely the *coherent information*, does not yield a single-letter formula for the quantum information capacity Q [2], [3]. Similarly, though the private capacity of a classical broadcast channel is known, and given by a single-letter formula [4], the private classical capacity of a quantum channel is not known.

The quantum capacity is given by [5], [6], [7]:

$$Q = \lim_{n \to \infty} \frac{1}{n} \max_{\phi_n} I^c \left(\mathcal{N}^{\otimes n}, \phi_n \right)$$
(1)

where

$$I^{c}(\mathcal{N},\phi) = I^{c}\left(I \otimes \mathcal{N}(|\phi^{AB}\rangle\!\langle\phi^{AB}|)\right) .$$
⁽²⁾

Here $|\phi^{AB}\rangle$ is a purification of ϕ and $I^{c}(\rho_{AB}) = S(\rho_{B}) - S(\rho_{AB})$ with $S(\rho) = -\operatorname{Tr}(\rho \log \rho)$. The private capacity is given by [6]

$$C_p(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} C_p^{(1)}(\mathcal{N}^{\otimes n})$$
(3)

where

$$C_p^{(1)}(\mathcal{N}) \equiv \sup_{\{p_x, |\varphi_x\rangle\}, X \to T} \left(I(T; B)_\omega - I(T; E)_\omega \right), \quad (4)$$

with $\omega_{ABE} = \sum_{x,t} p(t|x)p(x)|t\rangle\langle t|_A \otimes U_N|\varphi_x\rangle\langle\varphi_x|U_N^{\dagger}$ and U_N an isometric extension of \mathcal{N} (i.e., $\mathcal{N}(\rho) = \operatorname{Tr}_E U_N \rho U_N^{\dagger}$). The mutual information is defined, as usual, according to

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 $I(T; B)_{\omega_{TB}} = S(T)_{\omega_T} + S(B)_{\omega_B} - S(BT)_{\omega_{BT}}$, where we have used subscripts on the states to indicate which system they live on (e.g., $\omega_B = \text{Tr}_T \omega_{BT}$), and used the notation $S(B)_{\omega_B} = S(\omega_B)$. When it is clear which state we are referring to, we will omit the subscript on the entropy.

Since the formulas for these capacities involve maximizations over ever growing numbers of channel uses, we cannot evaluate them at all. This unsatisfying situation is a reflection of our lack of understanding of how to choose asymptotically good codes. Our best understanding is presented in [8].

Fortunately, we *can* evaluate capacity for some channels degradable ones [9]. In general, channels for which the coherent information is additive, *i.e.*

$$Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) \equiv \max_{\phi} I^{c}(\mathcal{N}, \phi), \tag{5}$$

of which degradable channels are an example, are much easier to deal with than arbitrary channels. Once we have an understanding of additive channels, we can use them to bound the capacities of other channels [10] [11]. Here we improve upon that work and develop new tighter and much simpler upper bounds.

First we will define the concepts of additive and degradable extensions to a channel and prove they have single-letter formulas for their capacities. We then use a particularly simple class of degradable extensions, which we call 'flagged extensions' to bound the quantum and private capacities. We also show that the best known previous techniques are special cases of our new bound. Finally, we bound the key rate of BB84 quantum key distribution [12] for a channel with bit error rate q by H(1/2 - 2q(1 - q)) - H(2q(1 - q)).

II. ADDITIVE AND DEGRADABLE EXTENSIONS

Definition 1 We call \mathcal{T} an additive extension of a quantum channel \mathcal{N} if there is a second channel \mathcal{R} such that $\mathcal{N} = \mathcal{R} \circ \mathcal{T}$ and $Q(\mathcal{T}) = Q^{(1)}(\mathcal{T})$.¹

A particularly nice type of additive extension is one which satisfies the following definition:

¹Of course, we could also define additive extensions which have $C_p^{(1)}(\mathcal{T}) = C_p(\mathcal{T})$ in order to find upper bounds on $C_p(\mathcal{N})$. However, since the only channels we know with $C_p^{(1)}(\mathcal{T}) = C_p(\mathcal{T})$ also have $C_p^{(1)}(\mathcal{T}) = Q^{(1)}(\mathcal{T})$, we will not pursue this approach here.

Definition 2 A channel \mathcal{N} , with isometric extension $U : A \to BE$ is called degradable if there is a degrading map \mathcal{D} such that $\mathcal{D} \circ \mathcal{N} = \widehat{\mathcal{N}}$, where $\widehat{\mathcal{N}}(\rho) = \operatorname{Tr}_B U \rho U^{\dagger}$. $\widehat{\mathcal{N}}$ is called the complementary channel of \mathcal{N} .

We call an additive extension of a quantum channel that is degradable a *degradable extension*. Degradable extensions have the additional property that their coherent information is an upper bound for the private classical capacity as well as the quantum capacity.

Our main tool will be the following simple theorem, which bounds the capacity of a quantum channel in terms of the capacity of its additive extensions.

Theorem 3 The quantum capacity of a channel N satisfies

$$Q(\mathcal{N}) \le Q^{(1)}(\mathcal{T}),\tag{6}$$

for all additive extensions, \mathcal{T} , of \mathcal{N} . Furthermore, if \mathcal{T} is degradable, the private classical capacity of \mathcal{N} satisfies

$$C_p(\mathcal{N}) \le Q^{(1)}(\mathcal{T}). \tag{7}$$

Proof: $Q^{(1)}(\mathcal{T}) = Q(\mathcal{T})$ and $Q(\mathcal{N}) \leq Q(\mathcal{T})$ follows immediately from the fact that \mathcal{N} can be obtained from \mathcal{T} by apply \mathcal{R} . If \mathcal{T} is degradable, it was shown in [11] that

$$C_p(\mathcal{T}) = Q^{(1)}(\mathcal{T}),\tag{8}$$

so that we have $C_p(\mathcal{N}) \leq C_p(\mathcal{T}) = Q^{(1)}(\mathcal{T}).$

III. KNOWN UPPER BOUNDS

In this section we show that the two strongest techniques for upper bounding the capacities of a quantum channel are encompassed by our approach. The first technique, established in [10], [13] and best for low noise levels, is to decompose the channel into a convex combination of degradable channels. The second, first studied in [14], [15], [16], is a no-cloning type argument that can sometimes be used to show that a very noisy channel has zero capacity.

A. Convex combinations of degradable channels

Lemma 4 Suppose we have

$$\mathcal{N} = \sum_{i} p_i \mathcal{N}_i,\tag{9}$$

where \mathcal{N}_i is degradable with degrading map \mathcal{D}_i . Then

$$\mathcal{T} = \sum_{i} p_i \mathcal{N}_i \otimes |i\rangle\!\langle i| \tag{10}$$

is a degradable extension of \mathcal{N} , and

$$Q(\mathcal{N}) \le \sum_{i} p_i Q^{(1)}(\mathcal{N}_i).$$
(11)

We will call \mathcal{T} a flagged degradable extension on \mathcal{N} since *i* keeps track of which \mathcal{N}_i actually occurred in the decomposition of \mathcal{N} .

Proof: First, let \mathcal{R} be the partial trace on the flagging system so that $\mathcal{N} = \mathcal{R} \circ \mathcal{T}$. To see that \mathcal{T} is degradable, note that the complementary channel of \mathcal{T} is

$$\widehat{\mathcal{T}} = \sum_{i} \widehat{\mathcal{N}}_{i} \otimes |i\rangle\!\langle i| \tag{12}$$

and that letting

$$\mathcal{D} = \sum_{i} \mathcal{D}_{i} \otimes |i\rangle\langle i|, \qquad (13)$$

where $\mathcal{D}_i \mathcal{T}_i = \widehat{\mathcal{T}}_i$, we have $\mathcal{D} \circ \mathcal{T} = \widehat{\mathcal{T}}$.

Finally, letting ϕ be the optimal input state for \mathcal{T} , we find

$$Q^{(1)}(\mathcal{T}) = S\left(\sum_{i} p_{i}\mathcal{N}_{i}(\phi) \otimes |i\rangle\langle i|\right)$$
$$-S\left(\sum_{i} p_{i}\widehat{\mathcal{N}}_{i}(\phi) \otimes |i\rangle\langle i|\right)$$
$$= \sum_{i} p_{i}\left(S\left(\mathcal{N}_{i}(\phi)\right) - S\left(\widehat{\mathcal{N}}_{i}(\phi)\right)\right)$$
$$\leq \sum_{i} p_{i}Q^{(1)}(\mathcal{N}_{i}),$$

so that by Theorem 2 the result follows.

B. No-cloning bounds

We next show that no-cloning bounds [15], [16] are a special case.

Suppose \mathcal{N} is antidegradable, meaning there is a channel \mathcal{D} such that $\mathcal{D} \circ \widehat{\mathcal{N}} = \mathcal{N}$. In this case, we can define a zerocapacity degradable extension of \mathcal{N} as follows. Let \mathcal{N} have isometric extension $U: A \to BE$, $d = \max(d_B, d_E)$, and F_1 and F_2 be d-dimensional spaces with $B \subset F_1$ and $E \subset F_2$. Then define isometry $V: A \to F_1F_2C_1C_2$ as

$$V|\phi\rangle = \frac{1}{\sqrt{2}}U|\phi\rangle|01\rangle_{C_1C_2} + \frac{1}{\sqrt{2}}(\mathrm{SWAP}_{\mathrm{F}_1\mathrm{F}_2}U|\phi\rangle)|10\rangle_{C_1C_2}$$

This gives a degradable extension of \mathcal{N} , $\mathcal{T}(\rho) = \operatorname{Tr}_{F_2C_2} V \rho V^{\dagger}$, which can be degraded to \mathcal{N} .

C. Convexity of bounds

We now show that if we have upper bounds for the capacity of two channels, both obtained from a degradable extension, the convex combination of the bounds is an upper bound for the capacity of the corresponding convex combination of the channels. More concretely, suppose \mathcal{T}_0 and \mathcal{T}_1 are degradable extensions of \mathcal{N}_0 and \mathcal{N}_1 , respectively. Then,

$$\mathcal{T} = p\mathcal{T}_0 \otimes |0\rangle\langle 0| + (1-p)\mathcal{T}_1 \otimes |1\rangle\langle 1| \tag{14}$$

is a degradable extension of $\mathcal{N} = p\mathcal{N}_0 + (1-p)\mathcal{N}_1$, and satisfies

$$Q^{(1)}(\mathcal{T}) = pI^{c}(\mathcal{T}_{0},\phi) + (1-p)I^{c}(\mathcal{T}_{1},\phi)$$
(15)

$$\leq pQ^{(1)}(\mathcal{T}_0) + (1-p)Q^{(1)}(\mathcal{T}_1).$$
 (16)

IV. BOUNDS ON SPECIFIC CHANNELS

In this section we will evaluate explicit upper bounds on the private classical and quantum capacities of the depolarizing channel and Pauli channels with independent amplitude and phase noise (which we also call "the BB84 channel", because of its relevance for BB84). In each case, we will use a flagged degradable extension of the channel of interest, based on the convex decomposition into degradable channels used in [10][11]. The advantage we obtain over this previous work is, essentially, due to the fact that our upper bound involves a maximization of the average coherent informations of the elements of our decomposition, all with respect to the same state. In [10][11], the corresponding bound is the average of the individual maxima, allowing *different* reference states for each channel in the decomposition, which generally leads to a weaker bound.

Throughout this section, we will use the following special property of coherent information for degradable channels, which was first proved in [17]. It will assist in the evaluation of coherent informations for specific degradable extensions below.

Lemma 5 Let \mathcal{N} be degradable. Then

 $pI^{c}(\mathcal{N},\phi_{0}) + (1-p)I^{c}(\mathcal{N},\phi_{1}) \leq I^{c}(\mathcal{N},p\phi_{0} + (1-p)\phi_{1}).$ In other words, $I^{c}(\mathcal{N},\phi)$ is concave as a function of ϕ .

Proof: Writing out the entropies involved explicitly, what we would like to prove is that

$$pS\left(\mathcal{N}(\phi_{0})\right) + (1-p)S\left(\mathcal{N}(\phi_{0})\right) \\ -S\left(p\mathcal{N}(\phi_{0}) + (1-p)\mathcal{N}(\phi_{1})\right) \\ \leq \\ pS\left(\widehat{\mathcal{N}}(\phi_{0})\right) + (1-p)S\left(\widehat{\mathcal{N}}(\phi_{0})\right) \\ -S\left(p\widehat{\mathcal{N}}(\phi_{0}) + (1-p)\widehat{\mathcal{N}}(\phi_{1})\right),$$
(17)

which, letting U be the isometric extension of \mathcal{N} and

$$\rho_{VBE} = p|0\rangle\!\langle 0|_V \otimes U\phi_0 U^{\dagger} + (1-p)|1\rangle\!\langle 1|_V \otimes U\phi_1 U^{\dagger},$$
(18)

is equivalent to

$$H(V|B)_{\rho_{VB}} \le H(V|E)_{\rho_{VE}}.$$
(19)

Noting that H(U|B) is nondecreasing under operations on B (which is a simple consequence of the strong subadditivity of quantum entropy), and there is a \mathcal{D} that maps B to E completes the proof.

We will also have use for $\mathcal{N}_{(u,v)}$, the most general degradable qubit channel (up to unitary operations on the input and output) [18]. $\mathcal{N}_{(u,v)}$ has Kraus operators

$$A_{+} = \begin{pmatrix} \cos(\frac{1}{2}(v-u)) & 0\\ 0 & \cos(\frac{1}{2}(v+u)) \end{pmatrix}$$
(20)

$$A_{-} = \begin{pmatrix} 0 & \sin(\frac{1}{2}(v+u)) \\ \sin(\frac{1}{2}(v-u)) & 0 \end{pmatrix}.$$
 (21)

In [13], $\mathcal{N}_{(u,v)}$ was shown to be degradable when $|\sin v| \le |\cos u|$.

A. Depolarizing Channel

The depolarizing channel of error probability p is given by

$$\mathcal{N}_p(\rho) = (1-p)\rho + \frac{p}{3}X\rho X + \frac{p}{3}Y\rho Y + \frac{p}{3}Z\rho Z$$

It is particularly nice to study since it has the property that for any unitary ${\cal U}$

$$\mathcal{N}_p(U\rho U^{\dagger}) = U\mathcal{N}_p(\rho)U^{\dagger} . \tag{22}$$

The following theorem, together with the subsequent corollary, provides the strongest upper bounds to date on the capacity of the depolarizing channel. We provide a proof of the theorem below, after establishing two essential lemmas.

Theorem 6 The capacity of the depolarizing channel with error probability *p* satisfies

$$Q(\mathcal{N}_p) \le \operatorname{co}\left[\Delta(p), 1 - 4p\right],\tag{23}$$

where

$$\Delta(p) = \min H\left[\frac{1}{2}[1+\sin u \sin v]\right] - H\left[\frac{1}{2}[1+\cos u \cos v]\right],$$

with the minimization over (u, v) such that $\cos^2(u/2)\cos^2(v/2) = 1 - p$, and $\cos[f_1(p), f_2(p) \dots f_n(p)]$ denotes the maximal convex function that is less than or equal to all $f_i(p)$, $i = 1 \dots n$.

Corollary 7

$$\begin{aligned} Q(\mathcal{N}_p) &\leq \operatorname{co}\left[1 - H(p), H(\frac{1 - \gamma(p)}{2}) - H(\frac{\gamma(p)}{2}), 1 - 4p\right], \\ \text{where } \gamma(p) &= 4\sqrt{1 - p}(1 - \sqrt{1 - p}). \end{aligned}$$

Proof: Corollary 7 follows from the theorem by noting that the first two terms inside the square brackets are special cases of Δ for values of (u, v) corresponding to amplitude damping and to dephasing channels, and using the fact that the true minimum is always bounded by particular cases.

To establish Theorem 6 we will use the following flagged degradable extension of the depolarizing channel:

$$\mathcal{T}_{(u,v)}^{\mathrm{dep}}(\rho) = \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} c^{\dagger} \mathcal{N}_{(u,v)}(c\rho c^{\dagger}) c \otimes |c\rangle\!\langle c|, \qquad (24)$$

where C is the set of unitaries which map $\{I, X, Y, Z\} \rightarrow \{I, X, Y, Z\}$ under conjugation (the Clifford group).

Lemma 8

$$Q^{(1)}(\mathcal{T}_{(u,v)}^{dep}) = H\left[\frac{1}{2}[1+\sin u \sin v]\right] - H\left[\frac{1}{2}[1+\cos u \cos v]\right]$$

Proof: The main step is to show that the coherent information of $\mathcal{T}_{(u,v)}^{dep}$ is maximized by the maximally mixed

state. To see this, first note that for any ϕ , we have

$$(X \otimes I)\mathcal{T}_{(u,v)}^{dep}(X\phi X)(X \otimes I)$$

$$= \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} Xc^{\dagger} \mathcal{N}_{(u,v)}(cX\phi Xc^{\dagger})cX \otimes |c\rangle\langle c|$$

$$= \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} Xc^{\dagger} \mathcal{N}_{(u,v)}(cX\phi Xc^{\dagger})cX \otimes V|cX\rangle\langle cX|V^{\dagger}$$

$$= (I \otimes V)\mathcal{T}_{(u,v)}^{dep}(\phi)(I \otimes V^{\dagger}),$$
(25)

where we have chosen unitary V such that $V|cX\rangle = |c\rangle$. Since

$$\widehat{\mathcal{T}}_{(u,v)}^{\mathrm{dep}}(\phi) = \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} c^{\dagger} \widehat{\mathcal{N}}_{(u,v)}(c\phi c^{\dagger}) c \otimes |c\rangle \langle c|, \qquad (26)$$

by an identical argument we also get that

$$\widehat{\mathcal{T}}_{(u,v)}^{\mathrm{dep}}(X\phi X) = (X\otimes V)\widehat{\mathcal{T}}_{(u,v)}^{\mathrm{dep}}(\phi)(X\otimes V^{\dagger}).$$
(27)

Thus, we have

$$S\left(\mathcal{T}_{(u,v)}^{\mathrm{dep}}(\phi)\right) = S\left(\mathcal{T}_{(u,v)}^{\mathrm{dep}}(X\phi X)\right)$$
(28)

$$S\left(\widehat{\mathcal{T}}_{(u,v)}^{\mathrm{dep}}(\phi)\right) = S\left(\widehat{\mathcal{T}}_{(u,v)}^{\mathrm{dep}}(X\phi X)\right), \qquad (29)$$

so that $I^c(\mathcal{T}_{(u,v)}^{\mathrm{dep}}, \phi) = I^c(\mathcal{T}_{(u,v)}^{\mathrm{dep}}, X\phi X)$, and similarly for Y and Z. Using the concavity of $I^c(\mathcal{T}_{(u,v)}^{\mathrm{dep}}, \phi)$ in ϕ , this gives us

$$I^c(\mathcal{T}^{\mathrm{dep}}_{(u,v)},\phi) \tag{30}$$

$$= \frac{1}{4} \sum_{P \in \mathcal{P}} I^c(\mathcal{T}_{(u,v)}^{\mathrm{dep}}, P\phi P^{\dagger})$$
(31)

$$\leq I^{c}\left(\mathcal{T}_{(u,v)}^{\mathrm{dep}}, \frac{1}{4}\sum_{P\in\mathcal{P}}P\phi P^{\dagger}\right)$$
(32)

$$= I^{c}\left(\mathcal{T}_{(u,v)}^{\mathrm{dep}}, \frac{1}{2}I\right), \qquad (33)$$

where we have let $\mathcal{P} = \{I, X, Y, Z\}$ denote the Pauli matrices. This shows that the maximum coherent information is achieved for the reference state I/2, where its value is

$$I^{c}\left(\mathcal{T}_{(u,v)}^{\mathrm{dep}},\frac{1}{2}I\right) = S(\mathcal{N}_{(u,v)}(I/2)) - S(\widehat{\mathcal{N}}_{(u,v)}(I/2))$$
$$= H\left(\frac{1}{2}[1+\sin u \sin v]\right)$$
$$-H\left(\frac{1}{2}[1+\cos u \cos v]\right).$$
(34)

The following lemma shows that $\mathcal{T}_{(u,v)}^{\text{dep}}$ can be degraded to a depolarizing channel, and computes the error probability of that channel as a function of u and v.

Lemma 9

$$\operatorname{Tr}_{2} \mathcal{T}_{(u,v)}^{\operatorname{dep}}(\rho) = \mathcal{N}_{p(u,v)}(\rho), \qquad (35)$$

where
$$p(u, v) = 1 - \cos^2(u/2) \cos^2(v/2)$$
.

Proof: First, note that for any channel \mathcal{N} , it was shown in [14] that

$$\widetilde{\mathcal{N}}(\rho) = \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} c^{\dagger} \mathcal{N}(c\rho c^{\dagger}) c \tag{36}$$

is a depolarizing channel with the same entanglement fidelity as \mathcal{N} . In other words,

$$\begin{split} \langle \phi^+ | (I \otimes \mathcal{N})(|\phi^+\rangle\langle \phi^+|) | \phi^+\rangle &= \langle \phi^+ | (I \otimes \widetilde{\mathcal{N}})(|\phi^+\rangle\langle \phi^+|) | \phi^+\rangle, \\ (37) \\ \text{where } |\phi^+\rangle &= (\frac{1}{\sqrt{2}})(|00\rangle + |11\rangle). \text{ As a result, letting } \mathcal{N}_{p(u,v)} = \\ \widetilde{\mathcal{N}}_{(u,v)} \text{ define } p(u,v), \text{ and using the fact that } 1 - p(u,v) = \\ \langle \phi^+ | (I \otimes \mathcal{N}_{p(u,v)})(|\phi^+\rangle\langle \phi^+|) | \phi^+\rangle, \text{ we have} \\ 1 - p(u,v) &= \langle \phi^+ | (I \otimes \mathcal{N}_{(u,v)})(|\phi^+\rangle\langle \phi^+|) | \phi^+\rangle \\ &= \langle \phi^+ | I \otimes A_+ | \phi^+\rangle\langle \phi^+ | I \otimes A_+^\dagger | \phi^+\rangle. \end{split}$$

Using the fact that

$$\langle \phi^+ | I \otimes A | \phi^+ \rangle \langle \phi^+ | I \otimes A^\dagger | \phi^+ \rangle = \left(\frac{1}{2} \operatorname{Tr} A\right) \left(\frac{1}{2} \operatorname{Tr} A^\dagger\right)$$

and A_{-} is traceless then gives

$$1 - p(u, v) = \left(\frac{1}{2} \operatorname{Tr} A_{+}\right)^{2}$$
(38)
$$= \left(\frac{1}{2} \left(\cos((v - u)/2) + \cos((v + u)/2)\right)\right)^{2}$$
$$= \cos^{2}(v/2) \cos^{2}(u/2).$$

Proof: (of Theorem 6) Let $1 - p(u, v) = \cos^2(u/2)\cos^2(v/2)$, and $\mathcal{T}_{(u,v)}^{dep}$ be the channel described in Eq. (24). Then, by Lemma 4, $\mathcal{T}_{(u,v)}^{dep}(\rho)$ is degradable, and by Lemma 9 tracing over the flag system degrades $\mathcal{T}_{(u,v)}^{dep}$ to $\mathcal{N}_{p(u,v)}$. Thus, $\mathcal{T}_{(u,v)}^{dep}$ is a degradable extension of $\mathcal{N}_{p(u,v)}$. As a result, by Theorem 3 we have

$$Q(\mathcal{N}_{p(u,v)}) \le Q^{(1)}(\mathcal{T}_{(u,v)}^{\mathrm{dep}}),\tag{39}$$

whereas by Lemma 8

$$Q^{(1)}(\mathcal{T}_{(u,v)}^{dep}) = H\left[\frac{1}{2}[1+\sin u \sin v]\right] - H\left[\frac{1}{2}[1+\cos u \cos v]\right].$$

Furthermore, it was shown in [15] that \mathcal{N}_p becomes antidegradable when p = 1/4, so that $Q(\mathcal{N}_{1/4}) = 0$.

Since both of these bounds are the result of arguments via a degradable extension, their convex hull is also an upper bound for the capacity of N_p , which completes the proof.

B. The BB84 Channel

Bennett-Brassard quantum key distribution [12] is the most widely studied and practically applied form of quantum cryptography. A simple bound on the achievable key rate is therefore quite useful. Here we evaluate the coherent information sharable though a degradable extension of the BB84 channel, which also bounds the secret key rate of this protocol.



Fig. 1. Bounds on the quantum capacity of the depolarizing channel with error probability p. The horizontal axis is the error probability, and the vertical axis is the rate. The lowest line is the achivable rate using hashing [14]. The top line is the minimum of 1 - H(p) [19] and 1 - 4p [14]. The middle line is our new bound from Corollary 7. While the bound from Theorem 6 is tighter everywhere, it, the bound plotted, and the bound from [10] all look essentially identical on this scale.

Define the following degradable channel

$$\mathcal{T}_{q}^{\mathrm{BB84}} = \frac{1}{2} \mathcal{N}_{\gamma(q)}^{\mathrm{ad}}(\rho) \otimes |0\rangle \langle 0| + \frac{1}{2} Y \mathcal{N}_{\gamma(q)}^{\mathrm{ad}}(Y \rho Y) Y \otimes |1\rangle \langle 1|,$$

with $\gamma(q)=4q(1-q),$ and $\mathcal{N}_{\gamma}^{\rm ad}$ the amplitude damping channel with Kraus operators

$$A_0 = \begin{pmatrix} 1 & 0\\ 0 & \sqrt{1-\gamma} \end{pmatrix} \tag{40}$$

$$A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \tag{41}$$

The channel $\mathcal{N}_{\gamma}^{\mathrm{ad}}$ is a special case of $\mathcal{N}_{(u,v)}$ with $u = v = \cos^{-1}(\sqrt{1-\gamma})$, which is degradable as long as $\gamma \leq 1/2$ [20].

By tracing out the flag system, we find that

$$\operatorname{Tr}_{2} \mathcal{T}_{q}^{\mathrm{BB84}}(\rho) = (1-q)^{2} \rho + q(1-q) X \rho X$$
$$+ q^{2} Z \rho Z + q(1-q) Y \rho Y.$$

With suitable unitary rotations on the input and output, this can be transformed to

$$\mathcal{N}_q^{\text{BB84}}(\rho) \equiv (1-q)^2 \rho + q(1-q) X \rho X \qquad (42)$$

$$+q(1-q)Z\rho Z + q^2 Y\rho Y. \tag{43}$$

The channel $\mathcal{N}_q^{\mathrm{BB84}}$ is such that its private classical capacity is equal to the maximal achievable key rate in BB84 with one-way postprocessing and quantum bit error rate q. Since $\mathcal{T}_q^{\mathrm{BB84}}$ is a degradable extension of an equivalent channel, its coherent information provides an upper bound on the key rate of this protocol.

The coherent information of $\mathcal{T}_q^{\text{BB84}}$ can readily be evaluated, giving the following upper bound.

Lemma 10

$$C_p(\mathcal{N}_q^{\text{BB84}}) \le H\left(\frac{1}{2} - 2q(1-q)\right) - H(2q(1-q))$$
 (44)

Proof: This follows, via Theorem 3, from the fact that $\mathcal{T}_q^{\text{BB84}}$ is a degradable extension of $\mathcal{N}_q^{\text{BB84}}$ together with Lemma 11, which evaluates this extension's coherent information.

Lemma 11

$$Q^{(1)}(\mathcal{T}_q^{\text{BB84}}) = H\left(\frac{1}{2} - 2q(1-q)\right) - H(2q(1-q)) \quad (45)$$

Proof: We would like to evaluate

$$\max_{\phi} I^c(\mathcal{T}_q^{\text{BB84}}, \phi). \tag{46}$$

For any ϕ , we have

$$I^{c}(\mathcal{T}_{q}^{\mathrm{BB84}},\phi) = I^{c}(\mathcal{T}_{q}^{\mathrm{BB84}},Y\phi Y)$$
(47)

so that, using the concavity of I^c in ϕ for degradable channels, we have

$$I^{c}\left(\mathcal{T}_{q}^{\mathrm{BB84}},\phi\right) \leq I^{c}\left(\mathcal{T}_{q}^{\mathrm{BB84}},\frac{1}{2}\phi+\frac{1}{2}Y\phi Y\right)$$
(48)

As a result, we may take the optimal ϕ to be of the form

$$\phi_{\alpha} = \frac{1}{2}I + \frac{\alpha}{2}Y.$$
(49)

Now,

$$I^{c}(\mathcal{T}_{q}^{\text{BB84}},\phi_{\alpha}) = S\left(\mathcal{N}_{4q(1-q)}(\phi_{\alpha})\right) - S\left(\mathcal{N}_{1-4q(1-q)}(\phi_{\alpha})\right),$$

which can be written more explicitly as

$$H\left(\frac{1}{2}\left(1-\sqrt{\gamma^{2}+\alpha^{2}(1-\gamma)}\right)\right)$$
(50)
$$-H\left(\frac{1}{2}\left(1-\sqrt{(1-\gamma)^{2}+\alpha^{2}\gamma}\right)\right),$$

from which we see that $I^c(\mathcal{T}_q^{\text{BB84}}, \phi_\alpha) = I^c(\mathcal{T}_q^{\text{BB84}}, \phi_{-\alpha})$. Using the concavity of I^c again, we see

$$I(\mathcal{T}_q^{\text{BB84}}) = I^c(\mathcal{T}_q^{\text{BB84}}, \frac{1}{2}I)$$
(51)

$$\geq I^{c}(\mathcal{T}_{q}^{\mathrm{BB84}},\phi_{\alpha}), \qquad (52)$$

and evaluating Eq. (50) for $\alpha = 0$ gives the result.

V. ADDITIVE EXTENSIONS AND SYMMETRIC ASSISTANCE

There is an entertaining connection to the capacity of a quantum channel with symmetric assistance [10]. We first briefly summarize the main finding of [10], using slightly more streamlined notation. Letting $\mathcal{H}' = \operatorname{span}\{|(i,j)\rangle\}_{i < j \in \mathbb{Z}^+}$ and $\mathcal{H} = \operatorname{span}\{|i\rangle\}_{i \in \mathbb{Z}^+}$, and defining the partial isometry $V : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{H}$ to act as $V|(i,j)\rangle = \frac{1}{\sqrt{2}}(|i\rangle|j\rangle - |j\rangle|i\rangle)$, we call the channel $\mathcal{A} : \mathcal{B}(\mathcal{H}') \to \mathcal{B}(\mathcal{H})$ that acts as $\mathcal{A}(\rho) = \operatorname{Tr}_2 V \rho V^{\dagger}$ the symmetric assistance channel. It was shown in [10] that for any channel \mathcal{N} , the quantum capacity of \mathcal{N} given free access to \mathcal{A} is given by

$$Q_{ss}(\mathcal{N}) = Q^{(1)}(\mathcal{N} \otimes \mathcal{A}), \tag{53}$$

where the single-letter nature of this expression comes from the non-obvious fact that

$$Q^{(1)}(\mathcal{N} \otimes \mathcal{M} \otimes \mathcal{A}) = Q^{(1)}(\mathcal{N} \otimes \mathcal{A}) + Q^{(1)}(\mathcal{M} \otimes \mathcal{A}) .$$
 (54)

We note that since \mathcal{A} is an infinite dimensional operator, \mathcal{A} is equally valuable for assistance to $\mathcal{A}^{\otimes n}$ Note also that $Q(\mathcal{A}) = Q^{(1)}(\mathcal{A}) = Q(\mathcal{A}^{\otimes n}) = Q^{(1)}(\mathcal{A}^{\otimes n}) = 0$ by a nocloning argument. Then

$$Q(\mathcal{N} \otimes \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n} \otimes \mathcal{A}^{\otimes n}) \quad (55)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(Q^{(1)}(\mathcal{N}^{\otimes n}\otimes\mathcal{A})+Q^{(1)}(\mathcal{A}^{\otimes(n-1)}\otimes\mathcal{A})\right) \quad (56)$$

$$\lim_{n \to \infty} \frac{1}{n} (nQ^{(1)}(\mathcal{N} \otimes \mathcal{A})) = Q^{(1)}(\mathcal{N} \otimes \mathcal{A}) .$$
 (57)

Therefore $\mathcal{N} \otimes \mathcal{A}$ is an additive extension of \mathcal{N} for any \mathcal{N} . JAS thanks ARO contract DAAD19-01-C-0056.

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