Universal Rateless Codes From Coupled LT Codes

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Abstract-It was recently shown that spatial coupling of individual low-density parity-check codes improves the beliefpropagation threshold of the coupled ensemble essentially to the maximum a posteriori threshold of the underlying ensemble. We study the performance of spatially coupled low-density generator-matrix ensembles when used for transmission over binary-input memoryless output-symmetric channels. We show by means of density evolution that the threshold saturation phenomenon also takes place in this setting. Our motivation for studying low-density generator-matrix codes is that they can easily be converted into rateless codes. Although there are already several classes of excellent rateless codes known to date, rateless codes constructed via spatial coupling might offer some additional advantages. In particular, by the very nature of the threshold phenomenon one expects that codes constructed on this principle can be made to be universal, i.e., a single construction can uniformly approach capacity over the class of binaryinput memoryless output-symmetric channels. We discuss some necessary conditions on the degree distribution which universal rateless codes based on the threshold phenomenon have to fulfill. We then show by means of density evolution and some simulation results that indeed codes constructed in this way perform very well over a whole range of channel types and channel conditions.

Index Terms—Spatial Coupling, LDGM, LDPC, LT codes, Rate-less Codes, Raptor Codes, LDPC Convolutional Codes

I. Introduction

THE idea of spatially coupling copies of a graphical model was introduced for the coding context in [1] in the form of convolutional LDPC ensembles. The performance of such ensembles was investigated in, among others, [2–4] and it was found to be very good. In particular the threshold of a coupled ensemble was consistently found to be significantly superior to the threshold of the underlying ensemble. It was then shown in [5, 6] why this is the case, and the phenomena was termed threshold saturation.

The key observation in the above papers is that the *belief propagation* (BP) threshold of the coupled ensemble is considerably improved and becomes close to the *maximum a posteriori* (MAP) threshold of the underlying ensemble while the MAP threshold of the coupled and underlying ensembles are close to each other. This phenomenon has also been observed in several other classes of graphical models [7, 8] and seems to be rather general: when we spatially couple, the dynamical threshold of the chain converges to the static threshold of the un-coupled model.

We study the coupling phenomenon for low-density generator-matrix (LDGM) codes. LDGM codes are closely related to LT codes. LT codes were originally designed for communication over the binary erasure channels (BEC) with

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unknown erasure probability [9]. For these codes the encoder generates an (in principle) infinite sequence of output symbols. The decoder collects as many output symbols as necessary to successfully recover all the information bits. LT codes are one of the first instances of *rateless* codes, see [10]. They are called rateless codes because the rate of the code is not fixed a priori and can vary from essentially zero to essentially one, depending on the channel condition.

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A typical application of rateless codes is a system where the actual channel is unknown to the encoder and chosen from a given uncertainty set. LT codes can asymptotically reach $1-\mu$ of the capacity of the BEC with unknown erasure probability, for any $\mu > 0$ [9]. In particular, LT codes are universal over the BEC. By adding a proper precoder to the LT codes, Shokrollahi introduced Raptor codes which exhibit an even better performance in terms of encoding/decoding complexity and error probability [11]. There is a considerable literature on rateless codes. Let us just mention a very small selection and refer the reader to some of the review articles for a more thorough literature review. The error performance of Raptor codes and LT codes over binary-input memoryless outputsymmetric (BMS) channels was investigated in [12]. Later in [10], the authors showed how to design μ -capacity achieving Raptor codes, for arbitrary $\mu > 0$, on the binary symmetric channel (BSC) and the binary additive white Gaussian noise channel (BAWGNC); the authors also proved that LT codes are not universal over the BSC and the BAWGN channel families.

The objective of this paper is to introduce a further alternative to the construction of rateless codes, namely to construct rateless codes via spatial coupling of LT ensembles. We show by means of density evolution that the threshold saturation also takes place in this setting. We provide some necessary conditions on the degree distributions in order for the constructed ensemble to be universal.

We describe the structure of coupled ensembles in Section II. There we also explain the relationship between LT and LDGM ensembles. The saturation phenomenon is investigated in Section III. We derive some necessary conditions for such an ensemble to be universal in Section IV. We also provide some simulation results for various channel types and rates which give further support for our conjecture.

II. RATELESS ENSEMBLES FROM COUPLED LT ENSEMBLES

We propose to construct rateless codes by spatially coupling LT codes. When the number of information bits and the number of output bits tends to infinity (at a fixed ratio), the performance of such a structure can in turn be assessed by analyzing an ensemble of spatially coupled LDGM codes. Let us start by recalling the definition of LT codes.

1) Structure of LT Ensemble: Let u_1, \ldots, u_m denote the information bits we want to transmit. For LT codes, in principle, an infinite stream of output symbols is generated from these m information bits. The receiver "listens" to as many of them as needed in order to decode the m information bits reliably.

More precisely, the encoder generates a sequence of output symbols as follows: First, an integer d, called the degree, is independently and randomly chosen according to a given degree distribution. This distribution is encoded by the polynomial $R(x) = \sum_{d=1}^{d_{\max}} R_d x^d$, where R_d is the probability of choosing d. Next, a d-tuple of information bits is uniformly picked from all $\binom{m}{d}$ distinct d-tuples, denote it by (i_1,\ldots,i_d) . Finally, the sum $u_{i_1}+\cdots+u_{i_d}$ is computed (also called the "output symbol") and it is transmitted over the channel. Here, we assumed that the transmitter and the receiver share randomness so that the choice of the degree as well as the choice of the indices is known both to the transmitted and to the receiver.

The receiver collects a number of output symbols (typically at least equal to the number of information bits) and starts the decoding process using the BP algorithm. If it cannot decode given this information, it collects further output symbols and retries. It continues in this manner until all m information bits are decoded. Assume that the receiver decodes all information bits using n output symbols. We then say that the code has rate $r = \frac{m}{n}$.

The received output symbols and information bits can be represented by a bipartite graph $\mathcal{G}(U,G;E)$. Here U denotes the set of information nodes and it has cardinality m. In the same manner, G denotes the set of *generator nodes* (output symbols). The set E denotes all edges; there is an edge between a generator node and an information node iff the corresponding bit was used in the computation of this output symbol.

2) Coupled LT Ensemble: Let us now discuss how to couple LT codes. Assume that the information bits are divided into L sets located at positions [0, L-1] and each having m information bits. Let these bits be labeled from 1 to mL. Let the generator nodes be located at positions [0, L + w - 2], where w is a *smoothing* parameter, $w \in \mathbb{N}$. To generate an output symbol, the encoder picks $i \in [0, L+w-2]$. This is the position of the next generator node which is being constructed. Next the degree d is chosen as in the uncoupled case, according to the distribution R(x), and independently from all previous choices. Then, each of the d connections is uniformly and independently chosen among the mw information bits in the range [i-w+1, i], see Fig. 1. For generator nodes situated close to the boundaries, if the position of a chosen information bit is not in the range [0, L-1] then the associated edge is omitted. Equivalently we can assume that it is connected to a bit outside of this range which is known both to the transmitter and the receiver and whose value can without loss of generality be assumed to be 0. Finally, the encoder sends the sum of the values of the connected information bits. As for the uncoupled case, we assume that shared randomness is available at the transmitter and the receiver so that the choice of positions, degrees, and connected bits is known on both sides. We call

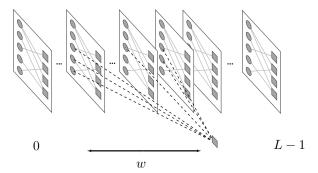


Fig. 1. Adding a new generator node to a coupled LDGM ensemble.

the resulting ensemble a spatially coupled LT ensemble.

3) LDGM Ensembles as Limits of LT Ensembles: Consider an uncoupled LT code. Since the degree of every generator node is chosen independently according to the distribution R(x), the empirical distribution of the degrees of the output symbols converges a.s. to R(x). Further, if we let m and n tend to infinity but fix their ratio, then the empirical degree distribution of the information bits converges a.s. to the Poisson distribution $\lambda(x) = e^{l_{\rm avg}(x-1)}$, where $l_{\rm avg} = \frac{n}{m}R'(1)$ is the average degree of an information bit. Therefore, in this sense (for increasing blocklengths) the resulting code tends to an instance of the LDGM $(e^{l_{\rm avg}(x-1)}, R(x))$ ensemble. Note also that for any fixed number of iterations density evolution is continuous in the degree distribution.

In order to study the threshold behavior of LT codes we can therefore study the threshold behavior of the equivalent LDGM ensemble. Only when we are interested in the finite-length scaling behavior do we need to take the small deviations of the degree distribution from the expected value into account. For coupled LT ensembles the same argument applies. Therefore, for the purpose of analysis, we consider L copies of $(e^{l_{avg}(x-1)}, R(x))$ LDGM ensembles spatially coupled in the same way as described above.

4) Design Rate: In the coupled setting, the design rate of the code is equal to the total number of non-trivial information bits, which is equal to mL, divided by the number of generator bits that are connected to at least one of the mL non-trivial information bits. We have,

Lemma 1 (Design Rate). Consider an $(\lambda(x), R(x), L, w)$ coupled LDGM ensemble such that the underlying ensemble has n generator nodes and m information bits. The design rate is $r = \frac{mL}{n(L-w+1)+2n\sum_{i=1}^{w-1}(1-R(\frac{i}{w}))}$.

We see that the design rate of the coupled ensemble is slightly decreased compared to the design rate m/n of the underlying ensemble. However, this rate loss vanishes with L at a speed of $\Theta(\frac{w}{L})$. Hence, we should not pick L too small in order to keep the rate loss at an acceptable level. On the other hand, picking L very large leads to very long codes. Hence, there is an inherent trade-off. For LDPC ensembles, various ways of reducing the rate loss were suggested in [6]. The same basic ideas can be applied in the present setting to substantially mitigate the rate loss. We will not pursue this topic further, although in a real setting it is important.

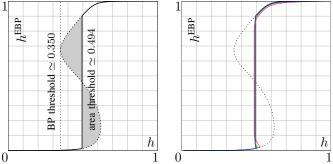


Fig. 2. Left: EBP GEXIT curve (dashed) and Maxwell curve (solid) of the $(e^{12.32(x-1)},R_1(x))$ LDGM ensemble with $R_1(x)=0.02x+0.6x^2+0.38x^{13}$ and r=0.5 for transmission over the BEC. The area under the Maxwell curve is r and the area threshold is 0.494. The BP threshold is 0.350. Right: The EBP GEXIT curves of the corresponding coupled ensembles for (L,w) equal to (64,5) (red curve), (128,5) (green curve), and (512,11) (blue curve). Note that these EBP GEXIT curves are all very close to the Maxwell curve of the underlying ensemble.

III. THRESHOLD SATURATION OF COUPLED LDGM ENSEMBLE

Let us consider as example the EBP GEXIT curve of the $(e^{12.32(x-1)},R_1(x))$ LDGM ensemble where $R_1(x)=0.02x+0.6x^2+0.38x^{13}$ and $r=\frac{1}{2}.^1$ In the left picture in Fig. 2 the EBP GEXIT curve is shown as a dashed line. For the given example it has the shape of an "S." Let us construct the Maxwell curve. We get the Maxwell curve by taking the EBP GEXIT curve and by cutting the "S" by means of a vertical line, where the line is located in such a way that the two gray areas are equal (see the left picture). The Maxwell curve then consists of the the vertical line plus the two connecting parts of the EBP GEXIT curve so that the total curve represents an increasing function. In the sequel we will refer to the entropy value where the vertical line is located as the area threshold (since the position of the vertical line is defined by an equality of areas). In Fig. 2 the Maxwell curve is shown as solid black curve and the area threshold is $h \approx 0.494$. This has to be compared to the BP threshold of this ensemble which can be seen to be around 0.35. The significance of the Maxwell curve is that for a wide range of ensembles and channels it is conjectured to characterize the performance of the MAP decoder, see e.g., [13]. ²

Let us now show by means of DE computations that the BP threshold of the coupled ensemble is very close to the area threshold of the underlying ensemble for a wide range of BMS channels (see Fig. 2). This observation suggests that the threshold saturation phenomenon also occurs in the current setting.

The first step in the evaluation of the asymptotic perfor-

mance is to write down the density evolution (DE) equations. To keep the notation at a manageable level, let us start with the case of the BEC.

Lemma 2 (DE Equations). Consider a coupled $(\lambda(x) = e^{l_{avg}(x-1)}, R(x), L, w)$ LDGM ensemble and transmission over the BEC with erasure probability ϵ . Let x_i , $i \in \mathbb{Z}$, denote the average erasure probability which is emitted by information nodes at position i and y_j , $j \in \mathbb{Z}$, denote the average erasure probability which is emitted by generator nodes at position j. The fixed point (FP) condition implied by DE is then

$$x_i = \lambda(\frac{1}{w} \sum_{j=i}^{i+w-1} y_j),$$
 (1)

$$y_j = 1 - (1 - \epsilon)\rho(1 - \frac{1}{w} \sum_{i=j-w+1}^{j} x_i),$$
 (2)

where $\rho(x) = R'(x)/R'(1)$ and $x_i = 0$ for $i \notin [0, L-1]$.

As mentioned before, since the information bits outside the interval [0,L-1] are known, we can assume that $x_i=0$ for $i\not\in [0,L-1]$. The decoding process starts with $x_i^{(0)}=1$ for $i\in [0,L-1]$. Let $x_i^{(l)}$ denote the average erasure probability which is emitted by information bits at position i at round l. If at each decoding round all $x_i^{(l)}$ are updated according to (1) and (2), then for each i the sequence $x_i^{(l)}$ is monotonically decreasing. Since the sequence is bounded from below it must converge. Call the limit $x_i^{(\infty)}$, $i\in [0,L-1]$. We call the vector $(x_0^{(\infty)},\cdots,x_{L-1}^{(\infty)})$ the forward DE FP for the erasure probability ϵ . From this we can compute the BP GEXIT value h_j at the generator nodes in position $j\in [0,L+w-2]$. It is defined by, $h_j(x_0,\cdots,x_{L-1})=1-R(1-\frac{1}{w}\sum_{i=j-w+1}^{j}x_i)$. The same analysis can be performed for general BMS channels by writing down the corresponding DE equations. Since this is rather routine, we skip this part.

Let us now illustrate the results of the DE analysis. Consider the LDGM ensemble in Fig. 2. As discussed above, the left picture shows its EBP GEXIT curve as well as the derived Maxwell curve. The right picture shows the EBP GEXIT curves of the corresponding coupled ensembles with various values of L and w. As we can see from the picture, all these curves are very close to the Maxwell curve of the underlying ensemble and seem to approach the closer the larger we choose L and w (as long as w is small compared to L). So, according to this numerical evidence, the threshold saturation phenomenon occurs for this ensemble as conjectured.

Fig. 3 shows the EBP GEXIT curves for several further examples. All pictures are for the degree distribution $R_2(x) = 0.360x^2 + 0.313x^3 + 0.327x^{22}$ and (L,w) equal to (32,3) and (64,4). The rows correspond to transmission over the BEC, BSC, and BAWGNC (top to bottom), respectively. The columns corresponds to rates 0.2, 0.5, and 0.8 (left to right), respectively. For all these cases we see that the threshold saturation phenomenon takes place. Also, the resulting thresholds are all very close to the Shannon capacity. Indeed, we see that this code is uniformly good over these classes of channels and a wide range of rates. This gives further evidence to our

¹In a nutshell, the EBP GEXIT curve is the curve of all fixed points (FP) of density evolution (DE) for the given ensemble. For an in-depth discussion we refer the reader to [13].

²Note that, even if we assume that the Maxwell curve characterizes the MAP performance, the area threshold defined above is not really the MAP threshold since there is an error floor (this code does not have a non-trivial MAP threshold). But if we assume, as it is e.g. the case for Raptor codes, that we are using a pre-code and that the error floor is sufficiently small, then this area threshold has an important operational significance. As a consequence of the error floor, this area threshold can even be slightly larger than the Shannon threshold.

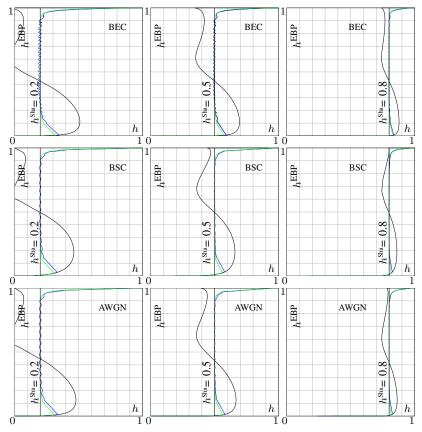


Fig. 3. EBP GEXIT curves of LDGM ensemble $(\lambda(x), R_2(x))$ (black curves) and the corresponding coupled ensembles for (L, w) equal to (32, 3) (blue curves) and (64, 4) (green curves) where $R_2(x) = 0.360x^2 + 0.313x^3 + 0.327x^{22}$. The ensembles are depicted for different rates 0.2, 0.5 and 0.8 in BEC, BSC and AWGN channel. For the rate 0.8, there is no BP threshold for the underlying LDGM ensembles. For all cases, the area threshold of individual ensembles is very close to the Shannon threshold.

conjecture that rateless codes constructed on coupling of LT codes can be made to be universal.

IV. NECESSARY CONDITIONS ON DEGREE DISTRIBUTION FOR UNIVERSALITY OF COUPLED LDGM ENSEMBLE

Although currently we do not know how to prove that coupled LT ensembles can be made universal, it is easy to derive some necessary conditions for this to happen.

- (i) Error Floor: Since we are dealing essentially with LDGM ensembles, our construction has generically a bit error floor. To achieve capacity, we have to ensure that this error floor tends to zero as the block-length grows large. This induces a constraint on average degree of the generator nodes, see [11].³
- (ii) Threshold Behavior: The premise of coupled ensembles is that their BP threshold is equal to their area threshold. Assuming the above premise, for a given generator degree distribution and a specific design rate, the corresponding coupled LDGM ensemble is therefore asymptotically (when L and w tend to infinity) capacity achieving for a family of channels if the area threshold of the underlying LDGM ensemble is equal to its Shannon threshold. If this property holds for any design rate, then we say that

the coupled ensemble with that degree distribution is universal on that family of channels. If it holds in addition for any channel family (within lets say the BMS channel family) then we say that the ensemble is universally capacity achieving.

We will see that this induces a constraint on R_1 and R_2 .

A. Error Floor

LDGM ensembles have in general a non-zero bit error probability below the "threshold" (which we call the error floor) and this error floor remains essentially unchanged by coupling.

Theorem 1 (Lower Bound on Error Floor). The error floor of the $(\lambda(x) = e^{l_{avg}(x-1)}, R(x))$ LDGM ensemble when transmission takes place over the BEC with erasure probability ϵ is lower bounded by

$$P_e \ge \frac{1}{2}\lambda(\epsilon)(1 + l_{avg}(1 - \epsilon)(1 - \frac{R'(1 - \lambda(\epsilon))}{R'(1)})).$$
 (3)

Hence, a necessary condition for this expression to tend to zero at a fixed erasure probability ϵ is that R'(1) tends to infinity.

B. Threshold Behavior

It is shown in [10] that in order for a $(e^{l_{avg}(x-1)}, R(x))$ LDGM ensemble (the asymptotic LT code with degree distri-

³Here we assume that we want to construct our sequence of capacity achieving ensembles in such a way that the rate of the outer code tends to one.

bution R(x) in their work) to be capacity achieving under BP decoding for a BMS channel with LLR density a, the following two conditions must be fulfilled:

- (i) $R_1=0$, (ii) $R_2=\frac{C({\sf a})}{2D({\sf a})}$, where $C({\sf a})$ denoted the capacity of the channel and $D({\sf a})=\int_{-\infty}^\infty a(l)\tanh(l/2){\sf d}l$.

Let us quickly review why these conditions are necessary. The first condition is due to the fact that if $R_1 > 0$, the probability that an information bit is connected to more than one generator node with degree one is strictly positive. Imagine that e.g. two generator nodes of degree one are connected to the same information node. With positive probability both of them are received. But then clearly one of the generator nodes is redundant. Hence we are bounded away from capacity.

To explain the second condition we will not follow the arguments used in [10] but rather use the language of EBP GEXIT curves. Assume that R_1 is very close to zero. Let H(a) = 1 - C(a) be the entropy associated to the channel with density a. The stability condition of LDGM ensembles [13] implies that the entropy value where the EBP GEXIT curve deviates from 1 occurs at the point $h = H(\hat{a})$ where $D(\hat{a}) =$ $\frac{r}{2R_2}$. Here r is the design rate.

If h > 1 - r, then the value of the EBP GEXIT curve at the Shannon threshold, 1-r, is strictly smaller than one (see e.g. Fig. 2). Consequently, by applying the area theorem, the area threshold (and hence the BP threshold) must be strictly below the Shannon threshold. Therefore, to achieve capacity, we need that $\hat{h} \leq 1 - r$. This implies that $R_2 \leq \frac{C(a)}{2D(a)}$.

If $\hat{h} < 1 - r$, then the EBP GEXIT curve deviates from 1 at h, which lies strictly below the Shannon threshold. Since the BP threshold cannot be greater than \hat{h} , also in this case we cannot achieve capacity (see e.g. Fig. 3; recall that currently we discuss uncoupled ensembles). Therefore, we must have $\hat{h} \geq 1-r.$ This implies that $R_2 \geq \frac{C(\mathsf{a})}{2D(\mathsf{a})}$

Combining these two conditions we see that we need equality for the uncoupled case. From this point of view it is immediately clear why in this framework we cannot construct universal codes - the right-hand side of the above equality depends on the channel!

Consider now the coupled case. Condition (i) is still necessary. What about condition (ii)? It is easy to see that if the EBP GEXIT curve deviates from 1 to the right of the threshold, then also in the coupled case we are bounded away from capacity. So the condition $\hat{h} \leq 1 - r$, or equivalently, $R_2 \leq \frac{C(\mathbf{a})}{2D(\mathbf{a})}$ still applies. But, due to the fact that the performance is now given by the Maxwell curve associated to the underlying ensembles, we are no longer bound by the second condition. Therefore, we might hope to find a value of R_2 which fulfills the inequality $R_2 \le \frac{C(\mathsf{a})}{2D(\mathsf{a})}$ for all BMS channels.

Lemma 3 (Minimum of Stability Condition). Over the class of densities a associated to BMS channels, $\inf_{\mathbf{a}} \frac{C(\mathbf{a})}{2D(\mathbf{a})} = \frac{1}{4\ln(2)}$, and the minimum is attained for the BSC with entropy 1.

Corollary 1 (Area Threshold of LDGM ensemble). In order for an $(e^{l_{avg}(x-1)}, R(x), L, w)$ coupled LDGM ensemble to be asymptotically universal over all BMS channels, it is necessary that: (i) $R_1 = 0$, (ii) $R_2 \le \frac{1}{4 \ln(2)} \approx 0.3606$.

Let us look back at the example in Fig. 3. This degree distribution was designed according to Corollary 1, i.e., we have $R_1 = 0$ and $R_2 \leq 0.3606$. Further, $R'(1) \approx 8.85$. We can see that the GEXIT values of the underlying ensembles are 1 for h > 1 - r and for all tested channels. Due to a rather small value for R_2 , the BP threshold of the underlying ensemble is quite small (in particular for low rates) and so the uncoupled ensemble itself would not be useful. But as we have seen, the performance for the coupled case is universally close to the Shannon threshold.

Let us summarize: Coupled LDGM codes have the potential advantage of being universal. This is indeed a nice property to have for typical applications of rateless codes. Further, since we are only concerned with the area threshold of the underlying ensemble, there are many more degrees of freedom in their design and typically only a small degree of irregularity suffices, making them potentially easier to implement.

To be fair, there is a price to be payed. Due to the coupled structure and the fact that L has to be chosen reasonably large in order to avoid a large rate loss, it is difficult to construct codes of very short length which perform well.

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