# Analysis of Sparse Representations Using Bi-Orthogonal Dictionaries 

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#### Abstract

The sparse representation problem of recovering an $N$ dimensional sparse vector $\boldsymbol{x}$ from $M<N$ linear observations $y=D x$ given dictionary $D$ is considered. The standard approach is to let the elements of the dictionary be independent and identically distributed (IID) zero-mean Gaussian and minimize the $l_{1}$-norm of $\boldsymbol{x}$ under the constraint $\boldsymbol{y}=\boldsymbol{D} \boldsymbol{x}$. In this paper, the performance of $l_{1}$-reconstruction is analyzed, when the dictionary is bi-orthogonal $D=\left[O_{1} O_{2}\right]$, where $O_{1}, O_{2}$ are independent and drawn uniformly according to the Haar measure on the group of orthogonal $M \times M$ matrices. By an application of the replica method, we obtain the critical conditions under which perfect $l_{1}$-recovery is possible with bi-orthogonal dictionaries.


## I. Introduction

The sparse representation (SR) problem has wide applicability, for example, in communications [1], [2], multimedia [3], and compressive sampling (CS) [4], [5]. The standard SR problem is to find the sparsest $\boldsymbol{x} \in \mathbb{R}^{N}$ that is the solution to the set of $M<N$ linear equations

$$
\begin{equation*}
\boldsymbol{y}=\boldsymbol{D} \boldsymbol{x} \tag{1}
\end{equation*}
$$

for a given dictionary or sensing matrix $\boldsymbol{D} \in \mathbb{R}^{M \times N}$ and observation $\boldsymbol{y}$. Finding such $\boldsymbol{x}$ is, however, non-polynomial (NP) hard. Thus, a variety of practical algorithms have been developed that solve the SR problem sub-optimally. The topic of the current paper is the convex relaxation approach where, instead of searching for the $\boldsymbol{x}$ having the minimum $l_{0}$-norm, the goal is to find the minimum $l_{1}$-norm solution of (1).

Let $K$ be the number of non-zero elements in $\boldsymbol{x}$ and assume that the convex relaxation method is used for recovery. The trade-off between two parameters $\rho=K / N$ and $\alpha=M / N$ is then of special interest since it tells how much the sparse signal can be compressed under $l_{1}$-reconstruction. An interesting question then arises: How does the sparsity-undersampling ( $\rho$ vs. $\alpha$ ) trade-off depend on the choice of dictionary $\boldsymbol{D}$ ?

The empirical study in [6, Sec. 15 in SI] gave evidence that the worst case $\rho$ vs. $\alpha$ trade-off is quite universal w.r.t different random matrix ensembles. Analysis in [7] further revealed that the typical conditions for perfect $l_{1}$-recovery are the same for all sensing matrices that are sampled from the rotationally invariant matrix ensembles. Dictionaries with independent identically distributed (IID) zero-mean Gaussian elements is one example of this. But correlations in $\boldsymbol{D}$ can degrade the performance of $l_{1}$-recovery [8], so it is not fully clear how the choice of $\boldsymbol{D}$ affects the $\rho$ vs. $\alpha$ trade-off.

Besides the random / unstructured dictionaries mentioned above, the information theoretic approach in [9] encompasses more general matrix ensembles but does not consider the $l_{1}$ reconstruction limit. Several studies in the literature have also considered the specific construction where $\boldsymbol{D}$ is formed by concatenating two orthogonal matrices [10]-[14]. Such biorthogonal dictionaries are easy to implement and can give elegant theoretical insights. Unfortunately, the "mutual coherence" based methods used in these papers provide pessimistic, or worst case, thresholds. Furthermore, the result are not easy to compare between the unstructured and bi-orthogonal cases.

We consider the analysis of the bi-orthogonal SR setup

$$
\boldsymbol{y}=\boldsymbol{D} \boldsymbol{x}=\left[\begin{array}{ll}
\boldsymbol{O}_{1} & \boldsymbol{O}_{2}
\end{array}\right]\left[\begin{array}{l}
\boldsymbol{x}_{1}  \tag{2}\\
\boldsymbol{x}_{2}
\end{array}\right]=\boldsymbol{O}_{1} \boldsymbol{x}_{1}+\boldsymbol{O}_{2} \boldsymbol{x}_{2}
$$

where the dictionary is constructed by concatenating two independent matrices $\boldsymbol{O}_{1}$ and $\boldsymbol{O}_{2}$, that are drawn uniformly according to the Haar measure on the group of all orthogonal $M \times M$ matrices. We use the non-rigorous replica method (see, e.g., [7], [15]-17] for related works) to assess $\rho$ for a given $\alpha$, up to which the $l_{1}$-recovery is successful. This allows a direct comparison between the random and bi-orthogonal dictionaries in average or typical sense. The main result of the paper is the sparsity-undersampling trade-off for the biorthogonal SR setup (2). We find that this matches the unstructured IID Gaussian dictionary when the non-zero components are uniformly distributed between the two blocks. Surprisingly, when the non-zero components are concentrated more on one block than the other, the bi-orthogonal dictionaries can cope with higher overall densities than the unstructured case. This extends to the case of general $T$-concatenated orthogonal dictionaries as reported elsewhere [18].

## II. Problem Setting

Consider the SR problem of finding the sparsest vector $\boldsymbol{x}=$ $\left[\begin{array}{ll}\boldsymbol{x}_{1}^{\top} & \boldsymbol{x}_{2}^{\top}\end{array}\right]^{\top} \in \mathbb{R}^{N}$, given the dense vector $\boldsymbol{y} \in \mathbb{R}^{M}$ and the dictionary $\boldsymbol{D}=\left[\begin{array}{ll}\boldsymbol{O}_{1} & \boldsymbol{O}_{2}\end{array}\right] \in \mathbb{R}^{M \times N}$. By definition $M / N=$ $1 / 2$ and $\boldsymbol{O}_{i}^{\top} \boldsymbol{O}_{i}=\boldsymbol{I}_{M}$ for this setup. Let $K_{1}$ and $K_{2}$ be the number of non-zero elements in $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$, respectively, so that $K=K_{1}+K_{2}$ is the total number of non-zero elements in $\boldsymbol{x}$. Denote $\rho=K /(2 M)$ for the overall sparsity of the source while $\rho_{1}=K_{1} / M$ and $\rho_{2}=K_{2} / M$ represent the signal densities of the two blocks.

It is important to note that $\boldsymbol{D}$ in (2) does not belong to the rotationally invariant matrix ensembles [7], and there are complex dependencies between the elements due to the orthogonality constraints. The fact that $\boldsymbol{O}_{1}^{\top} \boldsymbol{O}_{2} \neq \mathbf{0}$ makes the analysis of the setup highly non-trivial (for a sketch, see Appendices A and B). Thus, only the bi-orthogonal case is considered here and the analysis of general $T$-concatenated orthogonal dictionaries is reported elsewhere [18].

The system is assumed to approach the large system limit $M, K_{1}, K_{2} \rightarrow \infty$ where the signal densities $\rho_{1}, \rho_{2}$ are finite and fixed. We let $\left\{\boldsymbol{x}_{i}\right\}_{i=1}^{2}$ be independent sparse random vectors whose components are IID according to

$$
\begin{equation*}
p_{i}(x)=\left(1-\rho_{i}\right) \delta(x)+\rho_{i} \mathrm{e}^{-x^{2} / 2} / \sqrt{2 \pi}, \quad i=1,2 \tag{3}
\end{equation*}
$$

The convex relaxation of the original problem is considered and the goal is to find $\boldsymbol{x}=\left[\boldsymbol{x}_{1}^{\top} \boldsymbol{x}_{2}^{\top}\right]^{\top}$ that is the solution to

$$
\begin{equation*}
\min _{\boldsymbol{x}_{1}, \boldsymbol{x}_{2}}\left\|\boldsymbol{x}_{1}\right\|_{1}+\left\|\boldsymbol{x}_{2}\right\|_{1} \quad \text { s.t. } \quad \boldsymbol{y}=\boldsymbol{O}_{1} \boldsymbol{x}_{1}+\boldsymbol{O}_{2} \boldsymbol{x}_{2} \tag{4}
\end{equation*}
$$

Note that we do not consider the weighted $l_{1}$-reconstruction analyzed for the rotationally invariant $D$ in [15]. This corresponds to the scenario where the user has no prior knowledge about the relative statistics of the data blocks. In the next section we find the typical density $\rho=\left(\rho_{1}+\rho_{2}\right) / 2$ for which perfect $l_{1}$-reconstruction is possible under the constraint (2).

## III. Analysis

Let the postulated prior of the sparse vector $\boldsymbol{x}_{i}$ be

$$
\begin{equation*}
q_{\beta}\left(\tilde{\boldsymbol{x}}_{i}\right)=\mathrm{e}^{-\beta\left\|\tilde{\boldsymbol{x}}_{i}\right\|_{1}}, \quad i=1,2 \tag{5}
\end{equation*}
$$

where the components of $\tilde{\boldsymbol{x}}_{i} \in \mathbb{R}^{M}$ are IID. The inverse temperature $\beta$ is a non-negative parameter. Let $q_{\beta}(\tilde{\boldsymbol{x}})=$ $q_{\beta}\left(\tilde{\boldsymbol{x}}_{1}\right) q_{\beta}\left(\tilde{\boldsymbol{x}}_{2}\right)$ be the postulated prior of $\boldsymbol{x}$ in (2), and define a mismatched posterior mean estimator

$$
\begin{equation*}
\langle\tilde{\boldsymbol{x}}\rangle_{\beta}=Z_{\beta}(\boldsymbol{y}, \boldsymbol{D})^{-1} \int \tilde{\boldsymbol{x}} \delta(\boldsymbol{y}-\boldsymbol{D} \tilde{\boldsymbol{x}}) q_{\beta}(\tilde{\boldsymbol{x}}) \mathrm{d} \tilde{\boldsymbol{x}} \tag{6}
\end{equation*}
$$

Here $Z_{\beta}(\boldsymbol{y}, \boldsymbol{D})=\int \delta(\boldsymbol{y}-\boldsymbol{D} \tilde{\boldsymbol{x}}) q_{\beta}(\tilde{\boldsymbol{x}}) \mathrm{d} \tilde{\boldsymbol{x}}$, acts as the partition function of the system. Then, the zero-temperature estimate $\langle\tilde{\boldsymbol{x}}\rangle_{\beta \rightarrow \infty}$ is a solution (if at least one exists) to the original $l_{1}$-minimization problem (4).

Utilizing of one of the standard tools from statistical physics, namely the non-rigorous replica method, we study next the behavior of the estimator (6). We accomplish this by examining the so-called free energy density $f$ of the system in the thermodynamic limit $N \rightarrow \infty$. As a corollary, we obtain the critical compression threshold for the original optimization problem (4) when $\beta \rightarrow \infty$.

## A. Free Energy

As sketched in Appendix A the free energy density related to (6) reads under the replica symmetric (RS) ansatz

$$
\begin{align*}
f_{\mathrm{rs}} & =-\frac{1}{2} \lim _{\beta \rightarrow \infty} \frac{1}{\beta} \lim _{M \rightarrow \infty} \frac{1}{M} \lim _{u \rightarrow 0} \frac{\partial}{\partial u} \log \mathrm{E}_{\boldsymbol{y}, \boldsymbol{D}}\left\{Z_{\beta}^{u}(\boldsymbol{y}, \boldsymbol{D})\right\} \\
& =\frac{1}{2} \operatorname{cextr}_{\left\{\Theta_{1}, \Theta_{2}\right\}} \sum_{i=1}^{2} T\left(\Theta_{i}\right) \tag{7}
\end{align*}
$$

where

$$
\begin{align*}
& T\left(\Theta_{i}\right)=\frac{\rho_{i}-2 m_{i}+Q_{i}}{4 \chi_{i}}-\frac{Q_{i} \hat{Q}_{i}}{2}+\frac{\chi_{i} \hat{\chi}_{i}}{2}+m_{i} \hat{m}_{i} \\
& +\int\left(1-\rho_{i}\right) \phi\left(z \sqrt{\hat{\chi}_{i}} ; \hat{Q}_{i}\right)+\rho_{i} \phi\left(z \sqrt{\hat{m}_{i}^{2}+\hat{\chi}_{i}} ; \hat{Q}_{i}\right) \mathrm{D} z \tag{8}
\end{align*}
$$

$\Theta_{i}=\left\{Q_{i}, \chi_{i}, m_{i}, \hat{Q}_{i}, \hat{\chi}_{i}, \hat{m}_{i}\right\}$ is a set of parameters that take values on the extended real line, $\mathrm{D} z=(2 \pi)^{-1 / 2} \mathrm{e}^{-z^{2} / 2} \mathrm{~d} z$ is the Gaussian measure and

$$
\begin{equation*}
\phi(h ; \hat{Q})=\min _{x \in \mathbb{R}}\{\hat{Q} x / 2-h x+|x|\} \tag{9}
\end{equation*}
$$

In contrast to, e.g., [7], [15], here $\operatorname{cextr}_{\Theta} g(\Theta)$ is constrained extremization over the function $g(\Theta)$ when $\chi_{1}=\chi_{2}$, needs to be satisfied.
Remark 1. If the dictionary is sampled from the rotationally invariant matrix ensembles, the RS free energy density reads

$$
\begin{align*}
& f_{\mathrm{rs}}=\frac{1}{2} \operatorname{extr}_{\left\{\Theta_{1}, \Theta_{2}\right\}} \sum_{i=1}^{2}\left(\frac{\rho_{i}-2 m_{i}+Q_{i}}{2 \sum_{i=1}^{2} \chi_{i}}-\frac{Q_{i} \hat{Q}_{i}}{2}+\frac{\chi_{i} \hat{\chi}_{i}}{2}+m_{i} \hat{m}_{i}\right. \\
& \left.+\int\left(1-\rho_{i}\right) \phi\left(z \sqrt{\hat{\chi}_{i}} ; \hat{Q}_{i}\right)+\rho_{i} \phi\left(z \sqrt{\hat{m}_{i}^{2}+\hat{\chi}_{i}} ; \hat{Q}_{i}\right) \mathrm{D} z\right), \tag{10}
\end{align*}
$$

where extr is an unconstrained extremization w.r.t $\left\{\Theta_{1}, \Theta_{2}\right\}$.

## B. Constrained Extremization

Let us denote $Q(x)=\int_{x}^{\infty} \mathrm{D} z$ for the Q -function and define

$$
\begin{equation*}
r(h)=\sqrt{\frac{h}{2 \pi}} \mathrm{e}^{-\frac{1}{2 h}}-(1+h) Q\left(\frac{1}{\sqrt{h}}\right) \tag{11}
\end{equation*}
$$

After solving the integrals and the optimization problem in (9), the function (8) becomes

$$
\begin{align*}
T\left(\Theta_{i}\right)= & \frac{\rho_{i}-2 m_{i}+Q_{i}}{4 \chi_{i}}-\frac{Q_{i} \hat{Q}_{i}}{2}+\frac{\chi_{i} \hat{\chi}_{i}}{2}+m_{i} \hat{m}_{i} \\
& +\frac{1-\rho_{i}}{\hat{Q}_{i}} r\left(\hat{\chi}_{i}\right)+\frac{\rho_{i}}{\hat{Q}_{i}} r\left(\hat{m}_{i}^{2}+\hat{\chi}_{i}\right) \tag{12}
\end{align*}
$$

Introducing the Lagrange multiplier $\eta$ for the constraint $\chi_{1}=$ $\chi_{2}$, an alternative formulation for the free energy density reads

$$
\begin{equation*}
f_{\mathrm{rs}}=\frac{1}{2} \operatorname{extr}_{\left\{\Theta_{1}, \Theta_{2}, \eta\right\}}\left\{\eta\left(\chi_{1}-\chi_{2}\right)+T\left(\Theta_{1}\right)+T\left(\Theta_{2}\right)\right\} \tag{13}
\end{equation*}
$$

where the extremization is now an unconstrained problem. Taking partial derivatives w.r.t all optimization variables and setting the results to zero yields the identities

$$
\begin{equation*}
\hat{Q}_{i}=\hat{m}_{i} \quad \text { and } \quad \chi_{i}=\frac{1}{2 \hat{m}_{i}}, \quad i=1,2 \tag{14}
\end{equation*}
$$

We also find that the expressions

$$
\begin{align*}
\frac{1}{\hat{m}_{i}} & =\frac{2}{\hat{m}_{i}}\left[2\left(1-\rho_{i}\right) Q\left(\frac{1}{\sqrt{\hat{\chi}_{i}}}\right)+2 \rho_{i} Q\left(\frac{1}{\sqrt{\hat{m}_{i}^{2}+\hat{\chi}_{i}}}\right)\right]  \tag{15}\\
\hat{\chi}_{i} & =\frac{\rho_{i}-2 m_{i}+Q_{i}}{2 \chi_{i}^{2}}-\eta \frac{\partial}{\partial \chi_{i}}\left(\chi_{1}-\chi_{2}\right) \tag{16}
\end{align*}
$$

are satisfied by the extremum of 13]. Under perfect reconstruction in mean square error (MSE) sense (see, e.g., [7],
[15] for details), we have $\rho_{i}=Q_{i}=m_{i}$ and $\hat{m}_{i} \rightarrow \infty \Longrightarrow$ $\chi_{i} \rightarrow 0$. Hence, (15) simplifies to the condition

$$
\begin{equation*}
2\left(1-\rho_{i}\right) Q\left(\frac{1}{\sqrt{\hat{\chi}_{i}}}\right)+\rho_{i}=\frac{1}{2} . \tag{17}
\end{equation*}
$$

On the other hand, omitting the terms of the order $O\left(1 / \hat{m}^{3}\right)$, we have from the partial derivatives of $\hat{Q}_{i}$ and $\hat{m}_{i}$

$$
\begin{align*}
Q_{i} & =\rho_{i}-\frac{2 \rho_{i}}{\hat{m}_{i} \sqrt{2 \pi}}-\frac{2\left(1-\rho_{i}\right)}{\hat{m}_{i}^{2}} r\left(\hat{\chi}_{i}\right)+\frac{\rho_{i}}{\hat{m}_{i}^{2}}\left(1+\hat{\chi}_{i}\right)  \tag{18}\\
m_{i} & =\rho_{i}-\frac{\rho_{i}}{\hat{m}_{i} \sqrt{2 \pi}} \tag{19}
\end{align*}
$$

respectively, where we used 14 to simplify the expressions. Plugging the above to (16) and using again (14) yields

$$
\begin{equation*}
\hat{\chi}_{i}=(-1)^{i} \eta+2 \rho_{i}\left(1+\hat{\chi}_{i}\right)-4\left(1-\rho_{i}\right) r\left(\hat{\chi}_{i}\right) \tag{20}
\end{equation*}
$$

Before stating the final result, let us introduce a real parameter $\mu \in[0,1]$ and assume without loss of generality that $\rho_{1}=\mu \rho_{2}$. Then the per-block densities can be written as

$$
\begin{equation*}
\rho_{1}=\frac{2 \mu}{1+\mu} \rho \quad \text { and } \quad \rho_{2}=\frac{2}{1+\mu} \rho \tag{21}
\end{equation*}
$$

where $\rho=\rho(\mu)$ is the overall density of the source. The parameter $\mu$ determines thus how uniformly the non-zero components are distributed between the two blocks: $\mu=1$ means fully uniformly, $\mu=0$ implies that all non-zero components are in the second block.
Main Result. Let $\boldsymbol{x} \in \mathbb{R}^{2 M}, \boldsymbol{D} \in \mathbb{R}^{M \times 2 M}$ and $\boldsymbol{y}=\boldsymbol{D} \boldsymbol{x}$ as in (2). Given the parameter $\mu \in[0,1]$, the typical density $\rho(\mu)$ of the solution to the optimization problem

$$
\underset{\boldsymbol{x}=\left[\boldsymbol{x}_{1} \boldsymbol{x}_{2}\right]^{\top} \in \mathbb{R}^{2 M}}{\arg \min }\left\|\boldsymbol{x}_{1}\right\|_{1}+\left\|\boldsymbol{x}_{2}\right\|_{1} \quad \text { s.t. } \quad \boldsymbol{y}=\boldsymbol{D} \boldsymbol{x}
$$

is determined in the large system limit by the solutions of the following set of coupled equations

$$
\begin{align*}
\hat{\chi}_{1} & =\left[Q^{-1}\left(\frac{1}{4}-\frac{2 \mu \rho}{1+\mu}\left[\frac{1}{2}-Q\left(\frac{1}{\sqrt{\hat{\chi}_{1}}}\right)\right]\right)\right]^{-2},  \tag{22}\\
\eta & =\frac{4 \mu \rho}{1+\mu}\left[1+\hat{\chi}_{1}+2 r\left(\hat{\chi}_{1}\right)\right]-4 r\left(\hat{\chi}_{1}\right)-\hat{\chi}_{1},  \tag{23}\\
\hat{\chi}_{2} & =\frac{4 \rho}{1+\mu}\left[1+\hat{\chi}_{2}+2 r\left(\hat{\chi}_{2}\right)\right]-4 r\left(\hat{\chi}_{2}\right)+\eta,  \tag{24}\\
\rho & =(1+\mu)\left[\frac{1}{2}-2 Q\left(\frac{1}{\sqrt{\hat{\chi}_{2}}}\right)\right] /\left[2-4 Q\left(\frac{1}{\sqrt{\hat{\chi}_{2}}}\right)\right], \tag{25}
\end{align*}
$$

where $Q^{-1}$ is the functional inverse of the $Q$-function. For uniform sparsity, that is, $\mu=1$ and $\rho_{1}=\rho_{2}$, we have $\eta=0$, $\hat{\chi}_{1}=\hat{\chi}_{2}$ and $\chi_{1}=\chi_{2}$ always. The critical density is thus the same as for the dictionary that is drawn from the ensemble of rotationally invariant matrices.

## C. Numerical Examples

Given the dictionary $\boldsymbol{D}$ is drawn from the ensemble of rotationally invariant matrices, the critical density for $l_{1}$ recovery is known to be independent of the block densities $\left\{\rho_{1}, \rho_{2}\right\}$ and given by $\rho=0.19284483309074016 \ldots$ for all $\mu \in[0,1]$. For the bi-orthogonal $\boldsymbol{D}$, the threshold is the same


Fig. 1. Critical density for bi-orthogonal and rotationally invariant $\boldsymbol{D}$. The parameter $\mu \in[0,1]$ determines how uniformly the non-zero components are distributed between the two blocks ( $\mu=1$ fully uniform, $\mu=0$ all non-zero components are in the second block). The user has no knowledge about $\mu$.


Fig. 2. Critical density given $\mu=0$, that is, $\rho_{1}=0, \rho_{2}=2 \rho$ for finite sized systems. Here 'R' means rotationally invariant $\boldsymbol{D}$ and 'O' the bi-orthogonal case. Each point is averaged over $10^{6}$ realizations of the optimization problem. The filled markers at $x=0$ are the predictions given by the replica analysis.
only for the case of uniform sparsity $\mu=1$. For general $\mu$ we obtain different thresholds, as plotted in Fig. 1 Note that $\rho(\mu)$ is a decreasing function of $\mu$, implying that the more concentrated the non-zero components are in one block, the bigger the benefit of using the bi-orthogonal dictionary. We also carried out numerical simulations for the IID Gaussian and bi-orthogonal $\boldsymbol{D}$ using 'linprog' from Matlab Optimization Toolbox. The results are plotted in Fig. 2, where for each value of $N=16,18, \ldots, 50$, there are $10^{6}$ realizations of the SR problem. Cubic curves are fitted to the data using nonlinear least-squares regression. The critical density for the bi-orthogonal case is predicted by the replica method to be $\rho(0)=0.22666551758496698 \ldots$ and we observe that the simulations match the analysis up to the third decimal place.

## IV. Conclusions and Discussion

The sparsity-undersampling trade-off for the bi-orthogonal SR setup (2) was studied. For uniformly distributed non-zero components, there is no difference in compression ratio if we replace the rotationally invariant dictionary $\boldsymbol{D} \in \mathbb{R}^{M \times 2 M}$ by a concatenated matrix $\boldsymbol{D}=\left[\begin{array}{ll}\boldsymbol{O}_{1} & \boldsymbol{O}_{2}\end{array}\right] \in \mathbb{R}^{M \times 2 M}$, where $\boldsymbol{O}_{1}, \boldsymbol{O}_{2}$ are independent and drawn uniformly according to the Haar measure on the group of all orthogonal $M \times M$
matrices. For non-uniform block sparsities, however, the biorthogonal dictionaries were found to be beneficial compared to the unstructured random dictionaries.

## Appendix A

## Free Energy

Following [7], [15], we use the replica trick and write the free energy density as

$$
\begin{equation*}
f=-\frac{1}{2} \lim _{\beta \rightarrow \infty} \frac{1}{\beta} \lim _{u \rightarrow 0} \frac{\partial}{\partial u} \lim _{M \rightarrow \infty} \frac{1}{M} \log \Xi_{\beta, M}^{(u)} \tag{26}
\end{equation*}
$$

where denoting $\Delta \boldsymbol{x}_{i}^{[a]}=\boldsymbol{x}_{i}^{[0]}-\boldsymbol{x}_{i}^{[a]}, a=0,1, \ldots, u$,
$\boldsymbol{\Xi}_{\beta, M}^{(u)}=\mathrm{E} \lim _{\tau \rightarrow 0^{+}} \frac{1}{\tau^{\frac{u M}{2}}} \mathrm{E}\left\{\left.\mathrm{e}^{-\frac{1}{2 \tau} \sum_{a=1}^{u}\left\|\boldsymbol{O}_{1} \Delta \boldsymbol{x}_{1}^{[a]}+\boldsymbol{O}_{2} \Delta \boldsymbol{x}_{2}^{[a]}\right\|^{2}} \right\rvert\, \mathcal{X}\right\}$.
For $i=1,2$, the vectors $\left\{\boldsymbol{x}_{i}^{[a]}\right\}_{a=1}^{u}$ are IID conditioned on $\boldsymbol{D}$ and have the same density (5) as $\tilde{\boldsymbol{x}}_{i}$. Furthermore, the elements of the vectors $\boldsymbol{x}_{1}^{[0]}$ and $\boldsymbol{x}_{2}^{[0]}$ are independently drawn according to $p_{1}$ and $p_{2}$ as given in (3), and $\mathcal{X}=\left\{\boldsymbol{x}_{1}^{[a]}, \boldsymbol{x}_{2}^{[a]}\right\}_{a=0}^{u}$.

Let us concentrate on $\Xi_{\beta, M}^{(u)}$ and the inner expectation in (27), which is over the orthogonal matrices $\boldsymbol{O}_{1}$ and $\boldsymbol{O}_{2}$ given $\mathcal{X}$. Since $\boldsymbol{O}_{i}$ are orthogonal, the average affects only the crossterms of the form $\left(\boldsymbol{u}_{1}^{[a]}\right)^{\top} \boldsymbol{u}_{2}^{[a]}$ where $\boldsymbol{u}_{i}^{[a]}=\boldsymbol{O}_{i} \Delta \boldsymbol{x}_{i}^{[a]}$. Define matrices $\boldsymbol{S}_{i} \in \mathbb{R}^{u \times u}$ for $i=1,2$, whose $(a, b)$ th element

$$
\begin{equation*}
S_{i}^{[a, b]}=Q_{i}^{[0,0]}-Q_{i}^{[0, b]}-Q_{i}^{[a, 0]}+Q_{i}^{[a, b]}, \quad i=1,2 \tag{28}
\end{equation*}
$$

is the empirical covariance between the elements of $\Delta \boldsymbol{x}_{i}^{[a]}$ and $\Delta \boldsymbol{x}_{i}^{[b]}$, written in terms of the empirical covariances

$$
\begin{equation*}
Q_{i}^{[a, b]}=M^{-1}\left(\boldsymbol{x}_{i}^{[a]}\right)^{\top} \boldsymbol{x}_{i}^{[b]}, \quad a, b=0,1, \ldots, u \tag{29}
\end{equation*}
$$

between the components of the $a$ th and $b$ th replicas of $\boldsymbol{x}_{i}$. For analytical tractability, we make the standard replica symmetry (RS) assumption on the correlations 29, i.e., $r_{i}=Q_{i}^{[0,0]}$, $m_{i}=Q_{i}^{[0, b]}=Q_{i}^{[a, 0]} \forall a, b \geq 1, Q_{i}=Q_{i}^{[a, a]} \forall a \geq 1$ and $q_{i}=Q_{i}^{[a, b]} \forall a \neq b \geq 1$. The RS free energy density is denoted $f_{\mathrm{rs}}$ and we remark that it does not match $f$ if the system is replica symmetry breaking. Under the RS assumption,

$$
\begin{equation*}
\boldsymbol{S}_{i}=S_{i}^{[1,2]} \mathbf{1}_{u} \mathbf{1}_{u}^{\top}+\left(S_{i}^{[1,1]}-S_{i}^{[1,2]}\right) \boldsymbol{I}_{u}, \quad i=1,2 \tag{30}
\end{equation*}
$$

where $1_{u} \in \mathbb{R}^{u}$ is the vector of all-ones, and we may write the inner expectation in 27) as

$$
\begin{equation*}
\mathrm{e}^{-\frac{u M}{2 \tau}\left(S_{1}^{[1,1]}+S_{2}^{[1,1]}\right)} \mathrm{E}\left\{\left.\mathrm{e}^{-\frac{1}{\tau} \sum_{a=1}^{u}\left(\boldsymbol{u}_{1}^{[a]}\right)^{\top} \boldsymbol{u}_{2}^{[a]}} \right\rvert\, \mathcal{X}\right\} \tag{31}
\end{equation*}
$$

Using Lemma 2 and taking the limit $\tau \rightarrow 0^{+}$leads to

$$
\begin{equation*}
\Xi_{\beta, M}^{(u)}=\int \mathrm{e}^{-M G^{(u)}} \prod_{u=1}^{u} \mathrm{e}^{-\beta\left(\left\|\boldsymbol{x}_{1}^{[a]}\right\|_{1}+\left\|\boldsymbol{x}_{2}^{[a]}\right\|_{1}\right)} \mathrm{d} \boldsymbol{x}_{1}^{[a]} \mathrm{d} \boldsymbol{x}_{2}^{[a]} \tag{32}
\end{equation*}
$$

where $G^{(u)}=\lim _{\tau \rightarrow 0^{+}} G_{\tau}^{(u)}$. The function $G_{\tau}^{(u)}$ given in 33) at the top of the next page is implicitly a function of both $\boldsymbol{S}_{1}$ and $\boldsymbol{S}_{2}$. To obtain (33) we first used (45), then applied (39). Finally, some algebraic manipulations give the reported result.

The problem with the limit $G^{(u)}=\lim _{\tau \rightarrow 0^{+}} G_{\tau}^{(u)}$ is that it diverges and the free energy density grows without bound
which is an undesired result. To keep $G^{(u)}$ and the free energy density finite as $\tau \rightarrow 0^{+}$, we pose the constraints

$$
\begin{align*}
S_{1}^{[1,1]}-S_{1}^{[1,2]}+u S_{1}^{[1,2]} & =S_{2}^{[1,1]}-S_{2}^{[1,2]}+u S_{2}^{[1,2]}  \tag{34}\\
S_{1}^{[1,1]}-S_{1}^{[1,2]} & =S_{2}^{[1,1]}-S_{2}^{[1,2]} \tag{35}
\end{align*}
$$

on the elements of the replica symmetric matrices $\boldsymbol{S}_{1}, \boldsymbol{S}_{2}$. Given (34) and (35) are satisfied, we get in the limit $\tau \rightarrow 0^{+}$ the expression for $G^{(u)}=G_{1}^{(u)}+G_{2}^{(u)}$ in terms of

$$
\begin{align*}
G_{i}^{(u)}= & \frac{1}{4} \log \left(Q_{i}-q_{i}+u\left(r_{i}-2 m_{i}+q_{i}\right)\right) \\
& +\frac{u-1}{4} \log \left(Q_{i}-q_{i}\right), \quad i=1,2 \tag{36}
\end{align*}
$$

Comparing (36) to [7, eq. (A.4)] reveals that the corresponding terms for rotationally invariant and bi-orthogonal $\boldsymbol{D}$ match up to vanishing constants. Furthermore, in the limit $u \rightarrow 0$ the equalities (34) and (35) are equivalent to the condition $\chi_{1}=\chi_{2}$, where we denoted $\chi_{i}=\beta\left(Q_{i}-q_{i}\right)$ for notational convenience. This provides the relevant constraint for the evaluation of the RS free energy, as stated in Section III-A.

The next task would be to average (32) over the correlations (29) using the theory of large deviations and saddle-point integration. But since the effect of the bi-orthogonal sensing matrix $D$ has been reduced to the above constraint, we omit the calculations here due to space constraints. For details, see [7. Appendix A] and [18].

## Appendix B

## Matrix Integration

Lemma 1. Let $\boldsymbol{O}_{1}$ and $\boldsymbol{O}_{2}$ be independent and drawn uniformly according to the Haar measure on the group of all orthogonal $M \times M$ matrices as in (2). Given vectors $\boldsymbol{x}_{1}, \boldsymbol{x}_{2} \in \mathbb{R}^{M}$, denote $\left\|\boldsymbol{x}_{i}\right\|^{2}=M r_{i}$, for $i=1,2$. Then

$$
\begin{equation*}
I_{M}\left(r_{1}, r_{2} ; c\right)=\mathrm{E}_{\boldsymbol{O}_{1}, \boldsymbol{O}_{2}} \mathrm{e}^{c \boldsymbol{x}_{1}^{\top} \boldsymbol{O}_{1}^{\top} \boldsymbol{x}_{2} \boldsymbol{O}_{2}}=\mathrm{E}_{\boldsymbol{u}_{1}, \boldsymbol{u}_{2}} \mathrm{e}^{c \boldsymbol{u}_{1}^{\top} \boldsymbol{u}_{2}} \tag{37}
\end{equation*}
$$

where $c \in R$ and vectors $\boldsymbol{u}_{1}, \boldsymbol{u}_{2} \in \mathbb{R}^{M}$ are independent and uniformly distributed on the hyper-spheres at the boundaries of $M$ dimensional balls with radiuses $R_{1}=\sqrt{M r_{1}}$ and $R_{2}=$ $\sqrt{M r_{2}}$, respectively. Furthermore,

$$
\begin{align*}
& F\left(r_{1}, r_{2} ; c\right)=\lim _{M \rightarrow \infty} M^{-1} \log I_{M}\left(r_{1}, r_{2} ; c\right) \\
& =\frac{\sqrt{1+4 c^{2} r_{1} r_{2}}}{2}-\frac{1}{2} \log \left(\frac{1+\sqrt{1+4 c^{2} r_{1} r_{2}}}{2}\right)-\frac{1}{2}  \tag{38}\\
& \approx \sqrt{c^{2} r_{1} r_{2}}-\log \left(c^{2} r_{1} r_{2}\right) / 4, \quad \text { for } c^{2} r_{1} r_{2} \gg 1 \tag{39}
\end{align*}
$$

Proof: Let $\boldsymbol{u}_{i}=\boldsymbol{O}_{i} \boldsymbol{x}_{i}$ where $\left\{\boldsymbol{x}_{i}\right\}_{i=1}^{2}$ are fixed and $\left\{\boldsymbol{O}_{i}\right\}_{i=1}^{2}$ independent and drawn uniformly according to the Haar measure on the group of all orthogonal $M \times M$ matrices. Since $\left\|\boldsymbol{u}_{i}\right\|^{2}=M r_{i}$ and $\boldsymbol{O}_{i}$ rotate the vectors $\boldsymbol{u}_{i}$ uniformly in all directions, $\boldsymbol{u}_{i}$ is uniformly distributed on the hyper-sphere at the boundaries of an $M$ dimensional ball having radius $R_{i}=\sqrt{M r_{i}}$, providing the second equality in 37).

To assess the second part of the lemma, the joint measure of $\left(\boldsymbol{u}_{1}, \boldsymbol{u}_{2}\right)$ reads $p\left(\boldsymbol{u}_{1} ; r_{1}\right) p\left(\boldsymbol{u}_{2} ; r_{2}\right) \mathrm{d} \boldsymbol{u}_{1} \mathrm{~d} \boldsymbol{u}_{2}$, where

$$
\begin{equation*}
p(\boldsymbol{u} ; r)=Z(r)^{-1} \delta\left(\|\boldsymbol{u}\|^{2}-M\right) \tag{40}
\end{equation*}
$$

$$
\begin{align*}
G_{\tau}^{(u)}= & \frac{1}{2 \tau}\left(\sqrt{S_{1}^{[1,1]}-S_{1}^{[1,2]}+u S_{1}^{[1,2]}}-\sqrt{S_{2}^{[1,1]}-S_{2}^{[1,2]}+u S_{2}^{[1,2]}}\right)^{2}+\frac{u-1}{2 \tau}\left(\sqrt{S_{1}^{[1,1]}-S_{1}^{[1,2]}}-\sqrt{S_{2}^{[1,1]}-S_{2}^{[1,2]}}\right)^{2} \\
& +\frac{1}{4} \log \left[\left(S_{1}^{[1,1]}-S_{1}^{[1,2]}+u S_{1}^{[1,2]}\right)\left(S_{2}^{[1,1]}-S_{2}^{[1,2]}+u S_{2}^{[1,2]}\right)\right]+\frac{u-1}{4} \log \left[\left(S_{1}^{[1,1]}-S_{1}^{[1,2]}\right)\left(S_{2}^{[1,1]}-S_{2}^{[1,2]}\right)\right], \tag{33}
\end{align*}
$$

The normalization constant $Z(r)$ in (40) is the volume of the hypersphere in which $\boldsymbol{u}$ is constrained to. Using Stirling's formula for large $M$, we get up to a vanishing term $O(1 / M)$

$$
\begin{equation*}
Z(r)=(2 \pi \mathrm{e} r)^{M / 2} / \sqrt{\pi r} . \tag{41}
\end{equation*}
$$

With the help of Laplace transform, we write

$$
\begin{equation*}
\delta(x-a)=\frac{1}{4 \pi \mathrm{i}} \int_{\gamma-\mathrm{i} \infty}^{\gamma+\mathrm{i} \infty} \mathrm{e}^{-\frac{1}{2} s(x-a)} \mathrm{d} s, \quad \gamma \in \mathbb{R} \tag{42}
\end{equation*}
$$

so that using (40) - (42), the latter expectation in (37) becomes

$$
\begin{align*}
& \frac{(4 \pi \mathrm{i})^{-2}}{Z\left(r_{1}, r_{2}\right)} \int \mathrm{e}^{c \boldsymbol{u}_{1}^{\top} \boldsymbol{u}_{2}-\sum_{i=1}^{2}\left(\left\|\boldsymbol{u}_{i}\right\|^{2}-M r_{i}\right) s_{i} / 2} \prod_{i=1}^{2} \mathrm{~d} \boldsymbol{u}_{i} \mathrm{~d} s_{i} \\
& \quad=\frac{(4 \mathrm{i})^{-2} \sqrt{r_{1} r_{2}}}{\pi \mathrm{e}^{M}\left(r_{1} r_{2}\right)^{M / 2}} \int \frac{\mathrm{e}^{M \frac{s_{1} r_{1}+s_{2} r_{2}}{2}}}{\left(s_{1} s_{2}-c^{2}\right)^{M / 2}} \mathrm{~d} s_{1} \mathrm{~d} s_{2}, \tag{43}
\end{align*}
$$

where we used Gaussian integration to obtain (43). Since $M \rightarrow \infty$, we next apply saddle-point integration to solve the integrals w.r.t $s_{1}$ and $s_{2}$. After canceling the vanishing terms,

$$
\begin{align*}
& \lim _{M \rightarrow \infty} M^{-1} \log I_{M}\left(r_{1}, r_{2} ; c\right) \\
& =-1-\frac{1}{2} \sum_{i=1}^{2} \log r_{i}+\frac{1}{2} \operatorname{extr}_{s_{1}, s_{2}}\left\{\sum_{i=1}^{2} s_{i} r_{i}-\log \left(s_{1} s_{2}-c^{2}\right)\right\}, \tag{44}
\end{align*}
$$

and (38) follows by solving the extremization, and (39) by neglecting the terms that are of the order unity.

Lemma 2. Let $\left\{\boldsymbol{O}_{i}\right\}_{i=1}^{2}$ be as in Lemma 1] and $\Delta \boldsymbol{x}_{i}^{[a]}$ for $i=1,2$ and $a=1, \ldots, u$ as in 27). Then, under RS ansatz

$$
\begin{align*}
& \lim _{M \rightarrow \infty} M^{-1} \log \mathrm{E}_{\boldsymbol{O}_{1}, \boldsymbol{O}_{2}}\left\{\mathrm{e}^{c \sum_{a=1}^{u}\left(\boldsymbol{O}_{1} \Delta \boldsymbol{x}_{1}^{[a]}\right)^{\top}\left(\boldsymbol{O}_{2} \Delta \boldsymbol{x}_{2}^{[a]}\right)} \mid \mathcal{X}\right\} \\
& =F\left(S_{1}^{[1,1]}-S_{1}^{[1,2]}+u S_{1}^{[1,2]}, S_{2}^{[1,1]}-S_{2}^{[1,2]}+u S_{2}^{[1,2]} ; c\right) \\
& \quad+(u-1) F\left(S_{1}^{[1,1]}-S_{1}^{[1,2]}, S_{2}^{[1,1]}-S_{2}^{[1,2]} ; c\right), \tag{45}
\end{align*}
$$

where $c \in R$ and $F\left(r_{1}, r_{2} ; c\right)$ is given in 38.
Proof: Denote $\boldsymbol{u}_{i}^{[a]}=\boldsymbol{O}_{i} \Delta \boldsymbol{x}_{i}^{[a]}$ for all $i=1,2$ and $a=1, \ldots, u$. Given $\mathcal{X}, \boldsymbol{u}_{i}^{[a]}$ lie on the surfaces of hyperspheres as in the proof of Lemma 1. The RS ansatz guarantees that $\boldsymbol{u}_{i}^{[a]}$ can be expressed as $\left[\boldsymbol{u}_{i}^{[1]} \boldsymbol{u}_{i}^{[2]} \cdots \boldsymbol{u}_{i}^{[u]}\right]=$ $\left[\tilde{\boldsymbol{u}}_{i}^{[1]} \tilde{\boldsymbol{u}}_{i}^{[2]} \cdots \tilde{\boldsymbol{u}}_{i}^{[u]}\right] \boldsymbol{E}^{\boldsymbol{\top}}$, where $\left\{\tilde{\boldsymbol{u}}_{i}^{[a]}\right\}$ is a set of vectors that satisfies $M^{-1} \tilde{\boldsymbol{u}}_{i}^{[a]} \cdot \tilde{\boldsymbol{u}}_{i}^{[b]}=0$ if $a \neq b$ and

$$
\frac{1}{M} \tilde{\boldsymbol{u}}_{i}^{[a]} \cdot \tilde{\boldsymbol{u}}_{i}^{[b]}= \begin{cases}u S_{i}^{[1,2]}+\left(S_{i}^{[1,1]}-S_{i}^{[1,2]}\right) & \text { if } a=b=1  \tag{46}\\ S_{i}^{[1,1]}-S_{i}^{[1,2]} & \text { if } a=b \geq 2 .\end{cases}
$$

The matrix $\boldsymbol{E}=\left[u^{-1 / 2} \mathbf{1}_{u} \boldsymbol{e}_{2} \cdots \boldsymbol{e}_{u}\right]$ provides an orthonormal basis that is independent of index $i$. This indicates that the expectation in (45) can be assessed w.r.t. $\left\{\tilde{\boldsymbol{u}}_{i}^{[a]}\right\}$ instead of the
original non-orthogonal set $\left\{\boldsymbol{u}_{i}^{[a]}\right\}$. The orthogonality allows us to independently evaluate the expectation for each replica index $a$ when $u \ll M$. Using Lemma 1 and 46 completes the proof.

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