A Connection between Good Rate-distortion Codes and Backward DMCs

Curt Schieler, Paul Cuff

Dept. of Electrical Engineering, Princeton University, Princeton, NJ 08544. E-mail: {schieler, cuff}@princeton.edu

Abstract—Let $X^n \in \mathcal{X}^n$ be a sequence drawn from a discrete memoryless source, and let $Y^n \in \mathcal{Y}^n$ be the corresponding reconstruction sequence that is output by a good rate-distortion code. This paper establishes a property of the joint distribution of (X^n, Y^n) . It is shown that for D > 0, the input-output statistics of a R(D)-achieving rate-distortion code converge (in normalized relative entropy) to the output-input statistics of a discrete memoryless channel (dmc). The dmc is "backward" in that it is a channel from the reconstruction space \mathcal{Y}^n to source space \mathcal{X}^n . It is also shown that the property does not necessarily hold when normalized relative entropy is replaced by variational distance.

I. INTRODUCTION

Consider a discrete memoryless source with generic distribution P_X and a per-symbol distortion measure d(x, y). Given a distortion allowance D, the minimum achievable rate of compression (in bits per source symbol) is given by ratedistortion theory as

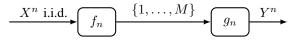
$$R(D) = \min_{P_{XY} \in \mathcal{P}(D)} I(X;Y),$$

where

$$\mathcal{P}(D) = \Big\{ P_{XY} : \sum_{y} P_{XY} = P_X \text{ and } \mathbb{E} d(X, Y) \le D \Big\}.$$

One intriguing achievability proof of this classic theorem was given by Wolfowitz in [1] (see also [2, Theorem 7.3]) and goes roughly as follows. A joint distribution $P_{XY} \in \mathcal{P}(D)$ gives rise to a random transformation $P_{X|Y}$ from the reproduction alphabet to the source alphabet. Using Feinstein's maximal code construction, create a *channel* code designed for the "backward" dmc $\prod_{i=1}^{n} P_{X|Y}(x_i|y_i)$; here, "backward" refers to the reversed flow of information from the reconstruction space to the source space. The resulting channel code can be transformed into a rate-distortion code by using the channel decoder as a source encoder and the channel encoder as a source decoder. In [1], it is shown that the distortion criterion is met as long as the channel code has large enough error probability, thus demonstrating that good rate-distortion codes can be constructed from certain channel codes.

In this paper, we explore another connection between lossy source coding and backward dmc's, one which involves the input-output statistics of good rate-distortion codes. Briefly, the result is as follows. Consider an arbitrary R(D)-achieving rate-distortion code¹ that maps source sequences X^n to recon-



(a) A rate-distortion code is a pair (f_n, g_n) that maps a source sequence X^n to a reconstruction codeword Y^n . The code induces a distribution $P_{X^nY^n}$ on the pair (X^n, Y^n) .



(b) Select a codeword \tilde{Y}^n uniformly at random from the codebook corresponding to (f_n, g_n) , then pass \tilde{Y}^n through a memoryless channel $P_{X|Y}$. The pair $(\tilde{X}^n, \tilde{Y}^n)$ induces a distribution $Q_{X^nY^n}$.

Fig. 1: Description of the true joint distribution $P_{X^nY^n}$ (Fig. 1a) and the approximating joint distribution $Q_{X^nY^n}$ (Fig. 1b).

struction codewords Y^n . The code induces a joint distribution $P_{X^nY^n}$ on the pair (X^n, Y^n) (see Figure 1a).

Using the corresponding codebook, select a codeword uniformly at random as the input to a backward dmc $\prod_{i=1}^{n} P_{X|Y}(x_i|y_i)$, where $P_{X|Y}$ is derived from the minimizer of R(D).² This channel coding operation induces a joint distribution $Q_{X^nY^n}$ on the pair $(\tilde{X}^n, \tilde{Y}^n)$, where \tilde{Y}^n is the randomly selected codeword and \tilde{X}^n is the channel output (see Figure 1b). We show that, provided some mild necessary conditions are satisfied,

$$\lim_{n \to \infty} \frac{1}{n} D(P_{X^n Y^n} || Q_{X^n Y^n}) = 0.$$
 (1)

That is, the input-output statistics of nearly all R(D)-achieving sequences of rate-distortion codes converge (in the sense of normalized relative entropy) to the output-input statistics of a backward dmc acting on the rate-distortion codebook.³

The property in (1) is analogous to the property of capacityachieving codes for memoryless channels established in [4, Theorem 15], namely that the channel output statistics converge (in normalized relative entropy) to a memoryless distribution. More precisely, a capacity-achieving sequence of codes

¹More precisely, a sequence of codes.

²Although the minimizer may not be unique, it is well-known that $P_{X|Y}$ is unique.

³We note that a similar claim appears in [3, Thm. 2]; however, their unconditional claim is not correct. Furthermore, their proof is brief and incorrect. We comment more on this during our proof.

satisfies

$$\lim_{n \to \infty} \frac{1}{n} D(P_{Y^n} || Q_{Y^n}) = 0,$$
(2)

where P_{Y^n} is the true distribution of the channel output and $Q_{Y^n} = \prod_{i=1}^n P_Y(y_i)$, where P_Y is the unique capacityachieving output distribution.

There are various properties of good rate-distortion codes that have been examined in the past (see, for example, [5] and [6]). Notably, [6] showed that the empirical kthorder distribution of a good rate-distortion code converges in distribution almost surely to the minimizer of the kth-order rate-distortion function (when that minimizer is unique). Note that the property in (1), in contrast, concerns the actual (not empirical) joint distribution and k = n. In some sense, (1) complements [6] in the same way that (2) complements the results in [7] on the kth-order empirical input distribution of good channel codes.

In order to show that good rate-distortion codes yield (1), we will first prove in Section II that the property holds for good empirical coordination codes. Empirical coordination, studied in [8], is similar to rate-distortion except for the distortion criterion, which is replaced by the requirement that the variational distance between the joint empirical distribution and a target joint distribution P_{XY} converges in probability. Thus, one aims to achieve coordination pairs $(R, P_{Y|X})$ instead of rate-distortion pairs (R, D). Upon demonstrating that (1) holds for good empirical coordination codes, we show in Section III that the property holds for good rate-distortion codes, as well. In Section IV, we show that the property can fail to hold when the distance measure is replaced by variational distance or unnormalized relative entropy.

Although we do not prove it here, we are able to use the property in (1) to solve a problem in information-theoretic secrecy relating to Yamamoto's "Rate-distortion theory of the Shannon cipher system" [9]. Specifically, one can use the property to show that the results of [10] can be achieved simply by using good rate-distortion codes, instead of the particular stochastic encoders that [10] asserts the existence of. It is likely that the property can provide a solution or give insight into other secrecy problems, as well.

II. GOOD EMPIRICAL COORDINATION CODES

We begin by introducing empirical coordination codes. All results in this paper will assume memoryless sources and finite alphabets. Furthermore, we assume for simplicity that the source satisfies $P_X(x) > 0$, $\forall x \in \mathcal{X}$. We first give the definition of a coordination code (see Figure 1a).

Definition 1. An (n, R_n) coordination code consists of an encoder-decoder pair (f_n, g_n) operating at rate R_n , where

$$f_n: \mathcal{X}^n \longrightarrow \{1, \dots, M\}$$
(3)

$$g_n: \{1, \dots, M\} \longrightarrow \mathcal{Y}^n \tag{4}$$

$$R_n = \frac{1}{n} \log M. \tag{5}$$

A coordination code acts on a memoryless source X^n with generic distribution P_X . For a fixed source sequence x^n , the code produces a codeword $y^n = g(f(x^n))$. The empirical distribution of the resulting pair (x^n, y^n) is defined for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ by

$$T_{x^n y^n}(x,y) \triangleq \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{(x_i, y_i) = (x,y)\}.$$
 (6)

The empirical distribution of the pair of random variables (X^n, Y^n) is itself a random variable and is denoted by $T_{X^nY^n}$. Variational distance, a measure of the distance between two distributions P and Q with common alphabet, is defined by

$$\|P - Q\| \triangleq \sup_{A} |P(A) - Q(A)|.$$
⁽⁷⁾

Definition 2. The pair $(R, P_{Y|X})$ is achievable if there exists a sequence of (n, R_n) coordination codes such that

$$\lim_{n \to \infty} R_n = R \tag{8}$$

and

$$||T_{X^nY^n} - P_{XY}|| \xrightarrow{i.p.} 0, \tag{9}$$

where $P_{XY} = P_X P_{Y|X}$.

Theorem 1 ([8]). The pair $(R, P_{Y|X})$ is achievable if and only if $R \ge I(X;Y)$.

The rate boundary in Theorem 1 justifies the following definition of a "good" coordination code.

Definition 3. Given a source P_X , a sequence of (n, R_n) coordination codes $\{(f_n, g_n)\}_{n=1}^{\infty}$ is good for $P_{Y|X}$ if

$$\lim_{n \to \infty} R_n = I(X;Y) \tag{10}$$

and

$$\|T_{X^nY^n} - P_{XY}\| \xrightarrow{i.p.} 0. \tag{11}$$

To each good sequence of coordination codes for $P_{Y|X}$, we associate two sequences of joint distributions $\{P_{X^nY^n}\}_{n=1}^{\infty}$ and $\{Q_{X^nY^n}\}_{n=1}^{\infty}$. The first, $P_{X^nY^n}$, is the distribution of the pair (X^n, Y^n) induced by the code. That is,

$$P_{X^nY^n} = P_{X^n} P_{Y^n|X^n},\tag{12}$$

where

$$P_{X^{n}}(x^{n}) = \prod_{i=1}^{n} P_{X}(x_{i})$$
(13)

is the memoryless source distribution and

$$P_{Y^{n}|X^{n}}(y^{n}|x^{n}) = \mathbf{1}\left\{y^{n} = g_{n}(f_{n}(x^{n}))\right\}$$
(14)

is the composition of the encoder with the decoder. The second distribution, $Q_{X^nY^n}$, is the distribution of the pair $(\tilde{X}^n, \tilde{Y}^n)$, where \tilde{Y}^n is a codeword selected uniformly at random and \tilde{X}^n is the output of the backward dmc when the input is \tilde{Y}^n . That is,

$$Q_{X^n Y^n} = Q_{Y^n} Q_{X^n | Y^n}, (15)$$

where

$$Q_{Y^n}(y^n) = \frac{|g_n^{-1}(y^n)|}{M}$$
(16)

is the uniform distribution over the codebook (which might to the lossy source coding theorem, we have contain duplicate codewords) and 1

$$Q_{X^{n}|Y^{n}}(x^{n}|y^{n}) = \prod_{i=1}^{n} P_{X|Y}(x_{i}|y_{i})$$
(17)

is the backward dmc with generic channel $P_{X|Y}$ derived from the joint distribution $P_{XY} = P_X P_{Y|X}$.

Our main result is the following theorem.⁴

Theorem 2. Let $P_{XY} \in A$, where

$$\mathcal{A} \triangleq \{ P_{XY} : P_{X|Y}(x|y) > 0, \forall (x,y) \}$$
(18)

Then, for any good sequence of coordination codes for $P_{Y|X}$, it holds that

$$\lim_{n \to \infty} \frac{1}{n} D(P_{X^n Y^n} || Q_{X^n Y^n}) = 0,$$
(19)

where $P_{X^nY^n}$ and $Q_{X^nY^n}$ are defined in (12)-(17). Furthermore, if $P_{XY} \notin A$, then there exists a good sequence of coordination codes for $P_{Y|X}$ such that

$$\lim_{n \to \infty} \frac{1}{n} D(P_{X^n Y^n} || Q_{X^n Y^n}) = \infty.$$
⁽²⁰⁾

Proof:

We will need the following property of variational distance, which is easily verified. Let $\varepsilon > 0$ and let f(x) be a function bounded by $b \in \mathbb{R}$. Then

$$||P - Q|| < \varepsilon \implies |\mathbb{E}_P f(X) - \mathbb{E}_Q f(X)| < \varepsilon b.$$
 (21)

We also need the following chain rule of relative entropy:

$$D(P_{X^nY^n} || Q_{X^nY^n}) = D(P_{X^nY^n} || P_{Y^n} Q_{X^n|Y^n}) + D(P_{Y^n} || Q_{Y^n}).$$
(22)

To begin the proof of Theorem 2, fix $P_{XY} \in \mathcal{A}$ and a good sequence of coordination codes for $P_{Y|X}$. We first show that such a sequence has the property⁵

$$\lim_{n \to \infty} \frac{1}{n} I(X^n; Y^n) = I(X; Y), \tag{23}$$

where $I(X^n; Y^n)$ is evaluated with respect to the true distribution $P_{X^nY^n}$. Throughout the proof, bear in mind that all expectations and mutual information expressions involving (X^n, Y^n) are evaluated with respect to the true distribution $P_{X^nY^n}$.

To show (23), we first introduce an auxiliary random variable $J \sim \text{Unif}\{1, \ldots, n\}$ independent of (X^n, Y^n) . Regurgitating some of the standard steps found in the converse

$$R_n = \frac{1}{n} \log M \tag{24}$$

$$\geq \frac{1}{n}H(Y^n) \tag{25}$$

$$\geq \frac{1}{n}I(X^n;Y^n) \tag{26}$$

$$\geq \frac{1}{n} \sum_{i=1}^{n} I(X_i, Y_i) \tag{27}$$

$$= I(X_{J}; Y_{J}|J)$$
(28)
(a) $I(X + Y - J)$ (20)

$$\cong I(X_J; Y_J, J) \tag{29}$$

$$\geq I(X_J; Y_J), \tag{30}$$

where (a) follows from $X_J \perp J$. If we can show that

$$\lim_{n \to \infty} I(X_J; Y_J) = I(X; Y), \tag{31}$$

then the proof of the property in (23) will be complete by (10) and the squeeze theorem. To that end, we use several observations from [8]. By the boundedness of variational distance, (11) implies

$$\lim_{n \to \infty} \mathbb{E} \left\| T_{X^n Y^n} - P_{XY} \right\| = 0.$$
(32)

Upon noting that

$$\mathbb{E} T_{X^n Y^n} = P_{X_J Y_J},\tag{33}$$

we have

$$||P_{X_J Y_J} - P_{XY}|| = ||\mathbb{E} T_{X^n Y^n} - P_{XY}||$$
(34)

$$\stackrel{(a)}{\leq} \mathbb{E} \|T_{X^n Y^n} - P_{XY}\|, \qquad (35)$$

where (a) follows from Jensen's inequality. Therefore,

$$\lim_{n \to \infty} \|P_{X_J Y_J} - P_{XY}\| = 0.$$
(36)

Since mutual information is continuous with respect to variational distance for finite alphabets (this follows from (21)), we see that (36) yields (31). Thus, the property in (23) holds.

We remark that the property in (36) underlies the reason that we are considering empirical coordination codes. In brief, it arises more naturally in an empirical coordination setting than in a rate-distortion setting. We will invoke (36) again shortly.

With (23) in hand, we now show that

$$\lim_{n \to \infty} \frac{1}{n} D(P_{X^n Y^n} || P_{Y^n} Q_{X^n | Y^n}) = 0.$$
(37)

To start, we have

n-

=

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \Big[\log \frac{\prod_{i=1}^{n} P_{X|Y}(X_i|Y_i)}{\prod_{i=1}^{n} P_X(X_i)} \Big]$$
$$= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \Big[\log \frac{P_{X|Y}(X_i|Y_i)}{P_X(X_i)} \Big]$$
(38)

$$= \lim_{n \to \infty} \mathbb{E} \left[\log \frac{P_{X|Y}(X_J|Y_J)}{P_X(X_J)} \right]$$
(39)

$$\stackrel{(a)}{=} \lim_{n \to \infty} \mathbb{E} \left[\log \frac{P_{X|Y}(X|Y)}{P_X(X)} \right]$$
(40)

$$= I(X;Y). \tag{41}$$

⁴We exclude the single pathological case $P_{XY}(x, y) = \frac{1}{|\mathcal{X}|} \mathbf{1}\{x = y\},\$ in which it is possible that there are some codebooks such that $P_{X^nY^n} = Q_{X^nY^n}$ and other codebooks such that $D(P_{X^nY^n} || Q_{X^nY^n}) = \infty$.

⁵In [3], the assertion is that the theorem follows from (23). However, this is not the case. It is necessary to establish the steps in (38)-(41), which rely on the property of coordination codes in (36).

To see how (a) follows, first note that the function

$$f(x,y) = \log \frac{P_{X|Y}(x|y)}{P_X(x)} \tag{42}$$

is bounded due to the restriction $P_{XY} \in \mathcal{A}$ (in fact, this is the only step where the restriction is needed). Then, use (36) along with (21). Continuing, we have

$$\lim_{n \to \infty} \frac{1}{n} D(P_{Y^n} P_{X^n | Y^n} | | P_{Y^n} Q_{X^n | Y^n})$$

$$= \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \Big[\log \frac{P_{X^n | Y^n} (X^n | Y^n)}{Q_{X^n | Y^n} (X^n | Y^n)} \Big]$$
(43)

$$= \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \Big[\log \frac{P_{X^n | Y^n}(X^n | Y^n)}{P_{X^n}(X^n)} \Big]$$
(44)

$$-\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[\log \frac{Q_{X^n | Y^n}(X^n | Y^n)}{P_{X^n}(X^n)} \right]$$
(45)

$$= \lim_{n \to \infty} \frac{1}{n} I(X^n; Y^n)$$
(46)

$$-\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \Big[\log \frac{\prod P_{X|Y}(X_i|Y_i)}{\prod P_X(X_i)} \Big]$$
(47)

$$\stackrel{(a)}{=} \lim_{n \to \infty} \frac{1}{n} I(X^n; Y^n) - I(X; Y) \tag{48}$$

$$\stackrel{(6)}{=} 0, \tag{49}$$

where (a) is due to (41) and (b) is due to (23). This proves the property in (37).

Finally, write

$$\lim_{n \to \infty} \frac{1}{n} D(P_{Y^n} || Q_{Y^n})$$
$$= \lim_{n \to \infty} \frac{1}{n} \sum_{y^n} P_{Y^n}(y^n) \log \frac{1}{Q_{Y^n}(y^n)}$$
(50)

$$-\lim_{n \to \infty} \frac{1}{n} H(Y^n) \tag{51}$$

$$\leq \lim_{n \to \infty} \frac{1}{n} \log M - \lim_{n \to \infty} \frac{1}{n} H(Y^n)$$
(52)

$$\stackrel{(a)}{=} 0 \tag{53}$$

where (a) follows from the squeeze theorem. To complete the first part of the theorem, invoke the chain rule of relative entropy in (22).

To show the second part of Theorem 2, fix $P_{XY} \notin A$ and a good sequence of coordination codes for the corresponding $P_{Y|X}$. The condition $P_{XY} \notin A$ implies the existence of a pair (x, y) such that $P_{X|Y}(x|y) = 0$. For every *n*, append a codeword y^n to the codebook and associate with it a sequence x^n such that

$$|i:(x_i,y_i)=(x,y)|>0.$$

Accordingly, modify f_n and g_n so that $y^n = g(f(x^n))$. Such a modification maintains the goodness of the code, but now $P_{X^nY^n}$ has support on (x^n, y^n) , while $Q_{X^nY^n}$ does not. Consequently, $\frac{1}{n}D(P_{X^nY^n}||Q_{X^nY^n})$ diverges.

III. GOOD RATE-DISTORTION CODES

In this section, we establish the counterpart to Theorem 2 for good rate-distortion codes. A rate-distortion code is defined according to Definition 1. The notion of good is also similar; in this case, a good code is an R(D)-achieving one.

Definition 4. Given a source P_X and a distortion measure d(x, y), a sequence of (n, R_n) rate-distortion codes $\{(f_n, g_n)\}_{n=1}^{\infty}$ is good for distortion D if

$$\lim_{n \to \infty} R_n = R(D) \tag{54}$$

and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} d(X_i, Y_i) \le D.$$
(55)

For a fixed per-letter distortion measure d(x, y), the rate-distortion function is defined for $D \ge D_{\min}$, where $D_{\min} = \mathbb{E}[\min_y d(X, y)]$. Without loss of generality, we assume that $D_{\min} = 0$.

In view of the restriction in Theorem 2 to $P_{XY} \in A$, the following lemma is useful.

Lemma 1 ([11, Ch. 2, Lemma 1]). Let D > 0. Any P_{XY} that minimizes R(D) is such that, if $P_{Y|X}(y|x) = 0$ for some (x, y), then $P_{Y|X}(y|x') = 0$ for all $x' \in \mathcal{X}$. Accordingly, the reproduction symbol y may be deleted from \mathcal{Y} without affecting R(D).

Thus, we have that for any D > 0 we can reduce the reproduction alphabet \mathcal{Y} , without penalty, to an alphabet $\mathcal{Y}^*(D)$ such that any P_{XY} minimizing R(D) satisfies $P_{XY}(x,y) > 0$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}^*$. In particular, $P_{XY} \in \mathcal{A}$. It is shown in [11] that this does not hold for D = 0. From this point on, we assume that \mathcal{Y} has been reduced according to Lemma 1, so that Theorem 2 can be invoked.

Although the minimizer of R(D) need not be unique, it turns out that the corresponding backward channel $P_{X|Y}$ is unique. This is analogous to the fact that the capacityachieving output distribution is unique, even though the input distribution is not.

Lemma 2 ([2, Problem 8.3]). If P_{XY} and Q_{XY} both minimize R(D), then $P_{X|Y} = Q_{X|Y}$.

We now state the counterpart to Theorem 2. The proof is immediate once we use the fact that good rate-distortion codes are good empirical coordination codes.

Theorem 3. Let D > 0, and assume that the reproduction alphabet has been reduced to $\mathcal{Y}^*(D)$. Then, for any good sequence of rate-distortion codes for D, it holds that

$$\lim_{n \to \infty} \frac{1}{n} D(P_{X^n Y^n} || Q_{X^n Y^n}) = 0,$$
 (56)

where

$$P_{X^nY^n}(x^n, y^n) = \prod_{i=1}^n P_X(x_i) \mathbf{1} \{ y^n = g_n(f_n(x^n)) \}$$
(57)

and

$$Q_{X^nY^n}(x^n, y^n) = \frac{|g_n^{-1}(y^n)|}{M} \prod_{i=1}^n P_{X|Y}(x_i|y_i), \qquad (58)$$

where $P_{X|Y}$ is the unique backward channel corresponding to D.

Proof: From [8, Theorem 11] or [6, Theorem 9], we have that a good rate-distortion code for D is a good empirical coordination code for some $P_{Y|X}$ minimizing R(D). Due to the reduction to $\mathcal{Y}^*(D)$, we have $P_{XY} \in \mathcal{A}$, which allows us to invoke Theorem 2.

IV. VARIATIONAL DISTANCE

In this section, we show that Theorem 2 does not hold when we replace normalized divergence by variational distance. From Pinsker's inequality, this implies that it does not hold in unnormalized relative entropy, either.

Theorem 4. There exists $P_{XY} \in A$ and a sequence of good coordination codes for the corresponding $P_{Y|X}$ such that

$$\lim_{n \to \infty} \left\| P_{X^n Y^n} - Q_{X^n Y^n} \right\| \neq 0, \tag{59}$$

where $P_{X^nY^n}$ and $Q_{X^nY^n}$ are defined in (12)-(17).

Proof: Let $P_{XY} \in \mathcal{A}$ be such that P_Y is an capacityachieving input distribution of the channel $P_{X|Y}$. Fix a sequence of good empirical coordination codes $\{(f_n, g_n)\}_{n=1}^{\infty}$ for $P_{Y|X}$ such that the decoder is bijective and

$$R_n = I(X;Y) + n^{-\frac{1}{2}+\delta},$$
 (60)

for some $\delta > 0$. This is possible by Theorem 1. By way of contradiction, suppose that

$$\lim_{n \to \infty} \|P_{X^n Y^n} - Q_{X^n Y^n}\| = 0.$$
 (61)

To reach a contradiction, we first define joint distributions $P_{X^nY^n\widehat{M}}$ and $Q_{X^nY^n\widehat{M}}$ by

$$\begin{aligned} & P_{X^{n}Y^{n}\widehat{M}}(x^{n}, y^{n}, \widehat{m}) = P_{X^{n}Y^{n}}(x^{n}, y^{n}) \, \mathbf{1}\big\{\widehat{m} = f_{n}(x^{n})\big\} \\ & Q_{X^{n}Y^{n}\widehat{M}}(x^{n}, y^{n}, \widehat{m}) = Q_{X^{n}Y^{n}}(x^{n}, y^{n}) \, \mathbf{1}\big\{\widehat{m} = f_{n}(x^{n})\big\}. \end{aligned}$$

Observe that $Q_{X^nY^n\widehat{M}}$ is the joint distribution governing the triple (X^n, Y^n, \widehat{M}) in the following channel coding setting:

$$\underbrace{\text{Unif}\{M\}}_{g_n} \underbrace{g_n}_{Y^n} \underbrace{P_{X|Y}}_{X|Y} \underbrace{X^n}_{f_n} \underbrace{\widehat{M}}_{F_n}$$

Thus, we have turned the rate-distortion code (f_n, g_n) into a channel code by identifying the channel encoder as the source decoder and the channel decoder as the source encoder. Because g_n is bijective, the error event for the channel coding is given by

$$\mathcal{E}_n = \left\{ g_n(\widehat{M}) \neq Y^n \right\},\tag{62}$$

and the probability of error is $Pr{Error(n)} \triangleq Q(\mathcal{E}_n)$.

On the other hand, notice that under the distribution $P_{X^nY^n\widehat{M}}$, it holds that

$${}_{n}(\widehat{M}) = g_{n}(f_{n}(X^{n})) = Y^{n}, \tag{63}$$

and thus $P(\mathcal{E}_n) = 0$.

Now, since variational distance has the property

$$||P_X P_{Y|X} - Q_X P_{Y|X}|| = ||P_X - Q_X||,$$
(64)

we have by (61) that

$$\lim_{n \to \infty} \|P_{X^n Y^n \widehat{M}} - Q_{X^n Y^n \widehat{M}}\|$$
(65)

$$= \lim_{n \to \infty} \|P_{X^n Y^n} - Q_{X^n Y^n}\|$$
(66)

Therefore, by the definition of variational distance,

$$\lim_{n \to \infty} \Pr\{\operatorname{Error}(n)\} = \lim_{n \to \infty} Q(\mathcal{E}_n) \tag{68}$$

$$= \lim_{n \to \infty} |Q(\mathcal{E}_n) - P(\mathcal{E}_n)| \qquad (69)$$

Thus, we have demonstrated a sequence of channel codes whose rates approach the channel capacity $slowly^6$ from above, yet whose probability of error vanishes. This is impossible due to the strong converse to the channel coding theorem (e.g., [12, Theorem 5.8.5]), yielding a contradiction.

V. ACKNOWLEDGEMENTS

This research was supported in part by the National Science Foundation under Grants CCF-1116013 and CCF-1017431, and also by the Air Force Office of Scientific Research under Grant FA9550-12-1-0196.

REFERENCES

- J. Wolfowitz, "Approximation with a fidelity criterion," in Proc. 5th Berkeley Symp. Math. Statist. Prob., vol. 1, 1966, pp. 565–573.
- [2] I. Csiszár and J. Körner, Information theory: coding theorems for discrete memoryless systems. Cambridge University Press, 2011.
- [3] S. S. Pradhan, "Approximation of test channels in source coding," Proc. Conf. Inf. Sys. Sci. (CISS), Mar. 2004.
- [4] T. S. Han and S. Verdu, "Approximation theory of output statistics," *IEEE Trans. Inf. Theory*, vol. 39, no. 3, pp. 752–772, May 1993.
- [5] A. Kanlis, S. Khudanpur, and P. Narayan, "Typicality of a good ratedistortion code," *Problems of Information Transmission*, vol. 32, no. 1, pp. 96–103, 1996.
- [6] T. Weissman and E. Ordentlich, "The empirical distribution of rateconstrained source codes," *IEEE Trans. Inf. Theory*, vol. 51, no. 11, pp. 3718–3733, Nov. 2005.
- [7] S. Shamai and S. Verdu, "The empirical distribution of good codes," *IEEE Trans. Inf. Theory*, vol. 43, no. 3, pp. 836–846, May 1997.
- [8] P. Cuff, H. Permuter, and T. Cover, "Coordination capacity," *IEEE Trans. Inf. Theory*, vol. 56, no. 9, pp. 4181–4206, Sept. 2010.
- [9] H. Yamamoto, "Rate-distortion theory for the Shannon cipher system," IEEE Trans. Inf. Theory, vol. 43, no. 3, pp. 827–835, May 1997.
- [10] P. Cuff, "A framework for partial secrecy," in *Proc. Global Telecomm. Conf. (GLOBECOM)*, Dec. 2010.
- [11] T. Berger, Rate-distortion theory. Prentice-Hall, 1971.
- [12] R. G. Gallager, Information theory and reliable communication. Wiley, 1968.

⁶Referring to the term $n^{-\frac{1}{2}+\delta}$ in (60).