A Generalization of Blahut-Arimoto Algorithm to Compute Rate-Distortion Regions of Multiterminal Source Coding Under Logarithmic Loss

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Abstract—In this paper, we present iterative algorithms that numerically compute the rate-distortion regions of two problems: the two-encoder multiterminal source coding problem and the Chief Executive Officer (CEO) problem, both under logarithmic loss distortion measure. With the clear connection of these models with the distributed information bottleneck method, the proposed algorithms may find usefulness in a variety of applications, such as clustering, pattern recognition and learning. We illustrate the efficiency of our algorithms through some numerical examples.

I. INTRODUCTION

The logarithmic loss (log-loss) function is a widely used penalty function that is particularly natural in settings in which reconstructions are allowed to be 'soft', rather than 'hard' or deterministic. That is, settings in which decoders or estimators output not only estimate values but also assessment of the levels of confidence in those values. More specifically, for a length-*n* vector or sequence $\mathbf{x} = (x_1, \ldots, x_n)$ with element x_i , $i = 1, \ldots, n$, in some alphabet \mathcal{X}_i , its reconstruction version or estimate is a vector $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_n)$ for which every component \hat{x}_i is a probability distribution on \mathcal{X}_i . The symbolwise distortion between x_i and \hat{x}_i is measured as

$$d(x_i, \hat{x}_i) = \log\left(\frac{1}{\hat{x}_i(x_i)}\right),\tag{1}$$

where $\hat{x}_i(x_i)$ represents the value of the probability distribution \hat{x}_i evaluated for the outcome x_i . Using this symbol-wise distortion, distortion between sequences is then defined as

$$d^{(n)}(\mathbf{x}, \hat{\mathbf{x}}) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, \hat{x}_i).$$

The logarithmic loss function (1) has many appreciable features. First, it is used as a penalty criterion in various contexts, including clustering and classification [1], pattern recognition, learning and prediction [2], image processing [3] and others. Second, it was recently shown in a remarkable paper by Courtade and Weissman [4] to admit key properties that allow to solve multiterminal source coding problems that are known to be difficult otherwise, in the sense that their solutions are still to be found for general distortion measures. Specifically, as mentioned in [4], the log-loss distortion measure admits a lower bound in the form of conditional entropy. Using this key finding, Courtade and Weissman successfully establish the single-letter characterization of the achievable rate-distortion (RD) region of the classical two-encoder multiterminal source coding problem [4, Theorem 6], as well as that of the Chief Executive Officer (CEO) problem [4, Theorem 3], both under log-loss distortion measure.



Fig. 1. Chief Executive Officer (CEO) source coding problem.

The computation of the RD regions of the aforementioned multiterminal source coding problems for general memoryless sources is important per-se; and even more considering the wide range of applications of lossy multiterminal source coding, including emerging applications in fields such as distributed learning and estimation [2, Chapter 9]. For example, the information bottleneck method [1] is an efficient data clustering algorithm, which essentially computes the RD region of a point-to-point rate-distortion problem, in which the distortion is measured under log-loss. Developing algorithms that allow to compute the RD region of multiterminal source coding problems can lead to efficient distributed algorithms for clustering and prediction.

Nonetheless, computing the RD region of multiterminal source-coding problems under log-loss for general memoryless sources is a difficult task, as it involves non-trivial optimization problems over distributions of auxiliary random variables. In this paper, we develop computational techniques for solving numerically the RD regions of the two-encoder multiterminal source coding problem and the CEO problem, both under logarithmic loss distortion measure. Our approach for the computation of both regions consists on first reexpressing the original RD region in terms of the union of simpler regions, whose boundary points can be expressed parametrically. Then, each boundary point can be computed numerically via an appropriate iterative minimization method that we develop here. The proposed method can be regarded as a generalization of the well known Blahut-Arimoto (BA) algorithm [5], [6] to the aforementioned multiterminal settings. For other generalizations of this algorithm, the reader may refer to related works on point-to-point [7], [8] and broadcast and multiple access multiterminal settings [9], [10].

II. THE CEO PROBLEM

Consider the discrete memoryless two-encoder CEO setup shown in Figure 1. In this setup, X is a discrete memoryless remote source with elements in some alphabet \mathcal{X} , and Y_1 and Y_2 are correlated memoryless observations or sources with elements in sets \mathcal{Y}_1 and \mathcal{Y}_2 , respectively. The joint probability mass function (pmf) of the triple (X, Y_1, Y_2) is P_{X,Y_1,Y_2} , which is assumed here to satisfy the Markov chain $Y_1 \oplus X \oplus Y_2$. The source Y_1 is observed at Encoder 1 and the source Y_2 is observed at Encoder 2. The encoders are connected to a decoder through error-free bit-pipes of capacities R_1 and R_2 , respectively. The decoder wants to reproduce an estimate \hat{X} of the remote source X to within some prescribed fidelity level D where the distortion is evaluated using the measure (1). That is, $\mathbb{E}[d(X, \hat{X})] \leq D$ with $d(\cdot)$ given by (1).

First, we recall the following theorem from [4, Theorem 3] which characterizes the RD region of the CEO problem under log-loss measure. We define $i^{c} \triangleq i \pmod{2} + 1$.

Theorem 1. [4, Theorem 3] The tuple $(R_1, R_2, D) \in \mathcal{RD}_{CEO}$ is achievable for the CEO problem under log-loss iff

$$R_i \ge I(U_i; Y_i | U_{i^c}, Q),$$
 for $i = 1, 2,$ (2)

$$R_1 + R_2 \ge I(U_1, U_2; Y_1, Y_2 | Q), \tag{3}$$

$$D \ge H(X|U_1, U_2, Q), \tag{4}$$

for some pmf $p(x)p(y_1|x)p(y_2|x)p(u_1|y_1,q)p(u_2|y_2,q)p(q)$, where $|\mathcal{U}_1| \leq |\mathcal{Y}_1|$, $|\mathcal{U}_2| \leq |\mathcal{Y}_2|$, and $|\mathcal{Q}| \leq 4$.

In this section, we develop a BA-type algorithm that allows to compute the convex region \mathcal{RD}_{CEO} for general memoryless sources. The outline of the proposed method is as follows. First, we rewrite the RD region \mathcal{RD}_{CEO} in terms of the union of two simpler regions in Proposition 1. The tuples lying on the boundary of each region are parametrically given in Theorem 2. Then, the boundary points of each simpler region are computed numerically via an alternating minimization method derived in Section II-B and detailed in Algorithm 1. Finally, the original RD region is obtained as the convex hull of the union of the tuples obtained for the two simple regions.

Due to the space limitations, some proofs are omitted or only outlined. The detailed proofs are relegated to the full version of this work [11].

A. Equivalent Parametrization of \mathcal{RD}_{CEO}

Define the two regions \mathcal{RD}_{CEO}^1 and \mathcal{RD}_{CEO}^2 for i=1,2 as

$$\mathcal{RD}^{i}_{\text{CEO}} = \{ (R_1, R_2, D) : D \ge D^{i}_{\text{CEO}}(R_1, R_2) \},\$$

with

$$D^{i}_{CEO}(R_1, R_2) \triangleq \min H(X|U_1, U_2)$$
s.t. $R_i \ge I(Y_i; U_i|U_{i^c})$ and $R_{i^c} \ge I(Y_{i^c}; U_{i^c})$, (5)

and the minimization is over set of joint measures P_{U_1,U_2,X,Y_1,Y_2} that satisfy $U_1 \twoheadrightarrow Y_1 \twoheadrightarrow X \twoheadrightarrow Y_2 \twoheadrightarrow U_2$.

As stated in the following proposition, the region \mathcal{RD}_{CEO} of Theorem 1 coincides with the convex hull of the union of the two regions \mathcal{RD}_{CEO}^1 and \mathcal{RD}_{CEO}^2 .

Proposition 1. The region \mathcal{RD}_{CEO} is given by

$$\mathcal{RD}_{\text{CEO}} = \operatorname{conv}(\mathcal{RD}_{\text{CEO}}^1 \cup \mathcal{RD}_{\text{CEO}}^2).$$
(6)

Proof. The outline of the proof is as follows. Let P_{U_1,U_2,X,Y_1,Y_2} and P_Q be such that $(R_1, R_2, D) \in \mathcal{RD}_{CEO}$. The polytope defined by the rate constraints (2)-(3), denoted by \mathcal{V} , forms a contra-polymatroid with 2! extreme points (vertices) [4], [12]. Given a permutation π on $\{1, 2\}$, the tuple

$$R_{\pi(1)} = I(Y_{\pi(1)}; U_{\pi(1)}), \ R_{\pi(2)} = I(Y_{\pi(2)}; U_{\pi(2)}|U_{\pi(1)}),$$

defines an extreme point of \mathcal{V} for each permutation. As shown in [4], for every extreme point $(\tilde{R}_1, \tilde{R}_2)$ of \mathcal{V} , the point $(\tilde{R}_1, \tilde{R}_2, D)$ is achieved by time-sharing two successive Wyner-Ziv (SWZ) strategies. The set of achievable tuples with such SWZ scheme is characterized by the convex hull of $\mathcal{RD}^{\pi(1)}_{\text{CEO}}$. Convexifying the union of both regions as in (6), we obtain the original RD region \mathcal{R}_{CEO} .

The main advantage of Proposition 1 it that it reduces the computation of region \mathcal{RD}_{CEO} to the computation of the two regions \mathcal{RD}_{CEO}^i , i = 1, 2, whose boundary can be efficiently parameterized, leading to an efficient computational method. In what follows, we concentrate on \mathcal{RD}_{CEO}^1 . The computation of \mathcal{RD}_{CEO}^2 follows similarly, and is omitted for brevity. Next theorem provides a parameterization of the boundary tuples of the region \mathcal{RD}_{CEO}^1 in terms, each of them, of an optimization problem over the pmfs $\mathbf{P} \triangleq \{P_{U_1|Y_1}, P_{U_2|Y_2}\}.$

Theorem 2. For each $\mathbf{s} \triangleq [s_1, s_2]$, $s_1 > 0$, $s_2 > 0$, define a rate-distortion tuple $(R_{1,\mathbf{s}}, R_{2,\mathbf{s}}, D_{\mathbf{s}})$ parametrically given by

$$D_{\mathbf{s}} = -s_1 R_{1,\mathbf{s}} - s_2 R_{2,\mathbf{s}} + \min_{\mathbf{P}} F_{\mathbf{s}}(\mathbf{P}), \tag{7}$$

$$R_{1,\mathbf{s}} = I(Y_1; U_1^* | U_2^*), \quad R_{2,\mathbf{s}} = I(Y_2; U_2^*), \tag{8}$$

where $F_{\mathbf{s}}(\mathbf{P})$ is given in (9); \mathbf{P}^* are the conditional pmfs yielding the minimum in (7) and U_1^*, U_2^* are the auxiliary variables induced by \mathbf{P}^* . Then, we have:

- Each value of s leads to a tuple (R_{1,s}, R_{2,s}, D_s) on the distortion-rate curve D_s = D¹_{CEO}(R_{1,s}, R_{2,s}).
- 2) For every point on the distortion-rate curve, there is an s for which (7) and (8) hold.

Proof. Suppose that \mathbf{P}^* yields the minimum in (7). For this \mathbf{P} we have $I(Y_1; U_1|U_2) = R_{1,\mathbf{s}}$ and $I(Y_2; U_2) = R_{2,\mathbf{s}}$. Then,

$$D_{\mathbf{s}} = -s_1 R_{1,\mathbf{s}} - s_2 R_{2,\mathbf{s}} + F_{\mathbf{s}}(\mathbf{P}^*)$$

= $-s_1 R_{1,\mathbf{s}} - s_2 R_{2,\mathbf{s}} + [H(X|U_1^*, U_2^*) + s_1 R_{1,\mathbf{s}} + s_2 R_{2,\mathbf{s}}]$
= $H(X|U_1^*, U_2^*) \ge D_{\text{CEO}}^1(R_{1,\mathbf{s}}, R_{2,\mathbf{s}}).$ (10)

Conversely, if \mathbf{P}^* is the solution to the minimization in (5), then $I(Y_1; U_1^*|U_2^*) \leq R_1$ and $I(Y_2; U_2^*) \leq R_2$ and for any s, $D_{\text{CEO}}^1(R_1, R_2) = H(X|U_1^*, U_2^*)$

$$F_{\mathbf{s}}(\mathbf{P}) \triangleq H(X|U_{1}, U_{2}) + s_{1}I(Y_{1}; U_{1}|U_{2}) + s_{2}I(Y_{2}; U_{2}) = H(X|U_{1}, U_{2}) + s_{1}[I(U_{1}; Y_{1}) - I(U_{1}; U_{2})] + s_{2}I(U_{2}; Y_{2})$$

$$= -\sum_{u_{1}u_{2}x} p(u_{1}, u_{2}, x) \log p(x|u_{1}, u_{2}) - s_{1} \sum_{u_{1}u_{2}} p(u_{1}, u_{2}) \log p(u_{1}, u_{2}) - s_{2} \sum_{u_{2}} p(u_{2}) \log p(u_{2})$$

$$+ s_{1} \sum_{u_{1}y_{1}} p(u_{1}|y_{1})p(y_{1}) \log p(u_{1}|y_{1}) + s_{2} \sum_{u_{2}y_{2}} p(u_{2}|y_{2})p(y_{2}) \log p(u_{2}|y_{2}) + s_{1} \sum_{u_{2}} p(u_{2}) \log p(u_{2}), \qquad (9)$$

$$F_{\mathbf{s}}(\mathbf{P}, \mathbf{Q}) \triangleq -\sum_{u_{1}u_{2}x} p(u_{1}, u_{2}, x) \log q(x|u_{1}, u_{2}) - s_{1} \sum_{u_{1}u_{2}} p(u_{1}, u_{2}) \log q(u_{1}, u_{2}) - s_{2} \sum_{u_{2}} p(u_{2}) \log q(u_{2})$$

$$+ s_{1} \sum_{u_{1}y_{1}} p(u_{1}|y_{1})p(y_{1}) \log p(u_{1}|y_{1}) + s_{2} \sum_{u_{2}y_{2}} p(u_{2}|y_{2})p(y_{2}) \log p(u_{2}|y_{2}) + s_{1} \sum_{u_{2}} p(u_{2}) \log p(u_{2}). \qquad (11)$$

$$\geq H(X|U_1^*, U_2^*) + s_1(I(Y_1; U_1^*|U_2^*) - R_1) + s_2(I(Y_2; U_2^*) - R_2) = D_{\mathbf{s}} + s_1(R_{1,\mathbf{s}} - R_1) + s_2(R_{2,\mathbf{s}} - R_2).$$

Given s, and hence $(R_{1,s}, R_{2,s}, D_s)$, letting $(R_1, R_2) = (R_{1,s}, R_{2,s})$ yields $D_{CEO}^1(R_{1,s}, R_{2,s}) \ge D_s$, which proves, together with (10), statement 1) and 2).

B. An iterative algorithm to compute \mathcal{RD}^1_{CEO}

In this section, we derive an algorithm to solve (7) for a given parameter value s. To that end, we express the optimization in (7) as a minimization of a function $F_{\mathbf{s}}(\mathbf{P}, \mathbf{Q})$, given in (11), over **P** and some auxiliary pmfs **Q**, defined as $\mathbf{Q} \triangleq \{Q_{X|U_1,U_2}, Q_{U_1,U_2}, Q_{U_2}\}$. We have the following result.

Proposition 2. For each $\mathbf{s} \triangleq [s_1, s_2]$, $s_1 > 0$, $s_2 > 0$, the rate-distortion tuple $(D_{\mathbf{s}}, R_{1,\mathbf{s}}, R_{2,\mathbf{s}})$ is given by

$$D_{\mathbf{s}} = -s_1 R_{1,\mathbf{s}} - s_2 R_{2,\mathbf{s}} + \min_{\mathbf{P}} \min_{\mathbf{Q}} F_{\mathbf{s}}(\mathbf{P}, \mathbf{Q}), \qquad (12)$$

where $R_{1,s}$ and $R_{2,s}$ are given in (8) and \mathbf{P}^* are the conditional pmfs yielding the minimum in (7).

Proof. Follows from Theorem 2 and Lemma 2 below.

Motivated by the BA algorithm [5], we propose an alternate optimization procedure over the set of pmfs \mathbf{P} and \mathbf{Q} as shown in Algorithm 1. The steps in the algorithm are derived from the following lemmas.

Lemma 1. $F_{s}(\mathbf{P}, \mathbf{Q})$ is convex in \mathbf{P} and convex in \mathbf{Q} . *Proof.* Follows from the log-sum inequality.

Lemma 2. For fixed **P**, there exists a unique **Q** that achieves the minimum $\min_{\mathbf{Q}} F_{\mathbf{s}}(\mathbf{P}, \mathbf{Q}) = F_{\mathbf{s}}(\mathbf{P})$, given by

$$Q_{X|U_1,U_2} = P_{X|U_1,U_2}, \ Q_{U_1,U_2} = P_{U_1,U_2}, \ Q_{U_2} = P_{U_2}.$$
 (13)

Proof. The proof follows from the relation

$$\begin{aligned} F_{\mathbf{s}}(\mathbf{P},\mathbf{Q}) &- F_{\mathbf{s}}(\mathbf{P}) \\ &= \sum_{u_1 u_2} p(u_1, u_2) D_{\mathrm{KL}}(p(x|u_1, u_2)) ||q(x|u_1, u_2)) \\ &+ s_1 D_{\mathrm{KL}}(p(u_1, u_2)) ||q(u_1, u_2)) + s_2 D_{\mathrm{KL}}(p(u_2)) ||q(u_2)) \geq 0, \end{aligned}$$

where equality holds if and only if (13) is satisfied.

Lemma 3. For fixed \mathbf{Q} , there exists a unique \mathbf{P} that achieves the minimum $\min_{\mathbf{P}} F_{\mathbf{s}}(\mathbf{P}, \mathbf{Q})$, where $P_{U_i|Y_i}$ is given by

$$p(u_i|y_i) = \frac{\exp[\rho_i(u_i, y_i)]}{\sum_{u_i} \exp[\rho_i(u_i, y_i)]}, \quad for \ i = 1, 2,$$
(14)

where $\rho_i(u_i, y_i)$, i = 1, 2, are defined in (15) given below.

Algorithm 1 BA-type algorithm to compute \mathcal{RD}_{CEO}^1

1: **input:** P_{X,Y_1,Y_2} , parameters s.

2: **output:** $P_{U_1|Y_1}^*$, $P_{U_2|Y_2}^*$; $(D_s, R_{1,s}, R_{2,s})$.

3: initialization Set n = 0. Choose $\mathbf{P}^{(0)}$ randomly.

Calculate $\mathbf{Q}^{(0)}$ by applying steps 6 and 8.

5:
$$n \leftarrow n+1$$
.

- 5: $n \leftarrow n + 1$. 6: Update $\mathbf{P}^{(n)}$ by using (14).
- 7: Update the following pmfs using $\mathbf{P}^{(n)}$.

$$p^{(n)}(u_i|x) = \sum_{y_i} p^{(n)}(u_i|y_i)p(y_i|x), \ i = 1, 2,$$

$$p^{(n)}(u_i) = \sum_{y_i} p(y_i)p^{(n)}(u_i|y_i), \quad i = 1, 2,$$

$$p^{(n)}(u_1, u_2, x) = p(x)p^{(n)}(u_1|x)p^{(n)}(u_2|x),$$

$$p^{(n)}(u_1, u_2) = \sum_{x} p^{(n)}(u_1, u_2, x).$$

8: Update $\mathbf{Q}^{(n)}$ by using (13).

9: until convergence.

Proof. We have that $F_{s}(\mathbf{P}, \mathbf{Q})$ is convex in **P** from Lemma 1. For a given **Q** and **s**, in order to minimize $F_{s}(\mathbf{P}, \mathbf{Q})$ over the convex set of pmfs **P**, let us define the Lagrangian as

$$\begin{split} \mathcal{L}(\mathbf{P}, \boldsymbol{\lambda}) \triangleq & F_{\mathbf{s}}(\mathbf{P}, \mathbf{Q}) + \sum_{y_1} \lambda_1(y_1) [1 - \sum_{u_1} p(u_1|y_1)] \\ &+ \sum_{y_2} \lambda_2(y_2) [1 - \sum_{u_2} p(u_2|y_2)], \end{split}$$

where $\lambda_1(y_1) \geq 0$ and $\lambda_2(y_2) \geq 0$ are the Lagrange multipliers corresponding the constrains $\sum_{u_i} p(u_i|y_i) = 1$, $y_i \in \mathcal{Y}_i$, i = 1, 2, of the pmfs $P_{U_1|Y_1}$ and $P_{U_2|Y_2}$, respectively. Due to the convexity of $F_{\mathbf{s}}(\mathbf{P}, \mathbf{Q})$, the KKT conditions are necessary and sufficient for optimality. From the KKT conditions

$$\frac{\partial \mathcal{L}(\mathbf{P}, \boldsymbol{\lambda})}{\partial p(u_1 | y_1)} = 0, \qquad \frac{\partial \mathcal{L}(\mathbf{P}, \boldsymbol{\lambda})}{\partial p(u_2 | y_2)} = 0,$$

we obtain (16) at the bottom of the page. Then, we proceeded by rearranging (16) as follows

$$p(u_i|y_i) = e^{\tilde{\lambda}_i(y_i)} e^{\rho_i(u_i, y_i)}, \quad i = 1, 2,$$
(17)

where $\rho_i(u_i, y_i)$, i = 1, 2, are given by (15) below, and we define $\tilde{\lambda}_1(y_1) \triangleq \lambda_1/[s_1p(y_1)] - 1$ and $\tilde{\lambda}_2(y_2) \triangleq [\lambda_2(y_2) - (s_1+s_2)p(y_2)]/s_2p(y_2)$. Note that $\tilde{\lambda}_i(y_i)$ contain all terms independent of u_i for i = 1, 2. Finally, the Lagrange multipliers $\lambda_i(y_i)$ satisfying the KKT conditions are obtained by finding $\tilde{\lambda}_i(y_i)$ such that $\sum_{u_i} p(u_i|y_i) = 1$, i = 1, 2. Substituting in (17), $p(u_i|y_i)$ can be found as in (14).

$$\rho_{1}(u_{1}, y_{1}) \triangleq \frac{1}{s_{1}} \sum_{u_{2}x} p(x|y_{1}) p(u_{2}|x) \log q(x|u_{1}, u_{2}) + \sum_{u_{2}x} p(x|y_{1}) p(u_{2}|x) \log q(u_{1}, u_{2}),$$

$$\rho_{2}(u_{2}, y_{2}) \triangleq \frac{1}{s_{2}} \sum_{u_{1}x} p(x|y_{2}) p(u_{1}|x) \log q(x|u_{1}, u_{2}) + \frac{s_{1}}{s_{2}} \sum_{u_{1}x} p(x|y_{2}) p(u_{1}|x) \log q(u_{1}, u_{2}) + \log q(u_{2}) - \frac{s_{1}}{s_{2}} \log p(u_{2}).$$
(15)

$$\log p(u_2|y_2) = \frac{1}{s_2} \sum_{u_1x} p(x|y_2) p(u_1|x) \log q(x|u_1, u_2) + \frac{s_1}{s_2} \sum_{u_1x} p(x|y_2) p(u_1|x) \log q(u_1, u_2) + \log q(u_2) - \frac{s_1}{s_2} \log p(u_2) + \frac{\lambda_2(y_2)}{s_2 p(y_2)} - \frac{s_1 + s_2}{s_2},$$

$$\log p(u_1|y_1) = \frac{1}{s_1} \sum_{u_2x} p(x|y_1) p(u_2|x) \log q(x|u_1, u_2) + \sum_{u_2x} p(x|y_1) p(u_2|x) \log q(u_1, u_2) + \frac{\lambda_1(y_1)}{s_1 p(y_1)} - 1.$$
(16)

At each iteration of Algorithm 1, $F_{s}(\mathbf{P}^{(n)}, \mathbf{Q}^{(n)})$ decreases until eventually it converges. However, since $F_{s}(\mathbf{P}, \mathbf{Q})$ is convex in each argument but not necessarily jointly convex, Algorithm 1 does not necessarily converge to the global optimal. In particular, next proposition shows that Algorithm 1 converges to a stationary point of the the minimization in (7).

Proposition 3. The sequence $\{\mathbf{P}^{(n)}, \mathbf{Q}^{(n)}\}, n \ge 0$ in Algorithm 1 converges to a stationary solution of the minimization problem in (12) for $n \to \infty$.

Proof. The convergence of the algorithm follows since due to Lemma (2) and Lemma (3), at the *n*-th iteration we have

$$F_{\mathbf{s}}(\mathbf{P}^{(n-1)}, \mathbf{Q}^{(n-1)}) \ge F_{\mathbf{s}}(\mathbf{P}^{(n)}, \mathbf{Q}^{(n-1)}) \ge F_{\mathbf{s}}(\mathbf{P}^{(n)}, \mathbf{Q}^{(n)})$$

which implies converge since the sequence is lower bounded. The convergence to a stationary point follows by noting that the proposed method is a maximization-minimization algorithm in which $F_{s}(\mathbf{P}, \mathbf{Q})$ is a surrogate function [13].

III. MULTITERMINAL SOURCE CODING PROBLEM

In this section, we derive a BA-type algorithm to compute the RD region of the classical two-encoder multiterminal source coding setup, following a similar approach to that in Section II. In this setup, we consider two correlated memoryless sources Y_1 and Y_2 with elements in sets \mathcal{Y}_1 and \mathcal{Y}_2 and distributed according the joint pmf P_{Y_1,Y_2} . The sources Y_1 and Y_2 are observed at Encoder 1 and 2, each connected to a decoder through an error-free bit-pipe of capacity R_1 and R_2 , respectively. The decoder wants to reproduce an estimate $\hat{Y_1}$ and $\hat{Y_2}$ of the sources Y_1 and Y_2 to within some prescribed fidelity levels D_1 and D_2 , respectively; where the distortions are evaluated using the log-loss measure (1), i.e., $\mathbb{E}[d(Y_1, \hat{Y}_1)] \leq D_1$ and $\mathbb{E}[d(Y_2, \hat{Y}_2)] \leq D_2$.

The RD region of the two encoder multiterminal source coding problem under log-loss measure is characterized in the following theorem from [4, Theorem 6].

Theorem 3. [4, Theorem 6] The tuple $(R_1, R_2, D_1, D_2) \in \mathcal{RD}_{BT}$ is achievable for the two encoder multiterminal source coding problem under log-loss iff

$$\begin{aligned} R_i &\geq I(U_i; Y_i | U_{i^c}, Q), & \text{for } i = 1, 2, \\ R_1 + R_2 &\geq I(U_1, U_2; Y_1, Y_2 | Q), \\ D_i &\geq H(Y_i | U_1, U_2, Q), & \text{for } i = 1, 2. \end{aligned}$$

for some pmf $p(y_1, y_2)p(u_1|y_1, q)p(u_2|y_2, q)p(q)$, where $|\mathcal{U}_1| \leq |\mathcal{Y}_1|, |\mathcal{U}_2| \leq |\mathcal{Y}_2|$, and $|\mathcal{Q}| \leq 5$.

Similarly to Section II, first we write \mathcal{RD}_{BT} in terms of the union of two simpler regions, and then, we propose an

algorithm to compute its boundary rate-distortion pairs. To that end, define the two RD regions \mathcal{RD}_{BT}^{i} , i = 1, 2, as

$$\mathcal{RD}^{i}_{\mathrm{BT}} \triangleq \{ (R_1, R_2, D_2, D_2) :$$

$$\alpha D_1 + \bar{\alpha} D_2 \ge D^{i}_{\mathrm{BT},\alpha}(R_1, R_2), \forall \alpha \in [0, 1] \},$$

where $\bar{\alpha} \triangleq 1 - \alpha$, and

$$D^{i}_{\mathrm{BT},\alpha}(R_{1}, R_{2}) \triangleq \min \ \alpha H(Y_{1}|U_{1}, U_{2}) + \bar{\alpha} H(Y_{2}|U_{1}, U_{2})$$

s.t. $R_{i} \ge I(Y_{i}; U_{i}|U_{i^{c}}) \text{ and } R_{i^{c}} \ge I(Y_{i^{c}}; U_{i^{c}}),$

where the optimization is over the set of joint pmfs P_{U_1,U_2,Y_1,Y_2} that satisfy $U_1 \rightarrow Y_1 \rightarrow Y_2 \rightarrow U_2$.

Proposition 4. $\mathcal{RD}_{BT} = \operatorname{conv}(\mathcal{RD}_{BT}^1 \cup \mathcal{RD}_{BT}^2).$

Now, similarly to Proposition 2, we provide a parametrization of \mathcal{RD}_{BT}^1 , which allows to compute each tuple on the boundary of the region as a double minimization over the conditional pmfs $\mathbf{P} = \{P_{U_1|Y_1}, P_{U_2|Y_2}\}$ and some auxiliary pmfs $\mathbf{Q} \triangleq \{Q_{Y_1|U_1,U_2}, Q_{Y_2|U_1,U_2}, Q_{U_1,U_2}, Q_{U_2}\}$, of an auxiliary function $F_{\beta}(\mathbf{P}, \mathbf{Q})$ defined in (18). We have the following result similar to Theorem 2, justified with Lemma 4 below.

Theorem 4. Each tuple on the boundary of $\mathcal{R}D_{BT}^1$ can be obtained from some $\beta \triangleq [s_1, s_2, \alpha], s_1 > 0, s_2 > 0, \alpha \in [0, 1]$ parametrically as $(R_{1,\beta}, R_{2,\beta}, D_{1,\beta}, D_{2,\beta})$ where

$$\begin{aligned} \alpha D_{1,\beta} &+ \bar{\alpha} D_{2,\beta} = -s_1 R_{1,\beta} - s_2 R_{2,\beta} + \min_{\mathbf{P}} \min_{\mathbf{Q}} F_{\beta}(\mathbf{P}, \mathbf{Q}), \\ D_{1,\beta} &= H(Y_1 | U_1^*, U_2^*), \qquad D_{2,\beta} = H(Y_2 | U_1^*, U_2^*), \\ R_{1,\beta} &= I(Y_1; U_1^* | U_2^*), \qquad R_{2,\beta} = I(Y_2; U_2^*), \end{aligned}$$

where \mathbf{P}^* , \mathbf{Q}^* are the pmfs yielding the minimization above.

We have the following lemmas.

Lemma 4. For fixed **P**, there exists a unique **Q** that achieves the minimum $\min_{\mathbf{Q}} F_{\beta}(\mathbf{P}, \mathbf{Q}) = F_{\beta}(\mathbf{P}, \mathbf{Q}^*)$, given by

$$Q_{Y_1|U_1,U_2} = P_{Y_1|U_1,U_2}, \quad Q_{Y_2|U_1,U_2} = P_{Y_2|U_1,U_2}, \quad (19)$$

$$Q_{U_1,U_2} = P_{U_1,U_2}, \quad Q_{U_2} = P_{U_2}.$$

Lemma 5. For fixed \mathbf{Q} , there exists a unique \mathbf{P} that achieves the minimum $\min_{\mathbf{P}} F_{\boldsymbol{\beta}}(\mathbf{P}, \mathbf{Q})$, where $P_{U_i|Y_i}$ is given by

$$p(u_i|y_i) = \frac{\exp[\mu_i(u_i, y_i)]}{\sum_{u_i} \exp[\mu_i(u_i, y_i)]}, \quad for \ i = 1, 2,$$
(20)

where $\mu_i(u_i, y_i)$, i = 1, 2, are defined in (21).

An immediate iterative optimization method follows from the two lemmas above as detailed in Algorithm 2. Similarly to Algorithm 1, Algorithm 2 converges to a stationary point.

Proposition 5. The sequence $\{\mathbf{P}^{(n)}, \mathbf{Q}^{(n)}\}$, $n \ge 0$ in Algorithm 2 converges to a stationary point of the minimization problem in Theorem 4 for $n \to \infty$.

$$F_{\boldsymbol{\beta}}(\mathbf{P}, \mathbf{Q}) \triangleq s_1 \sum_{u_1 y_1} p(u_1 | y_1) p(y_1) \log p(u_1 | y_1) + s_2 \sum_{u_2 y_2} p(u_2 | y_2) p(y_2) \log p(u_2 | y_2) + s_1 \sum_{u_2} p(u_2) \log p(u_2) - s_2 \sum_{u_2} p(u_2) \log q(u_2) - \alpha \sum_{u_1 u_2 y_1} p(u_1, u_2, y_1) \log q(y_1 | u_1, u_2) - \bar{\alpha} \sum_{u_1 u_2 y_2} p(u_1, u_2, y_2) \log q(y_2 | u_1, u_2) - s_1 \sum_{u_1 u_2} p(u_1, u_2) \log q(u_1, u_2).$$
(18)

 $\mu_{2}(u_{2}, y_{2}) \triangleq \frac{\alpha}{s_{2}} \sum_{u_{1}y_{1}} p(y_{1}|y_{2})p(u_{1}|y_{1}) \log q(y_{1}|u_{1}, u_{2}) + \frac{\bar{\alpha}}{s_{2}} \sum_{u_{1}} p(u_{1}|y_{2}) \log q(y_{2}|u_{1}, u_{2}) + \frac{s_{1}}{s_{2}} \sum_{u_{1}} p(u_{1}|y_{2}) \log q(u_{1}, u_{2}) + \log q(u_{2}) - \frac{s_{1}}{s_{2}} \log p(u_{2}),$ $\mu_{1}(u_{1}, y_{1}) \triangleq \frac{\alpha}{s_{1}} \sum_{u_{2}} p(u_{2}|y_{1}) \log q(y_{1}|u_{1}, u_{2}) + \frac{\bar{\alpha}}{s_{1}} \sum_{u_{2}y_{2}} p(y_{2}|y_{1})p(u_{2}|y_{2}) \log q(y_{2}|u_{1}, u_{2}) + \sum_{u_{2}} p(u_{2}|y_{1}) \log q(u_{1}, u_{2}).$ (21)

Algorithm 2 BA-type algorithm to compute \mathcal{RD}_{BT}^1

- 1: **input:** pmf P_{Y_1,Y_2} , parameter β .
- 2: **output:** $P_{U_1|Y_1}^*$, $P_{U_2|Y_2}^*$; $(R_{1,\beta}, R_{2,\beta}, D_{1,\beta}, D_{2,\beta})$.
- 3: initialization Set n = 0. Choose $\mathbf{P}^{(0)}$ randomly.
- 4: repeat Calculate $\mathbf{Q}^{(0)}$ by applying steps 6 and 8.
- 5: $n \leftarrow n+1$.
- 6: Update $\mathbf{P}^{(n)}$ by using (20).
- 7: Update the following pmfs.

$$p^{(n)}(u_i) = \sum_{y_i} p(y_i)p^{(n)}(u_i|y_i), \qquad i = 1, 2,$$

$$p^{(n)}(u_1, u_2, y_i) = p(y_i)p^{(n)}(u_1|y_i)p^{(n)}(u_2|y_i), \qquad i = 1, 2,$$

$$p^{(n)}(u_1, u_2) = \sum_{y_1y_2} p(y_1, y_2)p^{(n)}(u_1|y_1)p^{(n)}(u_2|y_2).$$

8: Update $\mathbf{Q}^{(n)}$ by using (19).

9: until convergence.



Fig. 2. The regions \mathcal{RD}^1_{CEO} and \mathcal{RD}^2_{CEO} of the CEO setup for crossover probability $\alpha_1 = \alpha_2 = 0.25$ and the tuples $(R, R, D) \in \mathcal{RD}_{CEO}$.

IV. NUMERICAL RESULTS

In this section, we focus on the computation of the RD region for a binary CEO setup in which X is a Bernoulli random variable distributed as $X \sim \text{Bern}(0.5)$; the channel between the source and Encoder *i* is modeled as a binary symmetric channel (BSC) with a crossover probability α_i for i = 1, 2, i.e., $Y_i = X \oplus Z_i$, where $Z_i \sim \text{Bern}(\alpha_i)$.

Figure 2 shows the rate-distortion tuples of regions \mathcal{RD}^1_{CEO} and \mathcal{RD}^2_{CEO} computed with Algorithm 1 for a symmetric setup in which $\alpha_1 = \alpha_2 = 0.25$ and different values of s. The region \mathcal{RD}_{CEO} can be obtained by computing the convex hull of these points. Additionally, the tuples of \mathcal{RD}_{CEO} achievable for $R_1 = R_2$, i.e., (R, R, D) are shown.

Figure 3 shows the rate-distortion tuples computed for $R_1 = R_2 = R$ and crossover probabilities $\alpha_1 = \alpha_2 = \alpha = \{0.01, 0.1, 0.25\}$. The results coincide with the rate-distortion pairs computed in [4, Fig. 3] for the same setup by exhaustive search over the conditional pmfs **P**.

Figure 4, illustrates the rate-distortion tuples of the regions \mathcal{RD}_{CEO}^1 and \mathcal{RD}_{CEO}^2 for a CEO setup with asymmetric crossover probabilities $\alpha_1 = 0.25$ and $\alpha_2 = 0.1$.

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Fig. 3. The RD region of the CEO setup for symmetric rates $R_1 = R_2 = R$ and crossover probability $\alpha = \{0.01, 0.1, 0.25\}$.



Fig. 4. The regions \mathcal{RD}_{CEO}^1 and \mathcal{RD}_{CEO}^2 of the CEO setup for crossover probabilities $\alpha_1 = 0.25$ and $\alpha_2 = 0.1$.

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