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Abstract—A decoding algorithm for polar codes with binary 16×16 kernels with polarization rate 0.51828 and scaling exponents 3.346 and 3.450 is presented. The proposed approach exploits the relationship of the considered kernels and the Arikan matrix to significantly reduce the decoding complexity without any performance loss. Simulation results show that polar (sub)codes with 16×16 kernels can outperform polar codes with Arikan kernel, while having lower decoding complexity.

I. INTRODUCTION

Polar codes are a novel class of error-correcting codes, which achieve the symmetric capacity of a binary-input discrete memoryless channel W, have low complexity construction, encoding and decoding algorithms [1]. However, the performance of polar codes of practical length is quite poor. The reasons for this are the presence of imperfectly polarized subchannels and the suboptimality of the successive cancellation (SC) decoding algorithm. To improve performance, successive cancellation list decoding (SCL) algorithm [2], as well as various code constructions were proposed [3], [4], [5].

Polarization is a general phenomenon, and is not restricted to the case of Arikan matrix [6]. One can replace it by a larger matrix, called *polarization kernel*, which can provide higher polarization rate. Polar codes with large kernels were shown to provide asymptotically optimal scaling exponent [7]. Many kernels with various properties were proposed [6], [8], [9], [10], but, to the best of our knowledge, no efficient decoding algorithms for kernels with polarization rate greater than 0.5 were presented, except [11], where an approximate algorithm was introduced. Therefore, polar codes with large kernels are believed to be impractical due to very high decoding complexity.

In this paper we present reduced complexity decoding algorithms for 16×16 polarization kernels with polarization rate 0.51828 and scaling exponents 3.346 and 3.45. We show that with these kernels increasing list size in the SCL decoder provides much more significant performance gain compared to the case of Arikan kernel, and ultimately the proposed approach results in lower decoding complexity compared to the case of polar codes with Arikan kernel with the same performance.

The proposed approach exploits the relationship between the considered kernels and the Arikan matrix. Essentially, the log-likelihood ratios (LLRs) for the input symbols of the considered kernels are obtained from the LLRs computed via the Arikan recursive expressions.

II. BACKGROUND

A. Channel polarization

I

Consider a binary input memoryless channel with transition probabilities $W\{y|c\}, c \in \mathbb{F}_2, y \in \mathcal{Y}$, where \mathcal{Y} is output alphabet. For a positive integer n, denote by [n] the set of n integers $\{0, 1, \ldots, n-1\}$. A polarization kernel K is a binary invertible $l \times l$ matrix, which is not upper-triangular under any column permutation. The Arikan kernel is given by $F_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. An $(n = l^m, k)$ polar code is a linear block code generated by k rows of matrix $G_m = M^{(m)}K^{\otimes m}$, where $M^{(m)}$ is a digit-reversal permutation matrix, corresponding to mapping $\sum_{i=0}^{m-1} t_i l^i \to \sum_{i=0}^{m-1} t_{m-1-i} l^i, t_i \in [l]$. The encoding scheme is given by $c_0^{n-1} = u_0^{n-1}G_m$, where $u_i, i \in \mathcal{F}$ are set to some pre-defined values, e.g. zero (frozen symbols), $|\mathcal{F}| = n - k$, and the remaining values u_i are set to the payload data.

It is possible to show that a binary input memoryless channel W together with matrix G_m gives rise to bit subchannels $W_{m,K}^{(i)}(y_0^{n-1}, u_0^{i-1}|u_i)$ with capacities approaching 0 or 1, and fraction of noiseless subchannels approaching I(W) [6]. Selecting \mathcal{F} as the set of indices of low-capacity subchannels enables almost error-free communication. It is convenient to define probabilities

$$V_{m,K}^{(i)}(u_0^i|y_0^{n-1}) = \frac{W_{m,K}^{(i)}(y_0^{n-1}, u_0^{i-1}|u_i)}{2W(y_0^{n-1})}$$
$$= \sum_{u_{i+1}^{n-1}} \prod_{i=0}^{n-1} W((u_0^{n-1}G_m)_i|y_i).$$
(1)

Let us further define $\mathbf{W}_{m}^{(j)}(u_{0}^{j}|y_{0}^{n-1}) = W_{m,K}^{(j)}(u_{0}^{j}|y_{0}^{n-1})$, where kernel K will be clear from the context. We also need probabilities $W_{t}^{(j)}(u_{0}^{j}|y_{0}^{l-1}) = W_{1,F_{2}^{\otimes t}}^{(j)}(u_{0}^{j}|y_{0}^{l-1})$ for Arikan matrix $F_{2}^{\otimes t}$. Due to the recursive structure of G_{n} , one has

$$\mathbf{W}_{m}^{(sl+t)}(u_{0}^{sl+t}|y_{0}^{n-1}) = \sum_{\substack{u_{sl+t+1}^{l(s+1)-1}}} \prod_{j=0}^{l-1} \mathbf{W}_{m-1}^{(s)}(\theta_{K}[u_{0}^{l(s+1)-1},j]|y_{j\frac{n}{l}}^{(j+1)\frac{n}{l}-1})$$
(2)

where $\theta_K[u_0^{(s+1)l-1}, j]_r = (u_{lr}^{l(r+1)-1}G_n)_j, r \in [s+1].$ A trellis-based algorithm for computing these values was presented in [12]. At the receiver side, one can successively estimate

$$\widehat{u}_{i} = \begin{cases} \arg \max_{u_{i} \in \mathbb{F}_{2}} \mathbf{W}_{m}^{(i)}(\widehat{u}_{0}^{i-1}.u_{i}|y_{0}^{n-1}), & i \notin \mathcal{F}, \\ \text{the frozen value of } u_{i} & i \in \mathcal{F}. \end{cases}$$
(3)

This is known as the successive cancellation (SC) decoding algorithm.

III. COMPUTING KERNEL INPUT SYMBOLS LLRS

A. General case

Our goal is to compute efficiently probabilities $\mathbf{W}_m^{(i)}(u_0^i|y_0^{n-1})$ for a given polarization transform $K^{\otimes m}$. Let us assume for the sake of simplicity that m = 1. The corresponding task will be referred to as *kernel processing*.

We propose to introduce approximate probabilities

$$\widetilde{\mathbf{W}}_{1}^{(j)}(u_{0}^{j}|y_{0}^{l-1}) = \max_{\substack{u_{j+1}^{l-1} \\ u_{j+1}^{l-1}}} \mathbf{W}_{1}^{(l-1)}(u_{0}^{l-1}|y_{0}^{l-1})$$
$$= \max_{\substack{u_{j+1}^{l-1} \\ u_{j+1}^{l-1}}} \prod_{i=0}^{l-1} W((u_{0}^{l-1}K)_{i}|y_{i}).$$
(4)

This is the probability of the most likely continuation of path u_0^j in the code tree, without taking into account possible freezing constraints on symbols $u_i, i > j$. Note that the same probabilities were introduced in [11], [13], and shown to provide substantial reduction of the complexity of sequential decoding of polar codes.

Decoding can be implemented using the log-likelihood ratios $\bar{\mathbf{S}}_{m,i} = \bar{\mathbf{S}}_m^{(i)}(u_0^{i-1}|y_0^{n-1}) = \ln \frac{\mathbf{W}_m^{(i)}(u_0^{i-1}.0|y_0^{n-1})}{\mathbf{W}_m^{(i)}(u_0^{i-1}.1|y_0^{n-1})}$. Hence, kernel output LLRs $\bar{\mathbf{S}}_{1,i}, i \in [l]$ can be approximated by

$$\begin{split} \bar{\mathbf{S}}_{1,i} &\approx \mathbf{S}_{1,i} = \ln \frac{\widetilde{\mathbf{W}}_{1}^{(i)}(u_{0}^{i-1}.0|y_{0}^{l-1})}{\widetilde{\mathbf{W}}_{1}^{(i)}(u_{0}^{i-1}.1|y_{0}^{l-1})} \\ &= \max_{u_{i+1}^{l-1}} \ln \mathbf{W}_{1}^{(l-1)}(u(0)^{i}|y_{0}^{l-1}) - \max_{u_{i+1}^{l-1}} \ln \mathbf{W}_{1}^{(l-1)}(u(1)^{i}|y_{0}^{l-1}), \end{split}$$
(5)

where $b(a)^i = (b_0^{i-1}.a.b_{i+1}^{l-1})$. The above expression means that $\mathbf{S}_{1,i}$ can be computed by performing ML decoding of the code, generated by last l-i+1 rows of the kernel K, assuming that all $u_j, i < j < l$, are equiprobable.

B. Binary algorithm

Straightforward evaluation of (5) for arbitrary kernel has complexity $O(2^l l)$. However, we have a simple explicit recursive procedure for computing these values for the case of the Arikan matrix $F_2^{\otimes t}$.

Let $l = 2^t$. Consider encoding scheme $c_0^{l-1} = v_0^{l-1} F_2^{\otimes t}$. Similarly to (4), define approximate probabilities

$$\widetilde{W}_{t}^{(i)}(v_{0}^{i}|y_{0}^{l-1}) = \max_{v_{i+1}^{l-1}} W_{t}^{(l-1)}(v_{0}^{l-1}|y_{0}^{l-1})$$

and modified log-likelihood ratios

$$S_{\lambda}^{(i)}(v_0^{i-1}, y_0^{l-1}) = \log \frac{\widetilde{W}_{\lambda}^{(i)}(v_0^{i-1}.0|y_0^{l-1})}{\widetilde{W}_{\lambda}^{(i)}(v_0^{i-1}.1|y_0^{l-1})}.$$

It can be seen that

$$S_{\lambda}^{(2i)}(v_0^{2i-1}, y_0^{N-1}) = Q(a, b)$$
(6)

$$S_{\lambda}^{(2i+1)}(v_0^{2i}, y_0^{N-1}) = P(a, b, v_{2i}), \tag{7}$$

where $N = 2^{\lambda}$, $a = S_{\lambda-1}^{(i)}(v_{0,e}^{2i-1} \oplus v_{0,o}^{2i-1}, y_{0,e}^{N-1})$, $b = S_{\lambda-1}^{(i)}(v_{0,o}^{2i-1}, y_{0,o}^{N-1})$, $Q(a,b) = \operatorname{sgn}(a)\operatorname{sgn}(b)\min(|a|, |b|)$, $P(a,b,c) = (-1)^{c}a + b$. Then the log-likelihood of a path (path score) v_{0}^{i} can be obtained as [14]

$$R(v_0^i|y_0^{l-1}) = \log \widetilde{W}_t^{(i)}(v_0^i|y_0^{l-1})$$

= $R(v_0^{i-1}|y_0^{l-1}) + \tau \left(S_t^{(i)}(v_0^{i-1}, y_0^{l-1}), v_i\right),$ (8)

where $R(\epsilon | y_0^{l-1})$ can be set to 0, ϵ is an empty sequence, and

$$\tau(S, v) = \begin{cases} 0, & \operatorname{sgn}(S) = (-1) \\ -|S|, & \text{otherwise.} \end{cases}$$

It can be verified that

$$\sum_{\beta=0}^{2^{j}-1} \tau(S_{0}^{(0)}(y_{\beta}), c_{\beta}) = \sum_{\beta=0}^{2^{j}-1} \tau(S_{j}^{(\beta)}(v_{0}^{\beta-1}, y_{0}^{2^{j}-1}), v_{\beta}), \quad (9)$$

where $c = v_0^{2^j - 1} F_2^{\otimes j}$.

It was suggested in [15] to express values $\mathbf{W}_1^{(i)}(u_0^i|y_0^{l-1})$ via $W_t^{(j)}(v_0^j|y_0^{l-1})$ for some j. One can represent the kernel Kas $K = TF_2^{\otimes t}$, where T is an $l \times l$ matrix. Let $v_0^{l-1} = u_0^{l-1}T$. Then, $c_0^{l-1} = v_0^{l-1}F_2^{\otimes t} = u_0^{l-1}K$, so that $u_0^{l-1} = v_0^{l-1}T^{-1}$.

Observe, that it is possible to reconstruct u_0^i from $v_0^{\tau_i}$, where τ_i is the position of the last non-zero symbol in the *i*-th row of T^{-1} . Recall that successive cancellation decoding of polar codes with arbitrary kernel requires one to compute values $\mathbf{W}_1^{(i)}(u_0^i|y_0^{l-1}), u_i \in \mathbb{F}_2$. However, fixing the values u_0^{i-1} may impose constraints on $v_j, j > \tau_i$, which must be taken into account while computing these probabilities.

Indeed, vectors u_0^{l-1} and v_0^{l-1} satisfy the equation

$$\Theta'(u_{l-1} \ldots u_1 \ u_0 \ v_0 \ v_1 \ \ldots \ v_{l-1})^T = 0,$$

where $\Theta' = (S \ I)$, and $l \times l$ matrix S is obtained by transposing T and reversing the order of columns in the obtained matrix. By applying elementary row operations, matrix Θ' can be transformed into a minimum-span form Θ , such that the first and last non-zero elements of the *i*-th row are located in columns *i* and z_i , respectively, where all z_i are distinct. This enables one to obtain symbols of vector *u* as

$$u_{i} = \sum_{s=0}^{i-1} u_{s} \Theta_{l-1-i,l-1-s} + \sum_{t=0}^{j_{i}} v_{t} \Theta_{l-1-i,l+t}, \quad (10)$$

where $j_i = z_{l-1-i} - l$. Let $h_i = \max_{0 \le i' \le i} j_{i'}$. It can be seen that¹

$$\mathbf{W}_{1}^{(j)}(u_{0}^{j}|y_{0}^{l-1}) = \sum_{\substack{v_{0}^{h_{j}} \in \mathcal{Z}_{j} \\ v_{0}^{h_{j}} \in \mathcal{Z}_{j}}} W_{t}^{(h_{j})}(v_{0}^{h_{j}}|y_{0}^{l-1})$$
$$= \sum_{\substack{v_{0}^{h_{j}} \in \mathcal{Z}_{j}}} \sum_{\substack{v_{h_{j}+1}^{l-1} \\ h_{j}+1}} W_{t}^{(l-1)}(v_{0}^{l-1}|y_{0}^{l-1}), \quad (11)$$

¹The method given in [10] is a special case of this approach.

where Z_j is the set of vectors $v_0^{h_j}$, such that (10) holds for $i \in [j]$. Similarly we can rewrite the above expression for the case of the approximate probabilities

$$\widetilde{\mathbf{W}}_{1}^{(j)}(u_{0}^{j}|y_{0}^{l-1}) = \max_{v_{0}^{h_{j}} \in \mathcal{Z}_{j}} \widetilde{W}_{t}^{(h_{j})}(v_{0}^{h_{j}}|y_{0}^{l-1})$$
$$= \max_{v_{0}^{h_{j}} \in \mathcal{Z}_{j}} \max_{v_{h_{j}+1}^{l-1}} W_{t}^{(l-1)}(v_{0}^{l-1}|y_{0}^{l-1}).$$
(12)

Let
$$\mathcal{Z}_{i,b} = \left\{ v_0^{h_i} | v_0^{h_i} \in \mathcal{Z}_i, \text{ where } u_i = b \right\}$$
. Hence, one obtains

$$\mathbf{S}_{1,i} = \max_{v_0^{h_i} \in \mathcal{Z}_{i,0}} R(v_0^{h_i} | y_0^{l-1}) - \max_{v_0^{h_i} \in \mathcal{Z}_{i,1}} R(v_0^{h_i} | y_0^{l-1}).$$
(13)

Observe that computing these values requires considering multiple vectors $v_0^{\hat{h}_i}$ of input symbols of the Arikan transform $F_2^{\otimes t}$. Let $\mathcal{D}_i = \{0, \ldots, h_i\} \setminus \{j_0, \ldots, j_i\}$ be a *decoding* window, i.e. the set of indices of Arikan input symbols $v_0^{h_i}$, which are not determined by symbols u_0^{i-1} . The number of such vectors, which determines the decoding complexity, is $2^{|\mathcal{D}_i|}$. In general, one has $|\mathcal{D}_i| = O(l)$ for an arbitrary kernel.

IV. Efficient processing of 16×16 kernels

To minimize complexity of proposed approach (13) one needs to find kernels with small decoding windows while preserving required polarization rate (> 0.5 in our case) and scaling exponent. By computer search, based on heuristic algorithm presented in [8], we found a 16×16 kernel

with BEC scaling exponent $\mu(K_1)$ 3.346 [8]. = Furthermore, to minimize the size of decoding windows, we derived another kernel $K_2 = P_{\sigma}K_1$, were P_{σ} is a permutation matrix corresponding to permutation [0, 1, 2, 7, 3, 4, 5, 6, 9, 10, 11, 12, 8, 13, 14, 15], with =scaling exponent $\mu(K_2)$ = 3.45. Both kernels have polarization rate 0.51828.

Table I presents the right hand side of expression (10) for each $i \in [16]$, as well as the corresponding decoding windows \mathcal{D}_i , for both kernels. It can be seen that the maximal decoding windows size for K_1 and K_2 is 4 and 3, respectively. Note that by applying the row permutation to K_1 , we have reduced decoding windows, but increased scaling exponent. Below we present efficient methods for computing some input symbol LLRs for these kernels.

TABLE I: Input symbols u_{ϕ} for kernels K_1, K_2 as functions of input symbols v for $F_2^{\otimes 4}$

ϕ	K_1			K_2		
	u_{ϕ}	\mathcal{D}_{ϕ}		u_{ϕ}	\mathcal{D}_{ϕ}	Cost
0	v_0	{}	15	v_0	{}	15
1	v_1	{}	1	v_1	{}	1
2	v_2	{}	3	v_2	{}	3
3	v_4	$\{3\}$	21	v_3	{}	1
4	v_8	$\{3, 5, 6, 7\}$	127	v_4	{}	7
5	$v_6\oplus v_9$	$\{3, 5, 6, 7\}$	48	v_8	$\{5, 6, 7\}$	67
6	$v_5 \oplus v_6 \oplus v_{10}$	$\{3, 5, 6, 7\}$	95	$v_6\oplus v_9$	$\{5, 6, 7\}$	24
7	v_3	$\{5, 6, 7\}$	1	$v_5 \oplus v_6 \oplus v_{10}$	$\{5, 6, 7\}$	47
8	v_{12}	$\{5, 6, 7, 11\}$	127	v_6	$\{5,7\}$	1
9	v_6	$\{5, 7, 11\}$	1	v_{10}	$\{7\}$	1
10	v_{10}	$\{7, 11\}$	1	v_7	{}	1
11	v_7	$\{11\}$	1	v_{11}	{}	1
12	v_{11}	{}	1	v_{12}	{}	7
13	v_{13}	{}	1	v_{13}	{}	1
14	v_{14}	{}	3	v_{14}	{}	3
15	v_{15}	{}	1	v_{15}	{}	1

A. Processing of kernel K_1 with $\mu = 3.346$

It can be seen that for $\phi \in \{0, 1, 2, 13, 14, 15\}$ one has $\mathbf{S}_{1,\phi} = S_4^{(\phi)}(v_0^{\phi}, y_0^{15})$, i.e. LLR for $F_2^{\otimes 4}$.

1) phase 3: In case of $\phi = 3$ expressions (13) and (10) imply that the decoding window $\mathcal{D}_3 = \{3\}$ and LLR for u_3 is given by

$$\mathbf{S}_{1,3} = \max_{v_3} R(\hat{v}_0 \hat{v}_1 \hat{v}_2 v_3 0 | y_0^{15}) - \max_{v_3} R(\hat{v}_0 \hat{v}_1 \hat{v}_2 v_3 1 | y_0^{15}),$$

where $\hat{v}_i = \hat{u}_i, i \in [3]$, are already estimated symbols.

- To obtain LLR $S_{1,3}$ one should compute:
- $S_4^{(3)}(v_0^2, y_0^{15})$ with 1 operation,
- $R(\hat{v}_0\hat{v}_1\hat{v}_2v_3|y_0^{15})$ for $v_3 \in [2]$. Observe that this can be done with 1 summation, since $R(\hat{v}_0\hat{v}_1\hat{v}_2v_3|y_0^{15}) =$ $R(\hat{v}_{0}\hat{v}_{1}\hat{v}_{2}|y_{0}^{15}) + \tau(S_{4}^{(3)}(v_{0}^{2},y_{0}^{15}),v_{3}) \text{ and there is } v_{3} \text{ such }$ as $\tau(S_{4}^{(3)}(v_{0}^{2},y_{0}^{15}),v_{3}) = 0,$
- $S_4^{(4)}(v_0^3, y_0^{15})$ for $v_3 \in [2]$ with 7 * 2 operations,
- $R(\hat{v}_0\hat{v}_1\hat{v}_2v_3v_4|y_0^{15})$ for $v_3, v_4 \in [2]$ with 2 operations,
- $\max_{v_3} R(\hat{v}_0 \hat{v}_1 \hat{v}_2 v_3 v_4 | y_0^{15})$ for $v_4 \in [2]$ with 2 operations, $\max_{v_3} R(\hat{v}_0 \hat{v}_1 \hat{v}_2 v_3 0 | y_0^{15}) \max_{v_3} R(\hat{v}_0 \hat{v}_1 \hat{v}_2 v_3 1 | y_0^{15})$ with 1 operation.

Total number of operations is given by 21.

2) phase 4: The decoding window is given by $\mathcal{D}_4 =$ $\{3, 5, 6, 7\}$ and

$$\mathbf{S}_{1,4} = \max_{v_0^8 \in \mathcal{Z}_{4,0}} R(v_0^8 | y_0^{15}) - \max_{v_0^8 \in \mathcal{Z}_{4,1}} R(v_0^8 | y_0^{l-1}),$$

where $\mathcal{Z}_{4,b}$ is given by the set of vectors $[\hat{v}_0 \hat{v}_1 \hat{v}_2 v_3 \hat{v}_4 v_5 v_6 v_7 b]$, $v_3, v_5, v_6, v_7 \in [2].$

Instead of exhaustive enumeration of vectors $v_0^{h_j}$ in (13), we propose to exploit the structure of K_1 to identify some common subexpressions (CSE) in formulas for $R(v_0^{h_i}|y_0^{l-1})$ and $S_{\lambda}^{(i)} = S_{\lambda}^{(i)}(v_0^{i-1}, y_0^{l-1})$, which can be computed once and used multiple times. In some cases computing these subexpressions reduces to decoding of well-known codes, which can be implemented with appropriate fast algorithms. Furthermore, we observe that the set of possible values of these subexpressions is less than the number of different $v_0^{h_j}$ to be considered. This results in further complexity reduction. More accurate and detailed description of CSE can be found

in [16]. To demonstrate this approach, we consider computing the LLR for u_4 of K_1 .

This requires considering 16 vectors v_0^7 satisfying (10). According to (8), one obtains $R(v_0^8|y_0^{15}) = R(v_0^7|y_0^{15}) +$ $au\left(S_4^{(8)}(v_0^7, y_0^{15}), v_8\right)$. Observe that $\left\{v_0^7 F_2^{\otimes 3} | v_0^7 \in \bar{Z}_4\right\}$ is a coset of Reed-Muller code RM(1,3), where \overline{Z}_4 is the set of vectors v_0^7 , so that (10) holds for $i \in [4]$. Furthermore,

$$R(v_0^7|y_0^{15}) = \frac{1}{2} \left(\sum_{i=0}^7 (-1)^{c_i} s_i - \sum_{i=0}^7 |s_i| \right)$$

where $s_i = S_1^{(0)}(\epsilon, (y_i, y_{i+8})), i \in [8], c_0^7 = v_0^7 F_2^{\otimes 3}$. Assume for the sake of simplicity that $v_i = 0, j \in \{0, 1, 2, 4\}$. Then the first term in this expression can be obtained for each $v_0^7 \in \bar{\mathcal{Z}}_4$ via the fast Hadamard transform (FHT) [17] of vector s, and the second one does not need to be computed, since it cancels in (13).

It remains to compute $S_4^{(8)}(v_0^7,y_0^{15}), v_0^7 \in \bar{\mathcal{Z}}_4$ and $|\bar{\mathcal{Z}}_4| =$ 16. In a straightforward implementation, one would recursively apply formulas (6) and (7) to compute $S_4^{(8)}$ for 16 vectors v_0^7 . It appears that there are some CSE arising in this computation.

At first, one needs to compute $\widetilde{S}_1^{(1)}(c_i, y_i, y_{i+8}), i \in [8], c_0^7 = v_0^7 F_2^{\otimes 3}$. Since $c_i \in \{0, 1\}$, the values $S_1^{(1)}(j, y_i, y_{i+8}), j \in \{0, 1\}, i \in [8]$ constitute the first set of CSE. We store them in the array L

$$L[4i+j] = S_1^{(1)}(j \mod 2, y_{i+\bar{j}}, y_{i+8+\bar{j}}),$$

where $i, j \in [4], \bar{j} = 4|j/2|$. Computing these values requires 16 summations only, instead of $16 \cdot 8 = 128$ summations in a straightforward implementation.

The next step is to compute the values $S_2^{(2)}((c_i, c_{i+4}), (y_i, y_{i+4}, y_{i+8}, y_{i+12})), i \in [4]$ which are equal to $Q(S_1^{(1)}(c_i, y_i, y_{i+8}), S_1^{(1)}(c_{i+4}, y_{i+4}, y_{i+12}))$. Since $(c_i, c_{i+4}) \in \mathbb{F}_2^2$, $S_2^{(2)}$ gives us the second set of CSE. One can use values stored in L to compute $S_2^{(2)}$ as

$$X[i][j] = Q(L[4i+j/2], L[4i+(j \mod 2)+2])], i, j \in [4].$$

Observe that for any $c_0^7 \in RM(1,3)$ one has $c_i^{i+4} \in$ $RM(1,2), i \in \{0,4\}$. That is, one needs to consider only vectors c_i^{i+4} of even weight while computing $S_3^{(4)}$. These values can be calculated as

$$Y[i][j+4k] = Q(X[i][j\oplus 3k], X[i+2][j]), i, k \in [2], j \in [4].$$

Finally, the values $S_4^{(8)}(v_0^7, y_0^{15})$ can be obtained as

$$Z[i+8j] = Q(Y[0][i\oplus 3j], Y[1][i]), i \in [8], j \in [2].$$

Each element of Z corresponds to some $c_0^7 \in RM(1,3)$. Finally, these values are used in (13) together with $R(v_0^7|y_0^{15})$ to calculate $S_{1,4}$.

Let us compute the number of operations required to process phase 4. One need to compute

- $R(v_0^7|y_0^{15})$ for $v_0^7 \in \bar{Z}_4$ via FHT with 24 operations, all different $S_1^{(1)}$ arising in CSE (array L) with 16 operations,

- all S₂⁽²⁾ in CSE (array X) with 16 operations,
 all S₃⁽⁴⁾ in CSE (array Y) with 16 operations,
 all S₄⁽⁸⁾ in CSE (array Z) with 16 operations,
- $R(v_0^8|y_0^{15})$ for $v_0^8 \in \mathbb{Z}_4$ with 16 operations,
- $\max_{v_0^8 \in \mathcal{Z}_{4,b}} R(v_0^8 | y_0^{15}), b \in [2]$, with 15 * 2 operations, $\mathbf{S}_{1,4}$ with 1 operation.

The overall complexity is given by 135 operation.

We also employ one observation to reduce the complexity of computing $\max_{v_0^8 \in \mathbb{Z}_{4,b}} R(v_0^8 | y_0^{15})$. Let s be a FHT of the vector s, where $s_i = S_1^{(0)}(\epsilon, (y_i, y_{i+8})), i \in [8]$. Observe that we can compute $\arg \max_{i \in [8]} |s_i|$ with 7 operations and obtain v_3, v_5, v_6, v_7 which gives us max $R(v_0^7|y_0^{15})$. Recall that $\tau(S,c)$ function is zero for one of $b \in [2]$, therefore, there is a value of $\hat{b}, b \in [2]$ such as $\max_{v_0^8 \in \mathbb{Z}_{4,b}} R(v_0^8 | y_0^{15}) =$ $\max R(v_0^7|y_0^{15})$. It implies that we remain need to compute $\max_{v_0^8 \in \mathcal{Z}_{4,1 \oplus \hat{b}}} R(v_0^8 | y_0^{15}).$ With this modification we have the complexity given by 127 operations.

3) phase 5: The decoding window D_5 remains the same as in the previous phase. According to expressions (7) and (8) to obtain $S_{1,5}$ one should compute:

- all $S_4^{(9)}$ with 16 operations, $R(v_0^9|y_0^{15})$ for $v_0^9 \in \mathbb{Z}_5$ with 16 operations,
- $\max_{v_0^9 \in \mathcal{Z}_{5,b}} R(v_0^9 | y_0^{15}), b \in [2]$, with 15 operations. Similarly to phase 4, there is a value of $\hat{b} \in [2]$ such as $\max_{v_0^9 \in \mathcal{Z}_{5,\hat{b}}} R(v_0^9 | y_0^{15}) = \max_{v_0^8 \in \mathcal{Z}_{4,v_4}} R(v_0^8 | y_0^{15}),$ • $\mathbf{S}_{1,5}$ with 1 operation.

Total complexity is given by 48 operations.

4) phase 6: The decoding window $\mathcal{D}_6 = \{3, 5, 6, 7\}$.

At this phase according to expressions (6)-(7) one should compute:

- 32 values of S₁⁽⁵⁾ in CSE with 32 operations,
 16 LLRs S₄⁽¹⁰⁾ with 16 operations,

- $R(v_0^{10}|y_0^{15})$ for $v_0^{10} \in \mathbb{Z}_6$ with 16 operations, $\max_{v_0^{10} \in \mathbb{Z}_{6,b}} R(v_0^{10}|y_0^{15}), b \in [2]$, with 30 operations,
- $\mathbf{S}_{1,5}$ with 1 operation.

Total complexity is given by 95 operations.

5) phase 7: At this phase the decoding window is reduced and given by $\mathcal{D}_7 = \{5, 6, 7\}$. Moreover, the value $h_7 = h_6 =$ v_{10} , which means that the values $R(v_0^{10}|y_0^{15})$ remains the same. We propose to use the following method: at phase 6 one should compute $\max_{\{v_0^{10}|v_0^{10}\in \mathcal{Z}_{7,b}, u_6=\bar{b}\}} R(v_0^{10}|y_0^{15})$ for $b \in [2], \bar{b} \in [2]$ [2] and obtain

$$\max_{v_0^{10}\in \mathcal{Z}_{6,b}} R(v_0^{10}|y_0^{15}) = \max_{\bar{b}\in [2]} \max_{\left\{v_0^{10}|v_0^{10}\in \mathcal{Z}_{7,b}, u_6=\bar{b}\right\}} R(v_0^{10}|y_0^{15}).$$

Once the value u_6 is determined, one can obtain $S_{1,7}$ with one operation directly from already computed values $\max_{v_0^{10} \in \mathcal{Z}_{7,b}} R(v_0^{10} | y_0^{15}).$

6) phase 8: At this phase the decoding window is increased and given by $\mathcal{D}_8 = \{5, 6, 7, 11\}$ and $h_8 = 12$. To obtain kernel input symbol LLR one should compute:

- + 8 LLRs $S_4^{(11)}$ with 8 operations,
- 16 path scores R(v₀¹¹|y₀¹⁵) with 8 operations,
 16 values S₂⁽³⁾ in CSE with 16 operations,

- 32 values S₃⁽⁶⁾ in CSE with 32 operations,
 16 values S₄⁽¹²⁾ with 16 operations,
- 32 path scores $R(v_0^{12}|y_0^{15})$ with 16 operations,
- $\max_{v_0^{12} \in \mathcal{Z}_{12,b}} R(v_0^{12}|y_0^{15}), b \ in[2]$ with 15 * 2 operations, • $\mathbf{S}_{1,12}$ with one operation.

Total complexity is given by 127 operations.

One can recursively apply approach described for phase 7, namely, construct tree of maximums of $R(v_0^{12}|y_0^{(15)})$, and obtain $\mathbf{S}_{1,\phi}$, $8 < \phi < 13$ with one operation.

B. Processing of kernel K_2 with $\mu = 3.45$

Below we briefly present a complete processing algorithm for kernel K_2 . It provides much better performancecomplexity tradeoff compared to K_1 . The algorithm uses the same CSE elimination techniques as described in section IV-A. After the pre-computation steps for $\phi \in \{5, 6, 7\}$, the LLR is obtained via (13).

- For $\phi \in \{0, 1, 2, 3, 4, 11, 12, 13, 14, 15\}$, compute $\mathbf{S}_{1,\phi}$ as LLRs $S_4^{(\phi)}(v_0^{\phi-1}, y_0^{15})$ for the Arikan transform $F_2^{\otimes 4}$. For $\phi = 5$:
- - 1) Since the set of vectors $c_0^3 = [\hat{v}_4 v_5 v_6 v_7] F_2^{\otimes 2}$ fixed $v_i, i \in \{0, 1, 2, 3, 4\}$, a coset of RM(1, 2), one can obtain 8 values of $R(v_4^7|y_0^{15})$ (recall that
 $$\begin{split} R(v_0^7|y_0^{15}) &= R(\hat{v}_0^3|y_0^{15}) + R(v_4^7|y_0^{15})) \text{ from the FHT} \\ \text{of the vector } s_i &= S_2^{(1)}(\bar{c}_i,(y_i,y_4,y_{i+8},y_{i+12})), i \in \mathcal{S}_2^{(1)}(\bar{c}_i,(y_i,y_4,y_{i+8},y_{i+12})), i \in \mathcal{S}_2^{(1)}(\bar{c}_i,(y_i,y_4,y_{i+8},y_{i+12}))$$
 $[4], \ \bar{c} = \hat{v}_0^3 F_2^{\otimes 2}.$
 - 2) Compute

 $L[i+8j] = S_1^{(1)}(j \mod 2, y_i, y_{i+8}), i \in [8], j \in [2].$

- 3) Since $\{(c_i, c_{i+4})\} = \{(0, 0), (1, 1)\}$, compute all possible $S_2^{(2)}$ values as $X[i][j] = Q(L[i+8j], L[i+4+8j]), i \in [4], j \in [2].$
- 4) Since $\{(c_i, c_{i+4}, c_{i+2}, c_{i+6})\}$ is a code generated by $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, compute all possible $S_3^{(4)}$ values as $Y[i][j] = Q(X[i][j/2], X[i+2][j \mod 2]), i \in$ $[2], j \in [4].$
- 5) For every $c_0^7 = v_0^7 F_2$ compute $S_4^{(8)}(v_0^7, y_0^{15})$ as $Z[i+4j] = Q(Y[0][i\oplus 3j], Y[1][i]), i \in [4], j \in [2].$
- For $\phi = 6$, compute $S_4^{(9)}(v_0^8, y_0^{15})$ as $Z[i+4j] = P(Y[0][i\oplus 3j], Y[1][i], v_8), i \in [4], j \in [2].$ • For $\phi = 7$:
 - 1) Compute $S_3^{(5)}$ as $\bar{Y}[0][j+4k] = P(X[0][j/2], X[2][j \text{ mod } 2], \bar{c}_{0,k}),$ $\bar{Y}[1][j+4k] = P(X[3][\bar{j}_k/2], X[1][\bar{j}_k \mod 2], \bar{c}_{1,k})$ and $\bar{c}_{0,k} = v_8 \oplus u_6 \oplus k$, $\bar{c}_{1,k} = u_6 \oplus k$, $\bar{j}_k = j \oplus 3$, $\begin{array}{l} i\in [2], j\in [4], k\in [2].\\ \text{2) Obtain } S_4^{(10)} \text{ as } Z[i]=Q(\bar{Y}[0][i],\bar{Y}[1][i]), i\in [8]. \end{array}$
- For $\phi \in \{8,9,10\}$: Use 16 already computed values of $R(v_0^{10}|y_0^{15})$ to obtain $\mathbf{S}_{1,\phi}$.

Remark 1. Let us comment the case of $\phi = 7$. In conventional Arikan SC for $F_2^{\otimes 4}$, after symbol v_9 is estimated, the LLRs $S_3^{(5)}$ for v_{10} are obtained by applying P function to $S_2^{(2)}$ and values $(v_8 \oplus v_9, v_8)$. In the case of K_2 , we do not have a fixed value for v_9 . Instead, we have a constraint $u_6 = v_6 + v_9$. Therefore,

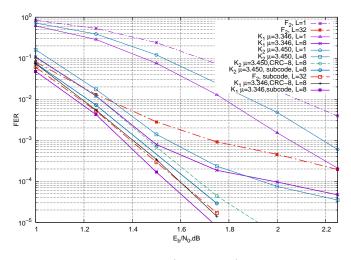


Fig. 1: Performance of (4096, 2048) polar codes

for each vector $v_0^9 \in \overline{Z}_7$ the value of v_9 is changed according to v_6 . This property is taken into account in the expressions for computing of $\overline{Y}[i][j]$.

For processing of K_2 we also used trick with simplified computation of path score maximums similarly to phase 5 of K_1 .

The cost, in terms of the total number of summations and comparisons, of computing $S_{1,\phi}$ using the proposed algorithm is shown in Table I. The overall processing complexity is 447 and 181 operations for kernels K_1 and K_2 respectively, while the trellis-based algorithm [12] requires 7557 and 9693 operations, respectively.

The above described techniques can be also used to implement an SCL decoder for polar codes with the considered kernels, using a straightforward generalization of the algorithm and data structures presented in [2].

V. NUMERIC RESULTS

We constructed (4096, 2048) polar codes with the considered kernels, and investigated their performance for the case of AWGN channel with BPSK modulation. The sets of frozen symbols were obtained by Monte-Karlo simulations.

Figure 1 illustrates the performance of plain polar codes, polar codes with CRC^2 and polar subcodes [4]. It can be seen that the codes based on kernels K_1 and K_2 with improved polarization rate $E(K_1) = E(K_2) = 0.51828$ provide significant performance gain compared to polar codes with Arikan kernel. Observe also that randomized polar subcodes provide better performance compared to polar codes with CRC. Moreover, polar subcodes with kernels K_1, K_2 under SCL with L = 8have almost the same performance as polar subcodes with Arikan kernel under SCL with L = 32. Observe also that the codes based on kernels with lower scaling exponent exhibit better performance.

Figure 2a presents simulation results for (4096, 2048) polar subcodes with different kernels under SCL with different L

²CRC length was selected to minimize FER with L = 8.

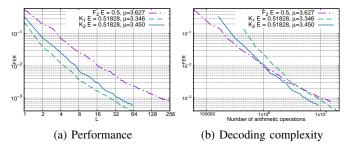


Fig. 2: SCL decoding of polar subcodes with different kernels

at $E_b/N_0 = 1.25$ dB. It can be seen that the kernels with polarization rate 0.51828 require significantly lower list size L to achieve the same performance as the code with the Arikan kernel. Moreover, this gap grows with L. This is due to improved rate of polarization, which results in smaller number of unfrozen imperfectly polarized bit subchannels. The size of the list needed to correct possible errors in these subchannels grows exponentially with their number (at least for the genieaided decoder considered in [18]). On the other hand, lower scaling exponent gives better performance with the same list L, but the slope of the curve remains the same for both kernels K_1, K_2 .

Figure 2b presents the same results in terms of the actual decoding complexity. Recall that proposed kernel processing algorithm uses only summations and comparisons. The SCL algorithm was implemented using the randomized order statistic algorithm for selection of the paths to be killed at each phase, which has complexity O(L). Observe that the polar subcode based on kernel K_2 can provide better performance with the same decoding complexity for FER $\leq 8 \cdot 10^{-3}$. This is due to higher slope of the corresponding curve in Figure 2a, which eventually enables one to compensate relatively high complexity of the LLR computation algorithm presented in Section IV.

Unfortunately, K_1 kernel, which provides lower scaling exponent, has greater processing complexity than K_2 , so that its curve intersects the one for the Arikan kernel only at FER= $2 \cdot 10^{-3}$.

VI. CONCLUSIONS

In this paper efficient decoding algorithms for some 16×16 polarization kernels with polarization rate 0.51828 were proposed. The algorithms compute kernel input symbols LLRs via the ones for the Arikan kernel, and exploit the structure of the codes induced by the kernel to identify and re-use the values of some common subexpressions. It was shown that in the case of SCL decoding with sufficiently large list size, the proposed approach results in lower decoding complexity compared to the case of polar (sub)codes with Arikan kernel with the same performance.

Extension of the proposed approach to the case of other kernels remains an open problem.

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