# Privacy Under Hard Distortion Constraints

Jiachun Liao, Oliver Kosut, Lalitha Sankar School of Electrical, Computer and Energy Engineering, Arizona State University

Email: {jiachun.liao,lalithasankar,okosut}@asu.edu

Flavio P. Calmon
School of Engineering and Applied Sciences
Harvard University
Email: fcalmon@g.harvard.edu

Abstract—We study the problem of data disclosure with privacy guarantees, wherein the utility of the disclosed data is ensured via a hard distortion constraint. Unlike average distortion, hard distortion provides a deterministic guarantee of fidelity. For the privacy measure, we use a tunable information leakage measure, namely maximal  $\alpha$ -leakage ( $\alpha \in [1,\infty]$ ), and formulate the privacy-utility tradeoff problem. The resulting solution highlights that under a hard distortion constraint, the nature of the solution remains unchanged for both local and non-local privacy requirements. More precisely, we show that both the optimal mechanism and the optimal tradeoff are invariant for any  $\alpha > 1$ ; i.e., the tunable leakage measure only behaves as either of the two extrema, i.e., mutual information for  $\alpha = 1$  and maximal leakage for  $\alpha = \infty$ .

Index Terms—Privacy-utility tradeoff, maximal  $\alpha$ -leakage, hard distortion, f-divergence.

#### I. INTRODUCTION

From social networks to medical databases, useful cloudbased services require some form of user data disclosure to a third party. Data disclosure, however, often incurs a privacy risk. In most non-trivial settings, there is a fundamental tradeoff between privacy and utility: on the one hand, disclosing data "as is" can lead to unwanted inferences of private information. On the other hand, perturbing or limiting the disclosed data can result in a reduced quality of service.

The exact nature of the privacy-utility tradeoff (PUT) will depend to varying degrees on the distribution of the underlying data, as well as the chosen metrics (e.g., differential privacy [1], mutual information (MI) [2], [3], f-divergence-based leakage measures [4], maximal leakage (MaxL) [5]). Furthermore, most information-theoretic PUTs capture utility as a statistical average of desired measures of fidelity [6]–[9]. This, in turn, simplifies the PUT to a single-letter optimization for independent and identically distributed (i.i.d.) datasets [10].

We measure utility in terms of a new *hard distortion* metric, which constrains the privacy mechanism so that the distortion function between original and released datasets is bounded with probability 1. This distortion metric is quite stringent, particularly when compared to average-case distortion constraints [10], but it has the advantage that it allows the data curator to make specific, deterministic guarantees on the fidelity of the disclosed dataset to the original one. This differs significantly from a probabilistic constraint, which does not allow the data

This material is based upon work supported by the National Science Foundation under Grant No. CCF-1350914 and CIF-1422358.

curator to make *any* guarantee that the realization of the disclosed dataset has any relationship to the original one.

We adopt maximal  $\alpha$ -leakage, which we introduced in [11], as an information leakage measure. Maximal  $\alpha$ -leakage is a tunable privacy metric defined via an  $\alpha$ -loss function with parameter  $\alpha \in [1,\infty]$ . For  $\alpha=1$ , this metric captures the inference gain by a (soft decision) belief-refining adversary after observing the disclosed data. As  $\alpha \to \infty$ , this metric captures the reduction in 0-1 loss or, equivalently, the gain of a (hard decision) adversary's guessing ability after data disclosure. These extreme points correspond to MI and MaxL, respectively. The tunable parameter  $\alpha$  allows continuous interpolation between the two extremal adversarial actions by determining how much weight an adversary gives to its posterior belief.

Using the aforementioned utility and privacy measures, we precisely quantify the PUT and show that: (i) the same privacy mechanism achieves the same optimal PUT for all  $\alpha>1$ , and both the optimal mechanism and the optimal PUT are independent of the distribution of original data; (ii) For  $\alpha=1$ , the optimal privacy mechanism depends on the distribution of original data. More generally, for the sake of completeness, we also consider a larger class of f-divergence-based information leakages and derive the optimal PUTs for this class.

The paper is organized as follows: in Sec. II, we review maximal  $\alpha$ -leakage. In Sec. III, we formulate and solve the PUT problems with maximal  $\alpha$ -leakage as well as its f-divergence-based variants as privacy measures, and using hard distortion as the utility measure. In Sec. IV, we illustrate our results via an example with binary data wherein the distortion function is the distance between types (empirical distributions) of the original and disclosed datasets.

## II. MAXIMAL $\alpha$ -LEAKAGE AND RELATED LEAKAGE MEASURES

Let X and Y represent the original and disclosed data, respectively, and let U represent an arbitrary (potentially random) function of X that the adversary (a curious or malicious observer of the disclosed data Y) is interested in learning. Maximal  $\alpha$ -leakage, introduced in [11], measures various aspects of leakage (ranging from the probability of correctly guessing to the posteriori distribution) about data U from the disclosed Y. We review the formal definition next.

**Definition 1** ([11, Def. 5]). Given a joint distribution  $P_{XY}$  on finite alphabets  $\mathcal{X} \times \mathcal{Y}$ , the maximal  $\alpha$ -leakage from X to Y is defined as

$$\mathcal{L}_{\alpha}^{max}(X \to Y)$$

$$\triangleq \sup_{U-X-Y} \lim_{\alpha' \to \alpha} \frac{\alpha'}{\alpha'-1} \log \frac{\max_{P_{\hat{U}|Y}} \mathbb{E}\left(\mathbb{P}(U=\hat{U}|U,Y)\frac{\alpha'-1}{\alpha'}\right)}{\max_{P_{\hat{U}}} \mathbb{E}\left(\mathbb{P}(U=\hat{U}|U)\frac{\alpha'-1}{\alpha'}\right)}, \quad (1)$$

where  $\alpha \in [1, \infty]$ , U represents any function of X and takes values from an arbitrary finite alphabet.

Given an  $\alpha$ -loss function,  $\alpha \in (1, \infty)$ ,

$$\ell_{\alpha}(u, y, P_{\hat{U}|Y}) \triangleq \frac{\alpha}{\alpha - 1} \left( 1 - P_{\hat{U}|Y}(u|y)^{1 - \frac{1}{\alpha}} \right), \tag{2}$$

in the limits of  $\alpha=1$  and  $\alpha=\infty$  we obtain the log-loss and the 0-1 loss functions, respectively. A related  $\alpha$ -gain function of (2) is  $1-\frac{\alpha-1}{\alpha}\ell_{\alpha}$ . Therefore, maximal  $\alpha$ -leakage in (1) measures the maximal multiplicative increase in the expected  $\alpha$ -gain for correctly inferring any function U of X when an adversary has access to Y [11]. The expression in (1) can be further simplified to obtain the following theorem.

**Theorem 1** ([11, Thm. 2]). For  $\alpha \in [1, \infty]$ , the maximal  $\alpha$ -leakage defined in (1) simplifies to

where in the supremum  $P_{\tilde{X}}$  is constrained to have the same support as X, and  $D_{\alpha}(\cdot)$  is the Rényi divergence [12] of order  $\alpha$  given by

$$D_{\alpha}(P_{\tilde{X}}P_{Y|X}||P_{\tilde{X}} \times Q_{Y}) = \frac{1}{\alpha - 1} \log \left( \sum_{xy} \frac{P_{\tilde{X}}(x)P_{Y|X}(y|x)^{\alpha}}{Q_{Y}(y)^{\alpha - 1}} \right)$$

and it is defined by its continuous extension for  $\alpha = 1$  or  $\infty$ .

The infimum over  $Q_Y$  in (3a) is exactly Sibson MI of order  $\alpha$  [13, Def. 4]. Note that for  $\alpha = 1$  and  $\alpha = \infty$ , the maximal  $\alpha$ -leakage simplifies to MI and MaxL, respectively. In [11], we show that maximal  $\alpha$ -leakage ( $\alpha \in [1, \infty]$ ) satisfies data processing inequalities and a composition theorem.

While we are mainly interested in maximal  $\alpha$ -leakage, our results apply to a broader class of information leakages derived from f-divergences. Recall that for a convex function  $f: \mathbb{R} \to \mathbb{R}$  such that f(1) = 0, an f-divergence  $D_f$  is a measure of the similarity between two distributions given by

$$D_f(P||Q) = \int dQ f\left(\frac{dP}{dQ}\right). \tag{4}$$

**Definition 2.** Given a joint distribution  $P_{XY} = P_{Y|X}P_X$  and a f-divergence  $D_f$ , a distribution-dependent leakage is defined as

$$\mathcal{L}_f(X;Y) = \inf_{Q_Y} D_f(P_{XY} || P_X \times Q_Y), \tag{5}$$

and a distribution-independent leakage is defined as

$$\mathcal{L}_f^{max}(X \to Y) = \sup_{P_{\tilde{Y}}} \inf_{Q_Y} D_f(P_{\tilde{X}} P_{Y|X} || P_{\tilde{X}} \times Q_Y), \quad (6)$$

where  $P_{\tilde{X}}$  is constrained to have the same support as  $P_X$ .

Recall that for  $\alpha=1$ , the maximal  $\alpha$ -leakage is MI and is a special case of  $\mathcal{L}_f(X;Y)$  in (5) with  $f(t)=t\log t$ . Furthermore, for  $\alpha>1$ , maximal  $\alpha$ -leakage has a one-to-one relationship with a special case of  $\mathcal{L}_f^{\max}$  in (6) for f given by

$$f_{\alpha}(t) = \frac{1}{\alpha - 1}(t^{\alpha} - 1),\tag{7}$$

such that  $D_f$  is the Hellinger divergence of order  $\alpha$  [14]. The following lemma makes precise this observation.

**Lemma 1.** For  $\alpha > 1$ , maximal  $\alpha$ -leakage can be written as

$$\mathcal{L}_{\alpha}^{max}(X \to Y) = \frac{1}{\alpha - 1} \log \left( 1 + (\alpha - 1) \mathcal{L}_{f_{\alpha}}^{max}(X \to Y) \right), \quad (8)$$

where  $\mathcal{L}_{f_{\alpha}}^{max}(X \to Y)$  is the  $\mathcal{L}_{f}^{max}(X \to Y)$  in (6) for  $f_{\alpha}$  given by (7) such that  $D_{f}$  is the Hellinger divergence of order  $\alpha$ .

## III. PRIVACY-UTILITY TRADEOFF WITH A HARD DISTORTION CONSTRAINT

We now consider PUT problems minimizing either maximal  $\alpha$ -leakage or its related f-divergence-based variants in Def. 2, subject to a *hard* distortion constraint. Such a constraint can be written as  $d(X,Y) \leq D$  with probability 1, where  $d(\cdot,\cdot)$  is a distortion function and D is the maximal permitted distortion. In other words, for any input  $x \in \mathcal{X}$ , the output y of the privacy mechanism must lie in a ball  $B_D(x)$  given by

$$B_D(x) \triangleq \{y : d(x,y) \le D\}. \tag{9}$$

We henceforth denote an optimal PUT as  $PUT_{HD,\mathcal{L}_*^*}$ , where HD and  $\mathcal{L}_*^*$  in the subscript indicate the utility and privacy measures, respectively. The following two theorems characterize  $PUT_{HD,\mathcal{L}_f}$  and  $PUT_{HD,\mathcal{L}_f^{max}}$  with detailed proofs in appendices A and B, respectively.

**Theorem 2.** For any distribution-dependent leakage  $\mathcal{L}_f$  in (5) and a distortion function  $d(\cdot, \cdot)$  with  $B_D(x)$  in (9), the optimal PUT is given by

$$PUT_{HD,\mathcal{L}_f}(D) = \inf_{P_{Y|X}:d(X,Y) \le D} \mathcal{L}_f(X;Y)$$
(10)

$$= f(0) + \inf_{Q_Y} \mathbb{E}\left(Q_Y(B_D(X))\left(f\left(\frac{1}{Q_Y(B_D(X))}\right) - f(0)\right)\right). \tag{11}$$

If there exists a distribution  $Q_Y^{\star}$  achieving the infimum in (11), an optimal mechanism  $P_{Y|X}^{\star}$  is given by

$$\frac{dP_{Y|X=x}^*}{dQ_Y^*}(y) = \frac{\mathbf{1}(d(x,y) \le D)}{Q_Y^*(B_D(x))}.$$
 (12)

 $^1$ The independence is with respect to the distribution of X. This "distribution-independent" measure depends on the distribution of X only through its support. In contrast, the distribution-dependent measure  $\mathcal{L}_f$  depends fully on the distribution of X. Both measures depend on the chosen mechanism  $P_{Y|X}$ .

F	$Y^n X^n$	T(0)	T(1)	T(2)	T(3)	T(4)	T(5)	T(6)	T(7)	T(8)	$P_{Y^n X^n}^*$
	T(0)	0	0	1							$\mathcal{X}^n \longrightarrow \mathcal{X}^n$
	T(1)	0	0	1	0						T(0)
	T(2)	0	0	1	0	0					$\vdots \qquad 1 \qquad y^n \in T(2)$
	T(3)		0	1	0	0	0				T(4)
	T(4)			1	0	0	0	0			
	T(5)				0	0	0	0	1		T(5)
	T(6)					0	0	0	1	0	1
	T(7)						0	0	1	0	$\vdots \qquad y^n \in T(7)$ $T(8)$
	T(8)							0	1	0	

Fig. 1: An optimal mechanism for  $\alpha>1$  with (n,m)=(8,2). Note that the hard distortion forces conditional probabilities of outputs outside the feasible ball of a given input to be zero. We highlight the conditional probabilities of feasible outputs in green, and give their values in the optimal mechanism.

**Theorem 3.** For any distribution-independent leakage  $\mathcal{L}_f^{max}$  in (6), a distortion function  $d(\cdot,\cdot)$  and  $B_D(x)$  in (9), the optimal PUT is given by

$$PUT_{HD,\mathcal{L}_f^{max}}(D) = \inf_{P_{Y|X}: d(X,Y) \le D} \mathcal{L}_f^{max}(X \to Y)$$
 (13)

 $=q^{\star}f((q^{\star})^{-1}) + (1-q^{\star})f(0), \qquad (14)$ 

with q\* defined as

$$q^* \triangleq \sup_{Q_Y} \inf_x Q_Y(B_D(x)).$$
 (15)

Moreover, if there exists  $Q_Y^*$  achieving the supremum in (15), an optimal mechanism  $P_{Y|X}^*$  is given by (12).

The PUTs in (11) and (14) simplify to finding an output distribution  $Q_Y$  that can be viewed as a "target" distribution, i.e., the optimal mechanism aims to produce this distribution as closely as possible, subject to the utility constraint. In particular, the resulting optimal mechanism (derived from (12)), for any input, distributes the outputs according to  $Q_Y$  while conditioning the output to be within a ball of radius D about the input. The optimization in (15) ensures that all inputs are uniformly masked while (11) provides average guarantees.

The following theorem characterizes the optimal tradeoff  $PUT_{HD,\mathcal{L}^{\max}_{\alpha}}$  for maximal  $\alpha$ -leakage. Recall that for  $\alpha=1$ ,  $\mathcal{L}^{\max}_1$  equals  $\mathcal{L}_f$  with  $f(t)=t\log t$ . For  $\alpha>1$ , from the one-to-one relationship between  $\mathcal{L}^{\max}_{\alpha}$  and  $\mathcal{L}^{\max}_{f_{\alpha}}$  in (8), we know that finding  $PUT_{HD,\mathcal{L}^{\max}_{f}}$  is equivalent to finding the optimal tradeoff  $PUT_{HD,\mathcal{L}^{\max}_{f}}$  in (13) for  $\mathcal{L}^{\max}_{f}=\mathcal{L}^{\max}_{f_{\alpha}}$ . Due to space constraints, we omit details.

**Theorem 4.** For maximal  $\alpha$ -leakage  $\mathcal{L}_{\alpha}^{max}$ , a distortion function  $d(\cdot, \cdot)$  and  $B_D(x)$  in (9), the optimal PUT is given by

$$PUT_{HD,\mathcal{L}_{\alpha}^{max}}(D) = \inf_{P_{Y|X}: d(X,Y) \le D} \mathcal{L}_{\alpha}^{max}(X \to Y)$$
 (16)

$$= \begin{cases} \inf_{Q_Y} \mathbb{E}\left(\log \frac{1}{Q_Y(B_D(X))}\right), & \alpha = 1 \text{ (17a)} \\ -\log q^*, & \alpha > 1 \text{ (17b)} \end{cases}$$

where  $q^*$  is defined in (15). Moreover, an optimal mechanism is given by (12), where for  $\alpha = 1$ ,  $Q_Y^*$  achieves the infimum

in (17a); and for  $\alpha > 1$ ,  $Q_V^*$  achieves the supremum in (15).

**Remark 1.** Note that subject to a hard distortion constraint, the optimal privacy mechanism is always given by (12). In particular, for maximal  $\alpha$ -leakage, the optimal mechanism as well as the optimal PUT are identical for all  $\alpha > 1$ .

### IV. EXAMPLE: HARD DISTORTION FOR BINARY TYPES

When considering dataset disclosure under privacy constraints, a reasonable goal is to design privacy mechanisms that preserve the statistics of the original dataset while preventing inference of each individual record (e.g., a sample or a row of the dataset). Since the type (empirical distribution) of a dataset captures its statistics, we quantify distortion as the distance between the type of the original and disclosed datasets. We use maximal  $\alpha$ -leakage to capture the gain of an adversary (with access to the disclosed dataset) in inferring any function of the original dataset.

Let  $X^n$  be a random dataset with n entries and  $Y^n$  be the corresponding disclosed dataset generated by a privacy mechanism  $P_{Y^n|X^n}$ . Entries of both  $X^n$  and  $Y^n$  are from the same alphabet  $\mathcal{X}$ . For a pair of input and output datasets  $(x^n,y^n)$  of  $P_{Y^n|X^n}$ , let  $P_{x^n}$  and  $P_{y^n}$  indicate the types, respectively. We define the distortion function as

$$d(x^{n}, y^{n}) = \max_{x \in \mathcal{X}} |P_{x^{n}}(x) - P_{y^{n}}(x)|,$$
 (18)

and therefore, obtain  $\operatorname{PUT}_{\operatorname{HD},\mathcal{L}_{\alpha}^{\max}}$  as in (16) but with datasets  $X^n,Y^n$  in place of single letters X,Y. Let the fraction  $\frac{m}{n}$   $(m\in[0,n])$  be the upper bound D in (16), where [0,n] indicates the set of integers from 0 to n.

We concentrate on binary datasets and let  $\mathcal{X} = \{0, 1\}$ . Note that for binary datasets, we can simply write  $d(x^n, y^n) = |P_{x^n}(0) - P_{y^n}(0)|$ . For a n-length binary dataset, the number of types is n+1. Therefore, all input and output datasets can be categorized into n+1 type classes defined as

$$T(i) \triangleq \{x^n : nP_{x^n}(0) = i\} = \{y^n : nP_{y^n}(0) = i\}, i \in [0, n].$$

**Theorem 5.** Given an arbitrary pair of  $(n,m) \in [1,\infty) \times [0,n]$ , the minimal leakage for  $\alpha > 1$  is

$$PUT_{HD,\mathcal{L}_{\alpha}^{max}}\left(\frac{m}{n}\right) = \log\left\lceil\frac{n+1}{2m+1}\right\rceil.$$
 (19)

An optimal privacy mechanism maps all input datasets in a type class to a unique output dataset which is feasible and belongs to a type class in the set  $T^*$  given by

$$\mathcal{T}^* \triangleq \left\{ T(j) : j = l + (2m+1)k, k \in [0, \lceil \frac{n+1}{2m+1} \rceil - 1] \right\}, \quad (20)$$

where 
$$l=m$$
 if  $m+\left(\lceil\frac{n+1}{2m+1}\rceil-1\right)(2m+1)\leq n$ , and otherwise,  $l=n-\left(\lceil\frac{n+1}{2m+1}\rceil-1\right)(2m+1)$ .

A detailed proof is in Appendix C. Let (n,m)=(8,2) such that from Thm. 5, we have  $\mathcal{T}^*=\{T(2),T(7)\}$ . Fig. 1 shows the optimal mechanism, which maps all input datasets in  $\{T(i): i\in [0,4]\}$  (resp.  $\{T(i): i\in [5,8]\}$ ) to a unique output dataset in T(2) (resp. T(7)) with probability 1.

We have explored PUTs in the context of hard distortion utility constraints. This utility constraint has the advantage that it allows the data curator to make specific, deterministic guarantees on the quality of the published dataset. Focusing on maximal  $\alpha$ -leakage and its f-divergence-based variants, under a hard distortion constraint, we have shown that: (i) for all  $\alpha>1$ , we obtain the same optimal privacy mechanism and optimal PUT, which are independent of the distribution of the original data (or datasets); (ii) for  $\alpha=1$ , the optimal mechanism differs and depends on the distribution of the original data (or data sets). In other words, for this distortion measure, the tunable privacy measure behaves as either MI or MaxL. Possible future directions include verifying whether the observed behavior holds for average distortion constraints and more complicated data models.

#### APPENDIX

### A. Proof of Theorem 2

The feasible ball  $B_D(x)$  around x is defined in (9). For the distribution dependent PUT in (10), we have

$$PUT_{HD,\mathcal{L}_f}(D)$$

$$= \inf_{P_{Y|X}:d(X,Y) < D} \inf_{Q_Y} D_f(P_{Y|X}P_X || P_X \times Q_Y)$$
(21)

$$=\inf_{Q_{Y}} \inf_{P_{Y|X}: d(X,Y) \le D} \int dP_{X} D_{f}(P_{Y|X=x} || Q_{Y})$$
 (22)

$$=\inf_{Q_Y} \int dP_X \inf_{\substack{P_Y \mid X = x \\ Y \in B_D(x)}} \int dQ_Y f\left(\frac{dP_{Y\mid X}(\cdot\mid x)}{dQ_Y}\right) \tag{23}$$

$$=\inf_{Q_Y} \int dP_X \inf_{\substack{P_Y \mid X = x \\ Y \in B_D(x)}} \left( \int_{u \in B_D(x)^c} dQ_Y f\left(\frac{dP_{Y \mid X}(\cdot \mid x)}{dQ_Y}\right) \right)$$

$$+Q_{Y}(B_{D}(x))\int_{y\in B_{D}(x)}\frac{dQ_{Y}}{Q_{Y}(B_{D}(x))}f\left(\frac{dP_{Y|X}(\cdot|x)}{dQ_{Y}}\right)\right)$$

$$\int_{y\in B_{D}(x)}\int_{y\in B_{D}(x)}dQ_{Y}\left(Q_{Y}(B_{D}(x))\right)f(0)$$

$$\geq \inf_{Q_Y} \int dP_X \inf_{\substack{P_Y \mid X = x \\ Y \in B_D(x)}} \left( Q_Y \left( B_D(x)^c \right) f(0) \right.$$
$$\left. + Q_Y \left( B_D(x) \right) f\left( \frac{1}{Q_Y \left( B_D(x) \right)} \right) \right)$$

$$+Q_Y(B_D(x))f\left(\frac{1}{Q_Y(B_D(x))}\right)$$

$$=f(0) + \inf_{Q_Y} \int dP_X\left(Q_Y(B_D(x))\left(f\left(\frac{1}{Q_Y(B_D(x))}\right) - f(0)\right)\right)$$
(24)

where

- (22) follows from the fact that  $D_f(P_{Y|X}||Q_Y|P_X)$  is convex in  $(P_{Y|X}, Q_Y)$  for fixed  $P_X$ ,
- (24) is from the Jensen's inequality and the equality holds if and only if there is a mechanism  $P_{Y|X}$  satisfying

$$\frac{dP_{Y|X}(y|x)}{dQ_Y(y)} = \frac{\mathbf{1}(y \in B_D(x))}{Q_Y(B_D(x))}.$$
 (25)

## B. Proof of Theorem 3

The feasible ball  $B_D(x)$  around x is defined in (9). For the distribution independent PUT in (13), we have

$$PUT_{HD,\mathcal{L}_{f}^{max}}(D)$$

$$= \inf_{P_{Y|X}:d(X,Y) \leq D} \sup_{P_{\tilde{X}}} \inf_{Q_{Y}} D_{f}(P_{\tilde{X}}P_{Y|X} || P_{\tilde{X}} \times Q_{Y}) \quad (26)$$

$$= \inf_{Q_{Y}} \sup_{P_{\tilde{X}}} \inf_{P_{Y|X}:d(X,Y) \leq D} D_{f}(P_{\tilde{X}}P_{Y|X} || P_{\tilde{X}} \times Q_{Y}) \quad (27)$$

$$= \inf_{Q_{Y}} \sup_{P_{\tilde{X}}} \inf_{P_{Y|X} \atop d(X,Y) \leq D} \int \left( dP_{\tilde{X}}(x) D_{f}(P_{Y|X=x} || Q_{Y}) \right) \quad (28)$$

$$=\inf_{Q_Y} \sup_{P_{\tilde{X}}} \int dP_{\tilde{X}}(x) \inf_{\substack{P_{Y|X=x}\\Y \in B_D(x)}} \int dQ_Y f\left(\frac{dP_{Y|X=x}}{dQ_Y}\right) \tag{29}$$

$$=\inf_{Q_Y} \sup_{P_{\tilde{X}}} \int dP_{\tilde{X}}(x) \inf_{\substack{P_Y \mid X = x \\ Y \in B_D(x)}} \left( \int_{B_D(x)} dQ_Y \right) f\left(\frac{dP_{Y \mid X = x}}{dQ_Y}\right) + \int_{B_D(x)^c} dQ_Y f(0)$$
(30)

$$= \inf_{Q_Y} \sup_{P_{\tilde{X}}} \int dP_{\tilde{X}}(x) \inf_{\substack{P_{Y|X=x} \\ Y \in B_D(x)}} \left( Q_Y(B_D(x)) \int_{B_D(x)} dx dx \right) dx$$

$$\frac{dQ_Y}{Q_Y(B_D(x))} f\left(\frac{dP_{Y|X=x}}{dQ_Y}\right) + Q_Y(B_D(x)^c) f(0)$$
(31)

$$\geq \inf_{Q_Y} \sup_{P_{\tilde{X}}} \int dP_{\tilde{X}}(x) \inf_{\substack{P_{Y|X=x}\\Y \in B_D(x)}} \left( Q_Y(B_D(x)) \right)$$

$$\cdot f\left(\frac{1}{Q_Y(B_D(x))}\right) + \left(1 - Q_Y(B_D(x))\right)f(0)$$
 (32)

$$=\inf_{Q_Y} \sup_{P_{\tilde{X}}} \int dP_{\tilde{X}}(x) g(Q_Y(B_D(x)))$$
 (33)

$$=\inf_{Q_Y} \sup_{x} g(Q_Y(B_D(x)))$$
 (34)

where

- (27) and (29) follow from the fact that  $D_f(P_{XY}||P_X \times Q_Y)$  is linear in  $P_X$  for fixed  $(P_{Y|X}, Q_Y)$  and convex in  $(P_{Y|X}, Q_Y)$  for fixed  $P_X$ ,
- (32) follows from the convexity of f and Jensen's inequality. The equality holds if and only if there exists a mechanism  $P_{Y|X}$  satisfying (25).
- (33) results from  $q \triangleq Q_Y(B_D(x))$  and

$$q(q) \triangleq q f(q^{-1}) + (1 - q) f(0).$$
 (35)

Due to the convexity of f, we have  $f(q^{-1}) - f(0) \le f'(q^{-1}) \left(q^{-1} - 0\right)$ , from which, the derivative  $g'(q) = f(q^{-1}) - q^{-1}f'(q^{-1}) - f(0) \le 0$ . Therefore, the function g in (35) is non-increasing, such that (34) is be simplified as  $g(q^*)$ , where  $q^*$  is given by

$$q^* \triangleq \sup_{Q_Y} \inf_x Q_Y(B_D(x)). \tag{36}$$

Define the feasible ball around an input dataset  $x^n$  as

$$B_D(x^n) \triangleq \left\{ y^n : |P_{x^n}(0) - P_{y^n}(0)| \le \frac{m}{n} \right\}.$$
 (37)

From Thm. 4, to find an optimal mechanism  $P_{Y^n|X^n}^*$ , we need to find an output distribution  $Q_{Y^n}^*$  which optimizes (15) with  $x^n$  and  $y^n$  in place of x, y.

Note that for the hard distortion  $|P_{x^n}(0) - P_{y^n}(0)| \leq \frac{m}{n}$ , all datasets in a type class share the same group of feasible output datasets, and this feasible group can be represented by the type classes. Therefore, for any  $x^n \in T(i)$   $(i \in [0,n])$ , we rewrite  $B_D(x^n)$  as

$$B_D(x^n) = B_D(T(i)) \triangleq \{T(j) : |i - j| \le m, j \in [0, n]\}.$$

We define an distribution  $Q_T$  of type classes for outputs as

$$Q_T(j) \triangleq \sum_{y^n \in T(j)} Q_{Y^n}(y^n), \text{ for } j \in [1, n],$$
 (38)

such that

$$q^* = \sup_{Q_T} \inf_{i \in [0,n]} Q_T(B_D(T(i))). \tag{39}$$

The optimal distribution  $Q_T$  is determined by both upper and lower bounding q\* in (39). The upper bound is determined by restricting the optimization in (39) to a judicious choice of a small set of input types. The lower bound is a constructive scheme. Let  $l=m+(2m+1)\left(\lceil\frac{n+1}{2m+1}\rceil-1\right)-n$ . We define an index set  $\mathcal{I}_T\subset[0,n]$  for types as

$$I_{T} \triangleq \begin{cases} \left\{ m + (2m+1)k : k \in \left[0, \left\lceil \frac{n+1}{2m+1} \right\rceil - 1 \right] \right\} & l \leq 0 \\ \left\{ l + (2m+1)k : k \in \left[0, \left\lceil \frac{n+1}{2m+1} \right\rceil - 1 \right] \right\} & l > 0 \end{cases}$$
 (40)

From the expression of  $\mathcal{I}_T$  in (40), we observe that: (i) for  $l \leq 0$  (resp. l > 0), the first (resp. last) element is m (resp. n); (ii) for  $l \leq 0$  (resp. l > 0), the last (resp. first) element is no less (resp. less) than n-m (resp. m+1); (iii) for both cases, the difference between adjacent elements is 2m+1. Therefore, it is not difficult to see that feasible balls of input type classes indexed by  $I_T$  are a partition of the set of all type classes, i.e.,

$$B_D(T(i_1)) \cap B_D(T(i_2)) = \emptyset \quad i_1, i_2 \in \mathcal{I}_T, \tag{41a}$$

$${T(j): j \in [0, n]} = \bigcup_{i \in \mathcal{I}_T} B_D(T(i)).$$
 (41b)

Therefore, the problem in (39) is upper bounded by

$$q^* \le \sup_{Q_T} \inf_{i \in \mathcal{I}_T} Q_T(B_D(T(i))) \tag{42}$$

$$\leq \sup_{Q_T} \frac{1}{|\mathcal{I}_T|} \sum_{i \in \mathcal{I}_T} Q_T(B_D(T(i))) \tag{43}$$

$$= \sup_{Q_T} \left( \left\lceil \frac{n+1}{2m+1} \right\rceil \right)^{-1} \sum_{j \in [1,n]} Q_T(T(j)) \tag{44}$$

$$= \left( \left\lceil \frac{n+1}{2m+1} \right\rceil \right)^{-1}. \tag{45}$$

Construct an distribution  $Q_T'$  as

$$Q_T'(j) = \left( \left\lceil \frac{n+1}{2m+1} \right\rceil \right)^{-1} \quad \text{for } j \in I_T, \tag{46}$$

and otherwise,  $Q'_T(j) = 0$ . By (41) for each  $i \in [0, n]$ , there is a *unique* k satisfying  $|i - I_T(k)| \le m$ , where  $I_T(k)$  is the  $k^{\text{th}}$  element<sup>2</sup> of  $I_T$ . Therefore, we lower bound (39) by

$$q^* \ge \inf_i Q_T'(B_D(T(i))) \tag{47}$$

$$=\inf_{i} Q_{T}' \Big( \bigcup_{\substack{|i-j| \le m \\ j \in \mathcal{I}_{T}}} T(j) \Big)$$
 (48)

$$=\inf_{k} Q_{T}'(T(I_{T}(k))) = \left( \left\lceil \frac{n+1}{2m+1} \right\rceil \right)^{-1}. \tag{49}$$

Thus,  $q^* = \left( \left\lceil \frac{n+1}{2m+1} \right\rceil \right)^{-1}$  and the  $Q_T'$  in (46) is optimal. From (38) and the  $Q_T'$ , we derive an optimal  $Q_{Y^n}^*$ , which assigns the same non-zero probability to only one dataset of each type classes indexed by  $I_T$ , i.e.,  $Q_{Y^n}^*(y^n) = q^*$  for one  $y^n \in T(j)$  for each  $j \in I_T$ . Therefore, from (12) we have the corresponding optimal privacy mechanism, which maps all input datasets in one input type class to one feasible output dataset with probability 1.

#### REFERENCES

[1] C. Dwork, "Differential privacy: A survey of results," in *Theory and Applications of Models of Computation: Lecture Notes in Computer Science.* New York:Springer, Apr. 2008.

[2] F. du Pin Calmon and N. Fawaz, "Privacy against statistical inference," in 2012 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton), 2012.

[3] L. Sankar, S. R. Rajagopalan, and H. V. Poor, "Utility-privacy trade-offs in databases: An information-theoretic approach," *IEEE Trans. on Inform. For. and Sec.*, vol. 8, no. 6, pp. 838–852, 2013.

[4] B. Rassouli and D. Gündüz, "Optimal utility-privacy trade-off with the total variation distance as the privacy measure," in arXiv:1801.02505v1 [cs.IT], 2018.

[5] I. Issa, S. Kamath, and A. B. Wagner, "An operational measure of information leakage," in 2016 Annual Conference on Information Science and Systems (CISS), 2016.

[6] J. C. Duchi, M. I. Jordan, and M. J. Wainwright, "Local privacy and statistical minimax rates," in 2013 IEEE 54th Annual Symposium on Foundations of Computer Science, 2013.

[7] Q. Geng, P. Kairouz, S. Oh, and P. Viswanath, "The staircase mechanism in differential privacy," *IEEE Journal of Selected Topics in Signal Processing*, vol. 9, no. 7, pp. 1176–1184, 2015.

[8] J. Liao, L. Sankar, V. Y. F. Tan, and F. P. Calmon, "Hypothesis testing under mutual information privacy constraints in the high privacy regime," *IEEE Transactions on Information Forensics and Security*, vol. 13, no. 4, pp. 1058–1071, 2018.

[9] H. Wang, M. Diaz, F. P. Calmon, and L. Sankar, "The utility cost of robust privacy guarantees," in arXiv:1801.05926v1, 2018.

[10] S. Asoodeh, F. Alajaji, and T. Linder, "Privacy-aware MMSE estimation," 2016 IEEE International Symposium on Information Theory (ISIT), pp. 1989–1993, 2016.

[11] J. Liao, O. Kosut, L. Sankar, and F. P. Calmon, "A tunable measure for information leakage." [Online]. Available: https://github. com/liachunLiao/ISIT2018

[12] A. Rényi, "On measures of entropy and information," in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*. The Regents of the University of California, 1961, pp. 547–561.

[13] S. Verdú, "α-mutual information," in 2015 Information Theory and Applications Workshop (ITA), 2015.

[14] F. Liese and I. Vajda, "On divergences and informations in statistics and information theory," *IEEE Transactions on Information Theory*, vol. 52, no. 10, pp. 4394–4412, Oct 2006.

<sup>2</sup>From (40),  $I_T(k)$  is either m + (2m+1)k or l + (2m+1)k.