Zero-Error Capacity of Multiple Access Channels via Nonstochastic Information

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Abstract—The problem of characterising the zero-error capacity region for multiple access channels even in the noiseless case has remained an open problem for over three decades. Motivated by this challenging question, a recently developed theory of nonstochastic information is applied to characterise the zero-error capacity region for the case of two correlated transmitters. Unlike previous contributions, this analysis does not assume that the blocklength is asymptotically large. Finally, a new notion of nonstochastic information is proposed for a noncooperative problem involving three agents. These results are preliminary steps towards understanding information flows in worst-case distributed estimation and control problems.

Index Terms—Nonstochastic information, multiple access channels, zero-error capacity, multi-agent systems

I. INTRODUCTION

The *multiple access channel* (MAC) was initially introduced by Shannon in his work [1]. The multiple access communication system consists of several senders that aim to transmit each an independent message reliably to a common receiver. This model corresponds indeed to various real-life scenarios such as multiple ground stations communicating with a satellite receiver, or the uplink phase of a cellular system. Clearly, the challenge in this case is not only the channel noise distorting the transmitted signal, but also the interference between the senders. The ordinary capacity region \mathscr{C} of MAC channels has been extensively studied in the literature [2]– [4], and by means of superposition coding, the single-letter characterization of this region was found by Slepian and Wolf [4]. It consists of the closure of the convex hull for all nonnegative rate tuples (R_0, R_1, R_2) satisfying

$$R_{1} \leq I(X_{1};Y|X_{2},U),$$

$$R_{2} \leq I(X_{2};Y|X_{1},U),$$

$$R_{1}+R_{2} \leq I(X_{1},X_{2};Y|U),$$

$$R_{0}+R_{1}+R_{2} \leq I(X_{1},X_{2};Y)$$
(1)

where $X_1 \leftrightarrow U \leftrightarrow X_2$ and $U \leftrightarrow X_1, X_2 \leftrightarrow Y$ form Markov chains. A further important notion in addition to the ordinary capacity is the so-called *zero-error capacity*. This parameter is defined as the least upper bound of rates leading to an error probability at the receiver which is *exactly* equal to zero [5]. The significance of zero-error capacity has recently been shown in worstcase control problems where strict, deterministic guarantees on performance must be met [12]. However, little is known about the zero-error capacity region of many simple MAC's. For instance, for deterministic binary adder channels, the best outer bound on this region has been found by Ordentlich and Shayevitz in [6] and presents a slight improvement on the result obtained by Urbanke and Li [7]. These studies mainly rely on combinatorics in order to tighten the outer bound of \mathscr{C}_0 .

In this paper we apply the concept of nonstochastic information [8] to obtain an intrinsic characterisation of the zeroerror capacity region of a general noisy MAC. A motivation for investigating such a problem arises from the study of decentralised control systems. In fact, the independent senders model the sensors reading the states of different plants, while the common decoder can be seen as the controller stabilising the system. Furthermore, the concept of zero-error capacity has increasingly gained more attention as it is an insightful parameter of the system worst-case performance. Contrary to communication systems, in control applications safety presents a crucial criterion, and hence, the plant performance must be guaranteed not only on average but rather at all times. Thus, in this case \mathscr{C}_0 can be considered a more useful figure of merit than the classical Shannon capacity & which allows an arbitrary small probability of error.

The rest of the paper is organised as follows. In Section II, some basic definitions related to the nonprobabilistic framework are introduced and the MAC model along with the zero-error coding scheme are presented. Next, the zero-error capacity region for the MAC channel for any given blocklength n is characterised in Section III, with converse and achievability proofs provided. A new notion of information in the MAC setting, namely the noncooperative NC-sense connectedness, is studied in Section IV. Finally, Section V concludes the article by summarising the main contributions and discussing possible future directions.

II. ZERO-ERROR COMMUNICATION OVER MACS IN THE NONSTOCHASTIC FRAMEWORK

In this section, we reformulate the problem of zero-error communication over multiple access channels (MACs) in

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terms of the nonstochastic framework of [8].

A. Uncertain Variables, Unrelatedness and Markovianity

First we briefly those elements of the nonstochastic framework of [9] that are needed for this section. We present further aspects as required in subsequent sections.

An uncertain variable (uv) X consists of a mapping from an underlying sample space Ω to a space \mathscr{X} of interest [9]. Each sample $\omega \in \Omega$ is hence mapped to a particular realization $X(\omega) \in \mathscr{X}$. For a pair of uv's X and Y, we denote their marginal, joint and conditional ranges as

$$[[X]] := \{X(\boldsymbol{\omega}) : \boldsymbol{\omega} \in \Omega\} \subseteq \mathscr{X}, \tag{2}$$

$$\llbracket X, Y \rrbracket := \{ (X(\boldsymbol{\omega}), Y(\boldsymbol{\omega})) : \boldsymbol{\omega} \in \Omega \} \subseteq \mathscr{X} \times \mathscr{Y}, \qquad (3)$$

$$\llbracket X|y \rrbracket := \{ X(\boldsymbol{\omega}) : Y(\boldsymbol{\omega}) = y, \boldsymbol{\omega} \in \Omega \} \subseteq \mathscr{X}.$$
(4)

The dependence on Ω will normally be hidden, with most properties of interest expressed in terms of operations on these ranges. As a convention, uv's are denoted by upper-case letters, while their realizations are indicated in lower-case. The family $\{ [X|y] : y \in [Y] \}$ of conditional ranges is denoted [X|Y].

Definition 1 (Unrelatedness [9]): The uvs $X_1, X_2, \dots X_n$ are said to be (mutually) unrelated if

$$\llbracket X_1, X_2, \cdots, X_n \rrbracket = \llbracket X_1 \rrbracket \times \llbracket X_2 \rrbracket \times \cdots \times \llbracket X_n \rrbracket.$$
 (5)

Remark: Unrelatedness, which is closely related to the notion of *qualitative independence* [13] between discrete sets, can be shown to be equivalent to the conditional range property

$$[\![X_k|x_{1:k-1}] = [\![X_k]\!], \ \forall x_{1:k-1} \in [\![X_{1:k-1}]\!], \ k \in [2:n].$$
(6)

Definition 2 (Markovianity [9]): The uvs X_1, X_2 and Y are said to form a *Markov uncertainty chain* $X_1 \leftrightarrow Y \leftrightarrow X_2$ if

$$[[X_1|y,x_2]] = [[X_1|y]], \quad \forall (y,x_2) \in [[Y,X_2]].$$
(7)

Remark: This can be shown to be equivalent to X_1 and X_2 being *conditionally unrelated given* Y, i.e.

$$\llbracket X_1, X_2 | y \rrbracket = \llbracket X_1 | y \rrbracket \times \llbracket X_2 | y \rrbracket, \ \forall y \in \llbracket Y \rrbracket.$$
(8)

By the symmetry of (8), $X_1 \leftrightarrow Y \leftrightarrow X_2$ iff $X_2 \leftrightarrow Y \leftrightarrow X_1$.

B. System Model

Consider a multiple access communication system with one receiver, two transmitters and three messages, as illustrated in Fig. (1). Assume the messages M^0 , M^1 and M^2 are mutually unrelated and finite-valued. Without loss of generality, for i = 0, 1, 2 let M^i take the integer values $[1 : \mu^i]$ for some integer $\mu^i \ge 1$. For a given block-length $n \ge 1$, the messages are encoded into channel input sequences $X_{1:n}^1$ and $X_{2:n}^2$ as

$$X_{1:n}^{j} = \gamma^{i}(M^{0}, M^{j}), \quad j = 1, 2,$$
(9)

where γ^1 and γ^2 are the coding functions at each transmitter. Observe that the *common message* M^0 is seen by both transmitters, while the *private messages* M^1 and M^2 are available only to their respective transmitters. The code rate for each message is defined as

$$R^{i} := (\log_{2} \mu^{i})/n, \quad i = 0, 1, 2.$$
(10)

Due to the common message, the two channel input sequences applied will typically be related. In the case where the common message can take only one value, so that $R^0 = 0$, each channel input is generated in isolation and is mutually unrelated with the other. At the other extreme, if the private messages can each take only one value so that $R^1 = R^2 = 0$, then the channel inputs are generated in complete cooperation.

The encoded data sequences are then sent through a stationary memoryless MAC as depicted in Fig. (1). The output $Y_k \in \mathscr{Y}$ of the MAC is given in terms of a fixed function $f : \mathscr{X}_1 \times \mathscr{X}_2 \times \mathscr{W} \to \mathscr{Y}$ as

$$Y_k = f(X_k^1, X_k^2, W_k) \in \mathscr{Y}, \quad k = 1, 2, \dots,$$
 (11)

where W_k is channel noise that is mutually unrelated with $W_{1:k-1}$, M^0 , M^1 , M^2 , and has constant range $[\![W_k]\!] = \mathcal{W}$.

At the receiver , the decoder δ produces message estimates \hat{M}^0 , \hat{M}^1 and \hat{M}^2 from the channel output sequence $Y_{1:n}$. Under a zero-error objective, these estimates must always be exactly equal to the original messages, regardless of channel noise or interference between X_k^1 and X_k^2 . In other words, $[M^i|y_{1:n}]$ is a singleton for each i = 0, 1, 2 and any $y_{1:n} \in [Y_{1:n}]$. For a given block-length n, we define the *zero-error n-capacity region* $\mathcal{C}_{0,n}$ of the MAC as the set of rate tuples $R = (R^i)_{i=0}^3$ for which this is possible by suitable choice of coding functions.

The system set-up above is inspired by that of [4]. The critical difference is that the messages and channel here are not assumed to have any statistical structure, and the aim is to recover the messages perfectly, not just with arbitrarily small error probability. In addition, we are interested in characterising the zero-error capacity region at finite n, not just as $n \rightarrow \infty$.

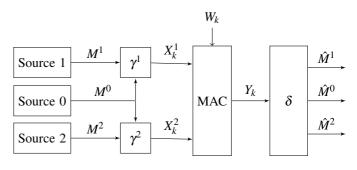


Fig. 1. The two-transmitter MAC system with a common message operating at time instant k.

III. NONSTOCHASTIC INFORMATION AND MAC ZERO-ERROR CAPACITY

In this section, we use the nonstochastic information concepts of [8], [9] to give an exact characterisation of the zeroerror capacity region of the multiple access channel (MAC) defined in the previous section.

A. Preliminaries on Nonstochastic Information

First we present some necessary background concepts. Throughout this subsection X, Y, Z, Z' and W denote uncertain variables (uv's).

Definition 3 (Overlap Connectedness [9]): Two points x and $x' \in [\![X]\!]$ are said to be $[\![X|Y]\!]$ -overlap connected, denoted $x \leftrightarrow x'$, if there exists a finite sequence $\{X|y_i\}_{i=1}^m$ of conditional ranges such that $x \in [\![X|y_1]\!], x' \in [\![X|y_m]\!]$ and $[\![X|y_i]\!] \cap [\![X|y_{i-1}]\!] \neq \emptyset$, for each $i \in [2, \dots, m]$.

Remarks: It is easy to see that overlap connectedness is both transitive and symmetric, i.e. it is an equivalence relation between x and x'. Thus it induces disjoint *equivalence classes* that cover [X] and form a unique partition. This is called the [[X|Y]]-overlap partition, denoted by $[[X|Y]]_*$.

Definition 4 (Nonstochastic Information [9]): The nonstochastic information between X and Y is given by

$$I_*[X;Y] = \log_2 | [X|Y]_* |.$$
(12)

Remark: This can be shown to be symmetric, i.e. $I_*[X;Y] = I_*[Y;X]$.

Definition 5 (Common Variables [10], [11]):

A uv Z is said to be a *common variable* (*cv*) for X and Y if there exist functions f and g such that Z = f(X) = g(Y).

It is further said to be a *maximal* cv if any other cv Z' admits a function h such that Z' = h(Z).

Remarks: In the context of random variables, these concepts were first discussed by Shannon [10], who used the term *common information element* for a maximal cv. Notice that no cv can take more distinct values than the maximal one.

The nonstochastic information $I_*[X;Y]$ is precisely the logcardinality of the range of a maximal cv between X and Y. This is because it can be shown that $\forall (x, y) \in \llbracket X, Y \rrbracket$, the partition set in $\llbracket X | Y \rrbracket_*$ that contains x also uniquely specifies the set in $\llbracket Y | X \rrbracket_*$ that contains y. Thus these overlap partitions define a cv for X and Y, with corresponding functions f and g given by the labelling. Furthermore, this cv can be proved to be maximal. See [8] for details.

Definition 6 (Conditional I_*): The conditional nonstochastic information between X and Y given W is

$$I_*[X;Y|W] := \min_{w \in \llbracket W \rrbracket} \log_2 |\llbracket X|Yw \rrbracket_*|, \qquad (13)$$

where for a given $w \in \llbracket W \rrbracket$, $\llbracket X | Yw \rrbracket_*$ is the overlap partition of $\llbracket X | w \rrbracket$ induced by the family $\llbracket X | Yw \rrbracket$ of conditional ranges $\llbracket X | yw \rrbracket$, $y \in \llbracket Y | w \rrbracket$ [9].

Remark: It can be shown that $I_*[X;Y|W]$ also has an important interpretation in terms of cv's: it is the maximum log-cardinality of the ranges of all cv's Z = f(X,W) = g(y,W) that are unrelated with W. See [9] for details.

B. MAC Zero-Error Capacity via Nonstochastic Information

We are now in a position to prove the main result of this paper.

Theorem 1: For a given block-length $n \ge 1$, let $\mathscr{R}(U, X_{1:n}^1, X_{1:n}^2)$ be the set of nonegative tuples (R_0, R_1, R_2) such that

$$nR_0 \le I_*[U;Y_{1:n}] \tag{14}$$

$$nR_1 \le I_*[X_{1:n}^1; Y_{1:n}|U] \tag{15}$$

$$nR_2 \le I_*[X_{1:n}^2; Y_{1:n}|U] \tag{16}$$

where $X_{1:n}^i$, i = 1, 2, are sequences of inputs to the multiple access channel (MAC) (11), $Y_{1:n}$ is the corresponding channel output sequence, and *U* is an auxiliary uncertain variable (uv).

Then the *zero-error n-capacity region* $\mathscr{C}_{0,n}$ of the MAC over n uses coincides with the union of the regions $\mathscr{R}(U, X_{1:n}^1, X_{1:n}^2)$ over all uv's $U, X_{1:n}^1, X_{1:n}^2$ that satisfy the Markov uncertainty chains $X_{1:n}^1 \leftrightarrow U \leftrightarrow X_{1:n}^2$ and $U \leftrightarrow (X_{1:n}^1, X_{1:n}^2) \leftrightarrow Y_{1:n}$.

Remarks: This result is the zero-error analogue of the Slepian-Wolf ordinary capacity region \mathscr{C} (1), in terms of nonstochastic rather than Shannon information. Although \mathscr{C} is *prima facie* given in 'single-letter' terms, it is operationally relevant only at large block-lengths *n*, to yield small probabilities of error. In contrast, the result above specifies all rates tuples that allow *exactly* zero errors to be achieved at a given finite *n*. This could potentially be of interest in safety-critical, low-latency applications in distributed networked control. If arbitrarily long blocks are permitted, then the relevant zero-error capacity region \mathscr{C}_0 is given by the convex hull of $\bigcup_{n>1} \mathscr{C}_{0,n}$.

Although (14)–(16) give a cuboidal rate region $\mathscr{R}(U, X_{1:n}^1, X_{1:n}^2)$, it is not clear if the zero-error capacity regions also have geometrically simple shapes, due to the unions over $U, X_{1:n}^1, X_{1:n}^2$ and *n*. We aim to investigate this in future work, for specific channels of interest.

1) Proof of Converse: Consider a zero-error code (9) with block-length *n* operating at rates R_0, R_1 and R_2 (10) over the MAC (11), and set $U = M^0$. As M^i , i = 0, 1, 2 are mutually unrelated, it follows from (9) that the codewords $X_{1:n}^1$ and $X_{1:n}^2$ are conditionally unrelated given M^0 , i.e. the first Markov uncertainty chain $X_{1:n}^1 \leftrightarrow U \leftrightarrow X_{1:n}^2$ is satisfied. Since the channel noise in (11) is unrelated with the messages and hence with the codewords, we also have the second Markov uncertainty chain $Y_{1:n} \leftrightarrow (X_{1:n}^1, X_{1:n}^2) \leftrightarrow U$.

As the messages are all errorlessly recovered at the receiver, there certainly exists a decoding function δ^0 such that $M^0 = \delta^0(Y_{1:n})$. Setting $U = M^0$, we see that M^0 is therefore a common variable (cv) between U and $Y_{1:n}$. By the maximal cv property of I_* ,

$$nR^{0} \equiv \log_{2} |\llbracket M^{0} \rrbracket| \le I_{*}[U; Y_{1:n}],$$
(17)

proving (14).

We next prove the remaining two inequalities. Observe that for a given realisation m^0 of the common message, there must be a unique message m^1 corresponding to each channel codeword $x_{1:n}^1$; otherwise, multiple values of m^1 would be associated with a single channel output sequence $y_{1:n}$, violating the zero-error requirement. Consequently, there must exist a mapping g such that $M^1 = g(X_{1:n}^1, M^0)$. Furthermore, by the zero-error property there also exists a function δ^1 such that $M^1 = \delta^1(Y_{1:n})$.

Thus M^1 is a cv between $(X_{1:n}^1, M^0)$ and $(Y_{1:n}, M^0)$. As by hypothesis it is also unrelated with $U = M^0$, the interpretation of conditional I_* in terms of maximal unrelated cv's allows us to conclude that

$$nR^{1} \equiv \log_{2} |\llbracket M^{1} \rrbracket| \le I_{*}[X_{1:n}^{1}; Y_{1:n} | M^{0}] = I_{*}[X_{1:n}^{1}; Y_{1:n} | U], \quad (18)$$



Fig. 2. Example of an overlap partition $[\![U|Y_{1:n}]\!]_*$. The horizontal lines represent to the different member-sets of each partition and the filled circles correspond to the selected points u_i .

proving (14). In a similar way, the bound on the rate R^2 stated in (16) can be shown.

2) Proof of Achievability: We now prove that if we have a block-length *n* and uv's *U*, $X_{1:n}^j$, j = 1,2 satisfying the requirements in Theorem 1), it is possible to construct a zeroerror coding scheme at rates achieving equality in (14)–(16).

a) Codebook Generation: First, set $nR^0 = I_*[U;Y_{1:n}]$ and pick one point in each of the disjoint sets of the overlap partition $[[U|Y_{1:n}]]_*$. With mild abuse of notation call these distinct points $u(m^0)$, $m^0 = 1, \ldots, 2^{nR_0}$.

Next, observe that since $nR^i = I_*[X_{1:n}^i; Y_{1:n}|U]$ for i = 1, 2, (13) implies that

$$2^{nR^{i}} \leq \left| [[X_{1:n}^{i}|Y_{1:n}, U = u(m^{0})]]_{*} \right|, \ i = 1, 2, m^{0} \in [1:2^{nR^{0}}].$$
(19)

For any m^0 , we may therefore pick 2^{nR^i} distinct codewords $x_{1:n}^i$ from $[X_{1:n}^i|U = u(m^0)]$ such that there is at most one codeword in each set of the overlap partition $[[X_{1:n}^i|Y_{1:n}, U = u(m^0)]]_*$. Denote these codewords by $\gamma^i(m^0, m^i)$, $m^i \in [1 : 2^{nR^i}]$. This gives us our coding laws (9).

b) Zero Error: To show that this code may be decoded with zero error, observe first that since $X_{1:n}^1 \leftrightarrow U \leftrightarrow X_{1:n}^2$, the joint conditional range $[X_{1:n}^1, X_{1:n}^2|U = u(m^0)]]$ is just the Cartesian product

$$[[X_{1:n}^1|, U = u(m^0)]] \times [[X_{1:n}^2|U = u(m^0)]].$$

Thus we are guaranteed that for every m^0 , all codeword pairs $(\gamma^1(m^0,m^1),\gamma^2(m^0,m^2))$, $m^i = 1,\ldots,2^{nR^i}$, i = 1,2, lie within the conditional joint range $[[X_{1:n}^1,X_{1:n}^2]U = u(m^0)]]$. In other words, for every combination of m^0,m^1 and m^2 , the triple $(\gamma^1(m^0,m^1),\gamma^2(m^0,m^2),u(m^0))$ is a valid point inside the joint range $[[X_{1:n}^1,X_{1:n}^2,U]]$.

The decoding proceeds in three stages. In the first stage, the common message m^0 is recovered. Recall that each of the 2^{nR^0} points $u(m^0)$ lies in a distinct set of the overlap partition $[\![U|Y_{1:n}]\!]_*$. By the common variable (cv) property of overlap partitions, this set is uniquely determined by the corresponding set of the matching overlap partition $[\![Y_{1:n}|U]\!]_*$ that contains the channel output sequence $y_{1:n}$. In this way, m^0 is uniquely decoded.

In the second stage, having recovered m^0 , the decoder calculates which distinct set of the conditional overlap partition $[[Y_{1:n}|X_{1:n}^1, U = u(m^0)]]_*$ contains $y_{1:n}$. Again by the cv property, this set uniquely determines the corresponding set of the matching conditional overlap partition $[[X_{1:n}^1|Y_{1:n}, U = u(m^0)]]_*$ that contains the codeword $\gamma^1(m^0, m^1)$. By construction, for each m^0 there is at most one codeword in each set of this latter conditional overlap partition; thus m^1 is uniquely recovered. In the third stage, the decoder repeats the second stage but with $X_{1:n}^2$ instead of $X_{1:n}^1$, and recovers m^2 uniquely in the same way.

IV. NONCOOPERATIVE CONNECTEDNESS AND INFORMATION

The notions of overlap connectedness and common variables (cv's) were critical in developing a characterisation of MAC zero-error capacity based on nonstochastic information. In this section, we consider a related but more basic problem, in which three uncertain variables X_1, X_2 and Y with joint range $[X_1, X_2, Y]$ are respectively observed by three agents. The agents observing X_1 and X_2 each wish to separately deduce as much as possible about Y, while the agent observing Y wishes to know exactly what the other two agents have deduced about it. In other words, we seek to characterise cv's of the form

$$Z = (f_1(X_1) \ f_2(X_2)) = (g_1(Y) \ g_2(Y)) \equiv g(Y)$$
(20)

In order to do so, we propose a new notion of connectedness and nonstochastic information.

Definition 7: (NC-Connectedness) A pair of points (x_1, x_2, y) and $(x'_1, x'_2, y') \in [[X_1, X_2, Y]]$ is called *noncooperatively* (NC-)connected, denoted $(x_1, x_2, y) \xrightarrow{NC} (x'_1, x'_2, y')$, if

- (i) $x_1 \leftrightarrow x'_1$ in $[X_1|Y]$,
- and
- (ii) $x_2 \leftrightarrow x'_2$ in $\llbracket X_2 | Y \rrbracket$,

where the symbol " refers to overlap connectedness.

Remark: It is clear that NC-connectedness inherits the symmetry and transitivity of overlap connectedness; thus it is an equivalence relation, which splits $[X_1, X_2, Y]$ into disjoint equivalence classes. Call this partition the *NC-partition* of $[X_1, X_2, Y]$.

From the definition, it can be shown that each set of the NC-partition is uniquely defined by a set in the product $[X_1|Y]_* \times [X_2|Y]_*$ of overlap partitions. By the common variable property of overlap partitions, it is also uniquely defined by a corresponding pair of sets in the matching overlap partitions $[Y|X_1]_*$ and $[Y|X_2]_*$. As both these latter partitions are of $[Y]_$, this pair of sets is uniquely defined by a more refined set in the pairwise intersection or *join* $[Y|X_1]_* \vee [Y|X_2]_*$.

Thus the overlap partitions $[\![X_1|Y]\!]_*$ and $[\![X_2|Y]\!]_*$ yield the functions $f_1(X_1)$ and $f_2(X_2)$ of (20), while $[\![Y|X_1]\!]_* \vee [\![Y|X_2]\!]_*$ yields the matching function g(Y).

A. Maximal Common Variable and Noncooperative I_*

Theorem 2: The functions f_1 , f_2 and g given respectively by the (labels of) the partitions $[\![X_1|Y]\!]_*$, $[\![X_2|Y]\!]_*$ and $[\![Y|X_1]\!]_* \lor$ $[\![Y|X_2]\!]_*$ yield a common variable (cv) Z_* in the sense (20) that is maximal. That is, any other cv

$$Z = (\bar{f}_1(X_1), \bar{f}_2(X_2)) = \bar{g}(Y)$$

admits a function *h* such that $Z = h(Z_*)$.

Proof. This statement can be proven by contradiction. Suppose that there is a set \mathscr{P} in the NC-partition of $[X_1, X_2, Y]$ that

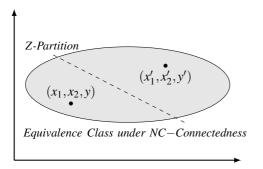


Fig. 3. Illustration of the scenario expressed by (21).

is not wholly contained inside any partition set induced by the cv Z. Then there must exist two admissible points (x_1, x_2, y) and (x'_1, x'_2, y') in \mathscr{P} that lie in different partition sets of Z, therefore yielding different values $z \neq z'$ of Z. That is,

$$\left(\bar{f}_1(x_1), \bar{f}_2(x_2)\right) \neq \left(\bar{f}_1(x_1'), \bar{f}_2(x_2')\right)$$
 (21)

Without loss of generality, say that $\bar{f}_1(x_1) \neq \bar{f}_1(x'_1)$. As Z is a cv in the sense (20), its first component is a cv $Z_1 = \bar{f}_1(X_1) = \bar{g}_1(Y)$ between X_1 and Y.

However, by the maximal cv property, Z_1 is a function of the set of the overlap partition $[X_1|Y]]_*$ that contains X_1 . Since both x_1 and x'_1 lie in the same set $[X_1|Y]]_*$, they must therefore yield the same value $\bar{f}_1(x_1) = \bar{f}_1(x'_1)$, contradicting (21).

With this result, it is then natural to take the log-cardinality of $[\![Z_*]\!]$ as a new measure of nonstochastic information,

$$I_*^{\rm NC}[X_1, X_2; Y] := \log_2 | \llbracket Y | X_1 \rrbracket_* \vee \llbracket Y | X_2 \rrbracket_* |.$$
(22)

V. CONCLUSION

In this paper, zero-error multiple access communication systems was analysed in a nonprobabilistic framework using uv's and nonstochastic information. These notions were used to characterise the zero-error capacity region of multiple access channels. The presented analysis is not only valid for asymptotically large blocklength but it also includes the case of finite n. Subsequently, the concept of noncooperative connectedness was introduced and used as a tool to extend the concept of nonstochastic information to non-cooperative situations.

Future work will consider the extension of this framework to include the general multi-user case (more than two input sequences) and MAC's with feedback. These scenarios represent the first steps of modelling information flows in distributed estimation and control systems using nonstochastic concepts.

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