# Joint Sensing and Communication over Memoryless Broadcast Channels 

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#### Abstract

A memoryless state-dependent broadcast channel (BC) is considered, where the transmitter wishes to convey two private messages to two receivers while simultaneously estimating the respective states via generalized feedback. The model at hand is motivated by a joint radar and communication system where radar and data applications share the same frequency band. For physically degraded BCs with i.i.d. state sequences, we characterize the capacity-distortion tradeoff region. For general BCs, we provide inner and outer bounds on the capacitydistortion region, as well as a sufficient condition when it is equal to the product of the capacity region and the set of achievable distortion. Interestingly, the proposed synergetic design significantly outperforms a conventional approach that splits the resource either for sensing or communication.


## I. Introduction

A key-enabler of future high-mobility networks such as Vehicle-to-Everything (V2X) is the ability to continuously track the dynamically changing environment, hereafter called the state, and to react accordingly by exchanging information between nodes. Although state sensing and communication have been designed separately in the past, power and spectral efficiency as well as hardware costs encourage the integration of these two functions, such that they are operated by sharing the same frequency band and hardware (see e.g. [1]). A typical example of such a scenario is joint radar parameter estimation and communication, where the transmitter equipped with a monostatic radar wishes to convey a message to a (already detected) receiver and simultaneously estimate the state parameters of interest such as velocity and range [2]. Motivated by such an application, the first information theoretical model for joint sensing and communication has been introduced in [3]. By modeling the backscattered signal as generalized feedback and designing carefully the input signal, the capacity-distortion tradeoff has been characterized for a single-user channel [3], while lower and upper bounds on the rate-distortion region over multiple access channel has been provided in [4].

The current paper extends [3] to the broadcast channel (BC), where the transmitter wishes to convey private messages to two receivers and simultaneously estimate their respective states. For simplicity, the state information is assumed known at each receiver. Although oversimplified, the scenario at hand relates to vehicular networks where a transmitter vehicle, equipped with a monostatic radar, sends (safety-related) messages to multiple vehicles and simultaneously estimates the parameters of these vehicles. The full characterization of the capacitydistortion region is very challenging, because the capacity
region of memoryless BCs with generalized feedback is generally unknown even without state sensing (see e.g. [5]). Therefore, we consider first physically degraded BCs where generalized feedback is only useful for state sensing, like for the single user channel. The capacity-distortion region is completely characterized for this class of BCs. Moreover, closed-form expressions of the region are provided for some binary examples. The numerical evaluations illustrate interesting tradeoffs between the achievable rates and distortions across two receivers. For general BCs, we provide a sufficient condition when the capacity-distortion region is simply the product of the capacity region and the set of all achievable distortions, thus no tradeoff between communication and sensing arises. Furthermore, we provide general inner and outer bounds on the capacity-distortion region, as well as a statedependent version of Dueck's BC. For all these kinds of BCs, we show though numerical examples that the synergetic design significantly outperforms the resource-sharing scheme that splits the resource either for sensing or communication.

The rest of the paper is organized as follows. Section II introduces our model and Section III presents some cases that yield no tradeoff between sensing and communication. Section IV] focuses on the physical degraded broadcast channel and provides some examples. Finally, upper and lower bounds for the general memoryless broadcast channel are provided along with an example in Section $\nabla$

## II. System Model

Consider a two-user state-dependent memoryless broadcast channel (SDMBC) with two private messages $W_{1}$ and $W_{2}$ as illustrated in Fig. 1. The model comprises a two-dimensional memoryless state sequence $\left\{\left(S_{1, i}, S_{2, i}\right)\right\}_{i \geq 1}$ whose samples at time $i$ are distributed according to a given joint law $P_{S_{1} S_{2}}$ over the state alphabets $\mathcal{S}_{1} \times \mathcal{S}_{2}$. Given input and output alphabets $\mathcal{X}, \mathcal{Y}_{1}, \mathcal{Y}_{2}, \mathcal{Z}$, input $X_{i}=x \in \mathcal{X}$ and state-realizations $S_{1, i}=$ $s_{1} \in \mathcal{S}_{1}$ and $S_{2, i}=s_{2} \in \mathcal{S}_{2}$, the SDMBC produces a triple of outputs $\left(Y_{1, i}, Y_{2, i}, Z_{i}\right) \in \mathcal{Y}_{1} \times \mathcal{Y}_{2} \times \mathcal{Z}$ according to a given time-invariant transition law $P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}\left(\cdot, \cdot, \cdot \mid s_{1}, s_{2}, x\right)$, for each time $i$. A SDMBC is thus entirely specified by the tuple of alphabets and (conditional) pmfs

$$
\begin{equation*}
\left(\mathcal{X}, \mathcal{Y}_{1}, \mathcal{Y}_{2}, \mathcal{Z}, P_{S_{1} S_{2}}, P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}\right) \tag{1}
\end{equation*}
$$

We will often describe a SDMBC only by the pair of pmfs ( $P_{S_{1} S_{2}}, P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}$ ), in which case, the corresponding alphabets should be clear from the context.


Fig. 1. Broadcast model for joint sensing and communication

A $\left(2^{n R_{1}}, 2^{n R_{2}}, n\right)$ code for an SDMBC $P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}$ consists of

1) two message sets $\mathcal{W}_{1}=\left[1: 2^{n R_{1}}\right]$ and $\mathcal{W}_{2}=\left[1: 2^{n R_{2}}\right]$;
2) a sequence of encoding functions $\phi_{i}: \mathcal{W}_{1} \times \mathcal{W}_{2} \times$ $\mathcal{Z}^{i-1} \rightarrow \mathcal{X}$, for $i=1,2, \ldots, n$;
3) for each $k=1,2$ a decoding function $g_{k}: \mathcal{S}_{k}^{n} \times \mathcal{Y}_{k}^{n} \rightarrow$ $\mathcal{W}_{k}$
4) for each $k=1,2$ a state estimator $h_{k}: \mathcal{X}^{n} \times \mathcal{Z}^{n} \rightarrow \hat{\mathcal{S}}_{k}^{n}$, where $\hat{\mathcal{S}}_{k}$ denotes the given reconstruction alphabet for state sequence $S_{k}^{n}=\left(S_{k, 1}, \cdots, S_{k, n}\right)$.
For a given code, we let the random messages $W_{1}$ and $W_{2}$ be uniform over the message sets $\mathcal{W}_{1}$ and $\mathcal{W}_{2}$ and the inputs $X_{i}=\phi_{i}\left(W_{1}, W_{2}, Z^{i-1}\right)$, for $i=1, \ldots, n$. The corresponding outputs $Y_{1, i} Y_{2, i}, Z_{i}$ at time $i$ are obtained from the states $S_{1, i}$ and $S_{2, i}$ and the input $X_{i}$ according to the SDMBC transition law $P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}$. Further, let $\hat{S}_{k}^{n}:=\left(\hat{S}_{k, 1}, \cdots, \hat{S}_{k, n}\right)=$ $h_{k}\left(X^{n}, Z^{n}\right)$ be the state estimates at the transmitter and let $\hat{W}_{k}=g_{k}\left(S_{k}^{n}, Y_{k}^{n}\right)$ be the decoded message by decoder $k$, for $k=1,2$.

The quality of the state estimates $\hat{S}_{k}^{n}$ is measured by a given per-symbol distortion function $d_{k}: \mathcal{S}_{k} \times \hat{\mathcal{S}}_{k} \mapsto[0, \infty)$, and we will be interested in the expected average per-block distortion

$$
\begin{equation*}
\Delta_{k}^{(n)} \triangleq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[d_{k}\left(S_{k, i}, \hat{S}_{k, i}\right)\right], \quad k=1,2 . \tag{2}
\end{equation*}
$$

For the decoded messages $\hat{W}_{1}$ and $\hat{W}_{k}$ we focus on their joint probability of error:

$$
\begin{equation*}
p^{n}(\text { error }):=\operatorname{Pr}\left(\hat{W}_{1} \neq W_{1} \quad \text { or } \quad \hat{W}_{2} \neq W_{2}\right) \tag{3}
\end{equation*}
$$

Definition 1. A rate-distortion tuple $\left(R_{1}, R_{2}, D_{1}, D_{2}\right)$ is said achievable if there exists a sequence (in $n$ ) of $\left(2^{n R_{1}}, 2^{n R_{2}}, n\right)$ codes that simultaneously satisfy

$$
\begin{align*}
\lim _{n \rightarrow \infty} p^{(n)}(\text { error }) & =0  \tag{4a}\\
\varlimsup_{n \rightarrow \infty} \Delta_{k}^{(n)} & \leq D_{k}, \quad \text { for } k=1,2 \tag{4b}
\end{align*}
$$

The closure of the union of all achievable rate-distortion tuples $\left(R_{1}, R_{2}, D_{1}, D_{2}\right)$ is called the capacity-distortion region and is denoted $\mathcal{C D}$. The current work aims at specifying the tradeoff between the achievable rates and distortions. As we will see in Sections III and $\mathbb{V}$, there is no such tradeoff in some cases, and the resulting region $\mathcal{C D}$ is the product of SDMBC's capacity region:
$\mathcal{C} \triangleq\left\{\left(R_{1}, R_{2}\right):\left(R_{1}, R_{2}, D_{1}, D_{2}\right) \in \mathcal{C D}\right.$ for $\left.D_{1}, D_{2} \geq 0\right\}$,
and its distortion region:
$\mathcal{D} \triangleq\left\{\left(D_{1}, D_{2}\right):\left(R_{1}, R_{2}, D_{1}, D_{2}\right) \in \mathcal{C D}\right.$ for $\left.R_{1}, R_{2} \geq 0\right\}$.(6)
Before presenting our results on the tradeoff region $\mathcal{C D}$ in the following sections, we describe the optimal choice of the estimators $h_{1}$ and $h_{2}$.

Lemma 1. For $k=1,2$ and any $i=1, \ldots, n$, whenever $X_{i}=x$ and $Z_{i}=z$, the optimal estimator $h_{k}$ that minimizes the average expected distortion $\Delta_{k}^{(n)}$ is given by

$$
\begin{equation*}
\hat{s}_{k, i}^{*}(x, z) \triangleq \arg \min _{s^{\prime} \in \hat{\mathcal{S}}_{k}} \sum_{s_{k} \in \mathcal{S}_{k}} P_{S_{k, i} \mid X_{i} Z_{i}}\left(s_{k} \mid x, z\right) d\left(s_{k}, s^{\prime}\right) \tag{7}
\end{equation*}
$$

In above definition (7), ties can be broken arbitrarily.
Notice that the lemma implies in particular that a symbolwise estimator that estimates $S_{k, i}$ only based on $\left(X_{i}, Z_{i}\right)$ is optimal; there is no need to resort to previous or past observations $\left(X^{i-1}, Z^{i-1}\right)$ or $\left(X_{i+1}^{n}, Z_{i+1}^{n}\right)$.

Proof of Lemma [7: Recall that $\hat{S}_{k}^{n}$ is a function of $X^{n}, Z^{n}$ and write for each $i=1, \cdots, n$ :

$$
\begin{align*}
& \mathbb{E}\left[d_{k}\left(S_{k, i}, \hat{S}_{k, i}\right)\right] \\
& \quad=\mathbb{E}_{X^{n}, Z^{n}}\left[\mathbb{E}\left[d_{k}\left(S_{k, i}, \hat{S}_{k, i}\right) \mid X^{n}, Z^{n}\right]\right]  \tag{8}\\
& \stackrel{(\mathrm{a})}{=} \sum_{x^{n}, z^{n}} P_{X^{n} Z^{n}}\left(x^{n}, z^{n}\right) \sum_{\hat{s}_{k} \in \mathcal{S}_{k}} P_{\hat{S}_{k, i} \mid X^{n} Z^{n}}\left(\hat{s}_{k} \mid x^{n}, z^{n}\right) \\
&  \tag{9}\\
& \quad \cdot \sum_{s_{k}} P_{S_{k, i} \mid X_{i} Z_{i}}\left(s_{k} \mid x_{i}, z_{i}\right) d\left(s_{k}, \hat{s}_{k}\right) \\
& \\
& \geq \sum_{x^{n}, z^{n}} P_{X^{n} Z^{n}}\left(x^{n}, z^{n}\right)  \tag{10}\\
& \quad \cdot \min _{\hat{s}_{k} \in \mathcal{S}_{k}} \sum_{s_{k}} P_{S_{k, i} \mid X_{i} Z_{i}}\left(s_{k} \mid x_{i}, z_{i}\right) d\left(s_{k}, \hat{s}_{k}\right) \\
& \\
& =\mathbb{E}\left[d\left(S_{k, i}, \hat{s}_{k, i}^{*}\left(X_{i}, Z_{i}\right)\right)\right]
\end{align*}
$$

where ( $a$ ) holds by the Markov chain

$$
\left(X^{i-1}, X_{i+1}^{n}, Z^{i-1}, Z_{i+1}^{n}, \hat{S}_{k, i}\right) \multimap\left(X_{i}, Z_{i}\right) \multimap S_{k, i}
$$

## III. Absence of Rate-Distortion Tradeoff

We first consider degenerate cases where the rate-distortion region $\mathcal{C D}$ is given by the Cartesian product between the capacity region $\mathcal{C}$ and the distortions region $\mathcal{D}$.

Proposition 2 (No Rate-Distortion Tradeoff). Consider a SDMBC $\quad\left(P_{S_{1} S_{2}}, P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}\right)$ and let $\left(X, S_{1}, S_{2}, Y_{1}, Y_{2}, Z\right) \sim P_{X} P_{S_{1} S_{2}} P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}$ for a given input law $P_{X}$. If there exist functions $\psi_{1}$ and $\psi_{2}$ with domain $\mathcal{Z}$ such that for all $P_{X}$ the Markov chains

$$
\begin{gather*}
\left(S_{k}, \psi_{k}(Z)\right) \perp X  \tag{11}\\
S_{k} \multimap \psi_{k}(Z) \multimap(Z, X), \quad k \in\{1,2\}, \tag{12}
\end{gather*}
$$

hold, then for the SDMBC under consideration:
$\mathcal{C D}=\mathcal{C} \times \mathcal{D}$.

In this case, there is no tradeoff between the achievable rate pairs $\left(R_{1}, R_{2}\right)$ and the achievable distortion pairs $\left(D_{1}, D_{2}\right)$.

Proof: Notice under the given Markov chain (12):

$$
\begin{equation*}
P_{S_{k, i} \mid X_{i} Z_{i}}\left(s_{k} \mid x, z\right)=P_{S_{k, i} \mid \psi_{k}\left(Z_{i}\right)}\left(s_{k} \mid \psi_{k}(z)\right) \tag{14}
\end{equation*}
$$

Trivially, $\mathcal{C D} \subseteq \mathcal{C} \times \mathcal{D}$. To see that also $\mathcal{C D} \supseteq \mathcal{C} \times \mathcal{D}$ holds, notice that by (14) and Lemma 1 the optimal estimators depend only on the sequences $\left\{\psi_{k}\left(Z_{i}\right)\right\}_{i=1}^{n}$, for $k=1,2$, and thus by (11) are independent of the considered coding scheme and the produced inputs.

In the following corollary, The following example satisfies conditions (11) and (12) in Proposition 2 for an appropriate choice of $\psi_{1}$ and $\psi_{2}$.

## A. Example: Erasure BC with Noisy Feedback

Let the joint law $\mathrm{P}_{S_{1} S_{2} E_{1} E_{2}}\left(s_{1}, s_{2}, e_{1}, e_{2}\right)$ over $\{0,1\}^{4}$ be arbitrary but given, and $\left(E_{1}, E_{2}, S_{1}, S_{2}\right) \sim \mathrm{P}_{S_{1} S_{2} E_{1} E_{2}}$. Consider the state-dependent erasure BC

$$
Y_{k}=\left\{\begin{array}{c}
X  \tag{15}\\
\text { if } S_{k}=0, \\
?
\end{array} \quad k \in\{1,2\},\right.
$$

where the feedback signal $Z=\left(Z_{1}, Z_{2}\right)$ is given by

$$
Z_{k}=\left\{\begin{array}{l}
Y_{k} \text { if } E_{k}=0,  \tag{16}\\
? \quad \text { if } E_{k}=1,
\end{array} \quad k \in\{1,2\}\right.
$$

Further consider the Hamming distortion measure $d_{k}(s, \hat{s})=$ $s \oplus \hat{s}$, for $k=1,2$. For the choice

$$
\psi_{k}(Z)= \begin{cases}1, & \text { if } Z_{k}=?  \tag{17}\\ 0, & \text { else }\end{cases}
$$

the described SDMBC satisfies the conditions in Proposition 2 and its capacity-distortion region is thus given by

$$
\begin{equation*}
\mathcal{C D}=\mathcal{C} \times \mathcal{D} \tag{18}
\end{equation*}
$$

Remark 1. For the case of output feedback $Z=\left(Y_{1}, Y_{2}\right)$ or $E_{1}=E_{2}=0$, the transmitter can perfectly estimate the state ( $S_{1}, S_{2}$ ), yielding $D_{1}=D_{2}=0$ regardless of the rate pair $\left(R_{1}, R_{2}\right) \in \mathcal{C}$. The capacity region $\mathcal{C}$ of the erasure broadcast channel with output feedback is still unknown in general.

## IV. Physically Degraded BCs

In this section, by focusing on the physically degraded SDMBC, we fully characterize the capacity-distortion region. Then, we discuss two binary physically degraded SDMBCs to illustrate the rate-distortion tradeoff between the two receivers.
Definition 2. An SDMBC $\left(P_{S_{1} S_{2}}, P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}\right)$ is called physically degraded if there are conditional laws $P_{Y_{1} \mid X S_{1}}$ and $P_{Y_{2} S_{2} \mid S_{1} Y_{1}}$ such that

$$
\begin{equation*}
P_{Y_{1} Y_{2} \mid S_{1} S_{2} X} P_{S_{1} S_{2}}=P_{S_{1}} P_{Y_{1} \mid S_{1} X} P_{Y_{2} S_{2} \mid S_{1} Y_{1}} \tag{19}
\end{equation*}
$$

That means for any arbitrary input $P_{X}$, if a tuple $\left(X, S_{1}, S_{2}, Y_{1}, Y_{2}\right) \sim P_{X} P_{S_{1} S_{2}} P_{Y_{1} Y_{2} \mid S_{1} S_{2} X}$, then it satisfies the Markov chain

$$
\begin{equation*}
X \multimap\left(S_{1}, Y_{1}\right) \multimap\left(S_{2}, Y_{2}\right) \tag{20}
\end{equation*}
$$

Proposition 3. The capacity-distortion region $\mathcal{C D}$ of $a$ physically degraded SDMBC is the closure of the set of all quadruples $\left(R_{1}, R_{2}, D_{1}, D_{2}\right)$ for which there exists a joint law $P_{U X}$ so that the tuple $\left(U, X, S_{1}, S_{2}, Y_{1}, Y_{2}, Z\right) \sim$ $P_{U X} P_{S_{1} S_{2}} P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}$ satisfies the two rate constraints

$$
\begin{align*}
& R_{1} \leq I\left(U ; Y_{1} \mid S_{1}\right)  \tag{21}\\
& R_{2} \leq I\left(X ; Y_{2} \mid S_{2}, U\right) \tag{22}
\end{align*}
$$

and the distortion constraints

$$
\begin{equation*}
\left.\mathbb{E}\left[d_{k}\left(S_{k}, \hat{s}_{k}^{*}(X, Z)\right)\right)\right] \leq D_{k}, \quad k \in\{1,2\} \tag{23}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{s}_{k}^{*}(x, z) \triangleq \arg \min _{s^{\prime} \in \hat{\mathcal{S}}_{k}} \sum_{s_{k} \in \mathcal{S}_{k}} P_{S_{k} \mid X Z}\left(s_{k} \mid x, z\right) d\left(s_{k}, s^{\prime}\right) \tag{24}
\end{equation*}
$$

Proof: The converse follows as a special case of Theorem 6 ahead where one can ignore constraints (35c) and (35d). Notice that constraint (35b is equivalent to (22) because ( $U, X$ ) is independent of $\left(S_{1}, S_{2}\right)$ and because for a physically degraded DMBC the Markov chain (20) holds.

Achievability is obtained by simple superposition coding and using the optimal estimator described in Lemma 1 .

We consider two binary state-dependent channels. For the binary states, we consider the Hamming distortion measure.

## A. Example: Binary BC with Multiplicative States

Consider the physically degraded SDMBC with binary input/output alphabets $\mathcal{X}=\mathcal{Y}_{1}=\mathcal{Y}_{2}=\{0,1\}$ and binary state alphabets $\mathcal{S}_{1}=\mathcal{S}_{2}=\{0,1\}$. The channel input-output relation is described by

$$
\begin{equation*}
Y_{k}=X \cdot S_{k}, \quad k=1,2 \tag{25}
\end{equation*}
$$

with the joint state pmf

$$
P_{S_{1} S_{2}}\left(s_{1}, s_{2}\right)= \begin{cases}1-q, & \text { if }\left(s_{1}, s_{2}\right)=(0,0)  \tag{26}\\ 0, & \text { if }\left(s_{1}, s_{2}\right)=(0,1) \\ q \cdot \gamma, & \text { if }\left(s_{1}, s_{2}\right)=(1,1) \\ q \cdot(1-\gamma) & \text { if }\left(s_{1}, s_{2}\right)=(1,0)\end{cases}
$$

for $\gamma, q \in[0,1]$. Notice that $S_{2}$ is a degraded version of $S_{1}$. We consider output feedback $Z=\left(Y_{1}, Y_{2}\right)$.
Corollary 4. The capacity-distortions region $\mathcal{C D}$ of the binary physically degraded SDMBC in (25)-(26) is the set of all quadruples $\left(R_{1}, R_{2}, D_{1}, D_{2}\right)$ satisfying

$$
\begin{align*}
& R_{1} \leq q \cdot H_{\mathrm{b}}(p) \cdot r  \tag{27a}\\
& R_{2} \leq \gamma \cdot q \cdot H_{\mathrm{b}}(p) \cdot(1-r)  \tag{27b}\\
& D_{1} \geq(1-p) \cdot \min \{q, 1-q\}  \tag{27c}\\
& D_{2} \geq(1-p) \cdot \min \{\gamma \cdot q, 1-\gamma \cdot q\} \tag{27d}
\end{align*}
$$

for some choice of the parameters $r, p \in[0,1]$.
Proof. It suffices to evaluate the rate-constraints (21) and (22) for $X=V \oplus U$ when $U$ and $V$ are independent Bernoulli distributed random variables. In (27), we choose the parameter


Fig. 2. Boundary of the capacity-distortion region $\mathcal{C D}$ for the example in Subsection IV-A
$p=\operatorname{Pr}[X=1]$ and $r=1-\frac{H(V)}{H_{b}(p)}$. To calculate the distortion, we determine the optimal estimator $\hat{s}_{k}^{*}\left(x, y_{1}, y_{2}\right)$ from (24) as

$$
\begin{align*}
\hat{s}_{k}^{*}\left(1, y_{1}, y_{2}\right) & =y_{k}  \tag{28a}\\
\hat{s}_{k}^{*}\left(0, y_{1}, y_{2}\right) & =\mathbf{1}\left\{P_{S_{k}}(1)>1 / 2\right\} \tag{28b}
\end{align*}
$$

Remark 2. Fixing $r=1$, the capacity-distortion region in (27) reduces to the capacity-distortion tradeoff of a single user channel [3] Proposition 1]. Similarly to the single-user case, we observe the tension between the minimum distortion by choosing $p=1$ (always sending $X=1$ ) and the maximum rate by choosing $p=1 / 2$. In the $B C$, the resource is shared between the two users via the time-sharing parameter $r$.

We evaluate the capacity-distortion region (27) for $\gamma=0.5$ and $q=0.6$. Fig. 2 shows in red colour the dominant boundary points of the projection of the tradeoff region $\mathcal{C D}$ onto the 3 -dimensional plane $\left(R_{1}, R_{2}, D_{1}\right)$. The tradeoff with $D_{2}$ is omitted because $D_{2}$ is a scaled version of $D_{1}$.

It is worth comparing the capacity-distortion region $\mathcal{C D}$, achieved by the proposed co-design scheme that uses a common waveform for both sensing and communication tasks, with the rate-distortion region achieved by a baseline scheme, called resource splitting, that separates the two tasks into two modes. In the sensing mode, the transmitter estimates the states via the feedback but does not communicate any messages to the two receivers. In the communication mode, it communicates with the receivers but without using the feedback. Moreover, in this second mode, it also estimates the states but again without accessing the feedback.

For the example at hand, the resource splitting scheme acts as follows. During the sensing mode, the transmitter always sends $X=1$ (which is equivalent to setting $p=1$ in (27)) so as to minimize the distortion. This achieves

$$
\begin{equation*}
\left(R_{1}, R_{2}, D_{1}, D_{2}\right)=(0,0,0,0) \tag{29}
\end{equation*}
$$

During the communication mode, the transmitter sets $p=0.5$
in 27 so as to maximize the communication rate and without using the feedback it estimates the states as $\hat{s}_{1}=\mathbf{1}\{q>0.5\}$ and $\hat{s}_{2}=\mathbf{1}\{q \cdot \gamma>0.5\}$. This achieves

$$
\begin{equation*}
\left(R_{1}, R_{2}, D_{1}, D_{2}\right)=\left(q \cdot r, \gamma \cdot q \cdot(1-r), D_{1, \max }, D_{2, \max }\right) \tag{30}
\end{equation*}
$$

where $D_{1, \max }=\min \{q, 1-q\}$ and $D_{2, \max }=\min \{\gamma \cdot q, 1-$ $\gamma \cdot q\}$, and where $r \in[0,1]$ denotes the time-sharing parameter between the two two communication rates. Fig. 2 shows the time-sharing region between the two modes (29) and (30) in blue colour.

Fig. 2 also shows the region achieved by a more sophisticated time-sharing scheme that combines the minimum distortion point $\left(R_{1}, R_{2}, D_{1}, D_{2}\right)=(0,0,0,0)$ of the capacitydistortion region $\mathcal{C D}$ with the maximum communication rate points of $\mathcal{C D},\left(R_{1}, R_{2}, D_{1}, D_{2}\right)=(q \cdot r, \gamma \cdot q \cdot(1-$ $\left.r), \frac{D_{1, \text { max }}}{2}, \frac{D_{2, \text { max }}}{2}\right)$ for $r \in[0,1]$.

We observe that both resource splitting and time sharing approaches fail to achieve the entire region $\mathcal{C D}$.

So far, there was no tradeoff between the two distortion constraints $D_{1}$ and $D_{2}$. This is different in the next example, which otherwise is very similar.

## B. Example: Binary BC with Flipping Inputs

Reconsider the same state pmf $P_{S_{1} S_{2}}$ as in the previous example, but now a SDMBC with transition law

$$
\begin{equation*}
Y_{1}=X \cdot S_{1}, \quad Y_{2}=(1-X) \cdot S_{2} \tag{31}
\end{equation*}
$$

Consider output feedback $Z=\left(Y_{1}, Y_{2}\right)$.
Corollary 5. The capacity-distortion region $\mathcal{C D}$ of the binary SDMBC with flipping inputs in (31) and output feedback is the set of all quadruples $\left(R_{1}, R_{2}, D_{1}, D_{2}\right)$ satisfying

$$
\begin{align*}
& R_{1} \leq q \cdot H_{\mathrm{b}}(p) \cdot r  \tag{32a}\\
& R_{2} \leq \gamma \cdot q \cdot H_{\mathrm{b}}(p) \cdot(1-r)  \tag{32b}\\
& D_{1} \geq(1-p) \cdot \min \{q(1-\gamma),(1-q)\}  \tag{32c}\\
& D_{2} \geq p \cdot q \min \{\gamma, 1-\gamma\} \tag{32d}
\end{align*}
$$

for some choice of the parameters $r, p \in[0,1]$.
Proof. To achieve this region, we can consider the same choices of $(U, X)$ as in the previous example. The optimal estimators are given by 28a) for receiver 1 and

$$
\begin{align*}
& \hat{s}_{2}^{*}\left(0, y_{1}, y_{2}\right)=y_{2}  \tag{33a}\\
& \hat{s}_{2}^{*}\left(1, y_{1}, y_{2}\right)=\mathbf{1}\left\{P_{S_{2}}(1)>1 / 2\right\} \tag{33b}
\end{align*}
$$

for receiver 2. Contrary to the previous example, we observe a tradeoff between the achievable distortions $D_{1}$ and $D_{2}$.

## V. General BCs

## A. General Bounds

Reconsider the general SDMBC (not necessarily physically degraded). We provide an inner and an outer bound on the capacity-distortion region.

[^0]\[

$$
\begin{align*}
R_{1} & \leq I\left(U_{0}, U_{1} ; Y_{1}, V_{1} \mid S_{1}\right)-I\left(U_{0}, U_{1}, U_{2}, Z ; V_{0}, V_{1} \mid S_{1}, Y_{1}\right)  \tag{34a}\\
& R_{2} \leq I\left(U_{0}, U_{2} ; Y_{2}, V_{2} \mid S_{2}\right)-I\left(U_{0}, U_{1}, U_{2}, Z ; V_{0}, V_{2} \mid S_{2}, Y_{2}\right)  \tag{34b}\\
R_{1}+ & R_{2} \leq I\left(U_{1} ; Y_{1}, V_{1} \mid U_{0}, S_{1}\right)+I\left(U_{2} ; Y_{2}, V_{2} \mid U_{0}, S_{2}\right)+\min _{i \in\{1,2\}} I\left(U_{0} ; Y_{i}, V_{i} \mid S_{i}\right)-I\left(U_{1} ; U_{2} \mid U_{0}\right) \\
- & I\left(U_{0}, U_{1}, U_{2}, Z ; V_{1} \mid V_{0}, S_{1}, Y_{1}\right)-I\left(U_{0}, U_{1}, U_{2}, Z ; V_{2} \mid V_{0}, S_{2}, Y_{2}\right)-\max _{i \in\{1,2\}} I\left(U_{0}, U_{1}, U_{2}, Z ; V_{0} \mid S_{i}, Y_{i}\right) \tag{34c}
\end{align*}
$$
\]

Theorem 6. If $\left(R_{1}, R_{2}, D_{1}, D_{2}\right)$ is achievable on a SDMBC $\left(P_{S_{1} S_{2}}, P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}\right)$, then there exists for each $k=$ 1,2 a conditional pmf $P_{U_{k} \mid X}$ such that the random tuple $\left(U_{k}, X, S_{1}, S_{2}, Y_{1}, Y_{2}, Z\right) \sim P_{U_{k} \mid X} P_{X} P_{S_{1} S_{2}} P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}$ satisfies the rate constraints

$$
\begin{align*}
R_{1} & \leq I\left(U_{1} ; Y_{1} \mid S_{1}\right)  \tag{35a}\\
R_{1}+R_{2} & \leq I\left(X ; Y_{1}, Y_{2} \mid S_{1}, S_{2}, U_{1}\right)  \tag{35b}\\
R_{1}+R_{2} & \leq I\left(X ; Y_{1}, Y_{2} \mid S_{1}, S_{2}, U_{2}\right)  \tag{35c}\\
R_{2} & \leq I\left(U_{2} ; Y_{2} \mid S_{2}\right) \tag{35d}
\end{align*}
$$

and the average distortion constraints

$$
\begin{equation*}
\left.\mathbb{E}\left[d_{k}\left(S_{k}, \hat{s}_{k}^{*}(X, Z)\right)\right)\right] \leq D_{k}, \quad k \in\{1,2\} \tag{36}
\end{equation*}
$$

where the function $\hat{s}_{k}^{*}(\cdot, \cdot)$ is defined in (24).
Proof: See Appendix A
Achievability results are easily obtained by combining existing achievability results for SDMBCs with generalized feedback with the optimal estimator in Lemma 1 For example, based on [5] we obtain:
Proposition 7. Consider a $\operatorname{SDMBC}\left(P_{S_{1} S_{2}}, P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}\right)$. For any (conditional) pmfs $P_{U_{0} U_{1} U_{2} X}$ and $P_{V_{0} V_{1} V_{2} \mid U_{0} U_{1} U_{2} Z}$ and tuple $\left(U_{0}, U_{1}, U_{2}, X, S_{1}, S_{2}, Y_{1}, Y_{2}, Z, V_{0}, V_{1}, V_{2}\right) \sim$ $P_{U_{0} U_{1} U_{2} X} P_{S_{1} S_{2}} P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X} P_{V_{0} V_{1} V_{2} \mid U_{0} U_{1} U_{2} Z}$, the convex closure of the set of all quadruples $\left(R_{1}, R_{2}, D_{1}, D_{2}\right)$ satisfying inequalities (34) on top of the this page and the distortion constraints

$$
\begin{equation*}
\left.\mathbb{E}\left[d_{k}\left(S_{k}, \hat{s}_{k}^{*}(X, Z)\right)\right)\right] \leq D_{k}, \quad k \in\{1,2\} \tag{37}
\end{equation*}
$$

for $\hat{s}_{k}^{*}(\cdot, \cdot)$ defined in (24), is achievable.
B. Example: Dueck's BC with Binary States


Fig. 3. A state-dependent version of Dueck's BC.

Consider the state-dependent version of Dueck's BC [6] in Figure 3 with input $X=\left(X_{0}, X_{1}, X_{2}\right) \in\{0,1\}^{3}$ and outputs

$$
\begin{equation*}
Y_{k}=\left(X_{0}, Y_{k}^{\prime}, S_{1}, S_{2}\right), \quad k \in\{1,2\} \tag{38}
\end{equation*}
$$

for states $S_{1}, S_{2} \in\{0,1\}$,

$$
\begin{equation*}
Y_{k}^{\prime}=S_{k}\left(X_{k} \oplus N\right), \quad k \in\{1,2\} \tag{39a}
\end{equation*}
$$

and $N$ a Bernoulli- $\frac{1}{2}$ noise independent of the inputs and the states. Assume i.i.d. states such that $P_{S_{1} S_{2}}\left(s_{1}, s_{2}\right)=$ $P_{S}\left(s_{1}\right) P_{S}\left(s_{2}\right)$ for a given pmf $P_{S}$. The feedback signal is

$$
\begin{equation*}
Z=\left(Y_{1}^{\prime}, Y_{2}^{\prime}\right) \tag{40}
\end{equation*}
$$

Notice that in this example, only the input bits $X_{1}$ and $X_{2}$ are corrupted by the state and the noise, but not $X_{0}$. This latter is thus completely useless for sensing. In fact, as we will show, for sensing it is optimal to choose $X_{0}$ arbitrary and depending on the state distribution either $X_{1}=X_{2}$ or $X_{1} \neq X_{2}$. In contrast, for communication without feedback, it is optimal to send uncoded bits using $X_{0}$ and to disregard the other two input bits $X_{1}$ and $X_{2}$. The baseline resource splitting scheme (where feedback is only used for sensing) thus orthogonalizes the inputs: $X_{0}$ is used for communication and $X_{1}, X_{2}$ are used for sensing. In a traditional resource splitting scheme, the two modes are never combined, which for this example is clearly suboptimal because both modes (sensing and communication) can be performed simultaneously without disturbing each other. As we will see, in certain cases (depending on the state distribution $P_{S}$ ) the simple approach that performs both resource splitting modes simultaneously is optimal when one insists on achieving the smallest possible distortions. For larger distortions, it can however be improved by also exploiting the feedback and the inputs $X_{1}$ and $X_{2}$ for communication. This is for example achieved by the scheme leading to Propostion 7 as we show in the following corollary and the subsequent numerical evaluation.

Corollary 8. The capacity-distortion region $\mathcal{C D}$ of Dueck's state-dependent $B C$ is included in the set of quadruples $\left(R_{1}, R_{2}, D_{1}, D_{2}\right)$ that for some choice of the parameters $p, q, \beta \in[0,1]$ satisfy the rate-constraints

$$
\begin{align*}
& R_{1} \leq 1-p  \tag{41a}\\
& R_{2} \leq p+\left(P_{S}(1)\right)^{2} \cdot H_{\mathrm{b}}(\beta)  \tag{41b}\\
& R_{1} \leq q+\left(P_{S}(1)\right)^{2} \cdot H_{\mathrm{b}}(\beta)  \tag{41c}\\
& R_{2} \leq 1-q \tag{41~d}
\end{align*}
$$

and for each $k \in\{1,2\}$ the distortion constraint

$$
\begin{align*}
D_{k} \geq & \frac{1}{2}(1-\beta) \cdot \min \left\{P_{S}(1), P_{S}(0) \cdot\left(1+P_{S}(0)\right)\right\} \\
& +\frac{1}{2} \beta P_{S}(1)\left[P_{S}(0)+\min \left\{P_{S}(0), P_{S}(1)\right\}\right] \tag{41e}
\end{align*}
$$

Moreover, depending on the values of $P_{S}(0)$ and $P_{S}(1)$, the following holds:

- When $P_{S}(1) \leq P_{S}(0)$, distortion constraint 41e simplifies to $D_{k} \geq \frac{1}{2} P_{S}(1)=D_{\min }$ and one can restrict to $\beta=1$ in above outer bound. In this case,

$$
\begin{equation*}
\mathcal{C D}=\mathcal{C} \times \mathcal{D} \tag{42}
\end{equation*}
$$

and the outer bound in (41) coincides with $\mathcal{C D}$.

- When $P_{S}(0)\left(1+P_{S}(0)\right) \geq P_{S}(1)>P_{S}(0)$, the smallest achievable distortion in (41e) (obtained for $\beta=1$ ) is $D_{\min }=P_{S}(1) P_{S}(0)$. Moreover, the region $\mathcal{C D}$ includes the set of all quadruples $\left(R_{1}, R_{2}, D_{1}, D_{2}\right)$ that for some $\beta, \gamma \in[0,1]$ satisfy the rate-constraints

$$
\begin{align*}
R_{k} & \leq 1, \quad k \in\{1,2\} \\
R_{1}+R_{2} & \leq 1+\gamma P_{S}(1)\left(H_{\mathrm{b}}\left(1-\frac{1-\beta}{\gamma}\right)-P_{S}(0)\right) \tag{43b}
\end{align*}
$$

and the distortion constraints in 41e), which simplify to

$$
\begin{equation*}
D_{k} \geq \frac{1}{2}(1-\beta) P_{S}(1)+\beta P_{S}(1) P_{S}(0), \quad k=1,2 \tag{44}
\end{equation*}
$$

- When $P_{S}(1)>P_{S}(0)\left(1+P_{S}(0)\right)$, the smallest achievable distortion in 41e) (obtained for $\beta=0$ ) is $D_{\text {min }}=$ $\frac{1}{2} P_{S}(0)\left(1+P_{S}(0)\right)$. Moreover, the region $\mathcal{C D}$ includes the set of all quadruples $\left(R_{1}, R_{2}, D_{1}, D_{2}\right)$ that for some $\gamma \in[0,1]$ and $\beta \in[0, \gamma]$ satisfy the rate-constraints

$$
\begin{align*}
R_{k} & \leq 1, \quad k \in\{1,2\}  \tag{45a}\\
R_{1}+R_{2} & \leq 1+\gamma P_{S}(1)\left(H_{\mathrm{b}}\left(\frac{\beta}{\gamma}\right)-P_{S}(0)\right) \tag{45b}
\end{align*}
$$

and the distortion constraints in (41e), which simplify to

$$
\begin{equation*}
D_{k} \geq \frac{1}{2}(1-\beta) P_{S}(0)\left(1+P_{S}(0)\right)+\beta P_{S}(1) P_{S}(0), \quad k=1,2 \tag{46}
\end{equation*}
$$

Proof: Based on Theorem 6 and Proposition 7 See Appendix B for details.

We evaluate the bounds for the state distribution $P_{S}(1)=$ $\frac{3}{4}$ and $P_{S}(0)=\frac{1}{4}$, which satisfies the condition $P_{S}(1) \geq$ $P_{S}(0)\left(1+P_{S}(0)\right)$. Specifically, we analyze the largest sumrates $R_{\Sigma}(D):=R_{1}+R_{2}$ that our inner and outer bounds admit under given symmetric distortion constraints $D_{1}=D_{2}=D$, and compare them to the baseline schemes. Notice first that for $P_{S}(1)=\frac{3}{4}$ and $P_{S}(0)=\frac{1}{4}$ the distortion constraint 41e) specializes to

$$
\begin{equation*}
D \geq \frac{1}{2}\left[(1-\beta) \frac{5}{16}+\beta \frac{6}{16}\right]=\frac{5+\beta}{32} \tag{47}
\end{equation*}
$$

and so the minimum distortion (obtained for $\beta=0$ )


Fig. 4. Upper and lower bounds of Corollary 8 on the maximum achievable sum-rate $R_{\Sigma}$ in function of the admissible distortion $D_{1}=D_{2}=D$ for the state-dependent Dueck BC when $P_{S}(1)=3 / 4$ and $P_{S}(0)=1 / 4$.
is $D_{\text {min }}=\frac{5}{32}$. For $\beta=1 / 2$ we obtain $D \geq \frac{11}{64}$. Turning back to the sum-rate $R_{\Sigma}$, for above state distribution, the outer bound (41) implies

$$
R_{\Sigma}(D) \leq \begin{cases}1+\left(\frac{3}{4}\right)^{2} H_{\mathrm{b}}(32 \cdot D-5), & \text { if } \frac{5}{32} \leq D \leq \frac{11}{64}  \tag{48}\\ \frac{25}{16}, & \text { if } D \geq \frac{11}{64}\end{cases}
$$

and the inner bound (45) implies

$$
\begin{align*}
& R_{\Sigma}(D) \geq \\
& \left\{\begin{array}{lr}
1+\max _{32 D-5 \leq \gamma \leq 1} \frac{3 \gamma}{4}\left(H_{\mathrm{b}}\left(\frac{32 D-5}{\gamma}\right)-\frac{1}{4}\right) \\
\frac{\text { if } \frac{5}{32} \leq D \leq \frac{11}{64}}{\frac{25}{16},} & \text { if } D \geq \frac{11}{64}
\end{array}\right. \tag{49}
\end{align*}
$$

Fig. 4 compares these two bounds to the maximum admissible sum-rates $R_{\Sigma}$ attained by the resource splitting baseline scheme, and by time-sharing the two points of our lower bound (49) that have minimum distortion $\left(R_{\Sigma}=1, D=D_{\min }=\right.$ $5 / 32$ ) and maximum rate ( $R_{\Sigma}=25 / 16, D=11 / 64$ ).

The resource splitting scheme achieves $\left(R_{\Sigma}=0, D=\right.$ $D_{\text {min }}=5 / 32$ ) during the sensing mode, by setting $X_{1}=X_{2}$ (either 0 or 1) and not using input $X_{0}$ at all. (This input is useless for state sensing.) Moreover, it achieves $\left(R_{\Sigma}=\right.$ $1, D=1 / 4$ ) in the communication mode, by completely ignoring the feedback, sending uncoded bits using inputs $X_{0}$, and estimating $\hat{S}_{1}=\hat{S}_{2}=1$. (This estimator is optimal without feedback because $P_{S}(1)>P_{S}(0)$.)

## VI. Conclusion

Motivated by a joint radar and communication system, we studied joint sensing and communication over memoryless
state-dependent broadcast channels (BC). First, we presented a sufficient condition under which there is no tradeoff between sensing and communication. Then, we characterized the capacity-distortion tradeoff region of the physically degraded BC. We further presented inner and outer bounds on the capacity-distortion region of general BCs with states and showed at hand of an example that they can be tight. Our numerical examples demonstrate that the proposed codesign schemes significantly outperforms the traditional coexist scheme where resources are split between communication and state sensing.

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## Appendix A <br> Proof of Theorem 6

Fix a sequence of $\left(2^{n R_{1}}, 2^{n R_{2}}, n\right)$ codes satisfying (4). Fix a blocklenth $n$ and start with Fano's inequality:

$$
\begin{align*}
R_{1} & =\frac{1}{n} H\left(W_{1}\right) \\
& \leq \frac{1}{n} \sum_{i=1}^{n} I\left(W_{1} ; Y_{1 i}, S_{1 i} \mid Y_{1}^{i-1}, S_{1}^{i-1}\right)+\epsilon_{n} \\
& \leq \frac{1}{n} \sum_{i=1}^{n} I\left(W_{1}, Y_{1}^{i-1}, S_{1}^{i-1} ; Y_{1, i}, S_{1, i}\right)+\epsilon_{n} \\
& =I\left(W_{1}, Y_{1}^{T-1}, S_{1}^{T-1} ; S_{1, T}, Y_{1, T} \mid T\right)+\epsilon_{n} \\
& \leq I\left(W_{1}, Y_{1}^{T-1}, S_{1}^{T-1}, T ; S_{1, T}, Y_{1, T}\right)+\epsilon_{n} \\
& =I\left(U ; Y_{1} \mid S_{1}\right)+\epsilon_{n} \tag{50}
\end{align*}
$$

where $T$ is chosen uniformly over $\{1, \cdots, n\}$ and independent of $X^{n}, Y_{1}^{n}, Y_{2}^{n}, W_{1}, W_{2}, S_{1}^{n}, S_{2}^{n} ; \epsilon_{n}$ is a function that tends to 0 as $n \rightarrow \infty ; U \triangleq\left(W_{1}, Y_{1}^{T-1}, S_{1}^{T-1}, T\right)$; and $Y_{1} \triangleq Y_{1, T}$ and $S_{1} \triangleq S_{1, T}$. Notice that $S_{1} \sim P_{S_{1}}$ and it is independent of $(U, X)$, where we define $X \triangleq X_{T}$.

Following similar steps, we obtain:
$R_{2}=\frac{1}{n} H\left(W_{2}\right)$

$$
\begin{align*}
& \leq \frac{1}{n} I\left(W_{2} ; Y_{2}^{n}, S_{2}^{n}\right)+\epsilon_{n} \\
& \stackrel{(a)}{\leq} \frac{1}{n} I\left(W_{2} ; Y_{1}^{n}, S_{1}^{n}, Y_{2}^{n}, S_{2}^{n} \mid W_{1}\right)+\epsilon_{n} \\
& =\frac{1}{n} \sum_{i=1}^{n} I\left(W_{2} ; Y_{1 i}, Y_{2 i}, S_{1 i}, S_{2 i} \mid Y_{1}^{i-1}, Y_{2}^{i-1}\right. \text {, } \\
& \left.S_{1}^{i-1}, S_{2}^{i-1}, W_{1}\right)+\epsilon_{n} \\
& \leq \frac{1}{n} \sum_{i=1}^{n} I\left(X_{i}, W_{2}, Y_{2}^{i-1}, S_{2}^{i-1} ; Y_{1, i}, Y_{2, i}, S_{1, i}, S_{2, i}\right. \\
& \left.\mid Y_{1}^{i-1}, S_{1}^{i-1}, W_{1}\right)+\epsilon_{n} \\
& =\frac{1}{n} \sum_{i=1}^{n} I\left(X_{i} ; Y_{1, i}, Y_{2, i}, S_{1, i}, S_{2, i} \mid Y_{1}^{i-1}, S_{1}^{i-1}, W_{1}\right)+\epsilon_{n} \\
& =I\left(X_{T} ; Y_{1 T}, Y_{2 T}, S_{1, T}, S_{2, T} \mid Y_{1}^{T-1},\right. \\
& \left.S_{1}^{T-1}, W_{1}, T\right)+\epsilon_{n} \\
& =I\left(X ; Y_{1}, Y_{2} \mid S_{1}, S_{2}, U\right)+\epsilon_{n}, \tag{51}
\end{align*}
$$

where (a) follows by the physically degradedness of the SDMBC and where we defined $Y_{2} \triangleq Y_{2, T}$ and $S_{2} \triangleq S_{2, T}$.

Recall that we assume the optimal estimators (7) in Lemma 1 Using the definitions of $T, X, S_{k}$ above and defining $Z \triangleq Z_{T}$, we can write the average expected distortions as:

$$
\begin{equation*}
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[d_{k}\left(S_{k, i}, \hat{s}_{k, i}^{*}\left(X_{i}, Z_{i}\right)\right]=\mathbb{E}\left[d_{k}\left(S_{k}, \hat{s}_{k, T}^{*}(X, Z)\right]\right.\right. \tag{52}
\end{equation*}
$$

Combining (50), (51), and (52) and letting $n \rightarrow \infty$, we obtain that there exists a limiting pmf $P_{U X}$ such that the tuple $\left(U, X, S_{1}, S_{2}, Y_{1}, Y_{2}, Z\right) \sim P_{U X} P_{S_{1} S_{2}} P_{Y_{1} Y_{2} Z \mid S_{1} S_{2} X}$ satisfies the rate-constraints

$$
\begin{align*}
& R_{1} \leq I\left(U ; Y_{1} \mid S_{1}\right)  \tag{53}\\
& R_{2} \leq I\left(X ; Y_{1}, Y_{2} \mid S_{1}, S_{2}, U\right) \tag{54}
\end{align*}
$$

and the distortion constraints

$$
\begin{equation*}
\mathbb{E}\left[d_{k}\left(S_{k}, \hat{S}_{k, T}(X, Z)\right] \leq D_{k}, \quad k=1,2\right. \tag{55}
\end{equation*}
$$

for a possibly probabilistic estimator $\hat{S}_{k, T}(X, Z)$. Similar to the proof of Lemma 1 one can however show optimality of the estimator in (24). This complete the proof.

## Appendix B <br> Proof of Corollary 8

## A. Proof of the Outer Bound

The outer bound is based on Theorem 6, as detailed out in the following. From (35a and 35b we obtain:

$$
\begin{align*}
R_{1} & \leq I\left(U_{1} ; Y_{1}^{\prime}, X_{0} \mid S_{1}, S_{2}\right) \\
& =H\left(U_{1}\right)-H\left(U_{1} \mid Y_{1}^{\prime}, X_{0}, S_{1}, S_{2}\right) \\
& =H\left(U_{1}\right)-H\left(U_{1} \mid X_{0}\right) \\
& =I\left(U_{1} ; X_{0}\right) \\
& =H\left(X_{0}\right)-H\left(X_{0} \mid U_{1}\right) \\
& \leq 1-p \tag{56}
\end{align*}
$$

where we defined $p:=H\left(X_{0} \mid U_{1}\right)$, and

$$
\begin{align*}
R_{2} \leq & I\left(X_{0}, X_{1}, X_{2} ; Y_{1}^{\prime}, Y_{2}^{\prime} \mid S_{1}, S_{2}, U_{1}\right) \\
\leq & H\left(X_{0} \mid U_{1}\right)+I\left(X_{1}, X_{2} ; Y_{1}^{\prime}, Y_{2}^{\prime} \mid S_{1}, S_{2}, U_{1}\right)  \tag{65a}\\
= & H\left(X_{0} \mid U_{1}\right)+I\left(X_{1}, X_{2} ; Y_{2}^{\prime} \mid S_{1}, S_{2}, U_{1}\right)  \tag{65b}\\
& \quad+I\left(X_{1}, X_{2} ; Y_{1}^{\prime} \mid S_{1}, S_{2}, Y_{2}^{\prime}, U_{1}\right) \\
\leq & H\left(X_{0} \mid U_{1}\right)+P_{S_{1} S_{2}}(1,1) \cdot H\left(X_{1} \oplus X_{2}\right) \\
= & p+P_{S_{1} S_{2}}(1,1) \cdot H_{b}(\beta) \tag{57}
\end{align*}
$$

where we defined $\beta:=\operatorname{Pr}\left[X_{1} \neq X_{2}\right]$.
In a similar way, we obtain (41c) and (41d) from (35c) and (35d).

Distortion constraint (41e) can be shown by evaluating the optimal estimator in (24) for this example, as we detail out in the following.

We first derive the optimal estimator $\hat{s}_{k}^{*}\left(\left(x_{1}, x_{2}\right), z\right)$ for a given realization of channel inputs and the feedback defined in (7). Denote the distortion resulting from this optimal estimator for a given triple $\left(x_{1}, x_{2}, z\right)$ by

$$
\begin{align*}
d_{k}^{\prime}\left(\left(x_{1}, x_{2}\right), z\right) & =P_{S_{k} \mid X_{1} X_{2} Z}\left(\hat{s}_{k}^{*}\left(\left(x_{1}, x_{2}\right), z\right) \oplus 1 \mid x_{1}, x_{2}, z\right) \text { (58) }  \tag{58}\\
& =\min _{s} \sum_{s_{k} \in\{0,1\}}\left(s_{k} \oplus s\right) P_{S_{k} \mid X_{1} X_{2} Z}\left(s_{k} \mid x_{1}, x_{2}, z\right)
\end{align*}
$$

The expected distortion can then be expressed as:

$$
\begin{equation*}
\sum_{x_{1}, x_{2}, z} P_{X_{1} X_{2} Z}\left(x_{1}, x_{2}, z\right) d_{k}^{\prime}\left(\left(x_{1}, x_{2}\right), z\right) \tag{60}
\end{equation*}
$$

In the following we identify $\hat{s}_{k}^{*}\left(\left(x_{1}, x_{2}\right), z\right)$.
Case $z=(1,1)$ : In this case, $S_{1}=A_{2}=1$ and

$$
\begin{equation*}
\hat{s}_{k}^{*}\left(\left(x_{1}, x_{2}\right),(1,1)\right)=1, \quad \forall x_{1}=x_{2}, k=1,2 \tag{61}
\end{equation*}
$$

which yields for any $k=1,2$ and $\left(x_{1}, x_{2}\right)$ :

$$
\begin{equation*}
d_{k}^{\prime}\left(\left(x_{1}, x_{2}\right),(1,1)\right)=0 \tag{62}
\end{equation*}
$$

Case $z=(1,0)$ : In this case, $S_{1}=1$ and the optimal estimator produces $\hat{s}_{1}^{*}\left(\left(x_{1}, x_{2}\right),(1,0)\right)=1$, irrespective of $x_{1}, x_{2}$. Consequently, as before, for any $\left(x_{1}, x_{2}\right)$ :

$$
\begin{equation*}
d_{1}^{\prime}\left(x_{1}, x_{2},(1,0)\right)=0 \tag{63}
\end{equation*}
$$

For receiver 2, we distinguish whether $x_{1}=x_{2}$ or $x_{1} \neq x_{2}$. When, $x_{1}=x_{2}$, then $S_{2}=y_{2}^{\prime}=0$ because in this case $x_{2} \oplus N=x_{1} \oplus N$ and this latter equals 1 because $y_{1}^{\prime}=1$. The optimal estimator thus sets $\hat{s}_{2}^{*}\left(\left(x_{1}, x_{2}\right), z\right)=0$ when $x_{1}=x_{2}$, which achieves 0 distortion $d_{2}^{\prime}\left(\left(x_{1}, x_{2}\right), z\right)=0$.

When $x_{1} \neq x_{2}$, then $x_{2} \oplus N=1 \oplus\left(x_{1} \oplus N\right)=0$ and the feedback $z$ is independent of the state $S_{2}$ because this latter is independent of state $S_{1}$. The optimal estimator for $x_{1} \neq x_{2}$ is thus $\hat{s}_{2}^{*}\left(\left(x_{1}, x_{2}\right),(1,0)\right)=\mathbf{1}\left\{P_{S}(1) \geq P_{S}(0)\right\}$. This yields the distortion for any $\left(x_{1}, x_{2}\right)$ :

$$
\begin{equation*}
d_{2}^{\prime}\left(x_{1}, x_{2},(1,0)\right)=\min \left\{P_{S}(0), P_{S}(1)\right\} \cdot \mathbf{1}\left\{x_{1} \neq x_{2}\right\} \tag{64}
\end{equation*}
$$

or when $x_{1} \oplus N=x_{2} \oplus N=1$ and $S_{\bar{k}}=0$, where $S_{\bar{k}}=1-S_{k}$. Since these are exclusive events and have total probability of $1 / 2+1 / 2 \cdot P_{S}(0)$, we obtain:

Case $z=(0,1)$ : This case is similar to the case $z=(1,0)$ but with exchanged roles for indices 1 and 2 . So,

$$
\begin{aligned}
& d_{1}^{\prime}\left(\left(x_{1}, x_{2}\right),(0,1)\right)=\min \left\{P_{S}(0), P_{S}(1)\right\} \cdot \mathbf{1}\left\{x_{1} \neq x_{2}\right\} \\
& d_{2}^{\prime}\left(\left(x_{1}, x_{2}\right),(0,1)\right)=0
\end{aligned}
$$

Case $z=(0,0)$ : We again distinguish the two cases $x_{1}=$ $x_{2}$ and $x_{1} \neq x_{2}$ and start by considering $x_{1}=x_{2}$. In this case, $x_{1} \oplus N=x_{2} \oplus N$, and so if $S_{k}=1$ then $Z=(0,0)$ only if $x_{1} \oplus N=x_{2} \oplus N=0$, which happens with probability $1 / 2$. By the independence of the states and the inputs we then have:

$$
\begin{aligned}
& P_{S_{k} \mid X_{1} X_{2} Z}\left(1 \mid x_{1}, x_{2},(0,0)\right) \\
& \quad=\frac{P_{S_{k}}(1) P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2} S_{k}}\left(0,0 \mid x_{1}, x_{2}, 1\right)}{P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)} \\
& \quad=\frac{P_{S}(1) 1 / 2}{P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)} .
\end{aligned}
$$

If $S_{k}=0$, then $z=(0,0)$ happens when $x_{1} \oplus N=x_{2} \oplus N=0$

$$
\begin{align*}
& P_{S_{k} \mid X_{1} X_{2} Z}\left(0 \mid x_{1}, x_{2},(0,0)\right)  \tag{59}\\
& \quad=\frac{P_{S_{k}}(0) P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2} S_{k}}\left(0,0 \mid x_{1}, x_{2}, 0\right)}{P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)}  \tag{66}\\
& \quad=\frac{P_{S}(0)\left(1 / 2+1 / 2 \cdot P_{S}(0)\right)}{P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)} . \tag{67}
\end{align*}
$$

We conclude that for $z=(0,0)$ and $x_{1}=x_{2}$, the optimal estimator is

$$
\begin{equation*}
\hat{s}_{k}^{*}\left(\left(x_{1}, x_{2}\right),(0,0)\right)=\mathbf{1}\left\{P_{S}(0)\left(1+P_{S}(0)\right)<P_{S}(1)\right\} \tag{68}
\end{equation*}
$$

and the corresponding distortion
$d_{k}^{\prime}\left(\left(x_{1}, x_{2}\right),(0,0)\right)=\frac{1}{2} \cdot \frac{\min \left\{P_{S}(0)\left(1+P_{S}(0)\right), P_{S}(1)\right\}}{P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)}$,
$x_{1}=x_{2}$.
We turn to the case $x_{1} \neq x_{2}$, where $x_{1} \oplus N=1 \oplus\left(x_{2} \oplus N\right)$. As before, if $S_{k}=1$, then $Y_{k}^{\prime}=0$ only if $x_{1} \oplus N=0$, which happens with probability $1 / 2$. Now this implies $x_{2} \oplus N=$ 1 , and thus $Y_{\bar{k}}^{\prime}=0$ only if $S_{\bar{k}}=0$, which happens with probability $P_{S}(0)$. We thus obtain for $x_{1} \neq x_{2}$ :

$$
\begin{align*}
& P_{S_{k} \mid X_{1} X_{2} Z}\left(1 \mid x_{1}, x_{2},(0,0)\right) \\
& \quad=\frac{P_{S_{k}}(1) P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2} S_{k}}\left(0,0 \mid x_{1}, x_{2}, 1\right)}{P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)}  \tag{70}\\
& \quad=\frac{P_{S}(1) 1 / 2 P_{S}(0)}{P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)} . \tag{71}
\end{align*}
$$

If $S_{k}=0$, then $z=(0,0)$ happens when $x_{\bar{k}} \oplus N=0$ or when $x_{\bar{k}} \oplus N=1$ and $S_{\bar{k}}=0$. Since these are exclusive events with total probability $1 / 2+1 / 2 \cdot P_{S}(0)$, we obtain:

$$
P_{S_{k} \mid X_{1} X_{2} Y_{1}^{\prime} Y_{2}^{\prime}}\left(0 \mid x_{1}, x_{2}, 0,0\right)
$$

$$
\begin{align*}
& =\frac{P_{S_{k}}(0) P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2} S_{k}}\left(0,0 \mid x_{1}, x_{2}, 0\right)}{P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)}  \tag{72}\\
& =\frac{P_{S}(0)\left(1 / 2+1 / 2 \cdot P_{S}(0)\right)}{P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)} . \tag{73}
\end{align*}
$$

We conclude that for $z=(0,0)$ and $x_{1} \neq x_{2}$, the optimal estimator is

$$
\begin{equation*}
\hat{s}_{k}^{*}\left(\left(x_{1}, x_{2}\right),(0,0)\right)=\mathbf{1}\left\{\left(1+P_{S}(0)\right)<P_{S}(1)\right\}, \quad x_{1} \neq x_{2}, \tag{74}
\end{equation*}
$$

and the corresponding distortion

$$
\begin{array}{r}
d_{k}^{\prime}\left(\left(x_{1}, x_{2}\right),(0,0)\right)=\frac{1}{2} \cdot \frac{P_{S}(0) \min \left\{\left(1+P_{S}(0)\right), P_{S}(1)\right\}}{P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)} \\
x_{1} \neq x_{2} \tag{75}
\end{array}
$$

We now turn to the conditional probabilities of the feedbacks given the inputs that are required to evaluate (60). Whenever the inputs $x_{1}=x_{2}$,

$$
\begin{equation*}
P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)=\frac{1+\left(P_{S}(0)\right)^{2}}{2} \tag{76}
\end{equation*}
$$

because $Y_{1}^{\prime}=Y_{2}^{\prime}=0$ happens only when either $N=x_{1}=x_{2}$ or when $N=x_{1} \oplus 1=x_{2} \oplus 1$ and $S_{1}=S_{2}=0$. These two are exclusive events and happen with total probability $1 / 2+1 / 2\left(P_{S}(0)\right)^{2}$. Whenever, $x_{1} \neq x_{2}$ :

$$
\begin{align*}
P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(0,1 \mid x_{1}, x_{2}\right) & =P_{Y_{1}^{\prime} Y_{2}^{\prime} \mid X_{1} X_{2}}\left(1,0 \mid x_{1}, x_{2}\right)  \tag{77}\\
& =\frac{P_{S}(1)}{2} \tag{78}
\end{align*}
$$

by symmetry and because for $x_{1} \neq x_{2}$ the event $Y_{1}^{\prime}=1$ and $Y_{2}=0$ happens only when $S_{1}=1$ and $N=x_{1} \oplus 1$. (Notice that since $x_{1} \neq x_{2}$, this latter condition implies that $N \oplus x_{2}=0$ and thus $Y_{2}^{\prime}=0$ independent of $S_{2}$.) Moreover, when $x_{1} \neq x_{2}$ :

$$
\begin{equation*}
P_{Y_{1}^{\prime}, Y_{2}^{\prime} \mid X_{1}, X_{2}}\left(0,0 \mid x_{1}, x_{2}\right)=P_{S}(0) \tag{79}
\end{equation*}
$$

because for $x_{1} \neq x_{2}$, the event $Y_{1}^{\prime}=0$ and $Y_{2}^{\prime}=0$ happens when either $N=x_{1}=x_{2} \oplus 1$ and $S_{2}=0$ or when $N=$ $x_{1} \oplus 1=x_{2}$ and $S_{1}=0$. These are exclusive events and happen with total probability $1 / 2 P_{S}(0)+1 / 2 P_{S}(0)=P_{S}(0)$.

Plugging (63), (64), (65), (69), (75) and (76)-(79) into (60) establishes the desired distortion constraint (41e) and concludes the proof of the outer bound.

## B. Proof of Achievability Results

The achievability results can be obtained by evaluating Proposition7for the following choices: $X_{0}, X_{1}, X_{2}$ Bernoulli$\frac{1}{2}$ with $X_{0}$ independent of $\left(X_{1}, X_{2}\right)$ and $X_{1}=X_{2}=x$ with probability $\frac{1-\beta^{\prime}}{2}$ for all $x \in\{0,1\} ; U_{i}=X_{i}$, for $i=0,1,2$; and one of the following three choices: $V_{1}=\left(X_{0}, X_{1}\right), V_{2}=$ $\left(X_{0}, X_{2}\right), V_{0}=X_{1} \oplus Y_{1}^{\prime}$ or $V_{1}=\left(X_{0}, X_{1}\right), V_{2}=\left(X_{0}, X_{2}\right)$, $V_{0}=X_{2} \oplus Y_{2}^{\prime}$ or $V_{1}=V_{2}=V_{0}=0$. The last choice corresponds to not using feedback for communication and achieves all quadruples ( $R_{1}, R_{2}, D_{1}, D_{2}$ ) satisfying $R_{1}+R_{2} \geq 1$ and
$D_{k} \geq D_{\text {min }}$, where the value of $D_{\text {min }}$ depends on the state probabilities $P_{S}(0)$ and $P_{S}(1)$ and is specified in the theorem.

More specifically, achievability of $\mathcal{C} \times \mathcal{D}$ when $P_{S}(1) \leq$ $P_{S}(0)$ can be established by time-sharing between the first two choices where we set $\beta^{\prime}=0$ in both of them. (That means we choose $X_{1}$ and $X_{2}$ to be independent.)

Achievability of (43) can be established by time-sharing one of the first two choices with parameter $\beta^{\prime}=1-\frac{1-\beta}{\gamma}$ over the fraction $\gamma$ of time with the third choice over the remaining fraction $1-\gamma$ of time.

Achievability of (45) can be established by time-sharing one of the first two choices with parameter $\beta^{\prime}=\frac{\beta}{\gamma}$ over the fraction $\gamma$ of time with the third choice over the remaining fraction $1-\gamma$ of time.


[^0]:    ${ }^{1}$ Recall that the capacity-distortion region in 27 is achieved without using the feedback for communication because the BC is physically degraded.

