# On Compressed Sensing Matrices Breaking the Square-Root Bottleneck

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#### Abstract

Compressed sensing is a celebrated framework in signal processing and has many practical applications. One of challenging problems in compressed sensing is to construct deterministic matrices having restricted isometry property (RIP). So far, there are only a few publications providing deterministic RIP matrices beating the square-root bottleneck on the sparsity level. In this paper, we investigate RIP of certain matrices defined by higher power residues modulo primes. Moreover, we prove that the widely-believed generalized Paley graph conjecture implies that these matrices have RIP breaking the square-root bottleneck.

## 1 Introduction

Matrices with restricted isometry property (RIP) have important applications to compressed processing. According to [7], by means of RIP matrices, it is possible to measure and recover sparse signals using significantly fewer measurements than the dimension of the signals.

**Definition 1** (Restricted isometry property, RIP). Let  $\Phi$  be a complex  $M \times N$  matrix. Suppose that  $K \leq M \leq N$  and  $0 \leq \delta < 1$ . Then  $\Phi$  is said to have the  $(K, \delta)$ -restricted isometry property (RIP) if

$$(1 - \delta)||\mathbf{x}||^2 \le ||\Phi\mathbf{x}||^2 \le (1 + \delta)||\mathbf{x}||^2$$
 (1)

for every N-dimensional complex vector  $\mathbf{x}$  with at most K non-zero entries. Here  $||\cdot||$  denotes the  $\ell_2$  norm.

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According to Candès [7], for applications to signal processing, it suffices to investigate the  $(K, \delta)$ -RIP matrix for some  $\delta < \sqrt{2} - 1$ . In addition, the *sparsity* K is expected to be as large as possible.

On the other hand, it is known ([2]) that the problem checking whether a given matrix has RIP is NP-hard. Thus many publications have attempted to look for deterministic constructions of matrices having RIP; see e.g. [20].

Throughout this paper, we assume that all matrices have column vectors with unit  $\ell_2$ -norm. Most of known constructions for RIP matrices are established via the *coherence*  $\mu(\Phi)$  of an  $M \times N$  matrix  $\Phi$  with column vectors  $\psi_1, \ldots, \psi_N$ , where

$$\mu(\Phi) := \max_{1 \le j \ne k \le N} |\langle \psi_j, \psi_k \rangle|, \tag{2}$$

and  $\langle \cdot, \cdot \rangle$  denotes the standard inner product in the Hilbert space  $\mathbb{C}^M$ . It can be proved (e.g. [6]) that if  $\mu(\Phi) = \mu$ , then  $\Phi$  has the  $(K, (K-1)\mu)$ -RIP, which implies the  $(K, \delta)$ -RIP with only  $K = O(\sqrt{M})$ , following from the well-known Welch bound (3) in [26].

$$\mu(\Phi) \ge \sqrt{\frac{N-M}{M(N-1)}}. (3)$$

This barrier on the magnitude of the order of K is popularly dubbed as the *square-root bottle-neck* or *quadratic bottleneck*. Accordingly, the following problem arises.

**Problem 2** ([6]). Construct an  $M \times N$  matrix  $\Phi$  having the  $(K, \delta)$ -RIP with  $K = \Omega(M^{\gamma})$  for some  $\gamma > 1/2$  and  $\delta < \sqrt{2} - 1$ .

To our best knowledge, the first (unconditional) solution to this problem was given by Bourgain, Dilworth, Ford, Konyagin and Kutzarova [6], later improved by Mixon [19]; see Table 1.

On the other hand, in [3], Bandeira, Fickus, Mixon and Wong conjectured that the Paley matrix, a  $(p+1)/2 \times (p+1)$  matrix defined by quadratic residues modulo an odd prime p, satisfies the  $(K, \delta)$ -RIP with  $K \geq C_1 \cdot p/\log^{C_2} p$  and some  $\delta < \sqrt{2} - 1$ , where  $C_1, C_2 > 0$  are universal constants. Under a number-theoretic conjecture in [10], Bandeira, Mixon and Moreira [4] proved that when  $p \equiv 1 \pmod{4}$ , the Paley matrix has the (K, o(1))-RIP with  $K = \Omega(p^{\gamma})$  for some  $\gamma > 1/2$ , which provides a conditional solution to Problem 2; recently, assuming that the Paley graph conjecture (see Remark 9) holds, Satake [21] extended the result in [4] for general odd primes p, and also gave some implications to a Ramsey-theoretic problem proposed by Erdős and Moser [11]. Also, under another type of number-theoretic conjecture, Arian and Yilmaz [1] proved that for a sufficiently large prime  $p \equiv 3 \pmod{4}$ , a  $(p+1)/2 \times p$  matrix obtained by deleting the last column from the Paley matrix (see Remark 14) has the  $(K, \delta)$ -RIP for any  $K < p^{5/7}/2$  and  $\delta < 1/\sqrt{2}$ . These results are summarized in Table 1.

Ref.	M	N	K	Comments
[6], [18]	large $M$	$N \le M^{1+\gamma_1}$	$\lfloor M^{1/2+\gamma_1'} \rfloor$	any $\gamma_1' < \gamma_1$ and $\gamma_1 = 5.5169 \times 10^{-28}$
[19]	p	$\Omega(p^{1+\eta})$	$\Omega(p^{1/2+\gamma_2'})$	some $\eta > 0$ , any $\gamma_2' < \gamma_2$ and $\gamma_2 = 4.4466 \times 10^{-24}$
[4]	$\frac{p+1}{2}$	p+1	$\Omega(p^{\gamma})$	some $\frac{1}{2} < \gamma < 1, p \equiv 1 \pmod{4}$
[21]	$\frac{p+1}{2}$	p+1	$\Omega(p^{\gamma})$	some $\frac{1}{2} < \gamma < 1, p \equiv 3 \pmod{4}$
[1]	$\frac{p+1}{2}$	p	$< \frac{p^{5/7}}{2}$	$p \equiv 3 \pmod{4}$
This paper	$\frac{p+k-1}{k}$	p	$\Omega(M^{\gamma})$	some $\frac{1}{2} < \gamma < 1$ , $k (p-1) \text{ s.t. } p^{\varepsilon_1} < k \le p^{\varepsilon_2}$ , any $0 \le \varepsilon_1 < \varepsilon_2 < \varepsilon_0$ , some small $\varepsilon_0 > 0$

Table 1: Main result and known solutions to Problem 2. Here p denotes an odd prime number.

In this paper, we aim to investigate RIP of certain matrices defined by higher power residues modulo primes. In particular, under the widely-believed *generalized Paley graph conjecture* formulated in Section 2, we prove that these matrices are new solutions to Problem 2, which forms our main theorem as described below.

**Theorem 3.** Suppose that the generalized Paley graph conjecture holds. Let  $\varepsilon_0 > 0$  be a small real number and  $\varepsilon_1, \varepsilon_2$  real numbers with  $0 \le \varepsilon_1 < \varepsilon_2 < \varepsilon_0$ . Then for a sufficiently large prime p such that p-1 contains a factor k with  $p^{\varepsilon_1} < k \le p^{\varepsilon_2}$ , there are  $M \times p$  matrices with  $(\Omega(M^{\gamma}), o(1))$ -RIP for some  $\gamma > 1/2$ , where M = (p+k-1)/k.

Theorem 3 provides a new conditional answer to Problem 2. As will be shown later, these matrices substantially contain the Paley matrix as a particular case, and realize the compression ratio N/M (e.g. [17]) significantly better than that from the Paley matrix in general; see Remarks 14 and 15, respectively.

The remainder of this paper is organized as follows. Section 2 introduces some fundamental terminologies and results of finite fields and character sums, as well as some key notions related to RIP and number-theoretic results on factors of shifted primes. Section 3 defines the matrix which we investigate in this paper, and then Section 4 proves Theorem 3.

## 2 Preliminaries

## 2.1 Finite fields and characters

Throughout this paper, let p denote a prime number. Let  $\mathbb{F}_p$  be a finite field with p elements which can be identified to the residue ring  $\mathbb{Z}/p\mathbb{Z}$ . It is well known that the *multiplicative group* of  $\mathbb{F}_p$ , denoted by  $\mathbb{F}_p^*$ , is a cyclic group of order p-1, consisting of all non-zero elements of  $\mathbb{F}_p$ .

The canonical additive character  $\psi$  of  $\mathbb{F}_p$  is a map from  $\mathbb{F}_p$  to the unit circle in  $\mathbb{C}$  such that  $\psi(x) := \exp(\frac{2\pi\sqrt{-1}}{p} \cdot x)$  for all  $x \in \mathbb{F}_p$ . Notice that for every pair of  $x, y \in \mathbb{F}_p$ , we have  $\psi(x+y) = \psi(x)\psi(y)$ . A multiplicative character  $\chi$  of  $\mathbb{F}_p$  is a map from  $\mathbb{F}_p^*$  to the unit circle in  $\mathbb{C}$  such that  $\chi(xy) = \chi(x)\chi(y)$  for every pair of  $x, y \in \mathbb{F}_p^*$ . We also adopt the convention that  $\chi(0) := 0$ . Note that if g is a generator of  $\mathbb{F}_p^*$ , then each multiplicative character  $\chi$  is a map such that  $\chi(x) := \exp(\frac{2\pi\sqrt{-1}}{p-1}st)$  for all  $x = g^t \in \mathbb{F}_p^*$  and some  $0 \le s \le p-2$ . The multiplicative character of  $\mathbb{F}_p$  for s = 0 is said to be trivial. The order of a multiplicative character  $\chi$  is the minimum positive integer k such that  $(\chi(x))^k = 1$  for all  $x \in \mathbb{F}_p^*$ . Notice that the order of the trivial multiplicative character of  $\mathbb{F}_p$  is 1. For each  $1 \le k \le p-2$  with k|(p-1), we can define a non-trivial multiplicative character  $\chi_k$  of  $\mathbb{F}_p$  of order k, that is,  $\chi_k(x) := \exp(\frac{2\pi\sqrt{-1}}{k}t)$  for all  $x = g^t \in \mathbb{F}_p^*$ . For each  $1 \le h \le k-1$ , define the multiplicative character  $\chi_k^{-h}$  of  $\mathbb{F}_p$  as

$$\chi_k^{-h}(x) := \exp\left(\frac{2\pi\sqrt{-1}}{k}(-ht)\right)$$

for all  $x = g^t \in \mathbb{F}_p^*$ . Notice that  $\chi_k^{-h}$  is non-trivial for every  $1 \le h \le k - 1$ .

#### 2.2 Gauss sums

This subsection provides two types of Gauss sums; for details, see e.g. [5].

**Definition 4** (k-th power Gauss sum). Let  $2 \le k \le p-2$ . Then for each  $a \in \mathbb{F}_p$ , the k-th power Gauss sum  $\mathcal{G}_k(a)$  is defined as

$$\mathcal{G}_k(a) := \sum_{x \in \mathbb{F}_p} \psi(ax^k). \tag{4}$$

**Definition 5** (Gauss sum). Let  $a \in \mathbb{F}_p$  and  $\chi$  a multiplicative character of  $\mathbb{F}_p$ . Then define the Gauss sum  $G(a,\chi)$  as

$$G(a,\chi) := \sum_{x \in \mathbb{F}_n} \chi(x)\psi(ax). \tag{5}$$

For simplicity, let  $G(\chi) := G(1, \chi)$ .

The following two lemmas are useful in this paper.

**Lemma 6.** ([5, Theorem 1.1.3 and (1.1.4)]) For each  $a \in \mathbb{F}_p^*$ , it holds that

$$\mathcal{G}_k(a) = \sum_{h=1}^{k-1} G(a, \chi_k^h) = \sum_{h=1}^{k-1} \chi_k^{-h}(a) G(\chi_k^h).$$
 (6)

**Lemma 7.** ([5, Theorem 1.1.4]) For each  $1 \le h \le k-1$ , it holds that

$$|G(\chi_k^h)| = \sqrt{p}. (7)$$

## 2.3 Generalized Paley graph conjecture

In this paper, we will make use of the following well-known generalized Paley graph conjecture; see e.g. [8], [9], [13], [15], [22], [27] and references therein.

Conjecture 8 (Generalized Paley graph conjecture). Let p be an odd prime and  $\chi$  a non-trivial multiplicative character of  $\mathbb{F}_p$ . For  $0 < \alpha \le 1$  and  $\beta > 0$ , we say that the property  $\mathcal{P}(\alpha, \beta)$  holds if for every pair of  $S, T \subset \mathbb{F}_p$  with  $|S|, |T| > p^{\alpha}$ ,

$$\left| \sum_{s \in S, t \in T} \chi(s - t) \right| \le p^{-\beta} |S| |T|. \tag{8}$$

Then for each  $0 < \alpha \le 1$ , there exist  $p(\alpha) > 0$  and  $\beta = \beta(\alpha) > 0$  such that  $\mathcal{P}(\alpha, \beta)$  holds for any prime  $p > p(\alpha)$ .

Remark 9. The Paley graph conjecture is Conjecture 8 for the case that  $\chi=\chi_2$ , where  $\chi_2$  is a non-trivial multiplicative character of  $\mathbb{F}_p$  of order 2. According to the observations by Lenstra (see [27]), it may be reasonable to believe that Conjecture 8 would be true for any non-trivial  $\chi$  if the Paley graph conjecture holds. Indeed, it is well-known (e.g. [15]) that Conjecture 8 is true for any non-trivial multiplicative character  $\chi$  of  $\mathbb{F}_p$  and any  $S, T \subset \mathbb{F}_p$  with  $|S| > p^{1/2+\alpha}$  and  $|T| > p^{\alpha}$ , where  $0 < \alpha \le 1/2$ . In [8], Chang made a significant progress towards Conjecture 8, confirming the conjecture for any non-trivial  $\chi$  and any  $S, T \subset \mathbb{F}_p$  such that  $|S| > p^{4/9+\alpha}$ ,  $|T| > p^{4/9+\alpha}$  with |T+T| < K|T| for some K > 0 and arbitrary  $\alpha > 0$ , where  $T + T := \{t_1 + t_2 \mid t_1, t_2 \in T\}$ . For further and related results, see e.g. [14], [23], [24], [25].

## 2.4 A theorem on factors of shifted primes

To prove Theorem 3, it is necessary to certify the existence of infinitely many primes p such that the *shifted prime* p-1 admits the prescribed factors. In this paper, we shall use the following theorem due to Ford [12, Theorem 7].

**Theorem 10** ([12]). For any  $0 \le \varepsilon_1 < \varepsilon_2 \le 1$  and any sufficiently large x > 0, there exists a constant  $C_{\varepsilon_1,\varepsilon_2} > 0$  depending only on  $\varepsilon_1$  and  $\varepsilon_2$  such that there exist at least  $C_{\varepsilon_1,\varepsilon_2} \cdot x/\log x$  primes  $p \le x$  so that p-1 contains at least one factor k with  $x^{\varepsilon_1} < k \le x^{\varepsilon_2}$ . In particular, there exist infinitely many primes p such that p-1 contains at least one factor k with  $p^{\varepsilon_1} < k \le p^{\varepsilon_2}$ .

#### 2.5 Flat restricted isometry property

The following notion, flat restricted isometry property (flat RIP), has been employed to verify the RIP of matrices in [3] and [6].

**Definition 11** (Flat RIP, [3, 6]). Let  $\Phi$  be an  $M \times N$  matrix with columns  $\psi_1, \ldots, \psi_N$ . Suppose that  $K \leq M \leq N$  and  $\theta > 0$ . Then  $\Phi$  is said to have the  $(K, \theta)$ -flat restricted isometry property (flat RIP) if

$$\left| \left\langle \sum_{i \in I} \psi_i, \sum_{j \in J} \psi_j \right\rangle \right| \le \theta \sqrt{|I||J|} \tag{9}$$

for every pair of disjoint subsets  $I, J \subset \{1, 2, \dots, N\}$  with  $|I|, |J| \leq K$ .

**Proposition 12** ([3, 6]). Suppose that each column of  $\Phi$  has unit  $\ell_2$  norm. Then  $\Phi$  has the  $(K, 150 \theta \log K)$ -RIP provided that  $\Phi$  has the  $(K, \theta)$ -flat RIP.

## 3 Construction of matrices

In this section, we define a type of matrices based on higher power residues modulo primes and later investigate their RIP in Section 4.

**Definition 13.** Let p be an odd prime and k|(p-1). Let  $R_p^{(k)} := \{x^k \mid x \in \mathbb{F}_p^*\}$  denote the set of all non-zero k-th powers of  $\mathbb{F}_p$ ; notice that  $|R_p^{(k)}| = (p-1)/k$ . Suppose that elements of  $\mathbb{F}_p$  and  $R_p^{(k)}$  are labelled as  $\mathbb{F}_p = \{0 = a_1, a_2, \dots, a_p\}$  and  $R_p^{(k)} = \{b_1, b_2, \dots, b_{(p-1)/k}\}$ , respectively. Recall that  $\psi$  denotes the canonical additive character of  $\mathbb{F}_p$ .

Then the matrix  $\Phi_p^{(k)}$  is defined as an  $M \times N$  complex matrix with  $M = \frac{p+k-1}{k}$  and N = p of the following form.

$$\Phi_p^{(k)} := \begin{bmatrix} \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{p}} & \dots & \frac{1}{\sqrt{p}} \\ \sqrt{\frac{k}{p}} & \sqrt{\frac{k}{p}} \psi(b_1 a_2) & \dots & \sqrt{\frac{k}{p}} \psi(b_1 a_p) \\ \sqrt{\frac{k}{p}} & \sqrt{\frac{k}{p}} \psi(b_2 a_2) & \dots & \sqrt{\frac{k}{p}} \psi(b_2 a_p) \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{k}{p}} & \sqrt{\frac{k}{p}} \psi(b_{\frac{p-1}{k}} a_2) & \dots & \sqrt{\frac{k}{p}} \psi(b_{\frac{p-1}{k}} a_p) \end{bmatrix}$$

Note that each column  $\phi_i$  of  $\Phi_p^{(k)}$  has  $\ell_2$ -norm 1 since for each  $1 \le i \le p$ ,

$$||\phi_i||^2 = \langle \phi_i, \phi_i \rangle$$

$$= \frac{1}{p} + \frac{k}{p} \sum_{l=1}^{\frac{p-1}{k}} \psi((a_i - a_i)b_l)$$

$$= \frac{1}{p} + \frac{k}{p} \cdot \frac{p-1}{k} = 1.$$

Remark 14. Let p be an odd prime. The *Paley matrix* in [4] and [21] is a  $(p+1)/2 \times (p+1)$  matrix obtained from the matrix  $\Phi_p^{(2)}$  joining the vector  $[(\sqrt{-1})^r, 0, \dots, 0]^T$  as the last column, where r = 0 if  $p \equiv 1 \pmod{4}$  and r = 1 if  $p \equiv 3 \pmod{4}$ . The matrix investigated in [1] is exactly  $\Phi_p^{(2)}$ .

**Remark 15.** In terms of applications to signal processing, it is desirable to construct an  $M \times N$  matrix with RIP whose compression ratio N/M (e.g. [17]) is as large as possible. By Definition 13, the compression ratio of  $\Phi_p^{(k)}$  is  $kp/(p+k-1) \approx k$  as long as k=o(p), while the compression ratio of the Paley matrix is only 2. Thus when  $k \geq 3$ ,  $\Phi_p^{(k)}$  has compression ratio significantly better than that from the Paley matrix in general.

## 4 Proof of theorems

This section aims to prove the main result Theorem 3. To that end, we shall prove the following Theorem 16.

**Theorem 16.** Assuming that Conjecture 8 is true, and

- 1) let  $0 < \alpha < 1/2$  be a real number;
- 2) let  $\beta_0 = \beta_0(\alpha) > 0$  be a real number such that  $\alpha + 2\beta_0 < 1/2$  and the property  $\mathcal{P}(\alpha, \beta_0)$  holds for any non-trivial multiplicative character of  $\mathbb{F}_q$  and any prime  $q > p(\alpha)$ , where  $p(\alpha) > 0$  is from Conjecture 8;
- 3) take real numbers  $\varepsilon_1$  and  $\varepsilon_2$  with  $0 \le \varepsilon_1 < \varepsilon_2 < \beta_0$ ;
- 4) let  $p > p(\alpha)$  be a prime such that there exists a factor k of p-1 with  $p^{\varepsilon_1} < k \le p^{\varepsilon_2}$ .

Then for any real number  $\tau$  with

$$\max\left\{\alpha + \beta_0, \frac{1 - \varepsilon_1}{2} - \beta_0\right\} < \tau < \frac{1}{2} - \varepsilon_2,\tag{10}$$

the  $(p+k-1)/k \times p$  matrix  $\Phi_p^{(k)}$  in Definition 13 has the  $(p^{\tau+\beta_0}, O(p^{\tau+\varepsilon_2-1/2+o(1)}))$ -RIP.

Remark 17. We remark that for each  $\alpha$ ,  $\beta_0$ ,  $\varepsilon_1$  and  $\varepsilon_2$  satisfying the conditions 1), 2) and 3), it is possible to take a real number  $\tau$  satisfying (10). Indeed, notice that  $\alpha + \beta_0 < 1/2 - \varepsilon_2$  holds since  $\alpha + \beta_0 < 1/2 - \beta_0 < 1/2 - \varepsilon_2$ , where the first and second inequalities follow from the conditions 2) and 3), respectively. Also  $(1 - \varepsilon_1)/2 - \beta_0 < 1/2 - \varepsilon_2$  is valid, since  $\varepsilon_2 - \varepsilon_1/2 < \beta_0$  holds for any  $\varepsilon_1$  and  $\varepsilon_2$  satisfying the condition 3).

Before proceeding to prove Theorem 16, we show that Theorem 3 can be immediately derived from Theorems 10 and 16.

*Proof of Theorem 3.* According to Theorem 16,

- let  $0 < \alpha < 1/2$  and  $\beta_0 = \beta_0(\alpha) > 0$  be a real number satisfying the condition 2);
- take  $\varepsilon_0 = \varepsilon_0(\alpha) = \beta_0$  and then choose  $\varepsilon_1$  and  $\varepsilon_2$  so that  $0 \le \varepsilon_1 < \varepsilon_2 < \varepsilon_0$ ;
- take a prime  $p > p(\alpha)$  satisfying the condition 4), which is doable by Theorem 10.

Note that based on the condition 4), we have  $p^{\varepsilon_1} < k \le p^{\varepsilon_2}$ , implying  $M = (p+k-1)/k = O(p^{1-\varepsilon_1})$ . Therefore Theorem 16 shows that the  $M \times p$  matrix  $\Phi_p^{(k)}$  has the (K, o(1))-RIP, where

$$K = p^{\tau + \beta_0} = \Omega(M^{\gamma})$$

and  $\gamma$  is a real number such that  $\gamma \geq (\tau + \beta_0) \cdot (1 - \varepsilon_1)^{-1} > 1/2$ . This proves Theorem 3.  $\square$ 

In order to prove Theorem 16, we need the following two key lemmas.

## 4.1 Two key lemmas

**Lemma 18.** Let  $0 < \alpha < 1/2$  and p be a prime with  $p > p(\alpha)$ . Let  $\chi$  be a non-trivial multiplicative character of  $\mathbb{F}_p$ . Suppose that there exists  $\beta = \beta(\alpha) > 0$  such that  $\alpha + \beta < 1/2$  and the property  $\mathcal{P}(\alpha, \beta)$  holds. Let  $\tau$  be an arbitrarily fixed real number with  $\alpha + \beta < \tau < 1/2$ . Then it holds that

$$\left| \sum_{s \in S, t \in T} \chi(s - t) \right| \le p^{\tau} \sqrt{|S||T|} \tag{11}$$

for every pair of  $S, T \subset \mathbb{F}_p$  with  $|S|, |T| \leq p^{\tau + \beta}$ .

*Proof.* Let  $S, T \subset \mathbb{F}_p$  with  $|S|, |T| \leq p^{\tau + \beta}$ . The proof is done by considering the following cases.

Case 1. If  $|S||T| \leq p^{2\tau}$ , then, by the trivial bound of  $|\sum_{s \in S, t \in T} \chi(s-t)|$ , we have

$$\left| \sum_{s \in S, t \in T} \chi(s - t) \right| \le |S||T| = \sqrt{|S||T|} \cdot \sqrt{|S||T|}$$
$$\le p^{\tau} \sqrt{|S||T|}.$$

Case 2. Next, suppose that  $|S||T|>p^{2\tau}$  and we may assume  $|S|>p^{\tau}$  without loss of generality.

Case 2.1. If  $|T| \leq p^{\alpha}$ , then the following inequalities hold by the assumption that  $|S| \leq p^{\tau+\beta}$ :

$$\begin{split} \Big| \sum_{s \in S, t \in T} \chi(s - t) \Big| &\leq |S||T| = \sqrt{|S||T|} \cdot \sqrt{|S||T|} \\ &\leq \sqrt{p^{\alpha} \cdot p^{\tau + \beta}} \sqrt{|S||T|} \end{split}$$

$$= p^{\frac{\tau + \alpha + \beta}{2}} \sqrt{|S||T|}$$
 
$$< p^{\tau} \sqrt{|S||T|},$$

where the last inequality follows from the assumption that  $\alpha + \beta < \tau$ .

Case 2.2. If  $|T| > p^{\alpha}$ , recall that  $|S| > p^{\tau} > p^{\alpha+\beta} > p^{\alpha}$ , by the assumption we have the property  $\mathcal{P}(\alpha,\beta)$  holds for S and T, that is,

$$\left| \sum_{s \in S, t \in T} \chi(s - t) \right| \le p^{-\beta} |S| |T|.$$

Together with the assumption that  $|S|, |T| \leq p^{\tau+\beta}$ , we obtain

$$\left| \sum_{s \in S, t \in T} \chi(s - t) \right| \le p^{-\beta} \sqrt{|S||T|} \cdot \sqrt{|S||T|}$$

$$\le p^{-\beta} \cdot \sqrt{p^{2(\tau + \beta)}} \cdot \sqrt{|S||T|}$$

$$= p^{\tau} \sqrt{|S||T|}.$$

**Lemma 19.** Let  $\phi_i$  be the *i*-th column of  $\Phi_p^{(k)}$ . Then, for each  $1 \leq i \neq j \leq p$ ,

$$\langle \phi_i, \phi_j \rangle = \frac{1}{p} \cdot \mathcal{G}_k(a_i - a_j) = \frac{1}{p} \sum_{h=1}^{k-1} \chi_k^{-h}(a_i - a_j) G(\chi_k^h).$$
 (12)

*Proof.* Recall that  $R_p^{(k)} = \{b_1, b_2, \dots, b_{(p-1)/k}\}$  is the set of all non-zero k-th powers of  $\mathbb{F}_p$ . Note that for every  $1 \le l \le \frac{p-1}{k}$ , the equation  $X^k \equiv b_l \pmod{p}$  has exactly k distinct non-zero solutions; see e.g. [5, p.11 and Lemma 10.4.1]. Then we have

$$\langle \phi_i, \phi_j \rangle = \frac{1}{p} + \frac{k}{p} \sum_{l=1}^{\frac{p-1}{k}} \psi((a_i - a_j)b_l)$$

$$= \frac{1}{p} + \frac{k}{p} \cdot \frac{1}{k} \sum_{x \in \mathbb{F}_p^*} \psi((a_i - a_j)x^k)$$

$$= \frac{1}{p} \sum_{x \in \mathbb{F}_p} \psi((a_i - a_j)x^k) = \frac{1}{p} \cdot \mathcal{G}_k(a_i - a_j),$$
(13)

which, together with Lemma 6, proves the lemma.

## 4.2 Proof of Theorem 16

Proof of Theorem 16. Suppose that Conjecture 8 holds. Then for each  $0 < \alpha < 1/2$  and each prime  $p > p(\alpha)$ , there exists some  $\beta = \beta(\alpha) > 0$  such that the property  $\mathcal{P}(\alpha, \beta)$  holds for any non-trivial multiplicative character of  $\mathbb{F}_p$ .

Now fix  $0 < \alpha < 1/2$  arbitrarily. If  $\alpha + 2\beta < 1/2$  holds, we may take  $\beta_0 = \beta$ . If  $\alpha + 2\beta \ge 1/2$ , choose  $\beta_0 < \beta$  so that  $\alpha + 2\beta_0 < 1/2$ ; note that the property  $\mathcal{P}(\alpha, \beta)$  implies the weaker property  $\mathcal{P}(\alpha, \beta_0)$  since  $\beta_0 < \beta$ . Now take real numbers  $\varepsilon_1$  and  $\varepsilon_2$  with  $0 \le \varepsilon_1 < \varepsilon_2 < \beta_0$ . Let  $p > p(\alpha)$  be a prime such that there exists a factor k of p-1 with  $p^{\varepsilon_1} < k \le p^{\varepsilon_2}$ , which is possible in the light of Theorem 10. Then according to Remark 17, we can pick a real number  $\tau$  such that  $\max\{\alpha + \beta_0, (1 - \varepsilon_1)/2 - \beta_0\} < \tau < 1/2 - \varepsilon_2$ .

It suffices to prove that the matrix  $\Phi_p^{(k)}$  satisfies the  $(p^{\tau+\beta_0}, (k-1)p^{\tau-1/2})$ -flat RIP. Since if this is true, Proposition 12 shows that  $\Phi_p^{(k)}$  has the  $(K, \delta)$ -RIP with  $K = p^{\tau+\beta_0}$  and  $\delta = 150 \cdot (k-1)p^{\tau-1/2} \cdot \log \left(p^{\tau+\beta_0}\right)$ , implying that  $\delta = O(p^{\tau+\varepsilon_2-1/2+o(1)})$  since  $k \leq p^{\varepsilon_2}$ . This gives the desired conclusion in Theorem 16.

To that end, recall that  $\phi_i$  is the *i*-th column of  $\Phi_p^{(k)}$  and  $\chi_k$  is a non-trivial multiplicative character of  $\mathbb{F}_p$  of order k. For every pair of disjoint subsets  $I, J \subset \{1, \dots, p\}$ , we have

$$\left| \left\langle \sum_{i \in I} \phi_{i}, \sum_{j \in J} \phi_{j} \right\rangle \right| = \frac{1}{p} \cdot \left| \sum_{i \in I, j \in J} \sum_{h=1}^{k-1} \chi_{k}^{-h} (a_{i} - a_{j}) G(\chi_{k}^{h}) \right|$$

$$= \frac{1}{p} \cdot \left| \sum_{h=1}^{k-1} G(\chi_{k}^{h}) \sum_{i \in I, j \in J} \chi_{k}^{-h} (a_{i} - a_{j}) \right|$$

$$\leq \frac{1}{p} \cdot \sum_{h=1}^{k-1} \left| G(\chi_{k}^{h}) \right| \cdot \left| \sum_{i \in I, j \in J} \chi_{k}^{-h} (a_{i} - a_{j}) \right|$$

$$= \frac{1}{\sqrt{p}} \sum_{h=1}^{k-1} \left| \sum_{i \in I, j \in J} \chi_{k}^{-h} (a_{i} - a_{j}) \right|,$$
(14)

where the first equality follows from Lemma 19; the inequality follows from the triangle inequality; and the last equality follows from Lemma 7. Recall that  $\chi_k^{-h}$  is a non-trivial multiplicative character of  $\mathbb{F}_p$  for every  $1 \leq h \leq k-1$ . Since the condition 2) implies that  $\alpha + \beta_0 < 1/2$ , if  $|I|, |J| \leq p^{\tau+\beta_0}$ , then (14) together with Lemma 18 yields

$$\left| \left\langle \sum_{i \in I} \phi_i, \sum_{j \in J} \phi_j \right\rangle \right| \le \frac{1}{\sqrt{p}} \cdot (k-1) \cdot p^{\tau} \sqrt{|I||J|}$$

$$= (k-1)p^{\tau - \frac{1}{2}} \sqrt{|I||J|}.$$
(15)

Therefore  $\Phi_p^{(k)}$  has the  $(p^{\tau+\beta_0},(k-1)p^{\tau-1/2})$ -flat RIP. This completes the proof.

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