

# Matched Quantized Min-Sum Decoding of Low-Density Parity-Check Codes

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**Abstract**—A quantized message passing decoding algorithm for low-density parity-check codes is presented. The algorithm relies on the min approximation at the check nodes, and on modelling the variable node inbound messages as observations of an extrinsic discrete memoryless channel. The performance of the algorithm is analyzed and compared to quantized min-sum decoding by means of density evolution, and almost closes the gap with the performance of the sum-product algorithm. A stability analysis is derived, which highlights the role played by degree-3 variable nodes in the stability condition. Finite-length simulation results confirm large gains predicted by the asymptotic analysis.

## I. INTRODUCTION

The deployment of high throughput communication links [1], [2] is motivating a revived interest in the design of low-complexity, high-speed channel code decoders. Recently, attention has been devoted to the design and the analysis of iterative decoders where the messages exchanged within the decoder are coarsely quantized. In the context of low-density parity-check (LDPC) codes, message passing algorithms exchanging of binary messages were presented in [3]. By introducing erasures, the performance of these algorithms is improved [4]. Finite-alphabet iterative decoders were also studied, for instance, in [5]–[9]. While coarse message quantization reduces the amount of information exchanged within the decoder, the decoding complexity can also be reduced by employing simplified update rules at the check nodes (CNs). Examples are the min-sum decoder [10], [11] and some of its variations (see, e.g., [12]–[14]), that limit the losses due to the min-approximation at the CNs by introducing simple corrections.

In this paper, we analyze and design quantized min-sum decoders [15]. At the CNs, we use the standard min-approximation rule. In contrast to the quantized min-sum (QMS) algorithm [15], the variable node (VN) decoder converts all incoming messages to log-likelihood ratios (LLRs) by modeling the extrinsic channel as a discrete memoryless channel (DMC), extending the approach introduced for binary message passing decoding in [5] to the case where messages are represented by  $b$  bits. The transition probabilities of the extrinsic DMCs are derived via density evolution (DE) analysis, which is developed for unstructured irregular LDPC ensembles. Because the VN inbound messages are matched to the reliability of the underlying extrinsic DMC, we refer to the proposed algorithm as *matched* quantized min-sum

(MQMS) decoding. A stability analysis is derived. As observed for the quantized message passing decoders of [5], [8], [9], the fraction of edges connected to degree-3 VNs plays an important role in the stability condition of the proposed decoding algorithm. The DE analysis shows how the proposed MQMS decoding algorithm improves the decoding thresholds of the QMS decoder [15], closing the gap to the performance achieved by the sum-product algorithm (SPA). The results are confirmed by finite-length simulations.

## II. PRELIMINARIES

### A. LDPC Codes

LDPC codes are binary linear block codes defined by an  $m \times n$  sparse parity-check matrix  $\mathbf{H}$ . The code dimension is  $k \geq n - m$ . The Tanner graph of an LDPC code is a bipartite graph  $G = (\mathcal{V} \cup \mathcal{C}, \mathcal{E})$  consisting of  $n$  VNs and  $m$  CNs. The set  $\mathcal{E}$  of edges contains the elements  $e_{ij}$ , where  $e_{ij}$  is an edge between VN  $v_j \in \mathcal{V}$  and CN  $c_i \in \mathcal{C}$ . Note that  $e_{ij} \in \mathcal{E}$  if and only if the parity-check matrix element  $h_{ij}$  is equal to 1. The sets  $\mathcal{N}(v_j)$  and  $\mathcal{N}(c_i)$  denote the neighbors of VN  $v_j$  and CN  $c_i$ , respectively. The degree of a VN  $v_j$  (CN  $c_i$ ) is the cardinality of the set  $\mathcal{N}(v_j)$  ( $\mathcal{N}(c_i)$ ). The edge-oriented degree distribution polynomials of an LDPC code graph are  $\lambda(x) = \sum_i \lambda_i x^{i-1}$  and  $\rho(x) = \sum_i \rho_i x^{i-1}$  where  $\lambda_i$  and  $\rho_i$  are, respectively, the fraction of edges incident to VNs and CNs with degree  $i$ . An unstructured irregular LDPC code ensemble  $\mathcal{C}_n^{\lambda, \rho}$  is the set of all LDPC codes with block length  $n$  defined by a bipartite graph with degree distributions  $\lambda(x)$  and  $\rho(x)$ .

### B. Channel Model

We consider the binary-input additive white Gaussian noise (biAWGN) channel with input alphabet  $\mathcal{X} = \{-1, +1\}$ . The channel output is  $Y = X + N$ , where  $N$  is Gaussian random variable (RV) with zero mean and variance  $\sigma^2$ . The channel signal-to-noise ratio (SNR) is  $E_b/N_0$ , where  $E_b$  is the energy per information bit and  $N_0$  is the single-sided noise power spectral density.

### C. Extrinsic Channels

Consider a binary-input  $M$ -ary output DMC with input alphabet  $\mathcal{X} = \{-1, +1\}$  and output alphabet  $\mathcal{Z} = \{-\frac{M-1}{2}, -\frac{M-3}{2}, \dots, 0, \dots, \frac{M-1}{2}\}$ , where  $M = 2^b - 1$  and

$b$  is a positive integer. For a generic channel output  $z$ , LLRs can be obtained as

$$L(z) = \ln \left[ \frac{P_{Z|X}(z|+1)}{P_{Z|X}(z|-1)} \right].$$

If the channel satisfies the symmetry constraint  $P_{Z|X}(-z|+1) = P_{Z|X}(z|-1)$  for all  $z \in \mathcal{Z}$ , we have

$$L(z) = \text{sign}(z)D_{|z|} \quad (1)$$

where  $\forall a \in \mathcal{Z}, a > 0$

$$D_a := \ln \left[ \frac{P_{Z|X}(a|+1)}{P_{Z|X}(-a|+1)} \right] \quad (2)$$

and where by convention the  $\text{sign}(x)$  function takes value 0 for  $x = 0$ . We refer to  $D_{|z|}$  as the *reliability* of  $z$ . The decomposition (1) will be instrumental to the development of a message-passing decoding algorithm for LDPC codes. In particular, we will focus on a decoding algorithm that exchanges quantized messages. In this case, a message sent from a CN to a VN can be modeled as the observation of the RV  $X$  after transmission over a binary-input  $M$ -ary output discrete memoryless extrinsic channel [16, Fig. 3], where  $M$  is the number of message quantization levels. While the transition probabilities of the extrinsic channel are in general unknown, accurate estimates can be obtained via DE analysis, as suggested in [5]. This observation will be used to derive the decoding algorithm presented in Section III.

#### D. Quantization

Throughout the paper, we consider uniform quantization of the messages. We denote by  $f : \mathbb{R} \rightarrow \mathcal{M}$  the quantization function of the exchanged messages, where the quantized message alphabet is  $\mathcal{M} = \{-S\Delta, -(S-1)\Delta, \dots, S\Delta\}$ . The function  $f$  is a  $b$ -bit uniform quantizer with step size  $\Delta$  and  $2^b - 1$  quantization levels. Formally, we have

$$f(x) := \text{sign}(x)\Delta \cdot \min \left\{ \left\lfloor \frac{|x|}{\Delta} + \frac{1}{2} \right\rfloor, S \right\} \quad (3)$$

where  $S = 2^{b-1} - 1$ .

For the channel output, we consider two cases: unquantized channel outputs and quantized channel outputs. For the latter case, the biAWGN channel output is quantized using a  $b_0$ -bit uniform quantizer with step size  $\Delta_0$ , where  $b_0$  and  $\Delta_0$  may in general differ from the corresponding parameters for the message quantization. The quantized channel output alphabet is  $\mathcal{M}_0 = \{-S_0\Delta_0, -(S_0-1)\Delta_0, \dots, S_0\Delta_0\}$  with  $S_0 = 2^{b_0-1} - 1$ , and the quantized version of  $y$  is denoted as  $m_{\text{ch}}$ .

### III. MATCHED QUANTIZED MIN-SUM DECODING

We denote by  $m_{c \rightarrow v}^{(\ell)}$  the message sent from CN  $c$  to its neighboring VN  $v$ . Similarly,  $m_{v \rightarrow c}^{(\ell)}$  is the message sent from VN  $v$  to CN  $c$  at the  $\ell$ -th iteration.

#### A. Unquantized Channel Output

Each VN computes the LLR of the corresponding channel output

$$L_{\text{ch}}(y) = \frac{2}{\sigma^2}y.$$

Then the VN passes a  $b$ -bit quantized value to its neighboring CNs. Thus,  $\forall c \in \mathcal{N}(v)$  we have

$$m_{v \rightarrow c}^{(0)} = f(L_{\text{ch}}(y))$$

where  $f$  is defined in (3) and we choose  $\Delta$  to minimize the iterative decoding threshold.

The min update rule is performed at the CNs. We have

$$m_{c \rightarrow v}^{(\ell)} = \min_{v' \in \mathcal{N}(c) \setminus v} \left| m_{v' \rightarrow c}^{(\ell-1)} \right| \prod_{v' \in \mathcal{N}(c) \setminus v} \text{sign} \left( m_{v' \rightarrow c}^{(\ell-1)} \right).$$

At the  $\ell$ -th iteration, each VN converts its channel message and the incoming CN messages to LLRs. The sum of these LLRs is then quantized into a  $b$ -bit message. Formally, we have

$$m_{v \rightarrow c}^{(\ell)} = f \left( L_{\text{ch}}(y) + \sum_{c' \in \mathcal{N}(v) \setminus c} L_{\text{ex}} \left( m_{c' \rightarrow v}^{(\ell)} \right) \right) \quad (4)$$

where

$$L_{\text{ex}} \left( m_{c' \rightarrow v}^{(\ell)} \right) := \text{sign} \left( m_{c' \rightarrow v}^{(\ell)} \right) D_{\left| m_{c' \rightarrow v}^{(\ell)} \right|}.$$

The final hard decision at each VN is

$$\hat{x}_v^{(\ell)} = \text{sign} \left( L_{\text{ch}}(y) + \sum_{c' \in \mathcal{N}(v)} L_{\text{ex}} \left( m_{c' \rightarrow v}^{(\ell)} \right) \right) \quad (5)$$

Note that the reliability of  $m_{c' \rightarrow v}^{(\ell)}$  depends on the iteration number and is in general unknown. In fact, the transition probabilities of the underlying extrinsic DMCs are not known. As proposed in [5], their values can be estimated via Monte Carlo simulations, or via DE analysis. The latter approach provides accurate results for moderate to large block lengths, as shown in [5], [8]. We hence follow this direction and use the DE presented in Section IV to estimate the message reliability at each iteration. For the special case of  $b = 2$ , we will obtain the ternary message passing (TMP) decoder introduced in [8].

#### B. Quantized Channel Output

If the channel output is quantized as described in Section II-D, we replace  $L_{\text{ch}}(y)$  in (4) and (5) by  $L_{\text{ch}}(m_{\text{ch}}) = \text{sign}(m_{\text{ch}})D_{|m_{\text{ch}}|}$ . We choose  $\Delta$  and  $\Delta_0$  to minimize the decoding threshold. As mentioned in Sec. II-C, the decoder's communication channel can be modeled as a binary-input  $|\mathcal{M}_0|$ -ary output DMC that satisfies the symmetry condition. The value of  $D_{|m_{\text{ch}}|}$  can then be computed from (2) by using the transition probabilities of the quantized communication channel.

$$q_i^{(\ell)} = \begin{cases} \frac{1}{2} \left[ \rho \left( \Phi_i^{(\ell-1)} + \Psi_i^{(\ell-1)} \right) + \rho \left( \Phi_i^{(\ell-1)} - \Psi_i^{(\ell-1)} \right) - \rho \left( \Phi_{i+1}^{(\ell-1)} + \Psi_{i+1}^{(\ell-1)} \right) - \rho \left( \Phi_{i+1}^{(\ell-1)} - \Psi_{i+1}^{(\ell-1)} \right) \right] & i > 0 \\ 1 - \rho \left( 1 - p_0^{(\ell-1)} \right) & i = 0 \\ \frac{1}{2} \left[ \rho \left( \Phi_{-i}^{(\ell-1)} + \Psi_{-i}^{(\ell-1)} \right) - \rho \left( \Phi_{-i}^{(\ell-1)} - \Psi_{-i}^{(\ell-1)} \right) - \rho \left( \Phi_{-i+1}^{(\ell-1)} + \Psi_{-i+1}^{(\ell-1)} \right) + \rho \left( \Phi_{-i+1}^{(\ell-1)} - \Psi_{-i+1}^{(\ell-1)} \right) \right] & i < 0 \end{cases} \quad (6)$$

$$p_i^{(\ell)} = \begin{cases} \sum_d \lambda_d \sum_{l_{\text{in}}} \mathbb{P} \left\{ L_{\text{in}}^{(\ell)} = l_{\text{in}} \right\} Q \left( \frac{(S-\frac{1}{2})\Delta + l_{\text{in}} + \mu_{\text{ch}}}{\sigma_{\text{ch}}} \right) & i = -S \\ \sum_d \lambda_d \sum_{l_{\text{in}}} \mathbb{P} \left\{ L_{\text{in}}^{(\ell)} = l_{\text{in}} \right\} Q \left( \frac{(S-\frac{1}{2})\Delta - l_{\text{in}} - \mu_{\text{ch}}}{\sigma_{\text{ch}}} \right) & i = S \\ \sum_d \lambda_d \sum_{l_{\text{in}}} \mathbb{P} \left\{ L_{\text{in}}^{(\ell)} = l_{\text{in}} \right\} \left[ Q \left( \frac{(i-\frac{1}{2})\Delta - l_{\text{in}} - \mu_{\text{ch}}}{\sigma_{\text{ch}}} \right) - Q \left( \frac{(i+\frac{1}{2})\Delta - l_{\text{in}} - \mu_{\text{ch}}}{\sigma_{\text{ch}}} \right) \right] & \text{otherwise} \end{cases} \quad (7)$$

#### IV. DENSITY EVOLUTION ANALYSIS

We provide a DE analysis of the MQMS algorithm for unstructured LDPC code ensembles. Due to symmetry, we may assume that the all-zeros codeword is transmitted. Let  $M_{v \rightarrow c}^{(\ell)}$  be the RV associated to VN to CN messages at the  $\ell$ -th iteration. Similarly,  $M_{c \rightarrow v}^{(\ell)}$  represents the RV associated to CN to VN messages. We denote by  $p_i^{(\ell)}$  the probability that  $M_{v \rightarrow c}^{(\ell)}$  takes the value  $\Delta i$ , with  $i \in \{-S, -(S-1), \dots, S\}$ . Similarly, we denote by  $q_i^{(\ell)}$  the probability that  $M_{c \rightarrow v}^{(\ell)}$  takes the value  $\Delta i$ . In the following,  $\ell_{\text{max}}$  denotes the maximum number of iterations. In the limit of  $n \rightarrow \infty$ , the evolution of the message distributions can be tracked as follows.

- 1) **Initialization.** Conditioned on  $X = +1$ , the channel LLRs are Gaussian RVs with mean  $\mu_{\text{ch}} = 4RE_b/N_0$  and variance  $\sigma_{\text{ch}}^2 = 2\mu_{\text{ch}}$ . Therefore, we have

$$p_i^{(0)} = \begin{cases} Q \left( \frac{(S-\frac{1}{2})\Delta + \mu_{\text{ch}}}{\sigma_{\text{ch}}} \right) & i = -S \\ Q \left( \frac{(S-\frac{1}{2})\Delta - \mu_{\text{ch}}}{\sigma_{\text{ch}}} \right) & i = S \\ Q \left( \frac{(i-\frac{1}{2})\Delta - \mu_{\text{ch}}}{\sigma_{\text{ch}}} \right) - Q \left( \frac{(i+\frac{1}{2})\Delta - \mu_{\text{ch}}}{\sigma_{\text{ch}}} \right) & \text{otherwise} \end{cases}$$

while if the channel output is quantized we have

$$p_i^{(0)} = \sum_{m_{\text{ch}}: f(L_{\text{ch}}(m_{\text{ch}})) = \Delta i} P_{M_{\text{ch}}|X}(m_{\text{ch}} + 1).$$

- 2) **For**  $\ell = 1, 2, \dots, \ell_{\text{max}}$

**Check to variable update.** For all  $j \in \{1, \dots, 2^{b-1}\}$ , we define  $\Phi_j^{(\ell)}$  and  $\Psi_j^{(\ell)}$  as

$$\begin{aligned} \Phi_j^{(\ell)} &:= \mathbb{P} \left\{ M_{v \rightarrow c}^{(\ell)} \geq \Delta j \right\} \\ \Psi_j^{(\ell)} &:= \mathbb{P} \left\{ M_{v \rightarrow c}^{(\ell)} \leq -\Delta j \right\}. \end{aligned}$$

The probabilities  $q_i^{(\ell)}$  can be computed as shown in (6).

**Variable to check update.** For  $i \in \{-S, -(S-1), \dots, S\}$ ,  $p_i^{(\ell)}$  can be computed from (7) for the unquantized channel output, while for the quantized one we have

$$p_i^{(\ell)} = \sum_d \lambda_d \sum_{m_{\text{ch}}} P_{M_{\text{ch}}|X}(m_{\text{ch}} + 1) \times \sum_{l_{\text{in}}: f(L_{\text{ch}}(m_{\text{ch}}) + l_{\text{in}}) = \Delta i} \mathbb{P} \left\{ L_{\text{in}}^{(\ell)} = l_{\text{in}} \right\}$$

where  $L_{\text{in}}^{(\ell)}$  is a RV associated to the sum of the LLRs of the  $d-1$  incoming CN messages at the  $\ell$ -th iteration. We have

$$\mathbb{P} \left\{ L_{\text{in}}^{(\ell)} = l_{\text{in}} \right\} = \sum_{\mathbf{v}} \binom{d-1}{v_{-S}, \dots, v_S} \prod_{i=-S}^S \left( q_i^{(\ell)} \right)^{v_i}$$

where the sum is over all integer vectors  $\mathbf{v}$  for which

$$\sum_{i=-S}^S v_i = d-1 \quad \text{and} \quad \sum_{i=1}^S (v_i - v_{-i}) D_{\Delta i}^{(\ell)} = l_{\text{in}}$$

where  $D_{\Delta i}^{(\ell)} := \ln \left( q_i^{(\ell)} / q_{-i}^{(\ell)} \right)$ . Note that the vector entry  $v_i$  represents the number of incoming CN messages with value  $\Delta i$ .

The ensemble iterative decoding threshold  $(E_b/N_0)^*$  is defined as the minimum  $E_b/N_0$  for which  $\lim_{\ell \rightarrow \infty} P_e^{(\ell)} = 0$  as  $n \rightarrow \infty$ , where

$$P_e^{(\ell)} = \sum_{i=-S}^0 p_i^{(\ell)}.$$

#### V. STABILITY ANALYSIS

We define

$$\mathbf{p}^{(\ell)} := \left[ p_{-S}^{(\ell)}, \dots, p_{S-1}^{(\ell)} \right]^T \quad \text{and} \quad \mathbf{q}^{(\ell)} := \left[ q_{-S}^{(\ell)}, \dots, q_{S-1}^{(\ell)} \right]^T.$$

We should determine the evolution of  $\mathbf{p}^{(\ell)}$  over one iteration in proximity of the fixed point  $\mathbf{p}^* = \mathbf{0}$ .<sup>1</sup> Note that, as  $\mathbf{p}^{(\ell)} \rightarrow \mathbf{0}$ ,  $\mathbf{q}^{(\ell)} \rightarrow \mathbf{0}$ . Thus, for the inbound VN extrinsic channel, we have  $D_{i\Delta}^{(\ell)} \rightarrow +\infty$  for  $i = S$  while  $D_{i\Delta}^{(\ell)} \rightarrow 0$  for  $i < S$ .

As  $\mathbf{q}^{(\ell)} \rightarrow \mathbf{0}$ , we have

$$p_{-S}^{(\ell)} = \sum_d \lambda_d \left[ \mathbb{P} \left\{ S_{\text{in}}^{(\ell)} = 0 \right\} \mathbb{P} \left\{ L_{\text{ch}} \leq -(S-0.5)\Delta \right\} + \mathbb{P} \left\{ S_{\text{in}}^{(\ell)} \leq -1 \right\} \right] \quad (8)$$

while for  $i \in \{-S+1, \dots, S-1\}$

$$p_i^{(\ell)} = \sum_d \lambda_d \mathbb{P} \left\{ S_{\text{in}}^{(\ell)} = 0 \right\} \mathbb{P} \left\{ (i-0.5)\Delta \leq L_{\text{ch}} \leq (i+0.5)\Delta \right\}$$

<sup>1</sup>We assume that the minimum VN degree is at least 2.

where  $S_{\text{in}}^{(\ell)}$  is a RV representing the difference between the number of incoming messages equal to  $S$  and the number of incoming messages equal to  $-S$  to a VN of degree  $d$ . We have

$$\mathbb{P}\{S_{\text{in}}^{(\ell)} = s_{\text{in}}\} = \sum_{\mathbf{v}} \binom{d-1}{v_{-S}, \dots, v_S} \prod_{i=-S}^S (q_i^{(\ell)})^{v_i} \quad (9)$$

where the sum is over all integer vectors  $\mathbf{v}$  for which

$$\sum_{i=-S}^S v_i = d-1, \quad (v_S - v_{-S}) = s_{\text{in}}.$$

We further have

$$\lim_{\mathbf{q}^{(\ell)} \rightarrow \mathbf{0}} \frac{\partial \mathbb{P}\{S_{\text{in}}^{(\ell)} = s_{\text{in}}\}}{\partial q_{-S}^{(\ell)}} = \begin{cases} d-1 & s_{\text{in}} = d-3 \\ -(d-1) & s_{\text{in}} = d-1 \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

while, for  $i \in \{-S+1, \dots, S-1\}$ ,

$$\lim_{\mathbf{q}^{(\ell)} \rightarrow \mathbf{0}} \frac{\partial \mathbb{P}\{S_{\text{in}}^{(\ell)} = s_{\text{in}}\}}{\partial q_i^{(\ell)}} = \begin{cases} d-1 & s_{\text{in}} = d-2 \\ -(d-1) & s_{\text{in}} = d-1 \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

From (6), for  $i, j \in \{-S, \dots, S-1\}$

$$\lim_{\mathbf{p}^{(\ell-1)} \rightarrow \mathbf{0}} \frac{\partial q_i^{(\ell)}}{\partial p_j^{(\ell-1)}} = \begin{cases} \rho'(1) & i = j \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

From (8), (9), (10), (11), (12), we obtain

$$\lim_{\mathbf{p}^{(\ell-1)} \rightarrow \mathbf{0}} \frac{\partial p_{-S}^{(\ell)}}{\partial p_{-S}^{(\ell-1)}} = \rho'(1) (2\lambda_3\alpha + \lambda_2) \quad (13)$$

for  $i, j \in \{-S+1, \dots, S-1\}$

$$\lim_{\mathbf{p}^{(\ell-1)} \rightarrow \mathbf{0}} \frac{\partial p_{-S}^{(\ell)}}{\partial p_i^{(\ell-1)}} = \lambda_2\alpha\rho'(1) \quad (14)$$

$$\lim_{\mathbf{p}^{(\ell-1)} \rightarrow \mathbf{0}} \frac{\partial p_i^{(\ell)}}{\partial p_{-S}^{(\ell-1)}} = 2\rho'(1)\lambda_3\gamma_i \quad (15)$$

$$\lim_{\mathbf{p}^{(\ell-1)} \rightarrow \mathbf{0}} \frac{\partial p_i^{(\ell)}}{\partial p_j^{(\ell-1)}} = \rho'(1)\lambda_2\gamma_i \quad (16)$$

where

$$\alpha = Q\left(\frac{(S - \frac{1}{2})\Delta + \mu_{\text{ch}}}{\sigma_{\text{ch}}}\right)$$

$$\gamma_i = Q\left(\frac{(i - 0.5)\Delta - \mu_{\text{ch}}}{\sigma_{\text{ch}}}\right) - Q\left(\frac{(i + 0.5)\Delta - \mu_{\text{ch}}}{\sigma_{\text{ch}}}\right)$$

if the channel output is unquantized while

$$\alpha = \sum_{m_{\text{ch}}: f(L_{\text{ch}}(m_{\text{ch}})) = -S\Delta} P_{M_{\text{ch}}|X}(m_{\text{ch}} + 1)$$

$$\gamma_i = \sum_{m_{\text{ch}}: f(L_{\text{ch}}(m_{\text{ch}})) = \Delta i} P_{M_{\text{ch}}|X}(m_{\text{ch}} + 1)$$

TABLE I  
DECODING THRESHOLDS  $(E_b/N_0)^*$  [dB] OF MQMS FOR QUANTIZED AND UNQUANTIZED CHANNEL OUTPUT AND FOR QMS.

$(d_v, d_c)$	$b$	MQMS		QMS	
		(unquant. channel)	(quant. channel)		
(3,6)	2		2	2.66	
		1.85	3	2.66	
			4	2.58	
	3		3	1.45	1.8
		1.32	4	1.34	1.8
	4		3	1.35	1.72
		1.21	4	1.24	1.65
	5		5	1.19	1.62
		1.18			
(4,8)	2		2	2.71	2.78
		2.11	3	2.22	2.4
			4	2.11	2.43
	3		3	1.85	2.17
		1.73	4	1.76	2.12
	4		3	1.77	2.14
		1.65	4	1.68	2.08
	5		5	1.64	2.06
		1.63			

if the channel output is quantized. The first order Taylor expansions via (13), (14), (15), (16) yield

$$\mathbf{p}^{(\ell)} = \mathbf{A} \cdot \mathbf{p}^{(\ell-1)}$$

where for  $i, j \in \{-S, \dots, S-1\}$

$$a_{i,j} = \lim_{\mathbf{p}^{(\ell-1)} \rightarrow \mathbf{0}} \frac{\partial p_i^{(\ell)}}{\partial p_j^{(\ell-1)}}.$$

Let  $r$  be the spectral radius of  $\mathbf{A}$ . The stability condition is fulfilled if and only if  $r < 1$ .

**Remark 1.** By a close inspection of (13) and (15), we see that the fraction of edges connected to degree-3 VNs plays an important role in the stability condition of the proposed decoding algorithm. This was already noted for the quantized decoders of [5], [8], [9], and in the analysis of saturated belief propagation decoding of [17]. The result hence shows that one should more strongly limit the use of degree-2 and degree-3 VNs for unstructured LDPC ensembles, with respect to the case of unquantized, non-saturated belief propagation decoding. From a practical viewpoint, it might be worth relaxing the definition of decoding threshold by adopting a suitably-low target decoding error probability, especially for codes tailored for moderate error rates.

## VI. NUMERICAL RESULTS

A first set of results deals with the asymptotic performance of MQMS decoding. Table I reports a comparison between the iterative decoding thresholds of MQMS for both quantized and unquantized channel outputs and QMS [15] for  $(d_v, d_c)$  regular LDPC ensembles and different values of  $b$  and  $b_0$ . MQMS decoding largely outperforms QMS, with gains up to 0.7 dB. Remarkably, for  $b = b_0 = 5$  the MQMS thresholds are within 0.1 dB of the unquantized belief propagation thresholds (which

are at  $(E_b/N_0)^* \approx 1.1$  dB for the regular (3,6) ensemble, and at  $(E_b/N_0)^* \approx 1.58$  dB for the regular (4,8) ensemble). Based on the DE analysis of Section IV, we designed a set of optimized irregular ensembles with various rates. For the design, we assumed a MQMS decoder with  $b = 4$  and unquantized channel output. We set the maximum VN degree to  $d_v^{\max} = 20$ . Due to space limitations, the optimized degree distributions, obtained via differential evolution are provided in [?]. We next considered the performance for rate 4/5 and 7/8 codes, designed for MQMS decoder and unquantized channel outputs, where we set  $b = 4, d_v^{\max} = 15, \ell_{\max} = 30$ . The codes have a block length  $n = 20000$  bits and their graphs were designed via the progressive edge-growth (PEG) algorithm [18]. The simulation results are shown in Fig. 1 in terms of frame error rate (FER) versus  $E_b/N_0$ . As a reference, we provide the simulation results of the optimized codes for MQMS under unquantized belief propagation (BP) decoding, MQMS for both 4 bit quantized and unquantized channel output and QMS with  $b = b_0 = 4$ , as well as the random coding union bound (RCU) of [19]. Observe that the MQMS algorithm outperforms the QMS decoder although they both use the same CN update rule. Admittedly, the VN update rule of MQMS is more complex than the one of the plain QMS decoder: An open question is whether the VN update rule in (4) can be efficiently implemented in approximate form (e.g., via look-up tables) without compromising the performance of the MQMS algorithm.

## VII. CONCLUSION

A quantized message passing decoding algorithm for LDPC codes was presented. The algorithm relies on the min approximation at the check nodes, and on modelling the variable node inbound messages as observations of an extrinsic channel output. The algorithm was analyzed via density evolution. Degree-3 variable nodes play an important role in the stability condition. The algorithm shows remarkable gains over quantized min-sum decoding, almost closing the gap with the performance of the SPA.

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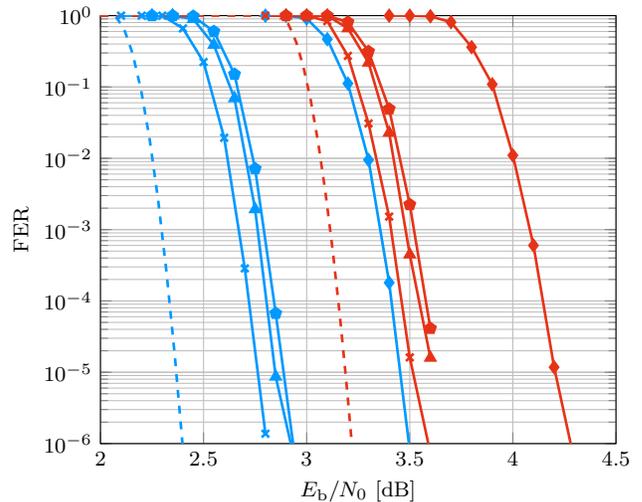


Fig. 1. FER versus  $E_b/N_0$  [dB] for unquantized SPA (—•—), MQMS for unquantized (—▲—) and quantized (—●—) channel output, QMS (—◆—), RCU bound (---, ---) for  $R = 4/5$  (—),  $R = 7/8$  (—).

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TABLE II  
 THRESHOLDS OF OPTIMIZED DEGREE DISTRIBUTIONS FOR THE MQMS DECODER FOR UNQUANTIZED CHANNEL WITH  $b = 4$  AND QUANTIZED CHANNEL  
 WITH  $b = b_0 = 4$ .

$R$	$\lambda(x)$	$\rho(x)$	$(E_b/N_0)^*$ [dB]	$(E_b/N_0)^*$ [dB] $b_0 = 4$	$(E_b/N_0)_{\text{Sh}}$ [dB]
2/3	$0.0317x + 0.489x^2 + 0.0374x^9 + 0.4419x^{19}$	$0.328x^{13} + 0.672x^{14}$	1.47	1.5	1.06
3/4	$0.0313x + 0.463x^2 + 0.0058x^9 + 0.4999x^{19}$	$0.5336x^{19} + 0.4664x^{20}$	1.96	2	1.62
4/5	$0.4961x^2 + 0.0051x^9 + 0.4988x^{19}$	$0.7907x^{25} + 0.2093x^{26}$	2.34	2.37	2.04
5/6	$0.0205x + 0.4646x^2 + 0.0534x^9 + 0.4616x^{19}$	$0.9926x^{30} + 0.0074x^{31}$	2.63	2.66	2.36
7/8	$0.4789x^2 + 0.0021x^4 + 0.032x^9 + 0.487x^{19}$	$0.3752x^{41} + 0.6248x^{42}$	3.08	3.11	2.85
9/10	$0.4442x^2 + 0.0403x^3 + 0.0025x^9 + 0.513x^{19}$	$0.6604x^{53} + 0.3396x^{54}$	3.42	3.44	3.2