# One-shot inner bounds for sending private classical information over a quantum MAC 

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#### Abstract

We provide the first inner bounds for sending private classical information over a quantum multiple access channel. We do so by using three powerful information theoretic techniques: rate splitting, quantum simultaneous decoding for multiple access channels, and a novel smoothed distributed covering lemma for classical quantum channels. Our inner bounds are given in the one shot setting and accordingly the three techniques used are all very recent ones specifically designed to work in this setting. The last technique is new to this work and is our main technical advancement. For the asymptotic iid setting, our one shot inner bounds lead to the natural quantum analogue of the best classical inner bounds for this problem.


## 1 Introduction

Private communication over a noisy channel is an important information processing and cryptographic primitive. Here, a sender Alice wants to send her message over a noisy channel $\mathfrak{C}$ so that the genuine receiver Bob can decode it with small error. At the same time, an eavesdropping receiver Eve should get almost no information about the transmitted message. There exist different ways of formalising the latter requirement as we will see very soon below.

The task of private communication over a noisy classical channel has an old history. Wyner [Wyn75], and Csiszár and Körner [CK78] first studied this problem for a point to point classical channel in the asymptotic setting of many independent and identical (iid) uses of the channel. Calling it the wiretap channel, they proved the following optimal bound:

$$
\begin{equation*}
R_{A}=\max _{P}\left(I(X: C)_{P}-I(X: E)_{P}\right), \tag{1}
\end{equation*}
$$

where the channel $\mathfrak{C}$ is modelled as a stochastic map from set $A$ to set $C \times E$ and the mutual information is measured with respect to the probability distribution $p(x) p(c e \mid x)$ where X is an

[^0]auxilliary random variable and the maximisation is done over all choices of the random variable $X$ and encoding maps $x \mapsto p(a \mid x)$. The above bound is obtained as follows. Let $n$ be the number of iid channel uses. Alice chooses a random code book of size $2^{n\left(R_{A}+r_{a}\right)}$ by independently sampling from the probability distribution $p\left(x^{n}\right)$ on $X^{n}$. This code book is divided into $2^{n R_{A}}$ blocks, each of size $2^{n r_{a}}$. To transmit the $m$ th message for an $m \in 2^{n R_{A}}$, Alice chooses a uniformly random codeword from the $m$ th block, say the $m^{\prime}$ th codeword $x\left(m, m^{\prime}\right)$, applies the stochastic encoding map to $x^{n}\left(m, m^{\prime}\right)$ to get a sample from a probability distribution on $A^{n}$ and feeds it to $n$ copies of the channel $\mathfrak{C}$. The output of the channel is a sample from the probability distribution $p\left(c^{n} e^{n} \mid x^{n}\left(m, m^{\prime}\right)\right)$. Bob can decode the pair $\left(m, m^{\prime}\right)$ if the rate $R_{A}+r_{a}$ per channel use is less than $I(X: C)_{P}$. On the other hand, Eve is 'obfuscated' if $r_{a} \geq I(X: E)_{P}$. This leads to the achievable rate $R_{A} \leq I(X: C)_{P}-I(X: E)_{P}$ for private classical communication. Here, different notions of secrecy lead to different notions of formalisation of the statement 'Eve is obfuscated'.

The multiple access channel (MAC) is arguably the simplest multiterminal communication channel where there are several independent senders but only one genuine receiver. Private communication over a MAC is an important cryptographic task modelling, for example, the secure communication of messages from multiple independent agents in the field to a base station. Here, the genuine receiver should be able to decode the entire transmitted message tuple with small error and an eavesdropping receiver should hardly get any information about the transmitted message tuple. In the last decade several authors have considered the problem of private classical communication over various types of classical multiple access channels in the asymptotic iid setting culminating in the work of Chen, Koyluoglu and Vinck [CKV16] who proved the following inner bound for a general classical discrete memoryless MAC in the asymptotic iid setting: the union of rate regions of the form

$$
\begin{align*}
R_{A} & \leq I(X: C Y \mid Q)_{P}-I(X: E \mid Q)_{P}, \\
R_{B} & \leq I(Y: C X \mid Q)_{P}-I(Y: E \mid Q)_{P},  \tag{2}\\
R_{A}+R_{B} & \leq I(X Y: C \mid Q)_{P}-I(X Y: E \mid Q)_{P},
\end{align*}
$$

where the mutual information is measured with respect to the probability distribution $p(q) p(x \mid q) p(y \mid q) p(c, e \mid x, y)$, $Q$ is an auxilliary 'time sharing' random variable, $X, Y$ are auxilliary random variables that are independent given $Q, X \rightarrow A, Y \rightarrow B$ are independent stochastic encoding maps and the channel is a stochastic map from $A \times B$ to $C \times E$. The union is taken over all probability distributions of the form $p(q) p(x \mid q) p(y \mid q) p(a \mid x) p(b \mid y)$.

Both Equations1 and 2 above use the asymptotically vanishing mutual information definition of secrecy viz. they require that $I\left(M_{A}: E^{n}\right) / n$ or $I\left(M_{A} M_{B}: E^{n}\right) / n$ approach zero as the number of iid channel uses $n \rightarrow \infty$ where ( $M_{A}, M_{B}$ ) denote the input messages distributed uniformly in $\left[2^{n R_{A}}\right] \times\left[2^{n R_{B}}\right]$. This definition is strictly weaker than the small leakage in trace distance definition of secrecy defined below that we will use in this paper. Nevertheless we will be able to reproduce the above bounds even under the stronger secrecy requirement.

The problem of private classical information over a point to point quantum channel was first studied by Devetak [Dev05] in the asymptotic iid setting. The channel $\mathfrak{C}$ is modelled as a completely positive trace preserving (CPTP) map from density matrices on the input Hilbert space $A$ to density matrices on the output Hilbert space $C \otimes E$. Here $A$ is the Hilbert spaces of a sender Alice, $C$ is the Hilbert space of the genuine receiver Charlie and $E$ is the Hilbert space of an eavesdropping
receiver Eve. Devetak proved that the natural regularised quantum analogue of Equation 1 is the optimal rate viz.

$$
\begin{equation*}
R_{A}=\lim _{n \rightarrow \infty} n^{-1} \max _{\rho}\left(I\left(X: C^{n}\right)_{\rho}-I\left(X: E^{n}\right)_{\rho}\right), \tag{3}
\end{equation*}
$$

where the mutual information is taken over all classical quantum states of the form $\rho^{X C^{n} E^{n}}=$ $\sum_{x} p(x)|X\rangle^{X}\langle x| \rho_{x}^{C^{n}} E^{n}$, and the maximisation is done over all random variables $X$ and encoding mappings $x \mapsto \sigma_{x}^{A^{n}}$. The state $\rho_{x}^{C^{n}} E^{n}$ is obtained by applying the channel $\mathfrak{C}^{\otimes n}$ to $\sigma_{x}^{A^{n}}$. Subsequently Renes and Renner [RR11], Radhakrishnan, Sen and Warsi [RSW17] and Wilde [Wil17] studied the quantum wiretap channel in the one shot setting culminating in the optimal bound

$$
\begin{equation*}
R_{A}=\max _{\rho}\left(I_{H}^{\varepsilon}(X: C)_{\rho}-I_{\max }^{\delta}(X: E)_{\rho}\right), \tag{4}
\end{equation*}
$$

where the one shot mutual informations (defined formally later on) are taken over all classical quantum states of the form $\rho^{X C E}=\sum_{x} p(x)|X\rangle^{X}\langle x| \rho_{x}^{C E}$, and the maximisation is done over all random variables $X$ and encoding mappings $x \mapsto \sigma_{x}^{A}$. The state $\rho_{x}^{C E}$ is obtained by applying the channel $\mathfrak{C}$ to $\sigma_{x}^{A}$. This one shot bound reduces to Devetak's bound in the asymptotic iid setting.

The above works behoove us to study the one shot private classical capacity of the quantum multiple access channel (QMAC). The channel $\mathfrak{C}$ is modelled as a CPTP map from input Hilbert space $A \otimes B$ to output Hilbert space $C \otimes E$. Here $A, B$ are to be thought of Hilbert spaces of two independent senders Alice and Bob. Alice gets a message $m_{a} \in\left[2^{R_{A}}\right]$ and Bob gets an independent message $m_{b} \in\left[2^{R_{B}}\right]$. Alice encodes $m_{a}$ into a density matrix $\sigma_{m_{a}}^{A}$ in the Hilbert space $A$ and Bob independently encodes $m_{b}$ into $\sigma_{m_{b}}^{B}$. Then $\sigma_{m_{a}}^{A} \otimes \sigma_{m_{b}}^{B}$ is fed into $\mathfrak{C}$ giving rise to a state $\rho_{m_{a}, m_{b}}^{C E}$ at the channel output. Let $0<\varepsilon, \delta<1$. We require that, averaged over the uniform probability distribution on $\left(m_{a}, m_{b}\right) \in\left[2^{R_{A}}\right] \times\left[2^{R_{B}}\right]$, Charlie should be able to recover ( $m_{a}, m_{b}$ ) with probability at least $\varepsilon$ from $\rho_{m_{a}, m_{b}}^{C}$, and Eve's state $\rho_{m_{a}, m_{b}}^{E}$ should be $\delta$-close to some fixed state $\bar{\rho}^{E}$ in trace distance. We then say that $\left(R_{A}, R_{B}\right)$ is an achievable rate pair for private classical communication over $\mathfrak{C}$ with error $\varepsilon$ and leakage $\delta$. Note that Equation 4 for the quantum wiretap channel above holds for the stronger definition of leakage in trace distance, thus improving even on the classical asymptotic iid wiretap results proved earlier. The trace distance leakage definition is stronger because $\delta_{n}$-leakage in trace distance implies asymptotically vanishing mutual information leakage if $\delta_{n} \rightarrow 0$. Continuing this tradition, in this paper we will aim for secrecy in the trace distance leakage sense only.

It is thus natural to ponder about private classical communication over a QMAC. A first attempt in this regard was made by Aghaee and Akhbari [AA20] all the way in the one shot setting, but their proof has the following serious gap. They use the single sender convex split lemma of Anshu, Devabathini and Jain [ADJ17] in order to guarantee individual secrecy for Alice and individual secrecy for Bob, but that does not guarantee joint secrecy. A natural way to get joint secrecy would be to use the tripartite convex split lemma of Anshu, Jain and Warsi [AJW18] instead. Indeed, Charlie can use the simultaneous QMAC decoder of Sen [Sen21] and Eve can be obfuscated via the tripartite convex split lemma in order to get the following achievable rate region of private classical communication over a QMAC: the union of rate regions of the form

$$
\begin{align*}
R_{A} & \leq I_{H}^{\varepsilon}(X: C Y \mid Q)_{\rho}-I_{\max }(X: E \mid Q)_{\rho}, \\
R_{B} & \leq I_{H}^{\varepsilon}(Y: C X \mid Q)_{\rho}-I_{\max }(Y: E \mid Q)_{\rho},  \tag{5}\\
R_{A}+R_{B} & \leq I_{H}^{e}(X Y: C \mid Q)_{\rho}-I_{\max }(X Y: E \mid Q)_{\rho},
\end{align*}
$$

where the mutual information is measured with respect to classical quantum state of the form

$$
\rho^{Q X Y C E}=\sum_{q, x, y} p(q) p(x \mid q) p(y \mid q)|q, x, y\rangle^{Q X Y}\langle q, x, y| \otimes \rho_{x y}^{C E},
$$

$Q$ is an auxilliary 'time sharing' random variable, $X, Y$ are auxilliary random variables that are independent given $Q, x \mapsto \sigma_{x}^{A}, y \mapsto \sigma_{y}^{B}$ are independent encoding maps, and the state $\rho_{x y}^{C E}$ is obtained by applying the channel $\mathfrak{C}$ to $\sigma_{x}^{A} \otimes \sigma_{y}^{B}$. The union is taken over all probability distributions of the form $p(q) p(x \mid q) p(y \mid q) p(a \mid x) p(b \mid y)$ and encoding maps $x \mapsto \sigma_{x}^{A}, y \mapsto \sigma_{y}^{B}$.

Though we will not formally prove the achievability of Equation 5 in this paper, the reader can easily do so using the techiques outlined here combined with the tripartite convex split lemma. However Equation 5 has a big drawback viz. the terms for Eve are stated in terms of the non-smooth max mutual information even though the terms for Charlie are stated in terms of the smooth hypothesis testing mutual information. Because of this drawback, we cannot conclude that in the asymptotic iid limit the one shot bounds lead to the natural quantum version of Equation 2. The drawback arises because the tripartite convex split lemma [AJW18] has only been proved for nonsmooth max mutual information. Proving it for smooth max mutual information is related to the simultaneous smoothing problem [DF13], a major open problem in quantum information theory.

In this work, we obtain an alternate one shot inner bound for private classical communication over a QMAC that is stated in terms of smooth mutual information quantities only. Our inner bound is contained inside the smooth version of the region of Equation 5. Nevertheless we are able to show that in the asymptotic iid setting, our one shot bound leads to the natural quantum version of Equation 2. Our inner bound holds for joint secrecy of Alice and Bob under the leakage in trace distance definition and is the first non-trivial inner bound for private classical communication over a QMAC.

We prove our inner bound by using three powerful information theoretic techniques. The first technique is the use of rate splitting, originally developed by Grant et al. [GRUW01] in the classical asymptotic iid setting, but recently extended to the one shot quantum setting by the present authors [CNS21]. Rate splitting allows us to split one sender, say Alice, into two independent senders Alice1 and Alice2. The two sender QMAC then becomes a three sender QMAC, the advantage of which will become clear very soon. The second technique is simultaneous decoding for sending classical information over a QMAC recently developed by Sen [Sen21]. Simultaneous decoding is used by Charlie to decode Alice's message block and codeword within the block, which has been split into Alice1's and Alice2's parts, and Bob's message block and codeword within the block, at any rate triple contained in the standard polyhedral achievable region of a three sender MAC. The three senders also have to ensure Eve's obfuscation which they do by randomising within a block as in the proof of the original classical wiretap channel result of Equation 1. The third and final technique that guarantees that this obfuscation strategy works is a novel result proved in this paper called the smoothed distributed covering lemma. This lemma is the main technical advancement of this work and should be useful elsewhere. It is proved by repeated applications of the single sender convex split lemma [ADJ17], which happens to hold for the smooth max mutual information. The lemma ensures the joint secrecy of Alice1, Alice2 and Bob with a rate region described by smooth max mutual information quantitites. Though this region is inferior to what one would get from a smoothed tripartite convex split lemma, it is nevertheless good
enough to lead to the desired region in the asymptotic iid setting. The advantage of splitting Alice into Alice1 and Alice2 now becomes clear because the split together with the distributed smoothed covering lemma gives more obfuscation rate tuples. This leads to a larger inner bound region for private classical communication than what one would obtain otherwise without rate splitting. In particular the region obtained without rate splitting seems to be insufficient to obtain the desired rate region in the asymptotic iid limit in the absence of a simultaneous smoothing result.

## 2 Preliminaries

All Hilbert spaces in this paper are finite dimensional. By $\mathcal{H}(A)$ we mean the Hilbert space associated with the system $A$. We will often use $\mathcal{H}(A)$ and $A$ interchangeably, in the sense that, when we say a state $\rho$ is defined on $A$, we mean the positive semidefinite matrix $\rho$ belongs to the Hilbert space $\mathcal{H}(A)$.

By the term 'cq state' we mean some classical-quantum state $\rho^{X B}$ which is of the form

$$
\rho^{X B}:=\sum_{x \in \mathcal{X}}|x\rangle\left\langle\left. x\right|^{X} \otimes \rho_{x}^{B}\right.
$$

Definition 2.1. Let $\rho$ and $\sigma$ be two states in the same Hilbert space. Then, given $0 \leq \varepsilon<1$ we define the smooth hypothesis testing relative entropy of $\rho$ with respect to $\sigma$ aa

$$
D_{H}^{\varepsilon}(\rho \| \sigma):=\max _{\Pi: \operatorname{Tr}[\Pi \rho] \geq 1-\varepsilon}-\log \operatorname{Tr}[\Pi \sigma]
$$

Definition 2.2. Given a state $\rho^{A B}$, the smooth hypothesis testing mutual information between $A$ and $B$ is defined as

$$
I_{H}^{\varepsilon}(A: B)_{\rho}:=D_{H}^{\varepsilon}\left(\rho^{A B} \| \rho^{A} \otimes \rho^{B}\right)
$$

We will require the notion of the purified distance, which, for any two states $\rho$ and $\sigma$ in the space Hilbert space, is defined as

$$
P(\rho, \sigma):=\sqrt{1-F^{2}(\rho, \sigma)}
$$

where $F(\rho, \sigma)$ is the fidelity between $\rho$ and $\sigma$. On occasion we will find it easier to use other metrics, such as the 1-norm. To that end, the Fuchs-Van de Graaf inequalities essentially prove that all these metrics are equivalent:

Fact 2.3. For any two states $\rho$ and $\sigma$ in the same Hilbert space, the following holds

$$
1-\frac{1}{2}\|\rho-\sigma\|_{1} \leq F(\rho, \sigma) \leq \sqrt{1-\frac{1}{4}\|\rho-\sigma\|_{1}^{2}}
$$

Definition 2.4. Given two states $\rho$ and $\sigma$ in the same Hilbert space, we define the max relative entropy of $\rho$ with respect to $\sigma$ as

$$
D_{\max }(\rho \| \sigma):=\inf \left\{\lambda \mid \rho \leq 2^{\lambda} \sigma\right\}
$$

Definition 2.5. Given the setting of Definition 2.4, the $\varepsilon$ smooth max relative entropy is defined as

$$
D_{\max }^{\varepsilon}(\rho \| \sigma):=\inf _{\rho^{\prime} \in B^{\varepsilon}(\rho)} D_{\max }(\rho \| \sigma)
$$

where $B^{\varepsilon}(\rho)$ is the $\varepsilon$ ball around $\rho$ with respect to the purified distance.
Definition 2.6. Given a state $\rho^{A B}$, the smooth max mutual information between $A$ and $B$ is defined as

$$
I_{\max }^{\varepsilon}(A: B):=D_{\max }^{\varepsilon}\left(\rho^{A B} \| \rho^{A} \otimes \rho^{B}\right)
$$

## 3 Our results

We study the single shot private capacity of the classical quantum multiple access channel. The problem is as follows: we are given a quantum multiple access channel along with two independent classical distributions $P_{X}$ and $P_{Y}$ on the inputs for the two senders, Alice and Bob. Suppose that the input distributions are supported on the classical alphabets $\mathcal{X}$ and $\mathcal{Y}$. The output states corresponding to each input tuple $(x, y)$ is a shared quantum state between the receiver Charlie and the eavesdropper Eve. This situation is usually modelled by the following so called control state:

$$
\begin{equation*}
\rho^{X Y C E}:=\sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} P_{X}(x) \cdot P_{Y}(y)|x\rangle\left\langle\left. x\right|^{X} \otimes \mid y\right\rangle\left\langle\left. y\right|^{Y} \otimes \rho_{x, y}^{C E}\right. \tag{6}
\end{equation*}
$$

The goal is for Alice and Bob to send messages $m$ and $n$ from the sets [ $M$ ] and [ $N$ ] via this channel to Bob in such a way that Eve does not gain any information about the message tuple that was sent, yet Charlie is able to decode both Alice an Bob's messages with high probability. To be precise, we require that, given $\varepsilon, \delta>0$ :

1. For all messages $m$ and $n$,

$$
\operatorname{Pr}[(\hat{m}, \hat{n}) \neq(m, n)] \leq \varepsilon
$$

where ( $\hat{m}, \hat{n}$ ) is Charlie's estimate of the messages sent by Alice and Bob. This is called the correctness condition.
2. There exists a state $\sigma^{E}$ such that, for all tuples ( $m, n$ )

$$
\left\|\rho_{f(m), g(n)}^{E}-\sigma^{E}\right\|_{1} \leq \delta
$$

where $\rho_{m, n}^{E}$ is the state induced on Eve's system when Alice and Bob send the messages $m$ and $n$ after encoding the messages into the input space of the channel via the maps $f:[M] \rightarrow$ $\mathcal{X}$ and $g:[N] \rightarrow \mathcal{Y}$. This is called the secrecy condition.

### 3.1 Previous Work

A simpler variant of this problem, formally known as the classical-quantum wiretap channel, has been studied before in the one-shot setting by Radhakrishnan-Sen-Warsi [RSW17]. The heart of the argument used in that paper is a technical tool called the covering lemma. To gain some understanding of the RSW argument, consider the following strategy:

1. Sender Alice chooses $2^{R}$ symbols $\left\{x_{1}, x_{2}, \ldots, x\left(2^{R}\right)\right\}$ iid from her input distribution $P_{X}$.
2. She then divides the list of $2^{R}$ symbols into blocks, each of size $2^{K}$.
3. Alice then assigns a block number to each message $m \in[M]$.
4. To send the message $m$, Alice first looks at the block of symbols corresponding to $m$, say $\left(x\left(i_{1}\right), x\left(i_{2}\right), \ldots,\left(i_{2^{K}}\right)\right)$. She then randomly picks an index $i_{\text {RAND }}$ from this block and sends the corresponding symbol through the channel.

Correctness : It is known [WR12] that as long as the rate $R-K$ is at most slightly less than the smooth hypothesis testing mutual information $I_{H}^{\varepsilon}(X: C)$, the decoding error is at most $\varepsilon$. Please note that the quantity $I_{H}^{\varepsilon}(X: C)$ is computed with respect to the control state corresponding to only a single sender for this channel. [AJ18]

Secrecy: To show that the secrecy condition holds, RSW proved a novel one-shot covering lemma. They showed that, as long as $K$ is slightly more than the smooth max mutual information $I_{\max }^{\delta}(X$ : $E)$ (again computed with respect to the single sender control state), then, for every message $m \in$ $[M]$, the following condition holds with high probability, over all choices of the codebook:

$$
\left\|\frac{1}{K} \sum_{j \in[K]} \rho_{i_{j}}^{E}-\rho^{E}\right\|_{1} \leq \delta
$$

where the indices $\left\{i_{j}\right\}$ belong to the block corresponding to message $m$, and $\rho^{E}$ is the marginal of the control state on $E$.

To see that this implies that privacy holds in the protocol, notice that the expression on the right inside the norm is precisely the state induced by Alice's encoding function on the system $E$.

### 3.1.1 The Single Shot Covering Lemma

The covering lemma proved by RSW goes via an operator Chernoff bound. While this style of argument gives a strong concentration bound for the secrecy condition, one caveat is that the rate $K$ becomes dependant on the dimension of the eavesdropper system $E$. To be precise, for the secrecy condition to hold, RSW require the following condition:

$$
K \geq I_{\max }^{O(\delta)}(X: E)-\log \delta+\log \log |E|+O(1)
$$

Strictly speaking, such a strong condition is not necessary to prove the covering lemma. One can show that the secrecy condition holds in expectation over the choice of symbols inside the block. To make things precise, consider the following fact:
Fact 3.1. Given the control state $\sum_{x \in \mathcal{X}} P_{X}(x)|x\rangle\left\langle\left. x\right|^{X} \otimes \rho^{E}\right.$ and $\delta>0$, let $\left\{x_{1}, x_{2}, \ldots, x_{K}\right\}$ be iid samples from the distribution $P_{X}$. Then, given the condition

$$
\log K \geq I_{\max }^{O(\delta)}(X: E)_{\rho}-\log \delta
$$

the following holds

$$
\mathbb{E}_{x_{1}, x_{2}, \ldots, x_{K}}\left\|\frac{1}{K} \sum_{i \in[K]} \rho_{x_{i}}^{E}-\rho^{E}\right\|_{1} \leq \delta
$$

This average version of the covering lemma is a direct consequence of the convex split lemma proved by Anshu, Devabathini and Jain [ADJ17], adapted to cq states. A proof of Fact 3.1 for the non-smooth max information can be found in [AJW19]. The smoothing argument is standard and can be easily adapted from the smooth version if the convex split lemma proved by Wilde [Wil17].

### 3.2 Our Contribution

As mentioned earlier we consider the problem of sending information privately over a classicalquantum multiple access channel in the single shot setting. The achievable rate region we would like to recover is as follows:

$$
\begin{aligned}
& \log M \lesssim I_{H}^{\varepsilon}(X: Y C)-I_{\max }^{\delta}(X: E) \\
& \log N \lesssim I_{H}^{\varepsilon}(Y: X C)-I_{\max }^{\delta}(Y: E) \\
& \log M+\log N \lesssim I_{H}^{\varepsilon}(X Y: C)-I_{\max }^{\delta}(X Y: E)
\end{aligned}
$$

where we have omitted the $\log \varepsilon$ and $\log \delta$ terms for clarity. To show that the above region is achievable, we will need the following technical tools:

1. A distributed covering lemma for multiple senders.
2. A decoder which can decode any message pair, which corresponds to a rate in the desired region.

### 3.2.1 A Smoothed Distributed Covering Lemma

We will address the second requirement later. For the distributed covering lemma, we wish to find the rate pairs $\left(K_{1}, K_{2}\right)$ such that, for $K_{1}$ and $K_{2}$ iid samples $\left\{x_{1}, x_{2}, \ldots, x_{K_{1}}\right\}$ and $\left\{y_{1}, y_{2}, \ldots, y_{K_{2}}\right\}$ from the distributions $P_{X}$ and $P_{Y}$ respectively, the following holds

Notice that a naïve extension of the single user covering lemma will not work. This is because, the total number of random bits required for the secrecy condition is at least $I_{\max }^{\delta}(X Y: E)$ bits while the naïve lemma would require only $\log K_{1}+\log K_{2} \geq I_{\max }^{\delta}(X: E)+I_{\max }^{\delta}(Y: E)$ random bits.

One way to prove the distributed covering lemma would be to appeal to a multipartite convex split lemma, and then exploit the connection between the convex split lemma for cq states and a covering lemma [AJW19]. Indeed such a non-smooth multipartite version of the convex split lemma does exist and is not hard to prove [AJW18]. However, this proof strategy will give us a region of the following kind:

$$
\begin{aligned}
& \log K_{1}>I_{\max }(X: E)-\log \delta \\
& \log K_{2}>I_{\max }(Y: E)-\log \delta \\
& \log K_{1}+\log K_{2}>I_{\max }(X Y: E)-\log \delta
\end{aligned}
$$

One can see that this region is described in terms of the non-smooth max information. Indeed, obtaining the above region in terms of the smooth max information is a major open problem in quantum information theory, and is known as the simultaneous smoothing conjecture [DF13]. In the absence of a smoothed region, we cannot hope to recover the desired rates in terms of the quantum mutual information in the asymptotic iid limit.

In this paper, we overcome this problem by taking a different approach. Instead of straightaway trying to show the secrecy property of the entire inverted pentagonal region (with two sides at infinity), we first prove a sequential covering lemma for a corner point of the region. We show that, if Alice randomises over a block of size $\log K_{1}>I_{\max }^{\delta}(X: E)$ and Bob randomises over a block of size $\log K_{2}>I_{\max }^{\delta}(Y: X E)$, then indeed Eq. (7) holds, albeit with a worse dependence in $\delta$. A similar statement holds for the other corner point as well. We call this a successive cancellation style covering lemma, since the strategy is similar in spirit to the successive cancellation style decoding for the multiple access channel.

To be precise, we prove the following lemma:
Lemma 3.2. Given the control state in Eq. (6), $\delta>0$ and $0<\varepsilon^{\prime}<\delta$ let $\left\{x_{1}, x_{2}, \ldots, x_{K_{1}}\right\}$ and $\left\{y_{1}, y_{2}, \ldots, y_{K_{2}}\right\}$ be iid samples from the distributions $P_{X}$ and $P_{Y}$. Then, if

$$
\begin{aligned}
& \log K_{1} \geq I_{\max }^{\delta-\varepsilon^{\prime}}(X: E)_{\rho}+\log \frac{3}{\varepsilon^{\prime 3}}-\frac{1}{4} \log \delta \\
& \log K_{2} \geq I_{\max }^{\delta-\varepsilon^{\prime}}(Y: E X)_{\rho}+\log \frac{3}{\varepsilon^{\prime 3}}-\frac{1}{4} \log \delta+O(1)
\end{aligned}
$$

the following holds

$$
\underset{\substack{x_{1}, x_{2}, \ldots, x_{K_{1}} \sim P_{X} \\ y_{1}, y_{2}, \ldots, y_{K_{2}} \sim P_{Y}}}{ }\left\|\frac{1}{K_{1} \cdot K_{2}} \sum_{i}^{K_{2}} \sum_{j}^{K_{1}} \rho_{x_{i}, y_{j}}^{E}-\rho^{E}\right\|_{1} \leq 20 \delta^{1 / 8}
$$

Remark 3.3. The proof of Lemma 3.2 can be extended to the case when there are more than two senders. The argument is a straightforward induction on the triangle inequality in the last step of the proof. The dependence of the expected error on $\delta$ worsens however, with the constant increasing from 20 to 40 in the case when there are three senders.

To recover the non-corner points in the idealised secrecy region, we use the idea of rate splitting. Rate splitting was first suggested by Grant, Rimoldi, Urbanke and Whiting [GRUW01] as an alternative to time sharing to achieve the non-corner points on the dominant face of the achievable pentagon, in the context of sending classical information over a classical multiple access channel in the asymptotic iid setting. Recently, Chakraborty, Nema and Sen [CNS21] adapted this technique to the one-shot fully quantum regime to derive entanglement transmission codes across a quantum multiple access channel.

The idea of rate splitting is roughly as follows : Given the input distribution $P_{X}$ corresponding to the sender Alice, we split the distribution into two independent distributions $P_{U}^{\theta}$ and $P_{V}^{\theta}$, with respect to a parameter $\theta \in[0,1]$. These two new distributions correspond to two new senders Alice $_{1}$ and Alice $2 . U^{\theta}$ and $V^{\theta}$ are independent random variables, each supported on the alphabet $\mathcal{X}$. This splitting is done by using a splitting function $f: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$, which has the following properties:

1. $f\left(U^{\theta}, V^{\theta}\right) \sim P_{X}$ for all $\theta \in[0,1]$.
2. For $\theta=0, P_{f\left(U^{\theta}, V^{\theta}\right) \mid U^{\theta}}=P_{X}$ and for $\theta=1, P_{f\left(U^{\theta}, V^{\theta}\right) \mid U^{\theta}}$ puts all its mass on one element.
3. For a fixed $u, P_{f\left(u^{\theta}, V^{\theta}\right) \mid u^{\theta}}$ is a continuous function of $\theta \in[0,1]$.

Grant et.al. proved that such a family of triples $\left\{\left(P_{U}^{\theta}, P_{V}^{\theta}, f\right)\right\}$ exists which obeys these properties. They did this via the following explicit construction:

For a fixed $\theta \in[0,1]$ and assuming that the elements of $\mathcal{X}$ have an ordering,

1. $\operatorname{Pr}\left[U^{\theta} \leq u\right]:=\theta \cdot \operatorname{Pr}[X \leq u]+1-\theta$
2. $\operatorname{Pr}\left[V^{\theta} \leq v\right]:=\frac{\operatorname{Pr}[X \leq v]}{\operatorname{Pr}\left[U^{\theta} \leq v\right]}$
3. $f(u, v):=\max (u, v)$

We will refer to this construction as the max contruction.
Using this split, we can rewrite the control state in Eq. (6) after splitting as follows:

$$
\begin{equation*}
\rho_{\theta}^{U V Y C E}:=\sum_{\substack{u, v \in \mathcal{X} \\ y \in \mathcal{Y}}} P_{U}^{\theta}(u) \cdot P_{V}^{\theta}(v) \cdot P_{Y}(y)|u\rangle\left\langle\left. u\right|^{U} \otimes \mid v\right\rangle\left\langle\left. v\right|^{V} \otimes \mid y\right\rangle\left\langle\left. y\right|^{Y} \otimes \rho(\theta)_{u, v, y}^{C E}\right. \tag{8}
\end{equation*}
$$

where for each $(u, v)$

$$
\rho(\theta)_{u, v, y}^{C E}:=\rho_{f(u, v), y}^{C E}
$$

Armed with this split state, we invoke the three sender version of Lemma 3.2 to prove the following theorem:

Theorem 3.4. Given the control state in Eq. (8), $\delta>0$ and $0<\varepsilon^{\prime}<\delta$ let $\left\{u_{1}, u_{2}, \ldots, u_{K_{1}}\right\},\left\{y_{1}, y_{2}, \ldots, y_{K_{2}}\right\}$ and $\left\{v_{1}, v_{2}, \ldots, v_{K_{3}}\right\}$ be iid samples from the distributions $P_{U}^{\theta}, P_{Y}$ and $P_{V}^{\theta}$ respectively, for a fixed $\theta \in[0,1]$. Then, if

$$
\begin{aligned}
& \log K_{1} \geq I_{\max }^{\delta-\varepsilon^{\prime}}\left(U^{\theta}: E\right)_{\rho_{\theta}}+\log \frac{3}{\varepsilon^{\prime 3}}-\frac{1}{4} \log \delta \\
& \log K_{2} \geq I_{\max }^{\delta-\varepsilon^{\prime}}\left(Y: E U^{\theta}\right)_{\rho_{\theta}}+\log \frac{3}{\varepsilon^{\prime 3}}-\frac{1}{4} \log \delta+O(1) \\
& \log K_{3} \geq I_{\max }^{\delta-\varepsilon^{\prime}}\left(V^{\theta}: E Y U^{\theta}\right)_{\rho_{\theta}}+\log \frac{3}{\varepsilon^{\prime 3}}-\frac{1}{4} \log \delta+O(1)
\end{aligned}
$$

the following holds

$$
\mathbb{E}_{\substack{u_{1}, u_{2}, \ldots, u_{K_{K}} \sim p_{u}^{\theta} \\ y_{1}, y_{2}, \ldots, y_{1} \sim M_{2} \\ v_{1}, v_{2}, \ldots, v_{K_{K}} \sim \mathcal{V}_{Y}}}\left\|\frac{1}{K_{V}^{\theta} \cdot K_{2} \cdot K_{3}} \sum_{i, j, k}^{K_{1}, K_{2}, K_{3}} \rho(\theta)_{u_{i}, y_{j}, v_{k}}^{E}-\rho^{E}\right\|_{1} \leq 40 \delta^{1 / 8}
$$

## Remark 3.5.

1. Note that by construction of the triple $\left(P_{U}^{\theta}, P_{V}^{\theta}, f\right)$,

$$
\rho^{E}=\rho_{\theta}^{E}
$$

2. Alice has to randomise over of total block of size of $K_{1} \cdot K_{3}$. This implies that, thinking of Alice as the combination of the two senders Alice $_{1}$ and Alice $_{2}$, the size of the block over which Alice has to randomize has to be at least

$$
I_{\max }^{\delta-\varepsilon^{\prime}}\left(U^{\theta}: E\right)_{\rho_{\theta}}+I_{\max }^{\delta-\varepsilon^{\prime}}\left(V^{\theta}: E Y U^{\theta}\right)_{\rho_{\theta}}+2 \log \frac{3}{\varepsilon^{\prime 3}}-\frac{1}{2} \log \delta+O(1)
$$

3. For $\theta=0$ and $\theta=1$, the expressions, $I_{\max }^{\delta-\varepsilon^{\prime}}\left(U^{\theta}: E\right)_{\rho_{\theta}}$ and $I_{\max }^{\delta-\varepsilon^{\prime}}\left(V^{\theta}: E Y U^{\theta}\right)_{\rho_{\theta}}$ take the value zero respectively. This can be easily seen from the properties of the max construction.
4. When $\theta \in\{0,1\}$, the secrecy region collapses to the two sender case. For $\theta=0$, the user Alice ${ }_{1}$ becomes trivial, and similarly for Alice 2 when $\theta=1$. These values of $\theta$ thus correspond to the corner points of the secrecy region.

As $\theta$ ranges from 0 to 1 , the point $\left(I_{\max }^{\delta-\varepsilon^{\prime}}\left(U^{\theta}: E\right)+I_{\max }^{\delta-\varepsilon^{\prime}}\left(V^{\theta}: E Y U^{\theta}\right), I_{\max }^{\delta-\varepsilon^{\prime}}\left(Y: E U^{\theta}\right)\right)$ traces out a curve between the corner points, which lies on or above the line joining the corner points. To show that this is true, we use the following properties of the smooth max mutual information:

Lemma 3.6. Given the control state in Eq. (6) and the post split state in Eq. (8) for some fixed $\theta \in[0,1]$, the following holds

$$
I_{\max }^{\varepsilon}\left(U^{\theta} V^{\theta} Y: E\right)_{\rho_{\theta}}=I_{\max }^{\varepsilon}(X Y: E)_{\rho}
$$

for any $\varepsilon>0$.
Lemma 3.7. Given a state $\varphi^{R A B}$, not necessarily pure, and $\varepsilon>0$, the following holds

$$
I_{\max }^{12 \varepsilon}(R: A B)_{\varphi} \leq I_{\max }^{\varepsilon-\gamma}(R: A)_{\varphi}+I_{\max }^{\varepsilon-\gamma}(R A: B)_{\varphi}+2 \log \frac{1}{\varepsilon}+\log \frac{3}{\gamma^{2}}
$$

These two lemmas together show that the boundary of the secrecy region between the corner points lies on or above the straight line $x+y=I_{\max }^{O(\varepsilon)}(X Y: E)_{\rho}$.

### 3.2.2 Decoding

We now turn our attention to the problem of Charlie decoding the messages sent by Alice and Bob. There are two kinds of decoders we can consider:

1. Successive Cancellation: One way to decode the messages would be a successive cancellation strategy, in which Charlie first decodes Alice ${ }_{1}$, then using Alice ${ }_{1}$ 's message as side information he decodes Bob, and finally using Alice ${ }_{1}$ and Bob's messages as side information he decodes Alice 2 . This gives us an achievable region which is the union over $\theta \in[0,1]$ over all rectangles subtended by the point

$$
\left(I_{H}^{\varepsilon}\left(U^{\theta}: C\right)+I_{H}^{\varepsilon}\left(V^{\theta}: C U^{\theta} Y\right), I_{H}^{\varepsilon}\left(Y: C U^{\theta}\right)\right)
$$

where we have neglected the additive $\log \varepsilon$ terms for brevity.

Remark 3.8. For reasons that will become clear shortly, instead of following the order of decoding given above, we actually would like to decode in the order Alice ${ }_{1}$-Bob-Alice ${ }_{1}$. This would give the rate point

$$
\left(I_{H}^{\varepsilon}\left(V^{\theta}: C\right)+I_{H}^{\varepsilon}\left(U^{\theta}: C V^{\theta} Y\right), I_{H}^{\varepsilon}\left(Y: C V^{\theta}\right)\right)
$$

2. Simultaneous Decoding: The other decoding strategy we consider is simultaneous decoding. Given a cq-mac and the control state in Eq. (6), a simultaneous decoder gives us the following achievable region:

$$
\begin{aligned}
& R_{1}<I_{H}^{\varepsilon}(X: Y C)-\log \frac{1}{\varepsilon} \\
& R_{2}<I_{H}^{\varepsilon}(Y: X C)-\log \frac{1}{\varepsilon} \\
& R_{1}+R_{2}<I_{H}^{\varepsilon}(X Y: C)-\log \frac{1}{\varepsilon}
\end{aligned}
$$

where $R_{1}$ and $R_{2}$ correspond to Alice and Bob's rates.

In the absence of chain rules for the smooth hypothesis testing mutual information, the rate region given by the successive cancellation decoder is a deformed version of the pentagonal region we expect.

This issue can be mitigated somewhat if we use the simultaneous decoder. Thus, we will use simultaneous decoding as our decoding strategy of choice. We elaborate on this in the next section.

### 3.2.3 Simultaneous Decoding

We will use the construction given by Sen in [Sen21]. Until recently, the existence of such a simultaneous decoder for the cq-mac which recovers the rate region given by Winter in [Win01] in the asymptotic iid setting was a major open problem. To be precise, Sen proved the following fact:

Fact 3.9. Given a cq mac and its associated control state Eq. (6), there exists an encoding and decoding scheme such that, all rate pairs $\left(R_{1}, R_{2}\right)$, where $R_{1}$ corresponds to Alice and $R_{2}$ corresponds to Bob, are achievable for transmission of classical information of the channel with error at most $49 \sqrt{\varepsilon}$ :

$$
\begin{aligned}
& R_{1} \leq I_{H}^{\varepsilon}(X: Y C)+\log \varepsilon-1 \\
& R_{2} \leq I_{H}^{\varepsilon}(Y: X C)+\log \varepsilon-1 \\
& R_{1}+R_{2} \leq I_{H}^{\varepsilon}(X Y: C)+\log \varepsilon-1
\end{aligned}
$$

The above lemma is easily generalised to the case when there are multiple senders. In our case, we use a three sender simultaneous decoder, which, for every fixed $\theta \in[0,1]$ gives us the following achievable region for Alice ${ }_{1}$, Bob and Alice ${ }_{2}$ :

$$
\begin{array}{|l|}
\hline R_{10}<I_{H}^{\varepsilon}\left(U^{\theta}: C V^{\theta} Y\right)+\log \varepsilon-1 \\
R_{2}<I_{H}^{\varepsilon}\left(Y: C U^{\theta} V^{\theta}\right)+\log \varepsilon-1 \\
R_{11}<I_{H}^{\varepsilon}\left(V^{\theta}: C U^{\theta} Y\right)+\log \varepsilon-1 \\
R_{10}+R_{11}<I_{H}^{\varepsilon}\left(U^{\theta} V^{\theta}: C Y\right)+\log \varepsilon-1 \\
R_{10}+R_{2}<I_{H}^{\varepsilon}\left(U^{\theta} Y: C V^{\theta}\right)+\log \varepsilon-1 \\
R_{2}+R_{11}<I_{H}^{\varepsilon}\left(Y V^{\theta}: C U^{\theta}\right)+\log \varepsilon-1 \\
R_{10}+R_{2}+R_{11}<I_{H}^{\varepsilon}\left(U^{\theta} Y V^{\theta}: C\right)+\log \varepsilon-1 \\
\hline
\end{array}
$$

Here $R_{10}, R_{2}$ and $R_{11}$ corresponds to Alice ${ }_{1}$, Bob and Alice ${ }_{2}$ respectively. The bound on the last term is equal to $I_{H}^{\varepsilon}(X Y: C)+\log \varepsilon-1$. The proof of this fact is the same as Lemma 3.6.

For every $\theta \in[0,1]$, we will project the above rate region to the 2 dimensional space which contains the achievable rate points for Alice and Bob. To obtain the full achievable region, we take a union bound over all $\theta$. to be precise, we show the following lemma:

Lemma 3.10. Given a 2 sender cq mac and the associated control state in Eq. (6), and its corresponding split state Eq. (8) for some fixed $\theta \in[0,1]$, the following rate region is achievable for sending classical information over the channel with error $\varepsilon^{1 / 8}$ is as follows:

$$
\begin{align*}
& R_{1} \leq I_{H}^{\varepsilon}\left(U^{\theta} V^{\theta}: C Y\right)+\log \varepsilon-1  \tag{9}\\
& R_{1} \leq I_{H}^{\varepsilon}\left(V^{\theta}: C U^{\theta} Y\right)+I_{H}^{\varepsilon}\left(U^{\theta}: C V^{\theta} Y\right)+2 \log \varepsilon-2 \\
& R_{2} \leq I_{H}^{\varepsilon}\left(Y: C U^{\theta} V^{\theta}\right)+\log \varepsilon-1 \\
& R_{2} \leq I_{H}^{\varepsilon}\left(Y V^{\theta}: C U^{\theta}\right)+\log \varepsilon-1 \\
& R_{2} \leq I_{H}^{\varepsilon}\left(U^{\theta} Y: C V^{\theta}\right)+\log \varepsilon-1 \\
& R_{1}+R_{2} \leq I_{H}^{\varepsilon}\left(V^{\theta}: C U^{\theta} Y\right)+I_{H}^{\varepsilon}\left(U^{\theta} Y: C V^{\theta}\right)+2 \log \varepsilon-2 \\
& R_{1}+R_{2} \leq I_{H}^{\varepsilon}\left(Y V^{\theta}: C U^{\theta}\right)+I_{H}^{\varepsilon}\left(U^{\theta}: C V^{\theta} Y\right)+2 \log \varepsilon-2 \\
& R_{1}+2 R_{2} \leq I_{H}^{\varepsilon}\left(U^{\theta} Y: C V^{\theta}\right)+I_{H}^{\varepsilon}\left(Y V^{\theta}: C U^{\theta}\right)+2 \log \varepsilon-2 \\
& R_{1}+R_{2} \leq I_{H}^{\varepsilon}\left(U^{\theta} Y V^{\theta}: C\right)+\log \varepsilon-1 \\
& \hline
\end{align*}
$$

where all the mutual information terms are computed with respect to Eq. (8).

### 3.2.4 The Private Capacity Region

Let us call the achievable region given by Lemma 3.10 as $\mathcal{S}_{\theta}$, for some fixed $\theta \in[0,1]$. For the same $\theta$, consider the block sizes $\left(K_{1}+K_{3}, K_{2}\right)$ given by Theorem 3.4. We define

$$
\mathcal{T}_{\theta}:=\left\{\left(\log K, \log K^{\prime}\right) \mid K \geq K_{1} \cdot K_{3}, K^{\prime} \geq K_{2}\right\}
$$

Then, we can have the following theorem, which gives an inner bound on the region for private transmission of classical information over the cq mac

Theorem 3.11. Given a classical quantum multiple access channel, the control state in Eq. (6) and a split $\left(P_{U}^{\theta}, P_{V}^{\theta}, f\right)$ of the distribution $P_{X}$, for some $\theta \in[0,1]$ the rate pairs in the following region, are achievable for private transmission of messages across the channel

$$
\left(\bigcup_{\theta \in[0,1]}\left(\mathcal{S}_{\theta}-\mathcal{T}_{\theta}\right)\right)^{+}
$$

with decoding error at most $49 \sqrt{\varepsilon}$ and privacy leakage at most $40 \delta^{1 / 8}$, where $\varepsilon, \delta>0$ and $0<\varepsilon<\delta$. All the information quantities above are computed with respect to the split state $\rho_{\theta}^{U V Y C E}$. Here, the operation $(A-B)^{+}$, where $A$ and $B$ are sets of real numbers is defined as $\{\max (a-b, 0) \mid a \in A, b \in B\}$.

To precisely describe the set $\mathcal{S}_{\theta}-\mathcal{T}_{\theta}$, let

$$
\begin{aligned}
& \delta^{\prime}:=\delta-\varepsilon^{\prime} \\
& c:=\log \frac{1}{\varepsilon}+\log \frac{1}{\varepsilon^{\prime 3}}-\frac{1}{4} \log \delta+O(1)
\end{aligned}
$$

and define the rates

$$
\begin{aligned}
& R_{A}:=R_{1}-\log K \\
& R_{B}:=R_{2}-\log K^{\prime}
\end{aligned}
$$

To ease the burden on notation, we drop the superscripts from the random variables $U^{\theta}$ and $V^{\theta}$. Then, for a fixed $\theta \in[0,1]$, the region $\mathcal{S}_{\theta}-\mathcal{T}_{\theta}$ looks like

$$
\begin{aligned}
& R_{A} \leq I_{H}^{\varepsilon}(U V: Y C)-I_{\max }^{\delta^{\prime}}(U: E)-I_{\max }^{\delta^{\prime}}(V: U Y E)+c \\
& R_{A} \leq I_{H}^{\varepsilon}(V: U Y C)+I_{H}^{\varepsilon}(U: V Y C)-I_{\max }^{\delta^{\prime}}(U: E)-I_{\max }^{\delta^{\prime}}(V: U Y E)+2 c \\
& R_{B} \leq I_{H}^{\varepsilon}(Y: U V C)-I_{\max }^{\delta^{\prime}}(Y: U E)+c \\
& R_{B} \leq I_{H}^{\varepsilon}(Y V: U C)-I_{\max }^{\delta^{\prime}}(Y: U E)+c \\
& R_{B} \leq I_{H}^{\varepsilon}(U Y: V C)-I_{\max }^{\delta^{\prime}}(Y: U E)+c \\
& R_{A}+R_{B} \leq I_{H}^{\varepsilon}(V: U Y C)+I_{H}^{\varepsilon}(U Y: V C)-I_{\max }^{\delta^{\prime}}(U: E)-I_{\max }^{\delta^{\prime}}(V: U Y E)-I_{\max }^{\delta^{\prime}}(Y: U E)+2 c \\
& R_{A}+R_{B} \leq I_{H}^{\varepsilon}(Y V: U C)+I_{H}^{\varepsilon}(U: V Y C)-I_{\max }^{\delta^{\prime}}(U: E)-I_{\max }^{\delta^{\prime}}(V: U Y E)-I_{\max }^{\delta^{\prime}}(Y: U E)+2 c \\
& R_{A}+2 R_{B} \leq I_{H}^{\varepsilon}(U Y: C V)+I_{H}^{\varepsilon}(Y V: U C)-I_{\max }^{\delta^{\prime}}(U: E)-I_{\max }^{\delta^{\prime}}(V: U Y E)-2 I_{\max }^{\delta^{\prime}}(Y: U E)+2 c \\
& R_{A}+R_{B} \leq I_{H}^{\varepsilon}(U Y V: C)-I_{\max }^{\delta^{\prime}}(U: E)-I_{\max }^{\delta^{\prime}}(V: U Y E)-I_{\max }^{\delta^{\prime}}(Y: U E)+c
\end{aligned}
$$

Remark 3.12. 1. To get an idea as to what the above region looks like, first note that for $\theta=$ $\{0,1\}$, the region in Lemma 3.10 is equivalent to the following region:

$$
\begin{aligned}
& R_{1} \leq I_{H}^{\varepsilon}(X: C Y)+O(\log \varepsilon) \\
& R_{2} \leq I_{H}^{\varepsilon}(Y: C X)+O(\log \varepsilon) \\
& R_{1}+R_{2} \leq I_{H}^{\varepsilon}(X Y: C)+O(\log \varepsilon)
\end{aligned}
$$

2. This essentially looks like the achievable rate region for the 2 -sender mac. In fact, from Lemma 3.10 we can see that as $\theta$ ranges from 0 to 1 , the corresponding rate regions $\mathcal{S}_{\theta}$ that we get are subsets of this pentagonal region.
3. Note that, if the smooth hypothesis testing mutual information obeyed a chain rule with equality, then the region in Lemma 3.10 would be equivalent to the pentagonal region in Item 1 for all values of $\theta \in[0,1]$.
4. On the other hand, the secrecy region given by Theorem 3.4 and following it, looks like an inverted pentagon in the first quadrant with two sides at infinity, and the dominant face slightly warped due the chain rule for the smooth max information Lemma 3.7.
5. The final secrecy region thus looks like a smaller pentagon, but with the dominant face warped inwards.

### 3.3 Extension to the Asymptotic IID Regime

In this we show that our one-shot techniques can be used to recover the expected private capacity region of the classical quantum multiple access channel in the limit of asymptotically many channel uses. To do this, we first note some facts about the asymptotic behaviour of the smoothed information quantities we have used so far:

Fact 3.13. Given a classical quantum state with $N$ classical inputs

$$
\rho^{X_{1} X_{2} \ldots X_{N} C}:=\sum_{x_{1} x_{2} \ldots x_{N}} \prod_{i}^{N} P_{X_{i}}\left(x_{i}\right)\left|x_{i}\right\rangle\left\langle\left. x_{i}\right|^{X_{i}} \otimes \rho_{x_{1} x_{2} \ldots x_{N}}^{C}\right.
$$

let $J \subseteq[N]$. Then, for some $\varepsilon>0$ and an integer $n \in \mathbb{N}$, the following holds true in the limit of $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ for all J,

$$
\begin{aligned}
& \lim _{\varepsilon \rightarrow 0} \lim _{n \rightarrow \infty} \frac{1}{n} I_{H}^{\varepsilon}\left(X_{J}^{n}: C^{n} X_{J^{c}}^{n}\right)_{\rho^{\otimes n}}=I\left(X_{J}: C X_{J^{c}}\right)_{\rho} \\
& \lim _{\varepsilon \rightarrow 0} \lim _{n \rightarrow \infty} \frac{1}{n} I_{\max }^{\varepsilon}\left(X_{J}^{n}: C^{n} X_{J^{c}}^{n}\right)_{\rho^{\otimes n}}=I\left(X_{J}: C X_{J^{c}}\right)_{\rho}
\end{aligned}
$$

where $X_{J}:=\prod_{j \in J} X_{j}$.

Fact 3.13 allows us to prove the following theorem from Theorem 3.11:

Theorem 3.14. Given a classical quantum multiple access channel, the control state in Eq. (6), the following rate region is achievable for private transmission of messages across the channel, when asymptotically many channel uses are allowed:

$$
\begin{aligned}
& R_{\mathrm{ALICE}}^{\text {private }}<I(X: Y C)-I(X: E) \\
& R_{\mathrm{BOB}}^{\text {private }}<I(Y: X C)-I(Y: E) \\
& R_{\mathrm{ALICE}}^{\text {private }}+R_{\mathrm{BOB}}^{\text {private }}<I(X Y: C)-I(X Y: E)
\end{aligned}
$$

### 3.4 A Generalisation

A generalisation of the theorems presented in the previous sections can be shown to be true using a time sharing random variable. To be precise, instead of the input distributions $P_{X}$ and $P_{Y}$ on the classical alphabets $\mathcal{X}$ and $\mathcal{Y}$, we will consider the joint distribution $P_{Q} \otimes P_{X \mid Q} \cdot P_{Y \mid Q}$ over the alphabet $\mathcal{Q} \times \mathcal{X} \times \mathcal{Y}$. Consider the control state

$$
\begin{equation*}
\rho^{Q X Y C E}:=\sum_{q, x, y} P_{Q}(q) P_{X \mid Q}(x \mid q) \cdot P_{Y \mid Q}(y \mid q)|q, x, y\rangle\left\langle q, x,\left.y\right|^{Q X Y} \otimes \rho_{x, y}^{C E}\right. \tag{10}
\end{equation*}
$$

Define

$$
\rho^{X Y C E \mid Q}:=\left(\rho^{Q} \otimes \mathbb{1}\right)^{-1} \frac{\rho^{Q X Y C E}}{\operatorname{rank}\left(\rho^{Q}\right)}\left(\rho^{Q} \otimes \mathbb{1}\right)^{-1}
$$

Using the above state, one can define the conditional smooth hypothesis testing mutual information and the smooth max information. A version of Fact 3.9 with respect to the above conditional control state was shown to be true in [Sen21]. It is also not hard to see that the successive cancellation covering lemma Lemma 3.2, can also be proved using this control state, since the operator inequalities used in the proof of that lemma are preserved by the above definition.

Before we go on to state the general theorem with respect to the state Eq. (10), we would like to remark that in order to get the most general version of the private capacity region, we consider a fully quantum or $q q$ multiple access channel $\mathfrak{C}$ which maps the systems $X^{\prime} Y^{\prime} \rightarrow C E$. To import this into the classical quantum setting, we introduce the classical alphabets $\mathcal{X}$ and $\mathcal{Y}$ and the maps $\mathfrak{F}: \mathcal{X} \rightarrow X^{\prime}$ and $\mathfrak{G}: \mathcal{Y} \rightarrow Y^{\prime}$ such that

$$
\begin{aligned}
& \mathfrak{F}(x):=\sigma_{x}^{X^{\prime}} \\
& \mathfrak{G}(y):=\sigma_{y}^{\gamma^{\prime}}
\end{aligned}
$$

where $\sigma_{x}^{X^{\prime}}$ and $\sigma_{y}^{Y^{\prime}}$ are states in the input Hilbert space of $\mathfrak{C}$.

Then define

$$
\begin{equation*}
\mathfrak{C}\left(\sigma_{x}^{X^{\prime}} \otimes \sigma_{y}^{Y^{\prime}}\right):=\rho_{x, y}^{C E} \tag{11}
\end{equation*}
$$

We are now ready to state the theorem:
Theorem 3.15. Given the channel $\mathfrak{C}: X^{\prime} Y^{\prime} \rightarrow C E$, the maps $\mathfrak{F}, \mathfrak{G}$ and the definition Eq. (11), consider the classical quantum control state given in Eq. (10). Then, given the split ( $\left.P_{U|Q| Q}^{\theta}, P_{V \mid Q}^{\theta}, f\right)$ with respect to the parameter $\theta \in[0,1]$, we have that the following region if achievable for private information transmission with error at most $\varepsilon^{1 / 8}$ and leakage at most $40 \delta^{1 / 8}$

$$
\begin{array}{|l}
R_{A} \leq I_{H}^{\varepsilon}(U V: Y C \mid Q)-I_{\max }^{\delta^{\prime}}(U: E \mid Q)-I_{\max }^{\delta^{\prime}}(V: U Y E \mid Q)+c \\
R_{A} \leq I_{H}^{\varepsilon}(V: U Y C \mid Q)+I_{H}^{\varepsilon}(U: V Y C \mid Q)-I_{\max }^{\delta^{\prime}}(U: E \mid Q)-I_{\max }^{\delta^{\prime}}(V: U Y E \mid Q)+2 c \\
R_{B} \leq I_{H}^{\varepsilon}(Y: U V C \mid Q)-I_{\max }^{\delta^{\prime}}(Y: U E \mid Q)+c \\
R_{B} \leq I_{H}^{\varepsilon}(Y V: U C \mid Q)-I_{\max }^{\delta^{\prime}}(Y: U E \mid Q)+c \\
R_{B} \leq I_{H}^{\varepsilon}(U Y: V C \mid Q)-I_{\max }^{\delta^{\prime}}(Y: U E \mid Q)+c \\
R_{A}+R_{B} \leq I_{H}^{\varepsilon}(V: U Y C \mid Q)+I_{H}^{\varepsilon}(U Y: V C \mid Q)-I_{\max }^{\delta^{\prime}}(U: E \mid Q)-I_{\max }^{\delta^{\prime}}(V: U Y E \mid Q)-I_{\max }^{\delta^{\prime}}(Y: U E \mid Q)+2 c \\
R_{A}+R_{B} \leq I_{H}^{\varepsilon}(Y V: U C \mid Q)+I_{H}^{\varepsilon}(U: V Y C \mid Q)-I_{\max }^{\delta^{\prime}}(U: E \mid Q)-I_{\max }^{\delta^{\prime}}(V: U Y E \mid Q)-I_{\max }^{\delta^{\prime}}(Y: U E \mid Q)+2 c \\
R_{A}+2 R_{B} \leq I_{H}^{\varepsilon}(U Y: C V \mid Q)+I_{H}^{\varepsilon}(Y V: U C \mid Q)-I_{\max }^{\delta^{\prime}}(U: E \mid Q)-I_{\max }^{\delta^{\prime}}(V: U Y \mid Q E)-2 I_{\max }^{\delta^{\prime}}(Y: U E \mid Q)+2 c \\
R_{A}+R_{B} \leq I_{H}^{\varepsilon}(U Y V: C \mid Q)-I_{\max }^{\delta^{\prime}}(U: E \mid Q)-I_{\max }^{\delta^{\prime}}(V: U Y E \mid Q)-I_{\max }^{\delta^{\prime}}(Y: U E \mid Q)+c \\
\hline
\end{array}
$$

and

$$
\begin{aligned}
& \delta^{\prime}:=\delta-\varepsilon^{\prime} \\
& c:=\log \frac{1}{\varepsilon}+\log \frac{1}{\varepsilon^{\prime 3}}-\frac{1}{4} \log \delta+O(1)
\end{aligned}
$$

where $\varepsilon, \delta>0,0<\varepsilon^{\prime}<\delta$ and all the information quantities are computed with respect to the split of the control state in Eq. (10).

## 4 Proofs of Important Lemmas

In this section we present the proof of all the lemmas and theorems stated in Section 3.2.

Proof of Lemma 3.2. Suppose we are given the cq state

$$
\rho^{X Y E}:=\sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} p_{X}(x) p_{Y}(y)|x\rangle\left\langle\left. x\right|^{X} \otimes \mid y\right\rangle\left\langle\left. y\right|^{Y} \rho_{x, y}^{E}\right.
$$

Consider the quantity

$$
\lambda:=\tilde{I}_{\max }^{\varepsilon}(Y: E X)_{\rho}:=\inf _{\left\|\rho^{\prime}-\rho\right\|_{1} \leq \varepsilon} D_{\max }\left(\rho^{\prime X Y E} \| \rho^{Y} \otimes \rho^{\prime X E}\right)
$$

Let $\tilde{\rho}^{X Y E}$ be the optimizer in the definition of $\tilde{I}_{\text {max }}^{\varepsilon}(Y: E X)$. Without loss of generality we can assume that $\tilde{\rho}^{X Y E}$ is a cq state. This is because, suppose the optimizer was a state $\rho^{*}$ which is not cq. By definition, $\rho^{*}$ obeys the following properties

$$
\begin{aligned}
& \rho^{*} \leq 2^{I_{\max }^{E}(Y: E X)} \rho^{Y} \otimes \rho^{* X E} \\
& \left\|\rho^{*}-\rho\right\|_{1} \leq \varepsilon
\end{aligned}
$$

Now we will measure the $X$ and $Y$ systems along the canonical bases $\{|x\rangle\}$ and $\{|y\rangle\}$, to get the cq state $\rho^{* *}$. Since measurement is CPTP it preserves the operator inequality. This also implies that $\rho^{* *}$ is in the $\varepsilon$ ball around $\rho$. Finally, it is easy to see that, $\rho^{* * X E}$ is the post measurement state on the systems $X E$. These observations imply that $\rho^{* *}$ is a cq state which is also an optimizer, proving the claim.

Next, suppose that

$$
\tilde{\rho}:=\sum_{x, y} \tilde{P}_{X Y}(x, y) x^{X} \otimes y^{Y} \otimes \tilde{\rho}_{x, y}^{E}
$$

where we have used the shorthand $x^{X}:=|x\rangle\left\langle\left. x\right|^{X}\right.$ and similarly for $y$. It is easy to see that the two following properties hold

$$
\begin{equation*}
\left\|\tilde{P}_{X Y}-P_{X} \cdot P_{Y}\right\|_{1} \leq \varepsilon \tag{12}
\end{equation*}
$$

Changing the Distributions We can infer from Eq. (12) that

$$
\left\|\tilde{P}_{x}-P_{X}\right\|_{1} \leq \varepsilon
$$

which implies that

$$
\begin{equation*}
\left\|\tilde{P}_{X Y}-\tilde{P}_{X} \cdot P_{Y}\right\|_{1} \leq 2 \varepsilon \tag{13}
\end{equation*}
$$

Eq. (13) can be written as

$$
\mathbb{E}_{\tilde{P}_{X} \cdot P_{Y}}\left[\left|\frac{\tilde{P}_{X Y}(X, Y)}{\tilde{P}_{X}(X) \cdot P_{Y}(Y)}-1\right|\right] \leq 2 \varepsilon
$$

Then, by Markov's inequality this implies that

$$
\begin{equation*}
\operatorname{Pr}_{\tilde{P}_{X} \cdot P_{Y}}\left[\left|\frac{\tilde{P}_{X Y}(X, Y)}{\tilde{P}_{X}(X) \cdot P_{Y}(Y)}-1\right| \geq \sqrt{\varepsilon}\right] \leq 2 \sqrt{\varepsilon} \tag{14}
\end{equation*}
$$

Now, by the definition of $\tilde{\rho}$ and using the classical nature of the $X Y$ system, we see that, for all $(x, y)$ the following holds

$$
\tilde{P}_{X Y}(x, y) \tilde{\rho}_{x, y}^{E} \leq 2^{\lambda} \tilde{P}_{X}(x) \cdot P_{Y}(y) \tilde{\rho}_{x}^{E}
$$

where

$$
\tilde{\rho}_{x}^{E}:=\sum_{y} \tilde{P}_{Y}(y \mid x) \tilde{\rho}_{x, y}^{E}
$$

Coupled with Eq. (14) this implies that with probability at least $1-2 \sqrt{\varepsilon}$ over the choice of $x$ from the distribution $\tilde{P}_{X}$ and $y$ from the distribution $P_{Y}$, the following holds

$$
\begin{equation*}
\tilde{\rho}_{x, y}^{E} \leq 2^{\lambda} \frac{1}{1-\sqrt{\varepsilon}} \tilde{\rho}_{x}^{E} \tag{15}
\end{equation*}
$$

The Set of GOOD $x$ 's Define the function $\mathbf{1}_{x, y}$ as the indicator, which is 1 when Eq. (15) holds. Further, define

$$
\mathbf{1}_{x}:=\sum_{y} P_{Y}(y) \mathbf{1}_{x, y}
$$

Intuitively, $\mathbf{1}_{x}$ is the probability that, for a fixed $x$, the pairs $(x, y)$ satisfy Eq. (15), over choice of $y$. We know from the discussion in the previous section that

$$
\sum_{x} \tilde{P}_{X}(x) \mathbf{1}_{x} \geq 1-2 \sqrt{\varepsilon}
$$

Then, another application of Markov's inequality implies that

$$
\begin{equation*}
\operatorname{Pr}_{\tilde{P}_{X}}\left[\left\{x \mid \mathbf{1}_{x} \geq 1-\varepsilon^{1 / 4}\right\}\right] \geq 1-2 \varepsilon^{1 / 4} \tag{16}
\end{equation*}
$$

We define

$$
\operatorname{NICE}_{X}:=\left\{x \mid \mathbf{1}_{x} \geq 1-\varepsilon^{1 / 4}\right\}
$$

What this implies is that, for any $x \in \operatorname{NICE}_{X}$, the probability over choice of $y$ that $(x, y)$ satisfies Eq. (15) is at least $1-\varepsilon^{1 / 4}$, and that the probability that a random $x$ is picked from $\operatorname{NICE}_{X}$ is at least $1-2 \varepsilon^{1 / 4}$ under the distribution $\tilde{P}_{X}$.

We will however require a few more conditions to define the good set. To that end, define

$$
\operatorname{NICER}_{X}:=\operatorname{NICE}_{X} \cap\left\{x \mid\left\|P_{Y \mid x}-P_{Y}\right\|_{1} \leq \sqrt{\varepsilon}\right\}
$$

From Eq. (12) we know that

$$
\mathbb{E}_{\tilde{P}_{X}}\left[\left\|\tilde{P}_{Y \mid X}-P_{Y}\right\|_{1}\right] \leq 2 \varepsilon
$$

By Markov's inequality we conclude that

$$
\operatorname{Pr}_{\tilde{P}_{X}}\left[\left\{x \mid\left\|P_{Y \mid x}-P_{Y}\right\|_{1} \leq \sqrt{\varepsilon}\right\}\right] \geq 1-2 \sqrt{\varepsilon}
$$

This implies that

$$
\underset{\tilde{P}_{X}}{\operatorname{Pr}}\left[\operatorname{NICER}_{X}\right] \geq 1-4 \varepsilon^{1 / 4}
$$

Since $\left\|\tilde{P}_{X}-P_{X}\right\|_{1} \leq \varepsilon$, this implies that

$$
\operatorname{Pr}_{P_{X}}\left[\operatorname{NICER}_{X}\right] \geq 1-5 \varepsilon^{1 / 4}
$$

Next, consider the state

$$
\rho^{\prime}:=\sum_{x, y} P_{X}(x) P_{Y}(y) x^{X} \otimes y^{Y} \otimes \tilde{\rho}_{x, y}^{E}
$$

Then,

$$
\begin{aligned}
\left\|\rho^{\prime}-\rho\right\|_{1} & \leq\left\|\rho^{\prime}-\tilde{\rho}\right\|_{1}+\|\tilde{\rho}-\rho\|_{1} \\
& =\left\|\tilde{P}_{X Y}-P_{X} \cdot P_{Y}\right\|_{1}+\|\tilde{\rho}-\rho\|_{1} \\
& \leq 3 \varepsilon
\end{aligned}
$$

Since

$$
\left\|\rho^{\prime}-\rho\right\|_{1}=\mathbb{E}_{P_{X}}\left[\left\|\sum_{y} P_{Y}(y) y^{\curlyvee} \otimes \tilde{\rho}_{x, y}^{E}-\sum_{y} P_{Y}(y) y^{\Upsilon} \otimes \rho_{x, y}^{E}\right\|_{1}\right]
$$

this implies that

$$
\operatorname{Pr}_{P_{X}}\left[\left\|\sum_{y} P_{Y}(y) y^{Y} \otimes \tilde{\rho}_{x, y}^{E}-\sum_{y} P_{Y}(y) y^{Y} \otimes \rho_{x, y}^{E}\right\|_{1} \geq \sqrt{\varepsilon}\right] \leq 3 \sqrt{\varepsilon}
$$

Call the event inside the last probability expression $B_{X}$. Finally, we define

$$
\operatorname{GOOD}_{X}:=\operatorname{NICER}_{X} \cap B_{X}
$$

This implies that,

$$
\operatorname{Pr}_{P_{X}}\left[\operatorname{GOOD}_{X}\right] \geq 1-10 \varepsilon^{1 / 4}
$$

The Covering Lemma Let us fix an an $x \in \operatorname{GOOD}_{X}$. Recall that, this implies

$$
\begin{gathered}
\left\|\tilde{P}_{Y \mid x}-P_{Y}\right\|_{1} \leq \sqrt{\varepsilon} \\
\text { or } \mathbb{E}_{P_{Y}}\left|\frac{\tilde{P}_{Y \mid x}(Y)}{P_{Y}(Y)}-1\right| \leq \sqrt{\varepsilon}
\end{gathered}
$$

Then, by Markov's inequality,

$$
\operatorname{Pr}_{P_{Y}}\left[\left|\frac{\tilde{P}_{Y \mid x}(Y)}{P_{Y}(Y)}-1\right| \geq \varepsilon^{1 / 4}\right] \leq \varepsilon^{1 / 4}
$$

Define the set

$$
\operatorname{GOOD}_{Y \mid x}:=\left\{y \mid(x, y) \text { s.t. Eq. (15) },\left|\frac{\tilde{P}_{Y \mid x}(Y)}{P_{Y}(Y)}-1\right| \leq \varepsilon^{1 / 4}\right\}
$$

Then, under the distribution $P_{Y}$,

$$
\operatorname{Pr}_{P_{Y}}\left[\operatorname{GOOD}_{Y \mid x}\right] \geq 1-2 \varepsilon^{1 / 4}
$$

Define the subdistribution $\bar{P}_{Y}$ as

$$
\begin{aligned}
\bar{P}_{Y \mid x} & =\tilde{P}_{Y \mid x} \quad y \in \operatorname{GOOD}_{Y \mid x} \\
& =0 \quad \text { otherwise }
\end{aligned}
$$

Then it holds that

$$
\begin{aligned}
& \bar{\rho}_{x}^{Y E}:=\sum_{y} \bar{P}_{Y \mid x}(y \mid x) y^{Y} \otimes \tilde{\rho}_{x, y}^{E} \leq 2^{\lambda} \frac{1}{1-\sqrt{\varepsilon}}\left(\sum_{y \in \mathrm{GOOD}}^{\mathrm{D}_{Y \mid x}} \tilde{P}_{Y \mid x}(y \mid x) y^{Y}\right) \otimes \tilde{\rho}_{x}^{E} \\
& \leq 2^{\lambda} \frac{1}{1-\sqrt{\varepsilon}}\left(\sum_{y \in \mathrm{GOOO}}^{Y \mid x}\right. \\
&\left.\tilde{P}_{Y \mid x}(y \mid x) y^{Y}+\sum_{y \in \notin \mathrm{GOOD} \mathrm{D}_{\Upsilon \mid x}} P_{Y}(y) y^{Y}\right) \otimes \tilde{\rho}_{x}^{E} \\
& \leq 2^{\lambda} \frac{1}{1-\sqrt{\varepsilon}}\left(\left(1+\varepsilon^{1 / 4}\right) \sum_{y \in \mathrm{GOOD} \mathrm{D}_{Y \mid x}} P_{Y}(y) y^{Y}+\sum_{y \in \notin \mathrm{GOOD} \mathrm{D}_{Y \mid x}} P_{Y}(y) y^{Y}\right) \otimes \tilde{\rho}_{x}^{E} \\
& \leq 2^{\lambda} \frac{1+\varepsilon^{1 / 4}}{1-\sqrt{\varepsilon}}\left(\sum_{y} P_{Y}(y) y^{Y}\right) \otimes \tilde{\rho}^{E} \\
&=2^{\lambda} \frac{1+\varepsilon^{1 / 4}}{1-\sqrt{\varepsilon}} \rho^{Y} \otimes \tilde{\rho}_{x}^{E}
\end{aligned}
$$

Also note that

$$
\begin{aligned}
\bar{\rho}_{x}^{E} & =\sum_{y} \bar{P}_{Y \mid x}(y \mid x) \tilde{\rho}_{x, y}^{E} \\
& \leq \sum_{y} \tilde{P}_{Y \mid x}(y \mid x) \tilde{\rho}_{x, y}^{E} \\
& =\tilde{\rho}_{x}^{E}
\end{aligned}
$$

Using the above fact in the proof of convex split lemma we get that

$$
\left\|\frac{1}{K} \sum_{i=1}^{K} \bar{\rho}_{x}^{Y_{i} E} \bigotimes \rho_{j \neq i}^{Y_{j}}-\tilde{\rho}_{x}^{E} \bigotimes_{i=1}^{K} \rho^{Y_{i}}\right\|_{1} \leq \sqrt{2^{\lambda} \frac{1+\varepsilon^{1 / 4}}{1-\sqrt{\varepsilon}} \frac{1}{K}}
$$

Next we bound the distance between $\rho_{x}^{Y E}$ and $\bar{\rho}_{x}^{Y E}$. To do this, consider the following triangle inequality

$$
\begin{aligned}
\|\rho-\bar{\rho}\|_{1} & \leq\left\|\bar{\rho}-\sum_{y} \tilde{P}_{Y \mid x}(y \mid x) y^{Y} \otimes \tilde{\rho}_{x, y}^{E}\right\|_{1} \\
& +\left\|\sum_{y} \tilde{P}_{Y \mid x}(y \mid x) y^{Y} \otimes \tilde{\rho}_{x, y}^{E}-\sum_{y} P_{Y}(y) y^{Y} \otimes \tilde{\rho}_{x, y}^{E}\right\|_{1} \\
& +\left\|\sum_{y} P_{Y}(y) y^{Y} \otimes \tilde{\rho}_{x, y}^{E}-\sum_{y} P_{Y}(y) y^{Y} \otimes \rho_{x, y}^{E}\right\|_{1} \\
& =\sum_{y: \bar{P}_{Y \mid x}(y \mid x)=0} \tilde{P}_{Y \mid x}(y)+\left\|\tilde{P}_{Y \mid x}-P_{Y}\right\|_{1}+\sum_{y} P_{Y}(y)\left\|\tilde{f}_{x, y}^{E}-\rho_{x, y}^{E}\right\|_{1}
\end{aligned}
$$

To bound the first term, observe that the summation is precisely over those $y$ 's which do not belong to $\operatorname{GOOD}_{Y \mid x}$. We already know that under the distribution $P_{Y}$ this set has probability at most $2 \varepsilon^{1 / 4}$. Thus by the definition of $\operatorname{GOOD}_{X}$ the first term is at most $\sqrt{\varepsilon}+2 \varepsilon^{1 / 4} \leq 3 \varepsilon^{1 / 4}$.

The second and third terms can be bounded similarly directly from the definition of $\mathrm{GOOD}_{X}$ by $\sqrt{\varepsilon}$ each. Thus,

$$
\left\|\rho_{x}^{Y E}-\bar{\rho}_{x}^{Y E}\right\|_{1} \leq 5 \varepsilon^{1 / 4}
$$

We will require one more triangle inequality to replace $\tilde{\rho}_{x}^{E}$ with $\rho_{x}^{E}$ :

$$
\begin{aligned}
\left\|\tilde{\rho}_{x}^{E}-\rho_{x}^{E}\right\|_{1} & =\left\|\sum_{y} \tilde{P}_{Y_{x}}(y \mid x) \tilde{\rho}_{x, y}^{E}-\sum_{y} P_{Y}(y) \rho_{x, y}^{E}\right\|_{1} \\
& \leq\left\|\sum_{y} \tilde{P}_{Y_{x}}(y \mid x) \tilde{\rho}_{x, y}^{E}-\sum_{y} P_{Y}(y) \tilde{\rho}_{x, y}^{E}\right\|_{1} \\
& +\left\|\sum_{y} P_{Y}(y) \tilde{\rho}_{x, y}^{E}-\sum_{y} P_{Y}(y) \rho_{x, y}^{E}\right\|_{1} \\
& \leq 2 \sqrt{\varepsilon}
\end{aligned}
$$

Collating all these arguments together and using the standard trick to get a covering lemma from the convex split lemma, we see that the following holds

$$
\mathbb{E}_{y_{1}, y_{2}, \ldots, y_{K} \sim P_{Y}}\left\|\frac{1}{K} \sum_{i}^{K} \rho_{x, y_{i}}^{E}-\rho_{x}^{E}\right\|_{1} \leq 8 \varepsilon^{1 / 4}+\sqrt{2^{\lambda} \frac{1+\varepsilon^{1 / 4}}{1-\sqrt{\varepsilon}} \frac{1}{K}}
$$

## The Successive Cancellation Step

Define $\varepsilon_{0}:=10 \varepsilon^{1 / 4}$. Suppose we sample $K^{\prime}$ times independently from the distribution $P_{X}$. Then, by Markov's inequality,

$$
\operatorname{Pr}\left[\sum_{i}^{K^{\prime}} I_{X_{i} \notin \mathrm{GOOD}_{X}} \geq \sqrt{\varepsilon_{0}} \cdot K^{\prime}\right] \leq \sqrt{\varepsilon_{0}}
$$

Suppose $x_{1} x_{2} \ldots x_{K^{\prime}}$ is a sequence which has at most $\sqrt{\varepsilon_{0}} K^{\prime}$ samples from $\operatorname{GOOD}_{X}^{c}$. Then, for this fixed sequence $x^{K^{\prime}}$, the following holds

$$
\begin{aligned}
\mathbb{E}_{y_{1}, y_{2}, \ldots, y_{K^{\prime}} \sim P_{Y}}\left\|\frac{1}{K \cdot K^{\prime}} \sum_{i}^{K^{\prime}} \sum_{j}^{K} \rho_{x_{i}, y_{j}}^{E}-\rho^{E}\right\|_{1} & \leq \mathbb{E}_{y_{1}, y_{2}, \ldots, y_{K} \sim P_{Y}}\left[\left\|\frac{1}{K \cdot K^{\prime}} \sum_{i}^{K^{\prime}} \sum_{j}^{K} \rho_{x_{i}, y_{j}}^{E}-\frac{1}{K^{\prime}} \sum_{i}^{K^{\prime}} \rho_{x_{i}}^{E}\right\|_{1}\right]+\left\|\frac{1}{K^{\prime}} \sum_{i}^{K^{\prime}} \rho_{x_{i}}^{E}-\rho^{E}\right\|_{1} \\
& \leq \frac{1}{K^{\prime}} \sum_{i}^{K^{\prime}} \mathbb{E}_{y_{1}, y_{2}, \ldots, y_{K} \sim P_{Y}}\left[\left\|\frac{1}{K} \sum_{j}^{K} \rho_{x_{i}, y_{j}}^{E}-\rho_{x_{i}}^{E}\right\|_{1}\right]+\left\|\frac{1}{K^{\prime}} \sum_{i}^{K^{\prime}} \rho_{x_{i}}^{E}-\rho^{E}\right\|_{1} \\
& \leq \frac{1}{K^{\prime}}\left(\left(1-\sqrt{\varepsilon_{0}}\right) K^{\prime} \cdot\left(8 \varepsilon^{1 / 4}+\sqrt{2^{\lambda} \frac{1+\varepsilon^{1 / 4}}{1-\sqrt{\varepsilon}}} \frac{1}{K}\right)+\sqrt{\varepsilon_{0}} K^{\prime} \cdot 2\right) \\
& +\left\|\frac{1}{K^{\prime}} \sum_{i}^{K^{\prime}} \rho_{x_{i}}^{E}-\rho^{E}\right\|_{1}
\end{aligned}
$$

We will now set the values of $K$ and $K^{\prime}$. We set $K$ and $K^{\prime}$ such that

$$
2^{\lambda} \frac{1+\varepsilon^{1 / 4}}{1-\sqrt{\varepsilon}} \frac{1}{K} \leq \varepsilon^{1 / 4}
$$

and

$$
\mathbb{E}_{x_{1}, x_{2}, \ldots, x_{K^{\prime}} \sim P_{X}}\left\|\frac{1}{K^{\prime}} \sum_{i}^{K^{\prime}} \rho_{x_{i}}^{E}-\rho^{E}\right\|_{1} \leq \varepsilon^{1 / 4}
$$

The second inequality can be set by using the smoothed version of the convex split lemma. Then,

$$
\begin{aligned}
\mathbb{E}_{\substack{x_{1}, x_{2}, \ldots, x_{K^{\prime}} \sim P_{X} \\
y_{1}, y_{2}, \ldots, y_{K} \sim P_{Y}}}\left\|\frac{1}{K \cdot K^{\prime}} \sum_{i}^{K^{\prime}} \sum_{j}^{K} \rho_{x_{i}, y_{j}}^{E}-\rho^{E}\right\|_{1} & \leq\left(\left(1-\sqrt{\varepsilon_{0}}\right) \cdot 9 \varepsilon^{1 / 4}+2 \sqrt{\varepsilon_{0}}\right) \cdot\left(1-\sqrt{\varepsilon_{0}}\right)+2 \sqrt{\varepsilon_{0}}+\varepsilon^{1 / 4} \\
& \leq 10 \varepsilon^{1 / 8}
\end{aligned}
$$

Proof of Lemma 3.7. Suppose that $|\varphi\rangle^{R A B C}$ is a purification of $\varphi^{R A B}$, where $C$ is the purifying register. Consider the following task : Let Alice possess the systems $A B C$ and $R$ be the reference. Alice wants to send the systems $A B$ to Bob. This is known as quantum state splitting. We will achieve this task in two steps. In Step 1, Alice will send the system $A$ to Bob while treating $B C$ as the purifying registers. This will require $\frac{1}{2} I_{\max }^{\varepsilon}(R: A)_{\varphi}+\log \frac{1}{\varepsilon}$ bits of quantum communication.

In Step 2, Alice will send the system $B$ to Bob, while while treating the system $C$ as the purifying register. This task will require $\frac{1}{2} I_{\max }^{\varepsilon}(R A: B)+\log \frac{1}{\varepsilon}$ bits of quantum communication. At the end of the protocol Alice will have successfully sent Bob the systems $A B$, with some $O(\varepsilon)$ error. We already know from [BCR11] that any one-way entanglement assisted protocol that achieves this task with $\varepsilon$ error requires at least $\frac{1}{2} I_{\text {max }}^{\varepsilon}(R: A B)_{\varphi}$ number of qubits. Collating these arguments together gives us the upper bound.

To achieve Step 1 and Step 2 above, we will use the smoothed convex split lemma, specifically the protocol in Theorem 1 of [ADJ17].

## Step 1

Let $\varphi^{\prime R A}$ be the optimiser for the expression $\tilde{I}_{\max }^{\varepsilon}(R: A)_{\varphi}$. Then the smoothed convex split lemma, along with two triangle inequalities shows us that

$$
P\left(\frac{1}{n} \sum_{i=1}^{n} \varphi^{R A_{i}} \bigotimes_{j \neq i} \varphi^{A_{j}}, \varphi^{R} \bigotimes_{i=1}^{n} \varphi^{A_{i}}\right) \leq 3 \varepsilon
$$

where $n>\tilde{I}_{\text {max }}^{\varepsilon}(R: A)_{\varphi}+2 \log \frac{1}{\varepsilon}$.
Armed with this relation, we can directly use the protocol in Theorem 1 to send the system $A$ to Bob and obtain a pure state $\left|\varphi^{\prime \prime}\right\rangle^{R A B C}$ such that

1. $P\left(\varphi^{\prime \prime R A B C}, \varphi^{R A B C}\right) \leq 3 \varepsilon$
2. The system $A$ is now with Bob.

Step 2 For the next part of the protocol, we recall that, whenever $m>\tilde{I}_{\max }^{\varepsilon}(R A: B)_{\varphi}+2 \log \frac{1}{\varepsilon}$

$$
P\left(\frac{1}{m} \sum_{i=1}^{m} \varphi^{R A B_{i}} \bigotimes_{j \neq i} \varphi^{B_{j}}, \varphi^{R A} \bigotimes_{i=1}^{m} \varphi^{B_{i}}\right) \leq 3 \varepsilon
$$

However, since we global state shared by Alice, Bob and Referee is $\varphi^{\prime \prime}$, we need a further triangle inequality to show that

$$
P\left(\frac{1}{m} \sum_{i=1}^{m} \varphi^{\prime \prime R A B_{i}} \bigotimes_{j \neq i} \varphi^{B_{j}}, \varphi^{\prime \prime R A} \bigotimes_{i=1}^{m} \varphi^{B_{i}}\right) \leq 9 \varepsilon
$$

Repeating the protocol in Theorem 1 to send the system $B$ to Bob, while Bob possesses the system $A$, we get a pure state $\tilde{\varphi}^{R A B C}$ such that

$$
\begin{aligned}
& P\left(\tilde{\varphi}^{R A B C}, \varphi^{\prime \prime R A B C}\right) \leq 9 \varepsilon \\
\Longrightarrow & P\left(\tilde{\varphi}^{R A B C}, \varphi^{R A B C}\right) \leq 12 \varepsilon
\end{aligned}
$$

Along with the lower bound, this implies that

$$
\tilde{I}_{\max }^{\varepsilon}(R: A)_{\varphi}+\tilde{I}_{\max }^{\varepsilon}(R A: B)_{\varphi}+4 \log \frac{1}{\varepsilon} \geq \tilde{I}_{\max }^{\varepsilon}(R: A B)_{\varphi}
$$

Finally, we use the bound that, for any state $\rho^{A B}$

$$
\tilde{I}_{\max }^{\varepsilon}(A: B)_{\rho} \leq I_{\max }^{\varepsilon-\gamma}(A: B)_{\rho}+\log \frac{3}{\gamma^{2}}
$$

to get the desired chain rule.

Proof of Lemma 3.10. The proof of this lemma is a simple Fourier-Motzkin elimination, with the extra condition that

$$
R_{1}=R_{10}+R_{11}
$$

and hence we omit it.

Proof of Theorem 3.11. Fix a $\theta \in[0,1]$ and fix the rate tuple for Alice ${ }_{1}$, Bob and Alice $_{2},\left(R_{10}, R_{2}, R_{10}\right)$ such that it belongs to the region given by Eq. (9). By Lemma 3.10, this ensures that the rate pair ( $R_{10}+R_{11}, R_{2}$ ) is achievable for classical message transmission across the cq-mac, with error at most $\varepsilon^{1 / 8}$. Recall that we use the definition $R_{1}=R_{10}+R_{11}$.

Again, fix $K_{1}, K_{2}$ and $K_{3}$ i.e. the block sizes over which Alice ${ }_{1}$, Bob and Alice ${ }_{2}$ randomise, as in Theorem 3.4. Define

$$
\begin{aligned}
& R_{\text {ALICE }}^{\text {private }}:=R_{1}-\log K_{1}-\log K_{2} \\
& R_{\text {ALICE }}^{\text {private }}:=R_{2}-\log K_{3}
\end{aligned}
$$

The code construction is as follows:

1. For Alice ${ }_{1}$ choose symbols $x(1), x(2), \ldots, x\left(R_{10}\right)$ iid from $P_{U}^{\theta}$.
2. For Bob choose symbols $y(1), y(2), \ldots, y\left(R_{2}\right)$ iid from $P_{Y}$.
3. For Alice $2_{2}$ choose symbols $z(1), z(2), \ldots, z\left(R_{11}\right)$ iid from $P_{V}^{\theta}$.
4. Divide Alice ${ }_{1}$ 's codebook into blocks, each of size $K_{1}$. Do the same for Bob and Alice ${ }_{2}$, with block sizes $K_{2}$ and $K_{3}$ respectively.
5. Alice ${ }_{1}$ maps her message set $\left[M_{1}\right]$ to codebook such that each message $m_{1} \in\left[M_{1}\right]$ corresponds to some block. Bob and Alice 2 do the same, for their message sets $[N]$ and $\left[M_{2}\right]$.
6. To send the message $m_{1}$, Alice $_{1}$ goes into the block corresponding to that message. Suppose that block contains the symbols $\left(x_{m_{1}}(1), x_{m_{1}}(2), \ldots, x_{m_{1}}\left(K_{1}\right)\right)$, Alice ${ }_{1}$ picks a symbol uniformly at random and transmits it,
7. Bob and Alice $_{2}$ do the same for their corresponding messages $n$ and $m_{2}$.

Decodability is guaranteed by the code specified rates and [Sen21]. Secrecy is guaranteed by the values of $K_{1}, K_{2}, K_{3}$ and Theorem 3.4. This argument implies that for any rate tuple ( $R_{1}, R_{2}$ ) that lies in the region given by Lemma 3.10, and for any $\left(\log K, \log K^{\prime}\right)$ such that

$$
\begin{aligned}
\log K \geq & \log K_{1}+\log K_{3} \\
& \log K^{\prime} \geq
\end{aligned}
$$

the rate pair $\left(R_{1}-\log K, R_{2}-\log K^{\prime}\right)$ is achievable for private transmission across the cq mac, assuming both coordinates are non-negative. This is precisely the definition of the set $\left(\mathcal{S}_{\theta}-\mathcal{T}_{\theta}\right)^{+}$. Repeating this procedure for all $\theta \in[0,1]$ and then taking a union bound over all the regions concludes the proof.

Proof of Theorem 3.14. The proof is easy and we only provide a brief sketch. First note that using Fact 3.13 , it is easy to see (using the chain rule for the mutual information and the data processing inequality) that for every $\theta \in[0,1]$, the region $\mathcal{S}_{\theta}$ is equivalent to the region

$$
\begin{aligned}
& R_{1}<I(X: Y C) \\
& R_{2}<I(Y: X C) \\
& R_{1}+R_{2}<I(X Y: C)
\end{aligned}
$$

Call this region $\mathcal{S}$. Along with Theorem 3.11, this implies that the private capacity region is given by

$$
\left(\mathcal{S}-\bigcup_{\theta \in[0,1]} \mathcal{T}_{\theta}\right)^{+}
$$

Using the continuity of the mutual information with respect to $\theta \in[0,1]$, and again via the chain rule for the mutual information, we see that the region $\underset{\theta \in[0,1]}{\bigcup} \mathcal{T}_{\theta}$ is equivalent to

$$
\begin{aligned}
& \log K \geq I(X: E) \\
& \log K^{\prime} \geq I(Y: E) \\
& \log K+\log K^{\prime} \geq I(X Y: E)
\end{aligned}
$$

Taking the difference of these two regions gives us the desired region for private transmission. This concludes the proof.

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