Neural Belief Propagation Decoding of Quantum LDPC Codes Using Overcomplete Check Matrices

Sisi Miao[†], Alexander Schnerring[‡], Haizheng Li[†], and Laurent Schmalen[†]

[†]Karlsruhe Institute of Technology (KIT), Communications Engineering Lab (CEL), 76187 Karlsruhe, Germany

[‡]now with German Aerospace Center (DLR), Institute of Solar Research, 04001 Almería, Spain

E-Mail: sisi.miao@kit.edu

Abstract—The recent success in constructing asymptotically good quantum low-density parity-check (QLDPC) codes makes this family of codes a promising candidate for error-correcting schemes in quantum computing. However, conventional belief propagation (BP) decoding of QLDPC codes does not yield satisfying performance due to the presence of unavoidable short cycles in their Tanner graph and the special degeneracy phenomenon. In this work, we propose to decode QLDPC codes based on a check matrix with redundant rows, generated from linear combinations of the rows in the original check matrix. This approach yields a significant improvement in decoding performance with the additional advantage of very low decoding latency. Furthermore, we propose a novel neural belief propagation decoder based on the quaternary BP decoder of QLDPC codes which leads to further decoding performance improvements.

I. INTRODUCTION

Quantum error correction (QEC) is an essential part of fault-tolerant quantum computing. Recent breakthroughs in designing asymptotically good quantum low-density paritycheck (QLDPC) codes with non-vanishing rate and minimal distance growing linearly with the code length [1]–[4] make QLDPC codes a promising candidate for future QEC schemes. In classical coding, low-density parity-check (LDPC) codes are typically decoded with a message-passing decoder such as the belief propagation (BP) decoder, which usually works well when the Tanner graph has a girth of at least 6. However, this condition cannot be fulfilled in the case of QLDPC codes, where 4-cycles cannot be avoided by construction. Therefore, to achieve good decoding performance for QLDPC codes similar to their classical counterparts, it is crucial to modify the BP decoder such that short cycles can be tolerated.

A variety of methods have been proposed to solve this issue. These can be grouped into two categories. The first category contains approaches that modify the BP decoder itself, for example, message normalization and offsets [5], [6], layered scheduling [7], [8], and matrix augmentation [9]. The second category contains methods that apply post-processing to the BP decoder output, e.g., ordered statistics decoding (OSD) [7], [10], random perturbation [11], enhanced feedback [12], and stabilizer inactivation [13]. However, most of these methods introduce additional decoding latency, which makes them less

appealing in QEC where decoding has to be performed with ultra low latency.

Neural belief propagation (NBP) has been shown to be effective in improving the decoding performance of classical linear codes without increasing the decoding latency, see, e.g., [14]–[16]. Using NBP to enhance the decoding of quantum stabilizer codes has been investigated in [19], [20] for the binary BP decoder (referred to as BP2 decoder), which decodes X and Z errors separately. Binary decoding is sub-optimal and has a higher error floor compared with a quaternary BP decoder (referred to as BP4 decoder), which takes the correlation between X and Z errors into account [6], [7]. The disadvantage of the conventional BP4 decoder is the high complexity due to the passing of vector messages instead of scalar messages as in BP2. The problem is solved by the recently proposed refined BP4 decoder with scalar messages [5], [6].

In this paper, we enhance the refined BP4 decoder with NBP for QLDPC codes. To avoid typical problems such as vanishing gradients in the training of very deep neural networks (NNs), our proposed NBP model is based on an overcomplete check matrix with redundant rows. It is inspired by the observation in classical coding theory that using an overcomplete parity check matrix (PCM) enables more node updates in parallel, which reduces the required number of decoding iterations and combats the effects of short cycles. Consequently, the trained NBP network effectively improves the decoding performance by orders of magnitude compared to conventional BP decoding for the considered codes. Moreover, the required number of decoding iterations is reduced significantly, yielding a low-latency decoder.

II. PRELIMINARIES

A. Stabilizer Formalism

Quantum stabilizer codes (QSCs) [17], [18] are the quantum analogs of classical linear codes. To define a QSC, we first need to define the Pauli operators. For simplicity, we ignore the global phase and consider the *n*-qubit Pauli group \mathcal{G}_n consisting of Pauli operators on *n* qubits $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \cdots \otimes \mathcal{P}_n$ where $\mathcal{P}_i \in \{I, X, Y, Z\}$ are Pauli operators on the *i*-th qubit. Without the risk of confusion, the tensor product \otimes and the identity operator I can be omitted. The weight w of a Pauli operator \mathcal{P} is the number of non-identity components in the tensor product. For the depolarizing channel with depolarizing

This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 101001899).

probability ϵ considered in this work, the errors X, Z and Y occur equally likely with probability $\frac{\epsilon}{3}$. To find a valid stabilizer code, it is crucial to find the stabilizer group S, which is an Abelian subgroup of \mathcal{G}_n . This can be transferred into a classical coding problem using the mapping of Pauli errors to binary strings $\mathcal{P} \mapsto (x_1 \cdots x_n | z_1 \cdots z_n) =: (\mathbf{p}_X | \mathbf{p}_Z)$ with $\mathcal{P}_i = X^{x_i} Z^{z_i}$. We can check that Pauli errors \mathcal{A} and \mathcal{B} commute if the symplectic product of their corresponding binary strings $(\mathbf{a}_X | \mathbf{a}_Z)$ and $(\mathbf{b}_X | \mathbf{b}_Z)$ is 0, i.e.

$$\sum_{i=1}^{n} a_{X,i} \cdot b_{Z,i} + \sum_{i=1}^{n} a_{Z,i} \cdot b_{X,i} = 0.$$
 (1)

Therefore, a stabilizer code can be constructed from a [2n, k] classical binary code given by its full rank PCM $H = (H_X | H_Z)$ of size m = 2n - k by 2n which fulfills the symplectic criterion:

$$\boldsymbol{H}_{\boldsymbol{X}}\boldsymbol{H}_{\boldsymbol{Z}}^{\mathsf{T}} + \boldsymbol{H}_{\boldsymbol{Z}}\boldsymbol{H}_{\boldsymbol{X}}^{\mathsf{T}} = \boldsymbol{0}. \tag{2}$$

Calderbank–Shor–Steane (CSS) codes are a special kind of stabilizer codes where

$$egin{array}{c|c} m{H} = \left(egin{array}{c|c} m{H}'_X & m{0} \ m{0} & m{H}'_Z \end{array}
ight). \end{array}$$

In this case, (2) holds if $H'_X H'_Z^{\top} = 0$ and the code construction can be simplified. QLDPC codes are defined as a family of stabilizer codes whose row and column weights are upper bounded by a relatively small constant independent of the block length, i.e., H is sparse. Most of the successful QLDPC codes constructed so far are CSS codes and the QLDPC codes considered in this paper are also CSS codes.

To decode the four types of Pauli errors jointly, it is convenient to consider the quaternary form of H, denoted as $S \in GF(4)^{m \times n}$, where GF(4) consist of the elements $\{0, 1, \omega, \overline{\omega}\}$. H can be converted to S using the mapping of binary strings to strings over GF(4) $(x_1 \cdots x_n | z_1 \cdots z_n) \mapsto (p_1 \cdots p_n) =: p$ with $p_i = x_i \omega + z_i \overline{\omega}$. Then we can check that for A and B, mapped to vectors a and b over GF(4), (1) is equivalent to $\langle a, b \rangle = \sum_{i=1}^{n} \langle a_i, b_i \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the *trace inner product* over GF(4).

The matrix S is called the *check matrix* and every row of S is called a *check*. The checks correspond to the m stabilizers which generate the stabilizer group S. An [[n, k, d]] stabilizer code is a 2^k -dimensional subspace of the n-qubit Hilbert space $(\mathbb{C}^2)^{\otimes n}$ defined as the common +1 eigenspace of S. Additionally, we define $\mathcal{N}(S)$ to be the normalizer of S and S^{\perp} to be the matrix containing the vectors corresponding to the 2n - m generators of $\mathcal{N}(S)$. The minimum distance d is defined as the lowest error weight in $\mathcal{N}(S) \setminus S$. A code is called *degenerate* if S contains errors of weight less than d.

B. Syndrome Decoding of Stabilizer codes

Let $\mathcal{E} \in \mathcal{G}_n$ be an occurred error and e be the corresponding error vector over GF(4). To obtain an error syndrome, measurements are performed on the *m* stabilizer generators. The results are mapped to 0 or 1 indicating whether the error commutes with the corresponding stabilizer generator or not, i.e, the error syndrome is $\boldsymbol{z} = (z_1, z_2, \dots, z_m) \in \{0, 1\}^m$, where $z_i = \langle \boldsymbol{e}, \boldsymbol{S}_i \rangle$ and \boldsymbol{S}_i is the *i*-th row of \boldsymbol{S} .

Let \hat{e} be the estimate of e by the decoder and \mathcal{E} be its corresponding Pauli error. The decoder aims to find an \hat{e} which yields the same syndrome z and such that $\hat{\mathcal{E}}\mathcal{E} \in S$. The latter can be checked by

$$\langle (\boldsymbol{e} + \hat{\boldsymbol{e}}), \boldsymbol{S}_i^{\perp} \rangle = 0$$
 (3)

for every row $i \in \{1, 2, ..., 2n - m\}$ of S^{\perp} . Two types of decoding failures may happen: One is when the decoder fails to find any \hat{e} which matches the syndrome z. Another is when an inferred error \hat{e} is found which matches the syndrome z but (3) does not hold. This is called an "unflagged-error" [19] and it leads to an undetectable erroneous state.

C. Belief Propagation Decoder

The task of inferring an error from a given syndrome for a QLDPC code can be carried out with a BP4 decoder. We use the log-domain refined BP4 decoder proposed in [6]. It exploits the fact that the syndrome of a stabilizer code is binary, indicating whether the error commutes with the stabilizer or not, which enables scalar message passing. We briefly review the algorithm here.

For every variable node (VN) v_i , where $i \in \{1, 2, ..., n\}$, we initialize the log-likelihood ratio (LLR) vector $\Gamma_{i \to j}$ as $\Lambda_i = \left(\Lambda_i^{(1)} \quad \Lambda_i^{(\omega)} \quad \Lambda_i^{(\bar{\omega})}\right) \in \mathbb{R}^3$ with

$$\Lambda_i^{(\zeta)} = \ln\left(\frac{P(e_i = 0)}{P(e_i = \zeta)}\right) = \ln\left(\frac{1 - \epsilon_0}{\frac{\epsilon_0}{3}}\right)$$

where $\zeta \in GF(4) \setminus \{0\}$ and ϵ_0 is the estimated physical error probability of the channel. To exchange scalar messages, a *belief-quantization operator* $\lambda_n : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$\lambda_{\eta}(\mathbf{\Lambda}_{i}) = \ln\left(\frac{P(\langle e_{i}, \eta \rangle = 0)}{P(\langle e_{i}, \eta \rangle = 1)}\right) = \ln\left(\frac{1 + e^{-\Lambda_{i}^{(\eta)}}}{\sum_{\zeta \neq 0, \zeta \neq \eta} e^{-\Lambda_{i}^{(\zeta)}}}\right).$$

The operator λ_{η} turns the LLR vector into a scalar LLR of the binary random variable $\langle e_i, \eta \rangle$ where η runs over the nonzero entries of S. The initial scalar VN messages are calculated as

$$\lambda_{i \to j} := \lambda_{S_{ii}}(\Gamma_{i \to j}) \tag{4}$$

and are passed to the neighboring check nodes (CNs).

The outgoing messages of CN c_j , $j \in \{1, 2, ..., m\}$, are calculated using

$$\Delta_{i \leftarrow j} = (-1)^{z_j} \cdot 2 \tanh^{-1} \left(\prod_{i' \in \mathcal{N}(j) \setminus i} \tanh \frac{\lambda_{i' \to j}}{2} \right) \quad (5)$$

where $\mathcal{N}(j)$ denotes the neighboring VNs of c_j .

At the VN update, we first calculate the LLR vector $\Gamma_{i \to j} = \left(\Gamma_{i \to j}^{(1)} \quad \Gamma_{i \to j}^{(\omega)} \quad \Gamma_{i \to j}^{(\bar{\omega})} \right)$ with

$$\Gamma_{i \to j}^{(\zeta)} = \Lambda_i^{(\zeta)} + \sum_{\substack{j' \in \mathcal{M}(i) \setminus j, \\ \langle \zeta, S_{j'i} \rangle = 1}} \Delta_{i \leftarrow j'}, \tag{6}$$

for all $\zeta \in GF(4) \setminus \{0\}$ with $\mathcal{M}(i)$ denoting the neighboring CNs of v_i . Then the outgoing messages $\lambda_{i \to j} = \lambda_{S_{ji}}(\Gamma_{i \to j})$ are calculated and passed to the neighboring CNs.

To estimate the error, a hard decision is performed at the VNs by calculating Γ_i for $i \in \{1, 2, ..., n\}$ with

$$\Gamma_i^{(\zeta)} = \Lambda_i^{(\zeta)} + \sum_{\substack{j \in \mathcal{M}(i)\\\langle \zeta, S_{ji} \rangle = 1}} \Delta_{i \leftarrow j}, \tag{7}$$

for all $\zeta \in GF(4) \setminus \{0\}$. If all $\Gamma_i^{(\zeta)} > 0$, then $\hat{e}_i = 0$, otherwise $\hat{e}_i = \operatorname{argmin}_{\zeta} \Gamma_i^{(\zeta)}$.

The iterative process is performed until the maximum number of iterations L is reached or the syndrome is matched.

III. CHECK MATRIX WITH REDUNDANT ROWS

To construct an overcomplete check matrix S_{oc} with redundant rows, we treat the two binary matrices H_X and H_Z separately and consider them as classical binary PCMs. For both, H_X and H_Z , redundant rows are generated by linear combinations of the rows of the original matrix. We try to keep the sparsity of the PCM by only generating low-weight rows. The task is equivalent to generating low-weight dual codewords in classical coding theory and we use Leon's probabilistic algorithm [26]. BP decoding is performed on the Tanner graph associated with the overcomplete S_{oc} .

It is important to note that using this method does not require additional syndrome measurements which would be costly. Let H be either H_X or H_Z . The PCM H_{oc} with redundant rows is obtained by $H_{oc} = MH$ with M being a binary matrix of size $m_{oc} \times m$. The original syndrome is calculated as $He^{T} = z$. The new syndrome associated with H_{oc} is given by

$$\boldsymbol{z}_{\mathrm{oc}} = \boldsymbol{H}_{\mathrm{oc}} \boldsymbol{e}^{\mathsf{T}} = \boldsymbol{M} \boldsymbol{H} \boldsymbol{e}^{\mathsf{T}} = \boldsymbol{M} \boldsymbol{z},$$

being a linear mapping of z using M.

The idea of performing BP decoding over an overcomplete parity-check matrix has been investigated in [16], [21]–[25] for classical linear codes. One of the initial motivations is to perform more node updates in parallel to reduce the effect of short cycles. For QLDPC codes, this approach resembles matrix augmentation [9], where a fraction of the rows of the check matrix are duplicated. The advantage is that the messages associated with the duplicated check node are magnified which helps in breaking the symmetry during decoding and leading to a (hopefully) correct error estimation [9]. We demonstrate the extra benefit of the proposed method with a toy example. Consider the [[7,1,3]] quantum Bose– Chaudhuri–Hocquenghem (BCH) code with both its H'_X and H'_Z being the PCM of a [7,4,3] BCH code:

$$m{H}_{
m BCH} = egin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \ 0 & 1 & 1 & 0 & 0 & 1 & 1 \ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Consider the error $\mathcal{E} = Y_7$. The syndrome z of e is

$$\boldsymbol{z} = \begin{pmatrix} 1 \ 1 \ 1 \ 1 \ 1 \ 1 \end{pmatrix}.$$



Fig. 1. BP decoding results for the [[7,1,3]] CSS code with original and overcomplete check matrix (L = 32).

Assume that we initialize the decoder with $\epsilon_0 = 0.1$. According to (4), all the initial VN messages are 2.64. Then, in the first CN update, all CN messages are -1.55 according to (5). Based on these messages, in the next hard decision step, the decoder estimates the error as $\hat{\boldsymbol{\mathcal{E}}} = \boldsymbol{Y}_3 \boldsymbol{Y}_5 \boldsymbol{Y}_6 \boldsymbol{Y}_7$ which produces the same syndrome as \boldsymbol{z} . However, (3) does not hold and we end up with an unflagged-error.

In this example, the decoder wrongly estimates the error because all CNs have a syndrome of 1. Therefore, there is a strong indication that the error has a high weight. Except for VNs v_1 , v_2 , and v_4 with degree 2, all VNs that are connected to more than 2 CNs are erroneously estimated in the first hard-decision step. Duplicating some rows of the check matrix does not help in this case¹.

Now let us use the overcomplete H_X and H_Z with 7 rows, i.e., we take all the linear combinations of the 3 rows of the original H_{BCH} except the trivial case. The new syndrome is

$$\boldsymbol{z}_{\rm oc} = (1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1)$$

Although z_{oc} contains the same amount of information as z with respect to the coset where the error \mathcal{E} is located, the former is easier for the decoder to interpret. In the first hard-decision step, the error is correctly estimated. Fig. 1 depicts the frame error rate (FER) curves corresponding to decoding using the overcomplete check matrix and the original check matrix for different ϵ_0 .

IV. NEURAL BELIEF PROPAGATION

In our NBP decoder, additional weights are introduced for each Λ_i and each CN. The update rules (5) and (6) are modified to

$$\Delta_{i \leftarrow j} = (-1)^{z_j} \cdot 2w_{\mathsf{c},j}^{(\ell)} \tanh^{-1} \left(\prod_{i' \in \mathcal{N}(j) \setminus i} \tanh \frac{\lambda_{i' \to j}}{2} \right)$$
(8)

and

$$\Gamma_{i \to j}^{(\zeta)} = w_{\mathsf{v},i}^{(\ell)} \Lambda_i^{(\zeta)} + \sum_{\substack{j' \in \mathcal{M}(i) \setminus j, \\ \langle \zeta, S_{j'i} \rangle = 1}} \Delta_{i \leftarrow j'}, \tag{9}$$

¹However, note that this problem could be solved by initializing the decoder with a very small ϵ_0 such as 0.001 at the cost of degrading the overall decoding performance (depicted in Fig. 1), as in most cases, the decoder has a tendency towards trivial errors.

where ℓ is the index of the decoding iteration and $w_{\mathsf{c},j}^{(\ell)}$ and $w_{\mathsf{v},i}^{(\ell)}$ denote the trainable weights. The model is relatively simple compared to conventional NBP, where weights are applied on each edge of the Tanner graph. For the codes considered in this work, using more model parameters did not improve the performance.

In [19], a loss function for BP2 decoding has been proposed which takes degeneracy into account. We extend this loss function to the BP4 case. Using Γ_i calculated by (7) in the hard-decision step, we now calculate

$$P(\langle \hat{e}_i, \eta \rangle = 1 | \boldsymbol{z}) = \left(1 + e^{-\lambda_{\eta}(\boldsymbol{\Gamma}_i)}\right)^{-1}$$
(10)

for $\eta \in GF(4) \setminus \{0\}$, indicating the estimated probability of the *i*-th error commuting with X, Z, and Y, respectively. Then proposed loss function can be written as

$$\mathcal{L}(\boldsymbol{\Gamma}; \boldsymbol{e}) = \sum_{j=1}^{2n-m} f\left(\sum_{i=1}^{n} P\left(\langle e_i + \hat{e}_i, S_{ji}^{\perp} \rangle = 1 | \boldsymbol{z}\right)\right) \quad (11)$$

where $f(x) = |\sin(\pi x/2)|$, as in [19]. The loss is summed up over all rows S_j^{\perp} of S^{\perp} . For each row j, we sum up the values $P\left(\langle e_i + \hat{e}_i, S_{ji}^{\perp} \rangle = 1 | \mathbf{z} \right)$ for all the elements S_{ji}^{\perp} in S_j^{\perp} , representing the probability of S_{ji}^{\perp} being unsatisfied after estimating e_i as \hat{e}_i . It can be calculated as $P\left(\langle \hat{e}_i, S_{ji}^{\perp} \rangle = 1 + \langle e_i, S_{ji}^{\perp} \rangle | \mathbf{z} \right)$. When $\langle e_i, S_{ji}^{\perp} \rangle = 0$, we directly calculate $P\left(\langle \hat{e}_i, S_{ji}^{\perp} \rangle = 1 | \mathbf{z} \right)$ using (10), with η being S_{ji}^{\perp} . When $\langle e_i, S_{ji}^{\perp} \rangle = 1$, we calculate $P\left(\langle \hat{e}_i, S_{ji}^{\perp} \rangle = 0 | \mathbf{z} \right) = 1 - P\left(\langle \hat{e}_i, S_{ji}^{\perp} \rangle = 1 | \mathbf{z} \right)$. As f(x) approaches 0 with x approaching any even number, the loss function (11) is minimized when (3) holds.

During training, we use multi-loss [14], [15] calculated as the average value of the loss function after every iteration until the loss is minimized. This helps to increase the magnitude of gradients corresponding to earlier decoding iterations.

V. NUMERICAL RESULTS

We assess the performance of the proposed decoder using Monte Carlo simulations. To ensure sufficiently accurate results, 300 frame errors are collected for each data point. We always use the refined BP4 decoder with a flooding schedule. The initial ϵ_0 is always set to 0.1, which yields a good decoding performance for the codes considered in this work. This improves the decoding performance when the actual physical error probability ϵ is too small [6]. Using a fixed initialization also avoids estimating the channel statistics, which is not always feasible.

We consider generalized bicycle (GB) codes proposed in [7]. They are constructed by $H'_X = (A \ B)$ and $H'_Z = (B^T \ A^T)$ where A and B are $\frac{n}{2} \times \frac{n}{2}$ square circulant matrices. For constructing the check matrix, m/2 rows are chosen from both A or B. However, all the rows of A or B are linearly dependent. Therefore, the GB codes naturally have an overcomplete set of checks. Further redundant rows are generated following the method described in Sec. III.



Fig. 2. FER vs. depolarizing probability curves for the [[48,6,8]] code



Fig. 3. FER vs. depolarizing probability curves for the [[46,2,9]] code

A. NBP Results

Figure 2 and Fig. 3 show the simulation results for the [[48,6,8]] and [[46,2,9]] GB codes (codes A3 and A4 in [7]). We compare the decoding performance for the two codes using refined BP decoding with different settings. For BP4 decoding over the original Tanner graph, 32 iterations are performed. For the [[48,6,8]] code, we construct the overcomplete check matrix using 48 rows of weight 8 and 1952 rows of weight 12 (no checks of weight 10 were found). For the [[46,2,9]] code, we use 46 rows of weight 8 and 754 rows of weight 10. We observe that in this case, the number of required decoding iterations can be greatly reduced. Only 3 iterations are necessary for the [[48.6.8]] code and 6 iterations for the [[46.2.9]] code. Further increasing the number of iterations does not improve the performance noticeably. We compare the FER results with the reference results taken from [7] where normalized minsum decoding with layered scheduling is followed by an OSD post-processing step. The reference outperforms the original BP decoding which suffers from a higher error floor. However, BP decoding using the overcomplete check matrix performs comparably to the reference.

The FERs can be further reduced by NBP. Using the loss function described in Sec. IV, the weights $w_{{\rm c},{\rm j}}^{(\ell)}$ and $w_{{\rm v},i}^{(\ell)}$ are



Fig. 4. FER vs. error weight curves for the [[48,6,8]] code



Fig. 5. FER vs. error weight curves for the [[46,2,9]] code

optimized using gradient descent. During training, the learning rate is 0.001 and the Adam optimizer [27] is used. We first pretrain the model on 1500 batches of batch size 100 consisting of random weight 2 and weight 3 errors. Then we train the model on 600 batches of batch size 100 consisting of random weight 3 to 9 errors. Weight 1 errors are omitted for both codes, as the loss value of weight 1 errors is always almost zero. This training approach is based on our observation that directly training the NN using random errors causes the loss to diverge. The conjectured reason is that too many uncorrectable errors increase the gradient noise which makes the model prone to be stuck in a local minimum. We observe that on average, pre-training yields higher weights on low-degree CNs, which is consistent with our experience that low-degree checks are more helpful in decoding. Subsequent training on high-weight errors further improves the decoding performance. After training, our NBP decoder outperforms the reference results, with the additional advantage of a very small number of required decoding iterations.

To show that the performance gain is not simply due to overfitting, i.e., the decoder simply remembering how to decode all errors of weight 2 and weight 3, we plot the FER results when decoding errors of different weights in Fig. 4 and Fig. 5. The figures show that the trained NBP decoder also outperforms conventional BP on high-weight errors.

B. Effect of Check Matrix with Redundant Rows

We show that adding redundant rows is often beneficial for improving the decoding performance. However, the added



Fig. 6. FER vs. depolarizing probability ϵ curves for the [[126, 28, 8]] (A2) and [[254, 28, d]] (A1) GB code with original and overcomplete check matrix (L = 32).

rows need to be chosen carefully. We observe that the performance improves with a large number of redundant rows if the maximum CN degree stays unchanged. We compare the FERs of two GB codes with parameters [[126, 28, 8]] (code A2 in [7]) and [[254, 28, d]] (code A1 in [7]) when decoding with different check matrices. Checks of low weight are always added first when constructing the overcomplete check matrix.

For the [[126, 28, 8]] code, we consider three check matrices: The original check matrix S with 98 rows of weight 10, an overcomplete check matrix $S_{oc,1}$ with 126 rows of weight 10, and another overcomplete check matrix $S_{oc,2}$ consisting of $S_{oc,1}$ and 126 rows of weight 16. Decoding with $S_{oc,1}$ outperforms the decoder based on S by nearly an order of magnitude at $\epsilon = 0.02$. However, decoding with $S_{oc,2}$ shows a performance degradation. For the [[254, 28]] code, similar results can be observed. Compared with the results of decoding with the original check matrix with 226 rows, adding rows of weight 10 ($m_{oc} = 254$) improves the decoding performance. However, further adding rows of weight 18 ($m_{oc} = 2000$) degrades the decoding performance.

VI. CONCLUSION

In this paper, BP4 decoding for QLDPC codes based on a check matrix with redundant rows is proposed and investigated. The method is shown to be effective for several QLDPC codes in improving the decoding results. Moreover, combined with NBP, the performance can be further improved. As a large number of node updates are performed in parallel, which reduces the number of required decoding iterations, the decoder has a very small decoding latency. Moreover, it is well-accepted that the BP decoding performance is impacted by the structure of the underlying Tanner graph. Due to the symplectic criterion, optimizing the structure of the Tanner graph for QLDPC codes is more difficult than for classical LDPC codes. Using a check matrix with redundant rows gives us some degree of freedom in optimizing the graph structure.

REFERENCES

- J.-P. Tillich and G. Zémor, "Quantum LDPC codes with positive rate and minimum distance proportional to the square root of the blocklength," *IEEE Transactions on Information Theory*, vol. 60, no. 2, pp. 1193– 1202, 2013.
- [2] P. Panteleev and G. Kalachev, "Quantum LDPC codes with almost linear minimum distance," *IEEE Transactions on Information Theory*, vol. 68, no. 1, pp. 213–229, 2021.
- [3] —, "Asymptotically good quantum and locally testable classical LDPC codes," Preprint, available online at https://arxiv.org/abs/2111.03654.
- [4] N. P. Breuckmann, and J. N. Eberhardt. "Balanced product quantum codes," *IEEE Transactions on Information Theory*, vol. 67, no. 10, pp. 6653–6674, 2021.
- [5] K.-Y. Kuo and C.-Y. Lai, "Refined belief propagation decoding of sparse-graph quantum codes," *IEEE Journal on Selected Areas in Information Theory*, vol. 1, no. 2, pp. 487–498, 2020.
- [6] C.-Y. Lai and K.-Y. Kuo, "Log-domain decoding of quantum LDPC codes over binary finite fields," *IEEE Transactions on Quantum Engineering*, vol. 2, pp. 1–15, 2021.
- [7] P. Panteleev and G. Kalachev, "Degenerate quantum LDPC codes with good finite length performance," *Quantum*, vol. 5, 2021.
- [8] N. Raveendran and B. Vasić, "Trapping sets of quantum LDPC codes," *Quantum*, vol. 5, pp. 1–19, 2021.
- [9] A. Rigby, J. C. Olivier, and P. Jarvis, "Modified belief propagation decoders for quantum low-density parity-check codes," *Physical Review A*, vol. 100, no. 1, 2019.
- [10] J. Roffe, D. R. White, S. Burton, and E. Campbell, "Decoding across the quantum low-density parity-check code landscape," *Physical Review Research*, vol. 2, no. 4, 2020.
- [11] D. Poulin and Y. Chung, "On the iterative decoding of sparse quantum codes," *Quantum Information and Computation*, vol. 8, no. 10, pp. 987– 1000, 2008.
- [12] Y. Wang, B. C. Sanders, B. Bai, and X. Wang., "Enhanced feedback iterative decoding of sparse quantum codes," *IEEE Transactions on Information Theory*, vol. 58, no. 2, pp. 1231–1241, 2012.
- [13] J. du Crest, M. Mhalla, and V. Savin, "Stabilizer inactivation for message-passing decoding of quantum LDPC codes," Preprint, available online at https://arxiv.org/abs/2205.06125.
- [14] E. Nachmani, Y. Be'ery, and D. Burshtein, "Learning to decode linear codes using deep learning," in *Proc. Annual Allerton Conference on Communication, Control, and Computing*, pp. 341–346, 2016.
- [15] E. Nachmani, E. Marciano, L. Lugosch, W. J. Gross, D. Burshtein, and Y. Be'ery, "Deep learning methods for improved decoding of linear codes," *IEEE Journal of Selected Topics in Signal Processing*, vol. 12, no. 1, pp. 119–131, 2018.
- [16] M. Lian, F. Carpi, C. Häger, and H. D. Pfister, "Learned beliefpropagation decoding with simple scaling and SNR adaptation," in *Proc. IEEE International Symposium on Information Theory (ISIT)*, 2019.
- [17] D. Gottesman, Stabilizer codes and quantum error correction, Ph.D. thesis, California Institute of Technology, 1997.
- [18] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane, "Quantum error correction via codes over GF(4)," *IEEE Transactions* on *Information Theory*, vol. 44, no. 4, pp. 1369–1387, 1998.
- [19] T. Liu and D. Poulin, "Neural belief-propagation decoders for quantum error-correcting codes," *Physical review letters*, vol. 122, no. 20, 2019.
- [20] X. Xiao, N. Raveendran, and B. Vasić, "Neural-net decoding of quantum LDPC codes with straight-through estimators," in *Proc. Information Theory and Applications Workshop (ITA)*, 2019.
- [21] A. Buchberger, C. Häger, H. D. Pfister, L. Schmalen, and A. Graell i Amat, "Pruning and quantizing neural belief propagation decoders," *IEEE Journal on Selected Areas in Communications*, vol. 39, no. 7, pp. 1957–1966, 2020.
- [22] T. R. Halford and K. M. Chugg, "Random redundant soft-in soft-out decoding of linear block codes," in *Proc. IEEE International Symposium* on Information Theory (ISIT), pp. 2230–2234, 2006.
- [23] M. Bossert and F. Hergert, "Hard- and soft-decision decoding beyond the half minimum distance—an algorithm for linear codes," *IEEE Transactions on Information Theory*, vol. 32, no. 5, pp. 709–714, 1986.
- [24] A. Kothiyal, O. Y. Takeshita, W. Jin, and M. Fossorier, "Iterative reliability-based decoding of linear block codes with adaptive belief propagation," *IEEE Communications Letters*, vol. 9, no. 12, pp. 1067–1069, 2005.

- [25] J. Jiang and K. R. Narayanan, "Iterative soft-input soft-output decoding of Reed-Solomon codes by adapting the parity-check matrix," *IEEE Transactions on Information Theory*, vol. 52, no. 8, pp. 3746–3756, 2006.
- [26] J. S. Leon, "A probabilistic algorithm for computing minimum weights of large error-correcting codes," *IEEE Transactions on Information Theory*, vol. 34, no. 5, pp. 1354-1359, 1988.
- [27] D. P. Kingma and J. L. Ba, "Adam: A method for stochastic optimization," in Proc. International Conference on Learning Representations (ICLR), 2015.