Erasure codes with symbol locality and group decodability for distributed storage

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Abstract—We introduce a new family of erasure codes, called group decodable code (GDC), for distributed storage system. Given a set of design parameters $\{\alpha, \beta, k, t\}$, where k is the number of information symbols, each codeword of an (α, β, k, t) group decodable code is a t-tuple of strings, called *buckets*, such that each bucket is a string of β symbols that is a codeword of a $[\beta, \alpha]$ MDS code (which is encoded from α information symbols). Such codes have the following two properties:

(P1) Locally Repairable: Each code symbol has locality $(\alpha, \beta - \alpha + 1)$.

(P2) Group decodable: From each bucket we can decode α information symbols.

We establish an upper bound of the minimum distance of (α, β, k, t) -group decodable code for any given set of $\{\alpha, \beta, k, t\}$; We also prove that the bound is achievable when the coding field \mathbb{F} has size $|\mathbb{F}| > {n-1 \choose k-1}$.

I. INTRODUCTION

Distributed storage systems (DSS) are becoming increasingly important due to the explosively grown demand for largescale data storage, including large files and video sharing, social networks, and back-up systems. Distributed storage systems store a tremendous amount of data using a massive collection of distributed storage nodes and, to ensure reliability against node failures, introduce a certain of redundancy.

The simplest form of redundancy is replication. DSS with replication are very easy to implement, but extremely inefficient in storage efficiency, incurring tremendous waste in devices and equipment. In recent years, some efficient schemes for distributed storage systems, such as erasure codes [1] and regenerating codes [2], are proposed. We focus on erasure codes in this paper.

MDS codes are the most efficient erasure codes in term of storage efficiency. When use an [n, k] MDS code, the data file that need to be stored is divided into k information packets, where each packet is a symbol of the coding field. These k information packets are encoded into n packets and stored in n storage nodes such that each node stores one packet. Then the original file can be recovered from any k out of the n coded packets. Although MDS code is storage optimal, it is not efficient for node repair. That is, when one storage node fails, we must download the whole file from some other k nodes to reconstruct the coded packet stored in it.

To construct erasure codes with more repair efficiency than MDS codes, the concepts of locality and locally repairable code (LRC) were introduced [3], [4], [5]. Let $1 \le \alpha \le k$ and $\delta \ge 2$. The *i*th code symbol c_i $(1 \le i \le n)$ in an [n, k]linear code C is said to have locality (α, δ) if there exists a subset $S_i \subseteq [n] = \{1, 2, \dots, n\}$ containing *i* and of size $|S_i| \le \alpha + \delta - 1$ such that the punctured subcode of C to S_i has minimum distance at least δ . We will call each subset $\{c_j; j \in S_i\}$ a *repair group*. Thus, if c_i has locality (α, δ) , then c_i can be computed from any $|S_i| - \delta + 1$ other symbols in the repair group $\{c_j; j \in S_i\}$. A code is said to have all-symbol locality (α, δ) (or is called an $(\alpha, \delta)_a$ code) if all of its code symbols have locality (α, δ) . Note that $|S_i| - \delta + 1 \le \alpha$. The code has a higher repair efficiency than MDS code if $\alpha < k$. The minimum distance of an $(\alpha, \delta)_a$ linear code is bounded by (See [4]) :

$$d \le n - k + 1 - \left(\left\lceil \frac{k}{\alpha} \right\rceil - 1 \right) (\delta - 1). \tag{1}$$

However, for the case that $(\alpha + \delta - 1) \nmid n$ and $\alpha \mid k$, there exists no $(\alpha, \delta)_a$ linear code achieving the above bound [6].

The most common case of $(\alpha, \delta)_a$ linear code is that n is divisible by $\alpha + \delta - 1$. For this case, in the constructions presented in the literature, all code symbols of an $(\alpha, \delta)_a$ linear code are usually divided into $t = \frac{n}{\alpha+\delta-1}$ mutually disjoint repair groups such that each repair group is a codeword of an $[\alpha+\delta-1,\alpha]$ MDS code. Fig. 1 illustrates a $(4,3)_a$ systematic linear code with n = 18 and k = 6, where x_1, \dots, x_6 are the information symbols and y_1, \dots, y_{12} are the parities. All code symbols are divided into three groups and each group is a codeword of a [6, 4] MDS code. By constructing the parities elaborately, the code can be distance optimal according to (1).

Group 1:	x_1	x_2	x_3	x_4	y_1	y_2
Group 2:	x_5	x_6	y_3	y_4	y_5	y_6
Group 3:	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}

Fig 1. Illustration of a systematic locally repairable code: The information symbols x_1, \dots, x_6 are encoded into $x_1, \dots, x_6, y_1, \dots, y_{12}$ that are divided into three groups. Each group is a codeword of a [6, 4] MDS code.

As pointed out in [10], in distributed storage applications there are subsets of the data that are accessed more often than the remaining contents (they are termed "hot data"). Thus, a desired property of a distributed storage system is that the subsets of hot data can be retrieved easily and by multiple ways. For example, for the storage system illustrated by Fig. 1, suppose x_1 is hot data. There are two "easy ways" to retrieve it from the system: Downloaded x_1 directly from the node where it is stored, or decode it from any four coded symbols in the first group. Another way is to decode it from some six coded symbols, but this is not an easy way because to decode x_1 , one has to decode the whole data file.



Fig 2. Illustration of a (4, 6, 6, 3)-group decodable code: x_1, \dots, x_6 are information symbols and z_1, \dots, z_6 are parities. Each codeword has 3 buckets and each bucket is a codeword of a [6, 4] MDS code that is encoded from 4 information symbols. Clearly, each bucket is a repair group.

In this work, we introduce a new family of erasure codes, called *group decodable code* (*GDC*), for distributed storage system, which can provide more options of easy ways to retrieve each information symbol than systematic codes. Given a set of design parameters $\{\alpha, \beta, k, t\}$, where k is the number of information symbols, each codeword of an (α, β, k, t) -group decodable code is a t-tuple of strings, called *buckets*, such that each bucket is a string of β symbols that is a codeword of a $[\beta, \alpha]$ MDS code (which is encoded from α information symbols). So such codes have the following two properties:

- (P1) Locally Repairable: Each code symbol has locality $(\alpha, \beta \alpha + 1)$.
- (P2) *Group decodable*: From each bucket we can decode α information symbols.

Fig. 2 illustrates a (4, 6, 6, 3)-group decodable code. There are six information symbols x_1, \dots, x_6 . Each codeword has 3 buckets and each bucket is a codeword of a [6, 4] MDS code that is encoded from 4 information symbols. Clearly, each bucket is a repair group. So each code symbol of this code has locality (4, 3). Moreover, this code provides more options of easy ways to retrieve each information symbol than the code in Fig. 1. For example, x_1 can be downloaded directly from two nodes or can be decoded from any four symbols in bucket 1 or any four symbols in bucket 2. In the case that x_1 is requested simultaneously by many users of the system, the can ensure that multiple read requests can be satisfied concurrently and with no delays.

A. Our contribution

We establish an upper bound of the minimum distance of group decodable code for any given set of parameters $\{\alpha, \beta, k, t\}$ (Theorem 4). We also prove that there exist linear codes of which the minimum distances achieve the bound, which proves the tightness of the bound (Theorem 5). Our proof gives a method to construct (α, β, k, t) -group decodable code on a field of size $q > \binom{n-1}{k-1}$, where $n = t\beta$ is the length of the code.

B. Related Work

Some existing works consider erasure codes for distributed storage that can provide multiple alternatives for repairing information symbols or all code symbols with locality.

In [7], the authors introduced the metric "local repair tolerance" to measure the maximum number of erasures that do not compromise local repair. They also presented a class of locally repairable codes, named pg-BLRC codes, with high local repair tolerance and low repair locality. However, they did not present any bound on the minimum distance of such codes.

In [8], the concept of $(\alpha, \delta)_c$ -locality was defined, which captures the property that there exist $\delta - 1$ pairwise disjoint local repair sets for a code symbol. An upper bound on the minimum distance for [n, k] linear codes with information $(\alpha, \delta)_c$ -locality was derived, and codes that attain this bound was constructed for the length $n \ge k(\alpha(\delta - 1) + 1)$. However, for $n < k(\alpha(\delta - 1) + 1)$, it is not known whether there exist codes attaining this bound. Upper bounds on the rate and minimum distance of codes with all-symbol $(\alpha, \delta)_c$ -locality was proved in [9]. However, no explicit construction of codes that achieve this bound was presented. It is still an open question whether the distance bound in [9] is achievable.

Another subclass of LRC is codes with (r, t)-locality: any set of t code symbols are functions of at most r other code symbols [11]. Hence, for such codes, any t failed code symbols can be repaired by contacting at most r other code symbols. An upper bound of the minimum distance of such codes similar to (1) is derived in [11].

C. Organization

The paper is organized as follows. In Section II, we present the related concepts and the main results of this paper. We prove the main results in section III and Section IV.

II. MODEL AND MAIN RESULT

Denote $[n] := \{1, \dots, n\}$ for any given positive integer n. Let \mathbb{F} be a finite field and k be a positive integer. For any $S = \{i_1, \dots, i_{\alpha}\} \subseteq [k]$, the *projection* of \mathbb{F}^k about S is a function $\psi_S : \mathbb{F}^k \to \mathbb{F}^\alpha$ such that for any $(x_1, \dots, x_k) \in \mathbb{F}^k$,

$$\psi_S(x_1,\cdots,x_k) = (x_{i_1},\cdots,x_{i_\alpha}). \tag{2}$$

We can define group decodable codes (GDC) as follows. Definition 1: Suppose $S = \{S_1, \dots, S_t\}$ is a collection of subsets of [k] and $\mathcal{N} = \{n_1, \dots, n_t\}$ is a collection of positive integers such that $\bigcup_{i=1}^t S_i = [k]$ and $n_i > k_i = |S_i|, \forall i \in [t]$. A linear code C is said to be an (\mathcal{N}, S) -group decodable code (GDC) if C has an encoding function f of the following form:

$$\begin{array}{lcl} f: \mathbb{F}^k & \to & \mathbb{F}^{n_1} \times \dots \times \mathbb{F}^{n_t} \\ x & \mapsto & (f_1(\psi_{S_1}(x)), \cdots, f_t(\psi_{S_t}(x))). \end{array}$$
(3)

where each $f_i : \mathbb{F}^{k_i} \to \mathbb{F}^{n_i}$ is an encoding function of an $[n_i, k_i]$ MDS code and the output of it is called a *bucket*.

By Definition 1, if C is an (\mathcal{N}, S) -group decodable code, then C has length $n = \sum_{i=1}^{t} n_i$. For any message vector $x = (x_1, \dots, x_k)$ and $i \in [t]$, the subset of k_i messages $\{x_j; j \in S_i\}$ are encoded into a bucket of n_i symbols by the function f_i . A codeword of C is the concatenation of these t buckets. Since f_i is an encoding function of an $[n_i, k_i]$ MDS code, each bucket is a repair group and we can decode the subset $\{x_j; j \in S_i\}$ from any k_i symbols of the *i*th bucket.—The term "group decodable code" comes from this observation.

For the special case that S_1, \dots, S_t are pairwise disjoint, an $(\mathcal{N}, \mathcal{S})$ -group decodable code \mathcal{C} is just the direct sum of the *t* buckets and the minimum distance of \mathcal{C} is $\min\{n_i - k_i + 1; i \in [t]\}$. In this work, we consider the most general case that S_1, \dots, S_t can have arbitrary intersection.

Definition 1 depends on the explicit collections S and N. We can also define GDC based on design parameters.

Definition 2: Let α, β, k, t be positive integers such that $\alpha < \min\{k, \beta\}$. A linear code C is said to be an (α, β, k, t) -group decodable code if C is an $(\mathcal{N}, \mathcal{S})$ -group decodable code for some $\mathcal{S} = \{S_1, \dots, S_t\}$ and $\mathcal{N} = \{n_1, \dots, n_t\}$ such that $S_i \subseteq [k], |S_i| = \alpha$ and $n_i = \beta$ for all $i \in [t]$.

If C is an (α, β, k, t) -group decodable code, then by Definition 2, the length of C is $n = t\beta$. Moreover, since $\bigcup_{i=1}^{t} S_i = [k]$ and $|S_i| = \alpha$, then $t\alpha = \sum_{i=1}^{t} |S_i| \ge k$, which implies that $\left|\frac{t\alpha}{k}\right| \ge 1$. So we have the following remark.

Remark 3: If \overline{C} is an (α, β, k, t) -group decodable code, then $n = t\beta$ and $|\frac{t\alpha}{k}| \ge 1$.

We will give a tight upper bound on the minimum distance d of an (α, β, k, t) -group decodable code C. Our main results are the following two theorems.

Theorem 4: Let $t\alpha = sk + r$ such that $s \ge 1$ and $0 \le r \le k - 1$. If C is an (α, β, k, t) -group decodable code, then

$$d \le s\beta - \left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil + 1. \tag{4}$$

Note that an (α, β, k, t) -group decodable code is an $(r, \delta)_a$ with the additional property (P2). So the bound (4) is looser than the bound (1). The sacrifice in minimum distance is resulted from the property (P2).

Theorem 5: If $|\mathbb{F}| > {\binom{n-1}{k-1}}$, then there exists an (α, β, k, t) -group decodable code over \mathbb{F} with d achieves the bound (4).

By Remark 3, $t\alpha \ge k$. So we always have $t\alpha = sk + r$ for some $s \ge 1$ and $0 \le r \le k - 1$. So Theorem 4 and 5 covers all possible sets of parameters $\{\alpha, \beta, k, t\}$.

III. PROOF OF THEOREM 4

In this section, we prove Theorem 4. We will use some similar discussions as in [16], [17], [18].

In the rest of this paper, we always assume that $S = \{S_1, \dots, S_t\}$ is a collection of subsets of [k] and $\mathcal{N} = \{n_1, \dots, n_t\}$ such that $\bigcup_{i=1}^t S_i = [k]$ and $n_i = \beta > |S_i| = \alpha$ for all $i \in [t]$. Moreover, let $n = t\beta$ and

$$J_i = \{(i-1)\beta + 1, (i-1)\beta + 2, \cdots, i\beta\}.$$
 (5)

Clearly, J_1, \dots, J_t are pairwise disjoint and $\bigcup_{i=1}^t J_i = [n]$.

Let ℓ be any positive integers and A be any $k \times \ell$ matrix. If $J \subseteq [\ell]$, we use A_J to denote the sub-matrix of A formed by the columns of A that are indexed by J. Moreover, we will use the following notations:

- 1) For $i \in [k]$ and $j \in [\ell]$, $R_A(i)$ and $C_A(j)$ are the support of the *i*th row and the *j*th column of A respectively. Meanwhile, $|R_A(i)|$ and $|C_A(j)|$ are called the weight of the *i*th row and the *j*th column of A respectively.
- 2) The minimum row weight of A is

$$w_{\min}(A) = \min_{i \in [k]} |R_A(i)|.$$
 (6)

The *i*th row is said to be minimal if $|R_A(i)| = w_{\min}(A)$. 3) The repetition number of the *i*th row, denoted by $\Gamma_A(i)$,

is the number of $i' \in [k]$ such that $R_A(i') = R_A(i)$. Let Φ_A be the set of indices of all minimal rows of A. We denote

$$\Gamma(A) = \max_{i \in \Phi_A} \Gamma_A(i). \tag{7}$$

Clearly, we always have $\Gamma(M) \ge 1$. The following example gives some explanation of the above notations.

Example 6: Consider the following 7×8 binary matrix

	1	0	1	0	0	0	1	0	
	0	1	0	1	0	1	0	0	
	0	0	1	0	1	0	1	1	
A =	1	0	0	0	1	1	0	1	
	0	1	0	1	0	1	0	0	
	1	0	1	0	0	0	1	0	
	0	1	0	1	1	0	0	1	

We have $R_A(1) = R_A(6)$. So $\Gamma_A(1) = \Gamma_A(6) = 2$. Similarly, $\Gamma_A(2) = \Gamma_A(5) = 2$ and the repetition number of all other rows are 1. Note that $w_{\min}(A) = 3$ and the minimal rows of A are indexed by $\{1, 2, 5, 6\}$. Then $\Gamma(A) = 2$.

To prove Theorem 4, we first give a description of $(\mathcal{N}, \mathcal{S})$ -group decodable codes using their generator matrix. To do this, we need the following two definitions.

Definition 7: Let $M = (m_{i,j})_{k \times n}$ be a binary matrix and $G = (a_{i,j})_{k \times n}$ be a matrix over \mathbb{F} . We say that G is supported by M if for all $i \in [k]$ and $j \in [n]$, $m_{i,j} = 0$ implies $a_{i,j} = 0$. If C is a linear code over \mathbb{F} and has a generator matrix G supported by M, we call M a support generator matrix of C.

Definition 8: Let M_0 be a $k \times t$ binary matrix and M be a $k \times n$ binary matrix such that $C_{M_0}(j) = S_j$ for all $j \in [t]$ and $C_M(j) = S_i$ for all $i \in [t]$ and $j \in J_i$. We call M_0 the incidence matrix of S and M the indicator matrix of (\mathcal{N}, S) .

Remark 9: Since $\bigcup_{i=1}^{t} S_i = [k]$ and $C_{M_0}(i) = S_i$ for all $i \in [t]$, then by Definition 8, each row of M_0 has at least one 1 and each column of M_0 has exactly α 1s. Moreover, by (5) and Definition 8, M is extended from M_0 by replicating each column of M_0 by β times. Hence, each row of M has at least β 1s and each column of M_0 has exactly α 1s.

Now, we can describe $(\mathcal{N}, \mathcal{S})$ -group decodable codes using their generator matrix.

Lemma 10: Let M be the indicator matrix of $(\mathcal{N}, \mathcal{S})$. Then \mathcal{C} is an $(\mathcal{N}, \mathcal{S})$ -group decodable code if and only if \mathcal{C} has a generator matrix G satisfying the following two conditions:

- (1) G is supported by M;
- (2) rank(G_J) = α for each i ∈ [t] and J ⊆ J_i with |J| = α.
 Proof: This lemma can be directly derived from Definition 1 and 8.

For any [n, k] linear code C, the well-known Singleton bound ([15, Ch1]) states that $d \le n-k+1$. On the other hand, we always have $d \ge 1$. So it must be that $d = n-k+1-\delta$ for some $\delta \in \{0, 1, \dots, n-k\}$. The following lemma describes a useful fact about d for any linear code [20].

Lemma 11: Let C be an [n, k, d] linear code and G be a generator matrix of C. Let $0 \le \delta \le n - k$. Then $d \ge n - k + 1 - \delta$ if and only if any $k + \delta$ columns of G has rank k.

Using this lemma, we can give a bound on the minimum distance of any linear code by its support generator matrix.

Lemma 12: Let $M = (m_{i,j})$ be a $k \times n$ binary matrix and $0 \le \delta \le n - k$. The following three conditions are equivalent:

- (1) There is an [n, k] linear code C over some field \mathbb{F} such that M is a support generator matrix of C and $d \ge n-k+1-\delta$.
- (2) $|\bigcup_{j \in J} C_M(j)| \ge \ell$ for any $\ell \in [k]$ and any $J \subseteq [n]$ of size $|J| = \ell + \delta$.
- (3) $|\bigcup_{i\in I} R_M(i)| \ge n k + |I| \delta$ for all $\emptyset \neq I \subseteq [k]$.

Moreover, if condition (2) or (3) holds, there exists an [n, k] linear code over the field of size $q > \binom{n-1}{k-1}$ with a support generator matrix M and minimum distance $d \ge n - k + 1 - \delta$.

Proof: The proof is given in Appendix A. For $(\mathcal{N}, \mathcal{S})$ -group decodable code, we have the following

two lemmas. Lemma 13: Suppose M is the indicator matrix of $(\mathcal{N}, \mathcal{S})$. If M satisfies condition (2) of Lemma 12, there exists an an $(\mathcal{N}, \mathcal{S})$ -group decodable code over the field of size $q > \binom{n-1}{k-1}$ with minimum distance $d \ge n - k + 1 - \delta$.

Proof: The proof is given in Appendix B.

Lemma 14: Let $M_0 = (m_{i,j})$ be the incidence matrix of S. For any (\mathcal{N}, S) -group decodable code C, we have

$$d \le w_{\min}(M_0)\beta - \Gamma(M_0) + 1.$$
(8)

Moreover, there exist an $(\mathcal{N}, \mathcal{S})$ -group decodable code over the field of size $q > \binom{n-1}{k-1}$ with $d = w_{\min}(M_0)\beta - \Gamma(M_0) + 1$.

Proof: The proof is given in Appendix C.

Now, we can prove Theorem 4.

Proof of Theorem 4: Suppose C is an (α, β, k, t) -group decodable code. By Definition 2, C is an (\mathcal{N}, S) -group decodable code for some $S = \{S_1, \dots, S_t\}$ and $\mathcal{N} = \{n_1, \dots, n_t\}$ such that $S_i \subseteq [k], |S_i| = \alpha$ and $n_i = \beta$ for all $i \in [t]$. Let M_0 be the incidence matrix of S. By Lemma 14, it is sufficient to prove $w_{\min}(M_0)\beta - \Gamma(M_0) + 1 \leq s\beta - \left\lceil \frac{k-r}{\binom{k}{s}} \right\rceil + 1$.

By Remark 9, each column of M_0 has exactly α ones. Then the total number of ones in M_0 is $N_{\text{one}} = t\alpha$. On the other hand, each row of M_0 has at least $w_{\min}(M_0)$ ones. So $N_{\text{one}} = t\alpha \ge kw_{\min}(M_0)$, which implies $w_{\min}(M_0) \le \frac{t\alpha}{k}$. Since $w_{\min}(M_0)$ is an integer, then we have

$$w_{\min}(M_0) \le \left\lfloor \frac{t\alpha}{k} \right\rfloor = s.$$
 (9)

Note that $\Gamma(M_0) \ge 1$. If $k - r \le {t \choose s}$, then we have $\left\lceil \frac{k-r}{t \choose s} \right\rceil = 1$, and (9) implies $w_{\min}(M_0)\beta - \Gamma(M_0) + 1 \le s\beta = s\beta - \left\lceil \frac{k-r}{t \choose s} \right\rceil + 1$. Thus, we only need to consider $k - r > {t \choose s}$. Again by (9), we have the following two cases:

Case 1: $w_{\min}(M_0) = s$. Let N_s be the number of rows of M_0 with weight s. Then M_0 has $k - N_s$ rows with weight at least s + 1. So the total number of ones in M_0 is $N_{\text{one}} = t\alpha = sk + r \geq sN_s + (s+1)(k-N_s) = ks + (k-N_s)$. Thus,

$$N_s \ge k - r. \tag{10}$$

If $\Gamma(M_0) < \left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil$, then the repetition number of each row of weight $w_{\min}(M_0) = s$ is at most $\left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil - 1$. Note that there are at most $\binom{t}{s}$ binary vector of length t and weight s. Then we have $N_s \leq \binom{t}{s} \left(\left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil - 1 \right) < k - r$, which contradicts to (10). So we have $\Gamma(M_0) \geq \left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil$. Thus, $w_{\min}(M_0)\beta - \Gamma(M_0) + 1 \leq s\beta - \left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil + 1$.

Case 2: $w_{\min}(M_0) \stackrel{(s')}{\leq} s - 1$. Note that $t\alpha = sk + r \geq sk$ and $\alpha \leq k$. Then we have $t \geq s$ and $\binom{t-1}{s-1} \geq 1$. Thus,

$$k - r \le k \le {\binom{t-1}{s-1}}k + \frac{r}{s}{\binom{t-1}{s-1}}$$
$$= \frac{sk+r}{s}{\binom{t-1}{s-1}} = \frac{t\alpha}{s}{\binom{t-1}{s-1}} = \alpha{\binom{t}{s}}.$$

So we have $\frac{k-r}{\binom{t}{s}} \leq \alpha$, which implies that

$$\left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil - 1 < \frac{k-r}{\binom{t}{s}} \le \alpha \le \beta.$$

Note that $\Gamma(M_0) \geq 1$. Then $w_{\min}(M_0)\beta - \Gamma(M_0) + 1 \leq w_{\min}(M_0)\beta \leq (s-1)\beta = s\beta - \beta \leq s\beta - \left\lceil \frac{k-r}{\binom{s}{s}} \right\rceil + 1$.

By above discussion, we proved
$$w_{\min}(M_0)\beta - \Gamma(M_0) + 1 \le s\beta - \left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil + 1$$
. By Lemma 14, $d \le s\beta - \left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil + 1$.

IV. PROOF OF THEOREM 5

In this section, we prove Theorem 5. We first give a lemma that will be used in our following discussion.

Lemma 15: Suppose $t\alpha = sk + r$, where $s \ge 1$ and $0 \le r \le k - 1$. If $k - r \le {t \choose s}$, then there exists a $k \times t$ binary matrix $M_0 = (m_{i,j})$ such that: (i) Each column of M_0 has exactly α 1s; (ii) $w_{min}(M_0) = s$ and $\Gamma(M_0) = 1$.

Proof: The proof is given in Appendix D. Now we can prove Theorem 5 Proof of Theorem 5: By Lemma 14, it is sufficient to construct a $k \times t$ binary matrix M_0 such that each column has exactly α 1s, $w_{\min}(M_0) = s$ and $\Gamma(M_0) = \left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil$. We have the following two cases:

Case 1: $k - r \leq {t \choose s}$. Then $\left\lceil \frac{k-r}{{t \choose s}} \right\rceil = 1$ and M_0 can be constructed by Lemma 15.

Case 2: $k - r > {t \choose s}$. In this case, we can assume

$$k - r = u \binom{t}{s} + v \tag{11}$$

where $u \ge 1$ and $0 \le v \le {t \choose s} - 1$. Since $t\alpha = sk + r$, then

$$t\left[\alpha - u\binom{t-1}{s-1}\right] = t\alpha - tu\binom{t-1}{s-1}$$
$$= t\alpha - su\binom{t}{s}$$
$$= t\alpha - s(k-r-v)$$
$$= (t\alpha - sk) + s(r+v)$$
$$= r + s(r+v).$$
(12)

Let M_1 be a $u\binom{t}{s} \times t$ binary matrix such that each binary vector of length t and weight s appears in M_1 exactly u times. Then each column of M_1 has exactly $u\binom{t-1}{s-1}$ 1s. We can further construct a $(r+v) \times t$ matrix M_2 and let $M_0 = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$. To do so, we need to consider the following two sub-cases:

So, we need to consider the following two sub-cases: Case 2.1: v = 0. By (12), $t \left[\alpha - u {t-1 \choose s-1} \right] = (s+1)r$. It is easy to construct an $r \times t$ binary matrix M_2 such that each column has exactly $\alpha - u {t-1 \choose s-1}$ 1s and each row has exactly s+1 1s. Let $M_0 = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$. Then M_0 is a $k \times t$ binary matrix and each column has exactly α 1s. Moreover, by the construction, we have $w_{\min}(M_0) = s$ and $\Gamma(M) = u = \begin{bmatrix} \frac{k-r}{r} \end{bmatrix}$.

we have $w_{min}(M_0) = s$ and $\Gamma(M) = u = \left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil$. Case 2.2: $v \neq 0$. Then $0 \leq r \leq v + r - 1$. Note that $0 \leq v \leq \binom{t}{s} - 1$ and by (12), $t \left\lceil \alpha - u\binom{t-1}{s-1} \right\rceil = s(r+v) + r$. By the same discussion as in Lemma 15, we can construct a $(r+v) \times t$ binary matrix M_2 such that: (i) Each column of M_2 has exactly $\alpha - u\binom{t-1}{s-1}$ 1s; (ii) $w_{min}(M_2) = s$ and $\Gamma(M_2) = 1$. Let $M_0 = \left\lceil \frac{M_1}{M_2} \right\rceil$. Then M_0 is a $k \times t$ binary matrix and each column has exactly α 1s. Moreover, by the construction, we have $w_{min}(M_0) = s$ and $\Gamma(M) = u + 1 = \left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil$.

Thus, we can always construct a $k \times t$ binary matrix M_0 such that each column has exactly α 1s, $w_{\min}(M_0) = s$ and $\Gamma(M_0) = \left\lceil \frac{k-r}{\binom{t}{s}} \right\rceil$. By Lemma 14, there exist $(\mathcal{N}, \mathcal{S})$ -group decodable code over the field of size $q > \binom{n-1}{k-1}$ with $d = \sqrt{\frac{n-1}{k-1}}$

$$w_{\min}(M_0)\beta - \Gamma(M_0) + 1 = s\beta - \left|\frac{k-r}{\binom{t}{s}}\right| + 1.$$

V. CONCLUSIONS

We introduce a new family of erasure codes, called group decodable codes (GDC), for distributed storage systems that allows both locally repairable and group decodable. Thus, such codes can be viewed as a subclass of locally repairable codes (LRC). We derive an upper bound on the minimum distance of such codes and prove that the bound is achievable for all possible code parameters. However, since GDC is a subclass of LRC, the minimum distance bound of GDC is smaller than the minimum distance bound of LRC in general.

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APPENDIX A PROOF OF LEMMA 12

The proof consists of three steps: In the first step, we prove condition (1) implies condition (2); In the second step, we prove condition (2) implies condition (3); In the third step, we prove that if condition (3) holds, then there exists an [n, k] linear code over the field of size $q > \binom{n-1}{k-1}$ with a support generator matrix M and minimum distance $d \ge n - k + 1 - \delta$.

Proof of Lemma 12: (1) \Rightarrow (2). Suppose condition (1) holds. Let $G = (a_{i,j})$ be a generator matrix of C supported by M. Then for any $i \in [k]$ and $j \in [n]$, $m_{i,j} = 0$ implies $a_{i,j} = 0$. Given any $\ell \in [k]$, since any $k + \delta$ columns of G has rank k (Lemma 11), then any $\ell + \delta$ columns of G has rank at least ℓ , i.e., rank $(G_J) \ge \ell$ for any $J \subseteq [n]$ of size $|J| = \ell + \delta$. So G_J has at most $k - \ell$ rows that are all zeros, which implies $|\bigcup_{j \in J} C_M(j)| \ge \ell$.

(2) \Rightarrow (3). We can prove this by contradiction. Suppose $\emptyset \neq I \subseteq [k]$ and $|\bigcup_{i \in I} R_M(i)| < n - k + |I| - \delta$. Let $J' = [n] \setminus \bigcup_{i \in I} R_M(i)$. Then $|J'| > k - |I| + \delta$ and $m_{i,j} = 0$ for all $i \in I$ and $j \in J'$. Let $\ell = k - |I| + 1$ and $J \subseteq J'$ such that $|J| = \ell$. Then $\bigcup_{j \in J} C_M(j) \subseteq [k] \setminus I$. So $|\bigcup_{j \in J} C_M(j)| \leq k - |I| = \ell - 1$, which contradicts to condition (2). Thus, it must be that $|\bigcup_{i \in I} R_M(i)| \geq n - k + |I| - \delta$.

(3) \Rightarrow (1). The key is to construct a $k \times n$ matrix G over a field \mathbb{F} of size $q > \binom{n-1}{k-1}$ such that G is supported by M and any $k + \delta$ columns of G has rank k.

Let $X = (x_{i,j})_{k \times n}$ such that $x_{i,j}$ is an indeterminant if $m_{i,j} = 1$ and $x_{i,j} = 0$ if $m_{i,j} = 0$. Let $f(\cdots, x_{i,j}, \cdots) = \prod_P \det(P)$, where the product is taken over all k by k submatrix P of X with $\det(P) \not\equiv O$. Note that each $x_{i,j}$ belongs to at most $\binom{n-1}{k-1}$ submatrix P and has degree at most 1 in each $\det(P)$. Then $x_{i,j}$ has degree at most $\binom{n-1}{k-1}$ in $f(\cdots, x_{i,j}, \cdots)$. Note that $f(\cdots, x_{i,j}, \cdots) = \prod_P \det(P) \not\equiv O$. By [14, Lemma 4], if $|\mathbb{F}| > \binom{n-1}{k-1}$, then there exist $a_{i,j} \in \mathbb{F}$ (for i, j where $m_{i,j} = 1$) such that $f(\cdots, a_{i,j}, \cdots) \neq 0$. Let $G = (a_{i,j})$ (for i, j where $m_{i,j} = 0$, we set $a_{i,j} = 0$). Then G is supported by M. We will prove $\operatorname{rank}(G_J) = k$ for any $J \subseteq [n]$ with $|J| = k + \delta$. By construction of G, it is sufficient to prove $\det(X_{J_0}) \not\equiv O$ for some $J_0 \subseteq J$ with $|J_0| = k$.

Let \mathcal{G}_{J} be the bipartite graph with vertex set $U \cup V$, where $U = \{u_i; i \in [k]\}, V = \{v_i; j \in J\}$ and $U \cap V = \emptyset$ such that (u_i, v_j) is an edge of \mathcal{G}_I if and only if $m_{i,j} = 1$. Then for each $u_i \in U$, the set of all neighbors of u_i is $N(u_i) = \{v_j; j \in R_M(i) \cap J\}$. So for all $I \subseteq [k]$, the set of all neighbors of the vertices in $S = \{u_i; i \in I\}$ is $N(S) = \{v_j; j \in (\bigcup_{i \in I} R_M(i)) \cap J\}$. By assumption, $\left|\bigcup_{i\in I} R_M(i)\right| \geq n-k+|I|-\delta$ and $|J|=k+\delta$. So we have $|N(S)| = \left|\bigcup_{i \in I} R_M(i) \cap J\right| \geq \left|\bigcup_{i \in I} R_M(i)\right| |[n] \setminus J| = |I| = |S|$. By Hall's Theorem ([16, p. 419]), \mathcal{G}_J has a matching which covers every vertex in U. Let $\mathcal{M} = \{(u_1, v_{\ell_1}), \cdots, (u_k, v_{\ell_k})\}$ be such a matching and $J_0 = \{\ell_1, \dots, \ell_k\}$. Let \mathcal{G}_{J_0} be the subgraph of \mathcal{G}_J generated by $U \cup \{v_j; j \in J_0\}$. Then \mathcal{M} is a perfect matching of \mathcal{G}_{J_0} and X_{J_0} is the Edmonds matrix of \mathcal{G}_{J_0} . It is well known ([17, p. 167]) that a bipartite graph has a perfect matching if and

only if the determinant of its Edmonds matrix is not identically zero. Hence $det(X_{J_0}) \neq O$.

By the construction of G, we have $det(G_{J_0}) \neq 0$ and $rank(G_J) = k$, where J is any subset of [n] and $|J| = k + \delta$. Let C be the [n, k] linear code generated by G. By Lemma 11, $d \geq n - k + 1 - \delta$. Note that we have proved that G is supported by M. So M is a support generator matrix of C.

Appendix B

Proof of Lemma 13

Proof of Lemma 13: Let G and C be constructed as in the proof of Lemma 12. We will prove that C is an $(\mathcal{N}, \mathcal{S})$ -group decodable code.

By Lemma 10, we need to prove $\operatorname{rank}(G_J) = \alpha$ for each $i \in [t]$ and each $J \subseteq J_i$ of size $|J| = \alpha$. To prove this, it is sufficient to construct a subset $J_0 \subseteq [n]$ such that $J \subseteq J_0$ and rank $(G_{J_0}) = k$. To simplify notations, without loss of generality, we can assume $J \subseteq J_1$, where J_1 is defined by (5). Since $\bigcup_{i=1}^{t} S_i = [k]$, we can always find a collection $\mathcal{S}' \subseteq \mathcal{S}$ (By proper naming, we can assume $S' = \{S_1, S_2, \cdots, S_r\}$.) such that $\bigcup_{i=1}^{r} S_i = [k]$ and $I_{\ell} = S_{\ell} \setminus \bigcup_{i=1}^{\ell-1} S_i \neq \emptyset, \ell =$ $2, \cdots, r$. Then $\{I_1, I_2, \cdots, I_r\}$ is a partition of [k], where $I_1 = S_1$. Let $J'_1 = J$ and for each $\ell \in \{2, \dots, r\}$, pick an $J'_{\ell} \subseteq J_{\ell}$ with $|J'_{\ell}| = |I_{\ell}|$. Let $J_0 = J'_1 \cup J'_2 \cup \cdots \cup J'_r$. Then $|J_0| = k$. Let \mathcal{G}_{J_0} be the bipartite graph with vertex set $U \cup V$, where $U = \{u_i; i \in [k]\}, V = \{v_j; j \in J_0\}$ and $U \cap V = \emptyset$ such that (u_i, v_j) is an edge of \mathcal{G}_{J_0} if and only if $m_{i,j} = 1$. By Definition 8, $m_{i,j} = 1$ for each $i \in I_{\ell}, j \in J'_{\ell}$ and $\ell \in [r]$. So each subgraph $\mathcal{G}_{I_{\ell},J'_{\ell}}$ is a complete bipartite graph and has a perfect matching, where $\mathcal{G}_{I_{\ell},J'_{\ell}}$ is generated by $\{u_i; i \in I_\ell\} \cup \{v_j; j \in J'_\ell\}$. So the bipartite graph \mathcal{G}_{J_0} has a perfect matching. By a similar discussion as in the proof of Lemma 12, $\operatorname{rank}(G_{J_0}) = k$. So $\operatorname{rank}(G_J) = \alpha$.

Moreover, by the proof of Lemma 12, G is supported by M and is a generator matrix of C. So by Lemma 10, C is an (\mathcal{N}, S) -group decodable code. By Lemma 12, $d \ge n-k+1-\delta$. So C is a code that satisfies our requirements.

As an example, let M_0 be the matrix A in Example 6. Then $\alpha = 3, k = 7$ and t = 8. Let $\beta = 5$. Then M is obtained from M_0 by replicating each column of M_0 by 5 times. By (5), $J_1 = \{1, \dots, 5\}, \dots, J_8 = \{36, \dots, 40\}$. Let $J = \{1, 3, 5\} \subseteq J_1$. We have $I_1 = S_1 = \{1, 4, 6\}, I_2 = S_2 \setminus S_1 = \{2, 5, 7\}$ and $I_3 = S_3 \setminus (S_1 \cup S_2) = \{3\}$. Moreover, we can pick $J'_1 = J, J'_2 = \{6, 7, 8\}$ and $J'_3 = \{11\}$. Then G_{J_0} is of the following form:

 *	*	*	0	0	0	*]
0	0	0	*	*	*	0
0	0	0	0	0	0	*
*	*	*	0	0	0	0
0	0	0	*	*	*	0
*	*	*	0	0	0	*
0	0	0	*	*	*	0

where stars denote the nonzero entries of G_{J_0} . Clearly, $\{(1,1), (2,4), (3,6), (4,2), (5,5), (6,7), (7,3)\}$ is a perfect matching of the corresponding bipartite graph \mathcal{G}_{J_0} . By construction of G, we have $\det(G_{J_0}) \neq 0$ and $\operatorname{rank}(G_{J_0}) = k = 7$.

Appendix C

Proof of Lemma 14

Proof of Lemma 14: Let M be the indicator matrix of $(\mathcal{N}, \mathcal{S})$. By Lemma 10, M is a support generator matrix of \mathcal{C} . Let δ_0 be the smallest number such that $|\bigcup_{j\in J} C_M(j)| \ge \ell$ for all $\ell \in [k]$ and all $J \subseteq [n]$ of size $|J| = \ell + \delta_0$. Then by Lemma 12, $d \le n - k + 1 - \delta_0$. By Lemma 12 and 13, there exists an $(\mathcal{N}, \mathcal{S})$ -group decodable code over the field \mathbb{F} of size $q > \binom{n-1}{k-1}$ with $d = n - k + 1 - \delta_0$. Thus, to prove this lemma, the key is to prove that $\delta_0 = n - w_{\min}(M_0)\beta - k + \Gamma(M_0)$.

By Definition 8, M_0 is a $k \times t$ binary matrix and M is a $k \times n$ binary matrix such that $C_{M_0}(i) = S_i$ for all $i \in [t]$ and $C_M(j) = S_i$ for all $i \in [t]$ and $j \in J_i$. For each $\ell \in [n]$, let

$$\xi_M(\ell) = \min_{J \subseteq [n], |J| = \ell} \left| \bigcup_{j \in J} C_M(j) \right|.$$
(13)

Then by definition of δ_0 , we have

$$\delta_0 = \min\{\delta; 0 \le \delta \le n - k, \xi_M(\ell + \delta) \ge \ell, \forall \ell \in [k]\}.$$
(14)

For each $i \in [t]$, let

$$\xi_{M_0}(i) = \min_{J \subseteq [n], |J| = i} \left| \bigcup_{j \in J} C_{M_0}(j) \right|.$$
(15)

Then we have the following four claims:

Claim 1: $\xi_{M_0}(i_0) = k - \Gamma(M_0) < k = \xi_{M_0}(i_0 + 1) = \cdots = \xi_{M_0}(t)$, where $i_0 = t - w_{\min}(M_0)$. Claim 2: For all $i \in [t]$ and $\ell \in J_i$, $\xi_M(\ell) = \xi_{M_0}(i)$. Claim 3: $\ell' - \xi_M(\ell') \le i_0\beta - \xi_M(i_0\beta)$, $\forall \ell' \in [i_0\beta]$. Claim 4: $\delta_0 = i_0\beta - \xi_{M_0}(i_0)$. Note that $n = t\beta$. Then Claims 1 and 4 imply that $\delta_0 = t\beta$.

 $n - w_{\min}(M_0)\beta - k + \Gamma(M_0), \text{ which completes the proof.} \blacksquare$ $Proof of Claim 1: \text{ Suppose } J \subseteq [t] \text{ and } i_0 + 1 \leq |J| \leq t.$ Then $\bigcup_{j \in J} C_{M_0}(j) = [k].$ Otherwise, there is an $\ell \in [k]$ such that $\ell \notin C_{M_0}(j)$ for all $j \in J$, which implies that $m_{\ell,j} = 0$ for all $j \in J$. So $R_{M_0}(\ell) \subseteq [t] \setminus J$ and $|R_{M_0}(\ell)| \leq |[t] \setminus J| = t - |J| \leq t - (i_0 + 1) = w_{\min}(M_0) - 1$, which contradicts to (6). Thus, we proved that $\bigcup_{j \in J} C_{M_0}(j) = [k].$ By (15), we have $\xi_{M_0}(i) = k$ for $i_0 + 1 \leq i \leq t$.

Now, suppose $J \subseteq [t]$ and $|J| = i_0 = t - w_{\min}(M_0)$. We have the following two cases:

Case 1: $J = [t] \setminus R_{M_0}(\ell)$ for some $\ell \in [k]$ such that $|R_{M_0}(\ell)| = w_{\min}(M_0)$. Then $|\bigcup_{j \in J} R_{M_0}(j)| = k - \Gamma_{M_0}(\ell)$. This can be proved as follows:

For each $\ell' \in [k]$ such that $R_{M_0}(\ell') = R_{M_0}(\ell)$, we have $m_{\ell',j} = m_{\ell,j} = 0$ for all $j \in J$. Thus, $\ell' \notin \bigcup_{j \in J} C_{M_0}(j)$.

For each $\ell' \in [k]$ such that $R_{M_0}(\ell') \neq R_{M_0}(\ell)$, since $|R_{M_0}(\ell)| = w_{\min}(M_0)$, then $R_{M_0}(\ell') \nsubseteq R_{M_0}(\ell)$. Note that $J = [t] \setminus R_{M_0}(\ell)$. Then $R_{M_0}(\ell') \cap J \neq \emptyset$ and $m_{\ell',j} \neq 0$ for some $j \in J$. So $\ell' \in C_{M_0}(j)$ and $\ell' \in \bigcup_{j \in J} C_{M_0}(j)$.

Thus, for each $\ell' \in [k]$, $\ell' \notin \bigcup_{j \in J} C_{M_0}(j)$ if and only if $R_{M_0}(\ell) = R_{M_0}(\ell)$. So $|\bigcup_{j \in J} C_{M_0}(j)| = k - \Gamma_M(\ell)$.

Case 2: $J \neq [t] \setminus R_{M_0}(\ell)$ for all $\ell \in [k]$ such that $|R_{M_0}(\ell)| = w_{\min}(M_0)$. Then $|\bigcup_{j \in J} C_{M_0}(j)| = k$. Otherwise, there is an $\ell' \in [k]$ such that $\ell' \notin C_{M_0}(j)$ for all $j \in J$, which implies that $m_{\ell',j} = 0$ for all $j \in J$, and hence $R_{M_0}(\ell') \subseteq [t] \setminus J$.

Note that $|J| = t - w_{\min}(M_0)$. Then $|R_{M_0}(\ell')| \leq |[t] \setminus J| = t - |J| = w_{\min}(M_0)$. Thus, $|R_{M_0}(\ell')| = w_{\min}(M) = t - |J|$ and $J = [t] \setminus R_{M_0}(\ell')$, which contradicts to assumption on J.

By the above discussion, we proved that for each $J \subseteq [n]$ of size $|J| = i_0$, either $|\bigcup_{j \in J} C_{M_0}(j)| = k - \Gamma_{M_0}(\ell)$ for some $\ell \in [k]$ with $|R_{M_0}(\ell)| = w_{\min}(M_0)$ or $|\bigcup_{j \in J} C_{M_0}(j)| = k$. Thus, by (15) and (7), $\xi_{M_0}(i_0) = k - \Gamma(M_0)$.

Proof of Claim 2: From Definition 8, we have

$$\bigcup_{j \in J} C_M(j) = \bigcup_{i' \in [t]: J \cap J_{i'} \neq \emptyset} S_{i'}, \ \forall J \subseteq [n].$$
(16)

Firstly, we prove $\left|\bigcup_{j\in J} C_M(j)\right| \ge \xi_{M_0}(i)$ for each $J \subseteq [n]$ of size $|J| = \ell$.

By (5), we have

$$(i-1)\beta + 1 \le |J| \le i\beta.$$

Note that by (5), $|J_i| = \beta$. Then the number of i' such that $J \cap J_{i'} \neq \emptyset$ is at least *i*. By (16) and (15), we have

$$\left| \bigcup_{j \in J} C_M(j) \right| = \left| \bigcup_{\substack{i' \in [t]: J \cap J_{i'} \neq \emptyset}} S_{i'} \right|$$
$$= \left| \bigcup_{\substack{i' \in [t]: J \cap J_{i'} \neq \emptyset}} C_{M_0}(i') \right|$$
$$\ge \xi_{M_0}(i).$$

The second equation holds because by Definition 8, for each $i' \in [t]$, $C_{M_0}(i') = S_{i'}$. So by (13), we have $\xi_M(\gamma) \ge \xi_{M_0}(i)$.

Secondly, we prove there exists a $J \subseteq [n]$ of size $|J| = \ell$ such that $\left|\bigcup_{i \in J} C_M(j)\right| = \xi_{M_0}(i)$.

By (15), there is a $\{j_1, \dots, j_i\} \subseteq [t]$ such that

$$\xi_{M_0}(i) = \left| \bigcup_{\lambda=1}^{i} C_{M_0}(j_\lambda) \right| = \left| \bigcup_{\lambda=1}^{i} S_{j_\lambda} \right|.$$
(17)

Since $\ell \in J_i$, then by (5), $\left|\bigcup_{\lambda=1}^{i-1} J_{j_\lambda}\right| = (i-1)\beta < \ell \leq \left|\bigcup_{\lambda=1}^{i} J_{j_\lambda}\right| = i\beta$. So we can always find a subset $J \subseteq [n]$ such that $\bigcup_{\lambda=1}^{i-1} J_{j_\lambda} \subsetneq J \subseteq \bigcup_{\lambda=1}^{i} J_{j_\lambda}$ and $|J| = \ell$. Then by (16) and (17), we have

$$\left| \bigcup_{j \in J} C_M(j) \right| = \left| \bigcup_{i' \in [t]: J \cap J_{i'} \neq \emptyset} S_{i'} \right|$$
$$= \left| \bigcup_{\lambda=1}^i S_{j_\lambda} \right|$$
$$= \xi_{M_0}(i).$$

Above discussion implies that $\xi_{M_0}(i) = \min_{J \subseteq [n], |J| = \ell} \left| \bigcup_{j \in J} C_M(j) \right|$. By (13), we have $\xi_M(\ell) = \xi_{M_0}(i)$.

Proof of Claim 3: We first prove

$$i\beta - \xi_M(i\beta) \le i_0\beta - \xi_M(i_0\beta), \ \forall i \in [i_0].$$
 (18)

For each $i \in \{1, 2, \cdots, t-1\}$, by (15), there exists a $J' \subseteq [t]$ of size |J'| = i such that

$$\xi_{M_0}(i) = \left| \bigcup_{j \in J'} C_{M_0}(j) \right|.$$

Pick a $j_0 \in [t] \setminus J'$ and let $J = J' \cup \{j_0\}$. Then by (15),

$$\xi_{M_0}(i+1) \le \left| \bigcup_{j \in J} C_{M_0}(j) \right|.$$

Above two equations imply that

$$\begin{aligned} \xi_{M_0}(i+1) - \xi_{M_0}(i) &\leq \left| \bigcup_{j \in J} C_{M_0}(j) \right| - \left| \bigcup_{j \in J'} C_{M_0}(j) \right| \\ &\leq |C_{M_0}(j_0)| = |S_{j_0}| = \alpha \\ &\leq \beta. \end{aligned}$$

Combining this with Claim 2, we have

$$i\beta - \xi_M(i\beta) = i\beta - \xi_{M_0}(i) \leq (i+1)\beta - \xi_{M_0}((i+1)\beta) = (i+1)\beta - \xi_M((i+1)\beta).$$

By induction, we have

$$\beta - \xi_M(\beta) \le 2\beta - \xi_M(2\beta) \le \cdots \le i_0\beta - \xi_M(i_0\beta),$$

which proves (18).

Now, we can prove Claim 3. Given $i \in [i_0]$ and $\ell' \in J_i$. Since by (5), $(i-1)\beta+1 \leq \ell' \leq i\beta$, and by Claim 2, $\xi_M(\ell') = \xi_{M_0}(i) = \xi_M(i\beta)$, then

$$\ell' - \xi_M(\ell') \le i\beta - \xi_M(i\beta).$$

Combining this with (18), we have

$$\ell' - \xi_M(\ell') \le i_0\beta - \xi_M(i_0\beta).$$

Note that by (5), $[i_0\beta] = \{1, 2, \cdots, i_0\beta\} = J_1 \cup J_2 \cup \cdots \cup J_{i_0}$. Thus, $\ell' - \xi_M(\ell') \le i_0\beta - \xi_M(i_0\beta), \ \forall \ell' \in [i_0\beta]\}.$

Proof of Claim 4: Denote $\delta'_0 = i_0\beta - \xi_{M_0}(i_0)$. We need to prove $\delta_0 = \delta'_0$. Since by Claim 2, $\xi_M(i_0\beta) = \xi_{M_0}(i_0)$, then we have $\delta'_0 = i_0\beta - \xi_M(i_0\beta)$.

Firstly, we prove $\xi_M(\ell + \delta'_0) \ge \ell$ for all $\ell \in [k]$.

Suppose $\ell \in [k]$. If $\ell + \delta'_0 \geq i_0\beta + 1$, then by (5), $\ell + \delta'_0 \in J_i$ for some $i \in \{i_0 + 1, \dots, t\}$. By Claim 1 and 2, $\xi_M(\ell + \delta'_0) =$ $\xi_{M_0}(i) = k \geq \ell, \forall \ell \in [k]$. If $\ell + \delta'_0 \leq i_0\beta$, by Claim 3, $(\ell + \delta'_0) - \xi_M(\ell + \delta'_0) \leq i_0\beta - \xi_M(i_0\beta) = \delta'_0$. So $\xi_M(\ell + \delta'_0) \geq \ell$. Thus, $\xi_M(\ell + \delta'_0) \geq \ell$ for all $\ell \in [k]$.

Secondly, we prove that if $\delta' < \delta'_0$, then $\xi_M(\ell + \delta'_0) < \ell$ for some $\ell \in [k]$. We can prove this by contradiction.

Suppose $\xi_M(\ell + \delta'_0) \ge \ell$ for all $\ell \in [k]$. We have the following two cases:

Case 1: $i_0\beta - \delta' \in [k]$. Note that $i_0\beta - \xi_M(i_0\beta) = \delta'_0 > \delta'$. Then $\xi_M(i_0\beta) < i_0\beta - \delta'$. Let $\ell = i_0\beta - \delta'$. Then $\ell \in [k]$ and $\xi_M(\ell + \delta') < \ell$, which contradicts to assumption.

Case 2: $i_0\beta - \delta' \notin [k]$. Since $i_0\beta - \xi_M(i_0\beta) = \delta'_0 > \delta'$, then $i_0\beta - \delta' > i_0\beta - \delta'_0 = \xi_M(i_0\beta) > 0$. So we have $i_0\beta - \delta' > k$ and $i_0\beta > k + \delta'$. By (13) and assumption, we have

$$\xi_M(i_0\beta) \ge \xi_M(k+\delta') \ge k.$$

By Claim 2, $\xi_M(i_0\beta) = \xi_{M_0}(i_0)$. Then above equation implies $\xi_M(i_0\beta) = \xi_{M_0}(i_0) \ge k$, which contradicts to Claim 1.

In both cases, we can derive a contradiction. Thus, we conclude that $\xi_M(\ell + \delta'_0) < \ell$ for some $\ell \in [k]$.

Above discussion shows that δ'_0 is the smallest number that satisfies the condition that $\xi_M(\ell + \delta'_0) \ge \ell$, $\forall \ell \in [k]$.

Thirdly, we prove $\delta'_0 \leq n-k$.

Let J_0 and \mathcal{G}_{J_0} be constructed as in the proof of Lemma 13 (We can denote $J_0 = \{j_1, j_2, \cdots, j_k\}$.). Then \mathcal{G}_{J_0} has a perfect matching. Thus, there exists a permutation (i_1, i_2, \cdots, i_k) of $(1, 2, \cdots, k)$ such that $m_{i_{\lambda}, j_{\lambda}} = 1$ for all $\lambda \in [k]$ and we have $|\bigcup_{j \in J'} C_M(j)| \ge |J'|$ for all $J' \subseteq J_0$. Now, for any $\ell \in [k]$ and $J \subseteq [n]$ of size $|J| = \ell + n - k$, since $|J_0| = k$, we have $|J \cap J_0| \ge \ell$. So $|\bigcup_{j \in J} C_M(j)| \ge |\bigcup_{j \in J \cap J_0} C_M(j)| \ge$ $|J \cap J_0| \ge \ell$. By (13), we have $\xi_M(\ell + n - k) \ge \ell$. Thus, we proved that $\delta' = n - k$ also satisfies the condition that $\xi_M(\ell + \delta') \ge \ell$, $\forall \ell \in [k]$.

Note that δ'_0 is the smallest number that satisfies the condition that $\xi_M(\ell + \delta'_0) \ge \ell$, $\forall \ell \in [k]$. So $\delta'_0 \le n - k$.

Finally, we prove $\delta'_0 \ge 0$.

By (15), there exists a $J \subseteq [t]$ such that $|J| = i_0$ and

 $\xi_{M_0}(i_0) = \left| \bigcup_{j \in J} C_{M_0}(j) \right| \le \sum_{j \in J} |C_{M_0}(j)| = \sum_{j \in J} |S_j| = i_0 \alpha \le i_0 \beta.$ So by Claim 2, $i_0 \beta - \xi_M(i_0 \beta) = i_0 \beta - \xi_{M_0}(i_0) \ge 0.$

Thus, we proved that $0 \le \delta'_0 \le n-k$ and δ'_0 is the smallest number that satisfies the condition that $\xi_M(\ell + \delta'_0) \ge \ell$, $\forall \ell \in [k]$. By (14), we have $\delta_0 = \delta'_0 = i_0\beta - \xi_{M_0}(i_0)$.

Appendix D

PROOF OF LEMMA 15

Proof of Lemma 15: Since $k-r \leq {t \choose s}$, we can construct a $k \times t$ binary matrix $M_0 = (m_{i,j})$ such that: 1) $R_{M_0}(i)$, $i = 1, \dots, k-r$, are mutually different and $|R_{M_0}(i)| = s$; 2) $|R_{M_0}(i)| = s + 1, i = k - r + 1, \dots, k$. Since $t\alpha = sk + r$ and $0 \leq r \leq k - 1$, the total number of 1s in M_0 is

$$N_{\text{one}} = (k - r)s + r(s + 1) = ks + r = t\alpha.$$

Clearly, M_0 satisfies condition (ii). We can further modify M_0 properly so that it satisfies conditions (i) and (ii).

Suppose there is a $j_1 \in [t]$ such that $|C_{M_0}(j_1)| < \alpha$. Since the total number of ones in M is $N_{\text{one}} = t\alpha$, there exists a $j_2 \in [t]$ such that $|C_{M_0}(j_2)| > \alpha$. We shall modify M_0 so that $|C_{M_0}(j_1)|$ increases by one and $|C_{M_0}(j_2)|$ decreases by one. To do this, let

$$I_1 = \{i; 1 \le i \le k - r, m_{i,j_1} = 1 \text{ and } m_{i,j_2} = 0\}$$

and

$$I_2 = \{i; 1 \le i \le k - r, m_{i,j_1} = 0 \text{ and } m_{i,j_2} = 1\}.$$

Then clearly, $I_1 \cap I_2 = \emptyset$ and $m_{i,j_1} = m_{i,j_2}$ for all $i \in \{1, \dots, k-r\} \setminus (I_1 \cup I_2)$. We have the following two cases:

Case 1: There is an $i \in \{k-r+1, \dots, k\}$ such that $m_{i,j_1} = 0, m_{i,j_2} = 1$ and $|R_{M_0}(i)| = s + 1$. Then we modify M by letting $m_{i,j_1} = 1, m_{i,j_2} = 0$. Then $|C_{M_0}(j_1)|$ increases by one and $|C_{M_0}(j_2)|$ decreases by one. Moreover, it is easy to see that M_0 still satisfies condition (ii).

Case 2: For all $i \in \{k - r + 1, \dots, k\}$, $m_{i,j_2} = 1$ implies $m_{i,j_1} = 1$. Note that $|C_{M_0}(j_1)| < \alpha < |C_{M_0}(j_2)|$, then we have $|I_1| < |I_2|$. For each $\ell \in I_2$, we modify M_0 by letting $m_{\ell,j_1} = 1$, $m_{\ell,j_2} = 0$ and the other entries of M_0 remain unchanged. Denote the resulted matrix by M_ℓ . Then $|C_{M_\ell}(j_1)|$

increases by one and $|C_{M_\ell}(j_2)|$ decreases by one. If there is an $\ell \in I_2$ such that M_ℓ does not satisfy condition (ii), it must be that $R_{M_\ell}(\ell) = R_{M_\ell}(\ell')$ for some $\ell' \in I_1$. Moreover, all such ℓ s and ℓ 's are in one to one correspondence. Note that $|I_1| < |I_2|$. Then there exists an $i_0 \in I_2$ such that M_{i_0} satisfies condition (ii). So we can let M_0 be M_{i_0} .

We can perform the above operation continuously until each column of M_0 has weight α . Thus, we obtain a matrix M_0 that satisfies conditions (i) and (ii).