# On the Secrecy Exponent of the Wire-tap Channel 

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#### Abstract

We derive an exponentially decaying upper-bound on the unnormalized amount of information leaked to the wiretapper in Wyner's wire-tap channel setting. We characterize the exponent of the bound as a function of the randomness used by the encoder. This exponent matches that of the recent work of Hayashi [12] which is, to the best of our knowledge, the best exponent that exists in the literature. Our proof (like those of [16], [17]) is exclusively based on an i.i.d. random coding construction while that of [12], in addition, requires the use of random universal hash functions.


## I. Introduction

Wyner [1] introduced the notion of the wire-tap channel (Fig. 1) in 1975: Alice wants to communicate a message $W \in\{1, \ldots, M\}$ to Bob through a communication channel $\mathrm{V}: \mathcal{X} \rightarrow \mathcal{Y}$. Eve also has access to what Alice transmits via a wire-tapper's channel $\mathrm{W}: \mathcal{X} \rightarrow \mathcal{Z}$ and the aim of Alice is to keep the message hidden from her while maximizing the rate of information transmitted to Bob, $R \triangleq \frac{1}{n} \log M$.


Fig. 1. The Wire-Tap Channel
To this end, Alice encodes $W$ as a codeword $\mathbf{X} \in \mathcal{X}^{n}$ and sends it via $n$ consecutive uses of the channel. Bob observes the output sequence of $\mathrm{V}, \mathbf{Y} \in \mathcal{Y}^{n}$, and estimates $W$ given $\mathbf{Y}$. On the other side, Eve has access to $\mathbf{Z} \in \mathcal{Z}^{n}$ (the output sequence of W ), and attempts to make an inference about $W$.

Wyner (in case when $W$ is degraded with respect to $V$ ) [1] and later Csiszár and Körner (in a more general context of $\vee$ being more capable than W ) [2] showed that, given any input distribution $P_{X}$, Alice can communicate reliably to Bob at any rate $R$ up to

$$
\begin{equation*}
I(X ; Y)-I(X ; Z) \tag{1}
\end{equation*}
$$

(when $(X, Y) \sim P_{X}(x) \vee(y \mid x)$ and $(X, Z) \sim P_{X}(x) \mathrm{W}(z \mid x)$ ) while keeping the rate of information leaked to Eve about $W$ as small as desired; i.e., guaranteeing

$$
\begin{equation*}
\frac{1}{n} I(W ; \mathbf{Z}) \leq \epsilon \tag{2}
\end{equation*}
$$

for any $\epsilon>0$, using sufficiently large $n$.
Wyner's measure of secrecy allows one to investigate the trade-off between the message rate and the information leakage rate but is too weak from the security point of view; even if the amount of information Eve learns about the message $W$
normalized to the number of channel uses vanishes asymptotically, the amount itself can grow unboundedly as the block-length increases. Therefore, it is natural to remove the normalization factor in (2) and ask for strong secrecy:

$$
\begin{equation*}
I(W ; \mathbf{Z}) \leq \epsilon \tag{3}
\end{equation*}
$$

Maurer and Wolf showed that the highest achievable rate (1) under strong secrecy requirement does not change [3].

Classical achievability constructions [1], [4] are based on associating each message $w \in\{1, \ldots, M\}$ with a sub-code of size $M^{\prime}=\exp \left(n R^{\prime}\right)$ and transmitting a randomly chosen codeword from that sub-code to communicate $w$. The reliability of the code is ensured by keeping the total rate $R^{\prime}+R$ below $I(X ; Y)$. Furthermore, by varying the rate $R^{\prime}$ from 0 to $I(X ; Z)$, the upper-bound on the information leakage rate, $\frac{1}{n} I(W ; \mathbf{Z})$, is controlled. Particularly, by choosing the rate $R^{\prime}$ just below $I(X ; Z)$, weak secrecy is established.

An alternative way to approach the secrecy problem is to establish secrecy through channel resolvability [5]-[7]. Given an input distribution $P_{X}$ that induces the distribution $P_{Z}$ at the output of a channel $\mathrm{W}: \mathcal{X} \rightarrow \mathcal{Z}$, a code of rate $I(X ; Z)$ or larger chosen from the i.i.d. $P_{X}$ random coding ensemble will, with high probability, induce an output distribution that approximates $P_{Z}^{n}$ when the index of the transmitted codeword is chosen uniformly at random. [6], [8]-[11].

For any fixed message $w \in\{1, \ldots, M\}$ the output of Eve's channel has distribution $P_{\mathbf{Z} \mid W=w}$. It is not difficult to see that the secrecy is guaranteed if $P_{\mathbf{Z} \mid W=w}$ 'well approximates' the product distribution $P_{Z}^{n}$ by setting the sub-codes' rate $R^{\prime}$ just above $I(X ; Z)$. In particular, if we measure the quality of approximation by asking the unnormalized Kullback-Leibler divergence between $P_{\mathbf{Z} \mid W=w}$ and $P_{Z}^{n}$ to be small, strong secrecy will be established. Indeed, in [6], [7] it has been shown that the information leakage, $I(W ; \mathbf{Z})$ will be exponentially small in $n$ provided that $R^{\prime}$ is above $I(X ; Z)$.

Definition 1. Given $R, R^{\prime}$ and W , a number $E$ is a secrecy exponent for the wire-tapper channel W , if there exist a sequence of reliable coding schemes of rate $R$, requiring the entropy rate $R^{\prime}$ at the encoder, for which $\liminf _{n \rightarrow \infty}-\frac{1}{n} \log [I(W ; \mathbf{Z})] \geq E$.

In [6], [7] the secrecy exponent is derived using i.i.d. random coding ensemble. More specifically, each message $w \in\{1, \ldots, M\}$ is associated with a sub-code whose codewords are independently (and independent of the codewords of the other sub-codes) sampled from the i.i.d. random coding ensemble. The exponent is derived by upper-bounding the
ensemble-expectation of $D\left(P_{\mathbf{Z} \mid W} \| P_{Z}^{n} \mid P_{W}\right)$ and then concluding that there exists a sequence of codes in the ensemble using which the information leakage decays at least as fast as $\mathbb{E}\left[D\left(P_{\mathbf{Z} \mid W} \| P_{Z}^{n} \mid P_{W}\right)\right]$ does. The secrecy exponent of Hou and Kramer in [7] is derived based on their resolvability proof of [8, Section III-A] which is simple but results in a small exponent. However, by applying the method described in [8, Section III-B] to the wire-tap channel setting a larger exponent can be obtained which is equal to that of Hayashi in [6].

In [12], Hayashi uses privacy amplification to improve the secrecy exponent based on a different construction than those of [6]-[8]. In addition to a code of size $M M^{\prime}$, whose codewords are sampled independently from the i.i.d. random coding ensemble, a hash function is sampled from the ensemble of universal hash functions from $\left\{1, \ldots, M M^{\prime}\right\}$ to $\{1, \ldots, M\}$ and revealed to Alice, Bob, and Eve. A message $m \in\{1, \ldots, M\}$ is communicated by sending a randomly chosen codeword from the code and, then, mapping the index of the sent codeword, using the hash function, to an element of $\{1, \ldots, M\}$. The expected information leakage (where the expectation is taken over both i.i.d. random coding and universal hash functions ensembles) is then upper-bounded to show that the exponent of the bound is a secrecy exponent.

In this paper, we derive an exponentially decaying upperbound on $\mathbb{E}\left[D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right)\right]$, where the expectation is taken over the i.i.d. random coding ensemble (i.e., the construction used in [6]-[8]), by analyzing the deviations of $P_{\mathbf{Z} \mid W=w}$ from its mean. It then follows (by standard expurgation arguments) that for $\forall \epsilon>0$, there exist a code of essentially the same rate $R$, using which $\max _{w} D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right) \leq$ $(1+\epsilon) \mathbb{E}\left[D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right)\right]$. As already noted in [7], this is a worst-case measure of secrecy in contrast to $I(W ; \mathbf{Z})$ which is an average-case measure of secrecy. In addition, this shows that our lower-bound on $\lim _{n \rightarrow \infty}-\frac{1}{n} \log \mathbb{E}\left[D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right)\right]$ is a secrecy exponent. This exponent matches that of [12] which is larger than those of [6]- [8].

## II. Notation

We use uppercase letters (like $X$ ) to denote a random variable and corresponding lowercase version $(x)$ for a realization of that random variable. The boldface letters denote sequences of length $n$. The $i$-th element of a sequence $\mathbf{x}$ is denoted as $x_{i}$. We denote finite sets by script-style uppercase letters like $\mathcal{S}$. The cardinality of set $\mathcal{S}$ is denoted by $|\mathcal{S}|$. For a positive integer $m, \llbracket m \rrbracket \triangleq\{1,2, \ldots, m\} . \mathbb{R}$ denotes the set of real numbers and $\overline{\mathbb{R}}=\mathbb{R} \cup\{-\infty,+\infty\}$ is the set of extended real numbers. We write $f(n) \doteq g(n)$ (resp. $f(n) \leq g(n)$ ) if $\lim _{n \rightarrow \infty} \frac{1}{n} \log \frac{f(n)}{g(n)}=0($ resp. $\leq 0)$.

We denote the set of distributions on alphabet $\mathcal{X}$ as $\mathcal{P}(\mathcal{X})$. If $P \in \mathcal{P}(\mathcal{X}), P^{n} \in \mathcal{P}\left(\mathcal{X}^{n}\right)$ denotes the product distribution $P^{n}(\mathbf{x}) \triangleq \prod_{i=1}^{n} P\left(x_{i}\right)$. Likewise, if $\mathrm{V}: \mathcal{X} \rightarrow \mathcal{Y}$ is a conditional distribution $\mathrm{V}^{n}: \mathcal{X}^{n} \rightarrow \mathcal{Y}^{n}$ denotes the conditional distribution $\mathrm{V}^{n}(\mathbf{y} \mid \mathbf{x})=\prod_{i=1}^{n} \mathrm{~V}\left(y_{i} \mid x_{i}\right)$.

We denote the type of a sequence $\mathrm{x} \in \mathcal{X}^{n}$ by $\hat{P}_{\mathbf{x}} \in \mathcal{P}(\mathcal{X})$ and the conditional type of $\mathbf{y} \in \mathcal{Y}^{n}$ given $\mathbf{x} \in \mathcal{X}^{n}$ by $\hat{\mathrm{V}}_{\mathbf{y} \mid \mathbf{x}}$ : $\mathcal{X} \rightarrow \mathcal{Y}$ (see [13, Chapter 2] for formal definitions).

A distribution $\hat{P} \in \mathcal{P}(\mathcal{X})$ is an n-type if $n \hat{P}(x) \in \mathbb{N}_{\geq 0}$ for $\forall x \in \mathcal{X}$. We denote the set of $n$-types on $\mathcal{X}$ as $\hat{\mathcal{P}_{n}}(\mathcal{X}) \subsetneq \mathcal{P}(\mathcal{X})$ and use the fact that $\left|\hat{\mathcal{P}}_{n}(\mathcal{X})\right|=O\left(n^{|\mathcal{X |}|}\right)$ [13, Lemma 2.2] repeatedly.

If $\hat{P} \in \hat{\mathcal{P}}_{n}(\mathcal{X})$, we denote the set of all sequences of type $\hat{P}$ as $\mathcal{T}_{\hat{P}} \subset \mathcal{X}^{n}$. If $\hat{\vee}: \mathcal{X} \rightarrow \mathcal{Y}$ is a conditional distribution, the $\hat{\mathrm{V}}$-shell of $\mathbf{x} \in \mathcal{X}^{n}$, is denoted as $\mathcal{T}_{\hat{\mathrm{V}}}(\mathbf{x}) \subset \mathcal{Y}^{n}$.

## III. Result

In the rest of the paper $(X, Z) \in \mathcal{X} \times \mathcal{Z}$ denotes the pair of random variables whose joint distribution is $P_{X, Z}(x, z)=$ $P_{X}(x) \mathrm{W}(z \mid x)$ where $P_{X}$ is a fixed input distribution. For simplicity (and with no essential loss of generality) we assume the $\operatorname{supp}\left(P_{X}\right)=\mathcal{X}$ and $\operatorname{supp}\left(P_{Z}\right)=\mathcal{Z} \square^{1}$
Following [4] we consider the following random code construction: for every message $w \in \llbracket M \rrbracket$, a codebook of size $M^{\prime} \triangleq \exp \left(n R^{\prime}\right)$, denoted by $\mathcal{C}_{w}$, is constructed by sampling $M^{\prime}$ codewords, $\mathbf{X}_{w, w^{\prime}}, w^{\prime} \in \llbracket M^{\prime} \rrbracket$ independently from the product distribution $P_{X}^{n}$. In order to communicate the message $w$, Alice picks $w^{\prime} \in \llbracket M^{\prime} \rrbracket$ uniformly at random and transmits $\mathbf{X}_{w, w^{\prime}}$. Given such a construction, for every $w \in \llbracket M \rrbracket$ and $\mathbf{z} \in \mathcal{Z}^{n}$, the conditional output distribution of W is

$$
\begin{equation*}
P_{\mathbf{Z} \mid W}(\mathbf{z} \mid w)=\frac{1}{M^{\prime}} \sum_{w^{\prime}=1}^{M^{\prime}} \mathrm{W}^{n}\left(\mathbf{z} \mid \mathbf{X}_{w, w^{\prime}}\right) \tag{4}
\end{equation*}
$$

which is an average of i.i.d. random variables and

$$
\begin{equation*}
\mathbb{E}\left[P_{\mathbf{Z} \mid W}(\mathbf{z} \mid w)\right]=P_{Z}^{n}(\mathbf{z}), \quad \forall w \in \llbracket M \rrbracket . \tag{5}
\end{equation*}
$$

Theorem 1. Using the aforementioned construction, for $\forall w \in$ $\llbracket M \rrbracket$,

$$
\mathbb{E}\left[D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right)\right] \dot{\leq} \exp \left[-n E_{\mathrm{s}}\left(P_{X}, \mathrm{~W}, R^{\prime}\right)\right]
$$

with

$$
\begin{equation*}
E_{\mathrm{s}}\left(P_{X}, \mathrm{~W}, R^{\prime}\right)=\max _{0 \leq \lambda \leq 1}\left\{\lambda R^{\prime}-F_{0}\left(P_{X}, \mathrm{~W}, \lambda\right)\right\}, \tag{6}
\end{equation*}
$$

where
$F_{0}\left(P_{X}, \mathrm{~W}, \lambda\right) \triangleq \log \left[\sum_{z \in \mathcal{Z}} P_{Z}(z) \sum_{x \in \mathcal{X}} P_{X \mid Z}(x \mid z)^{1+\lambda} P_{X}(x)^{-\lambda}\right]$.
Remark. $F_{0}\left(P_{X}, \mathrm{~W}, \lambda\right)$ is a convex function of $\lambda$ (cf. Appendix (E-B) passing through the origin with the slope

$$
\left.\frac{\partial}{\partial \lambda} F_{0}\left(P_{X}, \mathrm{~W}, \lambda\right)\right|_{\lambda=0}=I(X ; Z)
$$

Hence $E_{\mathrm{s}}\left(P_{X}, \mathrm{~W}, R^{\prime}\right) \geq 0$ with equality iff $R^{\prime} \leq I(X ; Z)$.
The only random quantity involved in the divergence $D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right)$ is the conditional distribution $P_{\mathbf{Z} \mid W=w}$ whose expectation is $P_{Z}^{n}$ as shown in (5). To prove Theorem 1 we shall analyze the deviations of the random variables $P_{\mathbf{Z} \mid W}(\mathbf{z} \mid w)$ from their mean, $P_{Z}^{n}(\mathbf{z})$.

As an immediate corollary to Theorem 1 we have:

[^0]Corollary 2. For any input distribution $P_{X}$ and a pair of rates $R$ and $R^{\prime}$, there exists a reliable code of rate $R$ using which, for any message distribution $P_{W}$,

$$
\begin{aligned}
P_{\mathrm{e}} & \dot{\leq} \exp \left[-n E_{\mathrm{r}}\left(P_{X}, \mathrm{~V}, R+R^{\prime}\right)\right] \\
I(W ; \mathbf{Z}) & \dot{\leq} \exp \left[-n E_{\mathrm{s}}\left(P_{X}, \mathrm{~W}, R^{\prime}\right)\right]
\end{aligned}
$$

where $P_{\mathrm{e}}$ denotes the decoding error probability of Bob and $E_{\mathrm{r}}$ is Gallager's random coding exponent [14 Chapter 5]. Hence, for $\left(R, R^{\prime}\right)$ such that $R+R^{\prime}<I(X ; Y)$, the $E_{\mathrm{s}}$ in Theorem 1 is a secrecy exponent.

Corollary 2 is proved in Appendix B
IV. Proof of Theorem 1

For $\forall w \in \llbracket M \rrbracket$ and $\forall \mathbf{z} \in \mathcal{Z}^{n}$ let

$$
\begin{equation*}
U_{n}(\mathbf{z} \mid w) \triangleq \frac{P_{\mathbf{Z} \mid W}(\mathbf{z} \mid w)}{P_{Z}^{n}(\mathbf{z})} \tag{7}
\end{equation*}
$$

Using (5), it is easy to see that $\mathbb{E}\left[U_{n}(\mathbf{z} \mid w)\right]=1$.
Using the linearity of expectation, we have:

$$
\begin{align*}
\mathbb{E} & {\left[D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right)\right] } \\
& =\sum_{\mathbf{z} \in \mathcal{Z}^{n}} \mathbb{E}\left[P_{\mathbf{Z} \mid W}(\mathbf{z} \mid w) \log \left(\frac{P_{\mathbf{Z} \mid W}(\mathbf{z} \mid w)}{P_{Z}^{n}(\mathbf{z})}\right)\right] \\
& =\sum_{\mathbf{z} \in \mathcal{Z}^{n}} P_{Z}^{n}(\mathbf{z}) \mathbb{E}\left[U_{n}(\mathbf{z} \mid w) \log \left(U_{n}(\mathbf{z} \mid w)\right)\right] \\
& =\sum_{\hat{P} \in \hat{\mathcal{P}}_{n}(\mathcal{Z})} \sum_{\mathbf{z} \in \mathcal{T}_{\hat{P}}} P_{Z}^{n}(\mathbf{z}) \mathbb{E}\left[U_{n}(\mathbf{z} \mid w) \log \left(U_{n}(\mathbf{z} \mid w)\right)\right] \tag{8}
\end{align*}
$$

To prove Theorem 1 we shall use the following result.
Lemma 3. For $P \in \mathcal{P}(\mathcal{Z})$, let

$$
\begin{align*}
& G_{0}\left(P_{X, Z}, P, \lambda\right) \\
& \quad \triangleq \sum_{z \in \mathcal{Z}} P(z) \log \left[\sum_{x \in \mathcal{X}} P_{X \mid Z}(x \mid z)^{1+\lambda} P_{X}(x)^{-\lambda}\right] \tag{9}
\end{align*}
$$

and

$$
\begin{equation*}
E_{t}\left(P_{X, Z}, R^{\prime}, P\right) \triangleq \max _{0 \leq \lambda \leq 1}\left\{\lambda R^{\prime}-G_{0}\left(P_{X, Z}, P, \lambda\right)\right\} \tag{10}
\end{equation*}
$$

Then, for every $w \in \llbracket M \rrbracket$,

$$
\begin{align*}
& \mathbb{E}\left[U_{n}(\mathbf{z} \mid w) \log \left(U_{n}(\mathbf{z} \mid w)\right)\right] \\
& \quad \dot{\leq} \exp \left[-n E_{t}\left(P_{X, Z}, R^{\prime}, \hat{P}_{\mathbf{z}}\right)\right] \tag{11}
\end{align*}
$$

Having proved Lemma 3, Theorem 1 follows by using (11) in (8) and [13, Lemma 2.6] to conclude

$$
\mathbb{E}\left[D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right)\right] \dot{\leq} \exp \left[-n E_{\mathrm{s}}\left(P_{X}, \mathrm{~W}, R^{\prime}\right)\right]
$$

where

$$
\begin{align*}
& E_{\mathrm{s}}\left(P_{X}, \mathrm{~W}, R^{\prime}\right) \\
& \quad \triangleq \min _{P \in \mathcal{P}(\mathcal{Z})}\left\{D\left(P \| P_{Z}\right)+E_{t}\left(P_{X, Z}, R^{\prime}, P\right)\right\} \tag{12}
\end{align*}
$$

Using (10), the equivalence of (12) and (6) is shown in Appendix (D This completes the proof of Theorem 1 .

Proof of Lemma 3. Pick any $\hat{P} \in \hat{\mathcal{P}}_{n}(\mathcal{Z})$ and observe that for $\mathbf{z} \in \mathcal{T}_{\hat{P}}$,

$$
\frac{\mathrm{W}^{n}(\mathbf{z} \mid \mathbf{x})}{P_{Z}^{n}(\mathbf{z})}=\exp \left[n\left(D\left(\hat{\mathrm{~V}}_{\mathbf{x} \mid \mathbf{z}} \| P_{X} \mid \hat{P}\right)-D\left(\hat{\mathrm{~V}}_{\mathbf{x} \mid \mathbf{z}} \| P_{X \mid Z} \mid \hat{P}\right)\right)\right]
$$

For every $P \in \mathcal{P}(\mathcal{Z})$ and stochastic matrix $\mathrm{Q}: \mathcal{Z} \rightarrow \mathcal{X}$ define

$$
\begin{equation*}
A_{X, Z}(P ; \mathrm{Q}) \triangleq D\left(\mathrm{Q} \| P_{X} \mid P\right)-D\left(\mathrm{Q} \| P_{X \mid Z} \mid P\right) \tag{13}
\end{equation*}
$$

Thus, using (4),

$$
\begin{equation*}
U_{n}(\mathbf{z} \mid w)=\frac{1}{M^{\prime}} \sum_{w^{\prime}=1}^{M^{\prime}} \exp \left[n A_{X, Z}\left(\hat{P} ; \hat{\mathrm{V}}_{\mathbf{X}_{w, w^{\prime}} \mid \mathbf{z}}\right)\right] \tag{14}
\end{equation*}
$$

Let

$$
\begin{equation*}
\tilde{\mathcal{A}} \triangleq\left\{A_{X, Z}(\hat{P} ; \hat{\mathrm{Q}}) \text { for all conditional types } \hat{\mathrm{Q}}\right\} \subset \overline{\mathbb{R}} \tag{15}
\end{equation*}
$$

and observe that $|\tilde{\mathcal{A}}|=O\left(n^{|\mathcal{X}||\mathcal{Z}|}\right)$. Set $\mathcal{A} \triangleq\{a \in \tilde{\mathcal{A}}: a>$ $-\infty\}$ and for each $a \in \mathcal{A}$ define

$$
\begin{equation*}
\mathcal{T}_{a}(\mathbf{z}) \triangleq \bigcup_{\hat{\mathbf{Q}}: A_{X, Z}(\hat{P} ; \mathbf{Q})=a} \mathcal{T}_{\hat{\mathbf{Q}}}(\mathbf{z}) \subseteq \mathcal{X}^{n} \tag{16}
\end{equation*}
$$

where $\mathcal{T}_{\hat{\mathbf{Q}}}(\mathbf{z})$ is the $\hat{\mathbf{Q}}$-shell of $\mathbf{z}$ and the union is over conditional types $\hat{\mathbb{Q}}: \mathcal{Z} \rightarrow \mathcal{X}$ (thus contains $O\left(n^{|\mathcal{X}||\mathcal{Z}|}\right)$ shells). Now we can rewrite (14) as $\square^{2}$

$$
\begin{equation*}
U_{n} \triangleq U_{n}(\mathbf{z} \mid w)=\frac{1}{M^{\prime}} \sum_{a \in \mathcal{A}} N_{a} \exp (n a) \tag{17}
\end{equation*}
$$

with $N_{a} \triangleq\left|\left\{w^{\prime}: \mathbf{X}_{w, w^{\prime}} \in \mathcal{T}_{a}(\mathbf{z})\right\}\right|$ denotes the number of codewords of $\mathcal{C}_{w}$ in $\mathcal{T}_{a}(\mathbf{z})$. Since the codewords are independent, $N_{a}$ is a $\operatorname{Binomial}\left(M^{\prime}, p_{a}\right)$ random variable where,

$$
\begin{align*}
p_{a} & =P_{X}^{n}\left(\mathcal{T}_{a}(\mathbf{z})\right)=\sum_{\hat{\mathbf{Q}}: A_{X, Z}(\hat{P} ; \hat{\mathbf{Q}})=a} P_{X}^{n}\left(\mathcal{T}_{\hat{\mathbf{Q}}}(\mathbf{z})\right) \\
& \doteq \exp \left[-n \min _{\hat{\mathrm{Q}}: A_{X, Z}(\hat{P} ; \hat{\mathbf{Q}})=a} D\left(\hat{\mathbf{Q}} \| P_{X} \mid \hat{P}\right)\right] \tag{18}
\end{align*}
$$

In the above, the second equality follows since $\hat{Q}$-shells are disjoint, the third equality follows from [13, Lemma 2.6] (a similar approach is used in [15] to express a quantity of interest as a weighted sum of Binomial random variables).

In Appendix C-A we compute the value of

$$
\begin{equation*}
E_{b}\left(P_{X, Z}, P, a\right) \triangleq \min _{\hat{\mathrm{Q}}: A_{X, Z}(P ; \hat{\mathrm{Q}})=a} D\left(\hat{\mathrm{Q}} \| P_{X} \mid P\right) \tag{19}
\end{equation*}
$$

and, in particular, show that

$$
\begin{equation*}
E_{b}\left(P_{X, Z}, P, a\right) \geq a \tag{20}
\end{equation*}
$$

with equality iff $a=D\left(P_{X \mid Z} \| P_{X} \mid P\right)$.
Partition $\mathcal{A}=\mathcal{A}_{1} \cup \mathcal{A}_{2}$ as

$$
\mathcal{A}_{1}=\left\{a \in \mathcal{A}: a \leq R^{\prime}\right\}, \quad \mathcal{A}_{2}=\left\{a \in \mathcal{A}: a>R^{\prime}\right\}
$$

[^1]

Fig. 2. The function $\psi(s)$ defined in 24) and the upper-bound in 25. In the figure $\overline{S_{n}} \triangleq \mathbb{E}\left[S_{n}\right]$.
and split (17) as

$$
U_{n}=\underbrace{\frac{1}{M^{\prime}} \sum_{a \in \mathcal{A}_{1}} N_{a} \exp (n a)}_{\triangleq S_{n}}+\underbrace{\frac{1}{M^{\prime}} \sum_{a \in \mathcal{A}_{2}} N_{a} \exp (n a)}_{\triangleq T_{n}}
$$

For non-negative $s$ and $t$ and $u \triangleq s+t$ we have

$$
\begin{aligned}
u \ln (u) & =s \ln (u)+t \ln (u) \\
& =s \ln (s)+s \ln (1+t / s)+t \ln (u) \\
& \leq s \ln (s)+t(1+\ln (u))
\end{aligned}
$$

where the inequality follows since $\ln (1+t / s) \leq t / s$. Hence,

$$
\begin{align*}
& \mathbb{E}\left[U_{n} \log \left(U_{n}\right)\right] \doteq \mathbb{E}\left[U_{n} \ln \left(U_{n}\right)\right] \\
& \quad \leq \mathbb{E}\left[S_{n} \ln \left(S_{n}\right)\right]+\mathbb{E}\left[T_{n}\left(1+\ln \left(U_{n}\right)\right)\right] \tag{21}
\end{align*}
$$

Moreover, since $U_{n} \leq 1 / P_{Z}^{n}(\mathbf{z})$, we have

$$
\ln \left(U_{n}\right) \leq \ln \left(1 / P_{Z}^{n}(\mathbf{z})\right) \leq n \ln \left(1 / p_{0}\right)
$$

where $p_{0} \triangleq \min _{z \in \mathcal{Z}} P_{Z}(z)>0$. Thus, from (21) we have

$$
\begin{align*}
\mathbb{E}\left[U_{n} \ln \left(U_{n}\right)\right] & \leq \mathbb{E}\left[S_{n} \ln \left(S_{n}\right)\right]+\left(n \ln \left(1 / p_{0}\right)+1\right) \mathbb{E}\left[T_{n}\right] \\
& \doteq \mathbb{E}\left[S_{n} \ln \left(S_{n}\right)\right]+\mathbb{E}\left[T_{n}\right] \tag{22}
\end{align*}
$$

We now upper-bound each of the above expectations to complete the proof.

First we note that for any constant $c \in \mathbb{R}$,

$$
\begin{equation*}
\mathbb{E}\left[S_{n} \ln \left(S_{n}\right)\right]=\mathbb{E}\left[S_{n} \ln \left(S_{n}\right)+c\left(S_{n}-\mathbb{E}\left[S_{n}\right]\right)\right] \tag{23}
\end{equation*}
$$

In particular,

$$
\mathbb{E}\left[S_{n} \ln \left(S_{n}\right)\right]=\mathbb{E}\left[\psi\left(S_{n}\right)\right]
$$

where

$$
\begin{equation*}
\psi(s) \triangleq s \ln (s)-\left(\ln \left(\mathbb{E}\left[S_{n}\right]\right)+1\right)\left(s-\mathbb{E}\left[S_{n}\right]\right) \tag{24}
\end{equation*}
$$

One can check that (see Fig. (2)

$$
\begin{equation*}
\psi(s) \leq \frac{\left(s-\mathbb{E}\left[S_{n}\right]\right)^{2}}{\mathbb{E}\left[S_{n}\right]}+\mathbb{E}\left[S_{n}\right] \ln \left(\mathbb{E}\left[S_{n}\right]\right) \leq \frac{\left(s-\mathbb{E}\left[S_{n}\right]\right)^{2}}{\mathbb{E}\left[S_{n}\right]} \tag{25}
\end{equation*}
$$

where the last inequality follows since $\mathbb{E}\left[S_{n}\right]=1-\mathbb{E}\left[T_{n}\right] \leq 1$ as $S_{n}$ and $T_{n}$ are both non-negative random variables.

Using (25) in (23) we conclude that

$$
\begin{equation*}
\mathbb{E}\left[S_{n} \ln \left(S_{n}\right)\right] \leq \frac{\operatorname{var}\left(S_{n}\right)}{\mathbb{E}\left[S_{n}\right]} \tag{26}
\end{equation*}
$$

We now have,

$$
\begin{align*}
\mathbb{E}\left[S_{n}\right] & =\sum_{a \in \mathcal{A}_{1}} p_{a} \exp (n a) \\
& \doteq \exp \left[-n \min _{a \in \mathcal{A}_{1}}\left\{E_{b}\left(P_{X, Z}, \hat{P}, a\right)-a\right\}\right] \tag{27}
\end{align*}
$$

where the last equality follows since $\left|\mathcal{A}_{1}\right|=O\left(n^{|\mathcal{X}||\mathcal{Z}|}\right)$. Furthermore,

$$
\begin{align*}
& \operatorname{var}\left(S_{n}\right)=\frac{1}{M^{\prime^{2}}} \sum_{\left(a, a^{\prime}\right) \in \mathcal{A}_{1}^{2}} \exp \left[n\left(a+a^{\prime}\right)\right] \operatorname{cov}\left(N_{a}, N_{a^{\prime}}\right) \\
& \quad \stackrel{(\text { a) }}{\leq} \frac{1}{{M^{\prime}}^{2}} \sum_{\left(a, a^{\prime}\right) \in \mathcal{A}_{1}^{2}} \exp \left[n\left(a+a^{\prime}\right)\right] \sqrt{\operatorname{var}\left(N_{a}\right)} \sqrt{\operatorname{var}\left(N_{a^{\prime}}\right)} \\
& \quad=\frac{1}{M^{\prime 2}}\left(\sum_{a \in \mathcal{A}_{1}} \exp [n a] \sqrt{\operatorname{var}\left(N_{a}\right)}\right)^{2} \\
& \quad \stackrel{\text { (b) }}{=} \frac{1}{M^{\prime 2}}\left(\max _{a \in \mathcal{A}_{1}}\left\{\exp [n a] \sqrt{\operatorname{var}\left(N_{a}\right)}\right\}\right)^{2} \\
& \quad=\max _{a \in \mathcal{A}_{1}}\left\{\frac{1}{M^{\prime^{2}}} \exp [2 n a] \operatorname{var}\left(N_{a}\right)\right\} \\
& \quad \stackrel{\text { (c) }}{\leq} \max _{a \in \mathcal{A}_{1}}\left\{\frac{1}{M^{\prime}} \exp [2 n a] p_{a}\right\} \\
& \quad \doteq \exp \left[-n \min _{a \in \mathcal{A}_{1}}\left\{R^{\prime}+E_{b}\left(P_{X, Z}, \hat{P}, a\right)-2 a\right\}\right] . \tag{28}
\end{align*}
$$

In the above,
(a) follows by Cauchy-Schwarz inequality,
(b) follows since $\left|\mathcal{A}_{1}\right|=O\left(n^{|\mathcal{X}||\mathcal{Z}|}\right)$,
(c) follows since $\operatorname{var}\left(N_{a}\right)=M^{\prime} p_{a}\left(1-p_{a}\right) \leq M^{\prime} p_{a}$,
and finally (28) follows from (18) and (19).
Similar to (27),

$$
\begin{equation*}
\mathbb{E}\left[T_{n}\right] \doteq \exp \left[-n \min _{a \in \mathcal{A}_{2}}\left\{E_{b}\left(P_{X, Z}, \hat{P}, a\right)-a\right\}\right] \tag{29}
\end{equation*}
$$

Putting (27) and (28) in (26) together with (29) in (22) we conclude that

$$
\begin{gather*}
E_{t}\left(P_{X, Z}, R^{\prime}, \hat{P}\right)=\min \left\{E_{1}\left(P_{X, Z}, R^{\prime}, \hat{P}\right)-\bar{E}_{2}\left(P_{X, Z}, R^{\prime}, \hat{P}\right)\right. \\
\left.E_{2}\left(P_{X, Z}, R^{\prime}, \hat{P}\right)\right\} \tag{30}
\end{gather*}
$$

where

$$
\begin{align*}
& E_{1}\left(P_{X, Z}, R^{\prime}, \hat{P}\right) \triangleq \min _{a \leq R^{\prime}}\left\{R^{\prime}+E_{b}\left(P_{X, Z}, \hat{P}, a\right)-2 a\right\},  \tag{31}\\
& E_{2}\left(P_{X, Z}, R^{\prime}, \hat{P}\right) \triangleq \min _{a>R^{\prime}}\left\{E_{b}\left(P_{X, Z}, P, a\right)-a\right\}  \tag{32}\\
& \bar{E}_{2}\left(P_{X, Z}, R^{\prime}, \hat{P}\right) \triangleq \min _{a \leq R^{\prime}}\left\{E_{b}\left(P_{X, Z}, P, a\right)-a\right\} \tag{33}
\end{align*}
$$

We now observe that:
i. lower-bounding $R^{\prime}$ by $a$ in (31) shows $E_{1}\left(P_{X, Z}, R^{\prime}, \hat{P}\right)-$ $\bar{E}_{2}\left(P_{X, Z}, R^{\prime}, \hat{P}\right) \geq 0$.
ii. by (20), one and only one of $E_{2}\left(P_{X, Z}, R^{\prime}, \hat{P}\right)$ or $\bar{E}_{2}\left(P_{X, Z}, R^{\prime}, \hat{P}\right)$ is zero.
Thus (30) simplifies to
$E_{t}\left(P_{X, Z}, R^{\prime}, \hat{P}\right)=\min \left\{E_{1}\left(P_{X, Z}, R^{\prime}, \hat{P}\right), E_{2}\left(P_{X, Z}, R^{\prime}, \hat{P}\right)\right\}$

In Appendix C-B we show that

$$
\begin{align*}
& E_{1}\left(P_{X, Z}, R^{\prime}, \hat{P}\right)=\max _{\lambda \leq 1}\left\{\lambda R^{\prime}-G_{0}\left(P_{X, Z}, \hat{P}, \lambda\right)\right\},  \tag{35a}\\
& E_{2}\left(P_{X, Z}, R^{\prime}, \hat{P}\right)=\max _{\lambda \geq 0}\left\{\lambda R^{\prime}-G_{0}\left(P_{X, Z}, \hat{P}, \lambda\right)\right\} . \tag{35b}
\end{align*}
$$

Using the above in (34) concludes the proof.

## V. Discussion

We derived a lower-bound on the secrecy exponent of the wire-tap channel using i.i.d. random codes. Comparing (6) with [12, Equation (12)], we see that our exponent is equal to that of [12] which is the best lower-bound on the secrecy exponent among those reported in [6], [7], [12]. However, our proof is based on a pure i.i.d. random coding construction and does not require the ensemble of universal hash functions as an additional tool. While this manuscript was in review, it came to our attention that in [16], [17] also alternative derivations of the same lower-bound are given based on pure i.i.d. random coding constructions.

Our proof is a generalization of that of [8, Section III-A]; instead of partitioning the set of output sequences $\mathcal{Z}^{n}$ into two classes of typical and atypical sequences, we partition it into $O\left(n^{|\mathcal{Z}|}\right)$ type-classes to upper-bound the expected unnormalized Kullback-Leibler divergence between the output distribution and the desired product distribution $P_{Z}^{n}$. In addition, in Lemma3, we bound the point-wise difference between those distributions at each $\mathbf{z} \in \mathcal{Z}^{n}$.

Furthermore, we believe that the method described here has merit in showing the doubly exponential nature of the concentration of the output distribution; as we see in (4), the output distribution $P_{\mathbf{Z} \mid W}(\mathbf{z} \mid w)$ is an average of $M^{\prime}$ i.i.d. random variables. If the distribution of the summands was independent of $M^{\prime}$, the average would have concentrated around its mean exponentially fast in $M^{\prime}$, that is doubly exponentially fast in $n$. Although this is not the case, we see in the proof of Lemma 3 that among polynomially many summands in (17), only the one corresponding to $a=D\left(P_{X \mid Z} \| P_{X} \mid \hat{P}_{\mathbf{z}}\right)$ has a significant contribution to the mean of $U_{n}(\mathbf{z} \mid w)$ (which is a normalized version of $P_{\mathbf{Z} \mid W}(\mathbf{z} \mid w)$ ); the rest all have exponentially small means. Applying the Chernoff bound to this particular term, we see that if $R^{\prime}>D\left(P_{X \mid Z} \| P_{X} \mid \hat{P}_{\mathbf{z}}\right)$ the dominant term concentrates around its mean doubly exponentially fast in $n$. In particular, there exists a class of wire-tapper channels for which $U_{n}(\mathbf{z} \mid w)$ consists only of this dominant term 3

The achievability constructions of [6]-[8], [12], [16], [17] are based on i.i.d. random codes. It is an open question whether random constant-composition codes [13] will lead to a better secrecy exponent. We believe that our method is easily adaptable to other types of random coding (some ideas presented in [18] can also be useful in this direction). Another important subject in the context of wire-tap channel is to derive non-trivial upper-bounds on the secrecy exponent.

The performance of a wire-tap code is measured via two quantities, the error probability and the information leakage,

[^2]which are both shown to be exponentially decaying as a function of the block-length $n$. The trade-off between these exponents has been recently studied in [19].

We conclude our discussion by remarking that, as shown in [2], for general channels V and W , any message rate up to

$$
I(V ; Y)-I(V ; Z)
$$

where $V \multimap-X \multimap(Y, Z)$ form a Markov chain, is achievable. Our results (and also those of others cited) are straightforwardly extensible to the case when the channels are prefixed with a channel $P_{X \mid V}$ and auxiliary random variable $V$ is used.

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## Appendix A

## Proof of (5)

The right-hand-side of (4) is the average of identically distributed random variables. The mean of each of them is

$$
\begin{aligned}
& \mathbb{E}\left[\prod_{i=1}^{n} \mathrm{~W}\left(z_{i} \mid X_{i}\right)\right]=\prod_{i=1}^{n} \mathbb{E}_{X \sim P_{X}}\left[\mathrm{~W}\left(z_{i} \mid X\right)\right] \\
& \quad=\prod_{i=1}^{n}\left[\sum_{x \in \mathcal{X}} P_{X}(x) \mathrm{W}\left(z_{i} \mid x\right)\right]=\prod_{i=1}^{n} P_{Z}\left(z_{i}\right)
\end{aligned}
$$

In the above, the first equality follows since the codewords are sampled from the product distribution $P_{X}^{n}$.

## Appendix B <br> Proof of Corollary 2

Let $M \triangleq \exp (n R)$ and construct $2 M$ i.i.d. codebooks of size $M^{\prime} \triangleq \exp \left(n R^{\prime}\right), \mathcal{C}_{w}, w \in \llbracket 2 M \rrbracket$ by sampling each codeword independently from the product distribution $P_{X}^{n}$. As we already described, in order to communicate $w \in \llbracket 2 M \rrbracket$, Alice picks $w^{\prime} \in \llbracket M^{\prime} \rrbracket$ uniformly at random and transmits $\mathbf{X}_{w, w^{\prime}}$ over the channel. The union of this codebooks $\mathcal{C} \triangleq$ $\bigcup_{w \in \llbracket 2 M \rrbracket} \mathcal{C}_{w}$ is a random i.i.d. codebook of rate $R^{\prime}+R+\frac{\log (2)}{n}$. Hence, using this ensemble for communicating over V , for each $w \in \llbracket 2 M \rrbracket$, the expected decoding error probability is upper-bounded as

$$
\begin{align*}
& \mathbb{E}[\operatorname{Pr}[\hat{W} \neq W \mid W=w]] \\
& \quad \leq \mathbb{E}\left[\operatorname{Pr}\left[\{\hat{W} \neq W\} \cup\left\{\hat{W}^{\prime} \neq W^{\prime}\right\} \mid W=w\right]\right. \\
& \quad \leq \exp \left[-n E_{\mathrm{r}}\left(P_{X}, \mathrm{~V}, R+R^{\prime}+o(1)\right)\right] \tag{36}
\end{align*}
$$

due to [14, Theorem 5.6.2]. In the above, $\hat{W}$ and $\hat{W}^{\prime}$ denote, respectively, the maximum likelihood estimations of $W$ and $W^{\prime}$ given $\mathbf{Y}$, the output sequence of V . Consequently,

$$
\begin{align*}
& \mathbb{E}\left[\frac{1}{2 M} \sum_{w=1}^{2 M} \operatorname{Pr}[\hat{W} \neq w \mid W=w]\right] \\
& \quad \dot{\leq} \exp \left[-n E_{\mathrm{r}}\left(P_{X}, \mathrm{~V}, R+R^{\prime}\right)\right] \tag{37}
\end{align*}
$$

Likewise, Theorem 1 implies

$$
\begin{array}{r}
\mathbb{E}\left[\frac{1}{2 M} \sum_{w=1}^{2 M} D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right)\right] \\
\leq \exp \left[-n E_{\mathrm{s}}\left(P_{X}, \mathrm{~W}, R^{\prime}\right)\right] \tag{38}
\end{array}
$$

Therefore, there exists a code $\mathcal{C}^{*}=\bigcup_{w \in \llbracket 2 M \rrbracket} \mathcal{C}_{w}^{*}$ in the ensemble using which we simultaneously haved:

$$
\begin{align*}
& \frac{1}{2 M} \sum_{w=1}^{2 M} \operatorname{Pr}[\hat{W} \neq w \mid W=w] \\
& \quad \dot{\leq} \exp \left[-n E_{\mathrm{r}}\left(P_{X}, \mathrm{~V}, R+R^{\prime}\right)\right]  \tag{39}\\
& \frac{1}{2 M} \sum_{w=1}^{2 M} D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right) \\
& \quad \leq \exp \left[-n E_{\mathrm{s}}\left(P_{X}, \mathrm{~W}, R^{\prime}\right)\right] \tag{40}
\end{align*}
$$

Since each of the summands in (39) is positive, there exist a subset $\mathcal{W}_{1} \subset\{1, \ldots, 2 M\}$ of cardinality $\left|\mathcal{W}_{1}\right| \geq \frac{3}{2} M$ such that, for $\forall w \in \mathcal{W}_{1}$,

$$
\begin{equation*}
\operatorname{Pr}[\hat{W} \neq w \mid W=w] \dot{\leq} 4 \exp \left[-n E_{\mathrm{r}}\left(P_{X}, \mathrm{~V}, R+R^{\prime}\right)\right] \tag{41}
\end{equation*}
$$

Similarly since the summands in (40) are positive, there exists a subset $\mathcal{W}_{2} \subset\{1, \ldots, 2 M\}$ of cardinality $\left|\mathcal{W}_{2}\right| \geq \frac{3}{2} M$ such that, for $\forall w \in \mathcal{W}_{2}$,

$$
\begin{equation*}
D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right) \dot{\leq} 4 \exp \left[-n E_{\mathrm{s}}\left(P_{X}, \mathrm{~W}, R^{\prime}\right)\right] \tag{42}
\end{equation*}
$$

Since $\left|\mathcal{W}_{1} \cap \mathcal{W}_{2}\right| \geq M$ there exist a subset $\mathcal{W} \subseteq \mathcal{W}_{1} \cap$ $\mathcal{W}_{2}$ of cardinality $|\mathcal{W}|=M$. The sub-code defined by the messages in $\mathcal{W}, \bigcup_{w \in \mathcal{W}} \mathcal{C}_{w}^{*}$ has rate $R$ and, using that, for any message distribution $P_{W}$ on $\mathcal{W}$, we have:

$$
\begin{aligned}
P_{\mathrm{e}} & =\sum_{w \in \mathcal{W}} P_{W}(w) \operatorname{Pr}[\hat{W} \neq w \mid W=w] \\
& \leq \exp \left[-n E_{\mathrm{r}}\left(P_{X}, \mathrm{~V}, R+R^{\prime}\right)\right]
\end{aligned}
$$

due to (41), and

$$
\begin{aligned}
I(W ; \mathbf{Z}) & =D\left(P_{\mathbf{Z} \mid W} \| P_{Z}^{n} \mid P_{W}\right)-D\left(P_{\mathbf{Z}} \| P_{Z}^{n}\right) \\
& \leq \sum_{w \in \mathcal{W}} P_{W}(w) D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right) \\
& \leq \exp \left[-n E_{s}\left(R^{\prime}, P_{X}, \mathrm{~W}\right)\right]
\end{aligned}
$$

due to (42).

## Appendix C <br> Derivation of Exponents for The Proof of Lemma 3

## A. Derivation of $E_{b}$ and It's Properties

Proposition 4. Let $E_{b}\left(P_{X, Z}, P, a\right)$ be defined as in (19). Then,

$$
\begin{equation*}
E_{b}\left(P_{X, Z}, P, a\right)=a+\max _{\rho \in \mathbb{R}}\left\{\rho a-G_{0}\left(P_{X, Z}, P, \rho\right)\right\} \tag{43}
\end{equation*}
$$

where $G_{0}$ is defined in (9).

$$
\begin{aligned}
& { }^{4} \text { Markov inequality implies for at least } \frac{2}{3} \text { of the codes in the ensemble, } \\
& \frac{1}{2 M} \sum_{w=1}^{2 M} \operatorname{Pr}[\hat{W} \neq w \mid W=w] \leq 3 \mathbb{E}\left[\frac{1}{2 M} \sum_{w=1}^{2 M} \operatorname{Pr}[\hat{W} \neq w \mid W=w]\right]
\end{aligned}
$$

Similarly for at least $\frac{2}{3}$ of the codes in the ensemble,

$$
\frac{1}{2 M} \sum_{w=1}^{2 M} D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right) \leq 3 \mathbb{E}\left[\frac{1}{2 M} \sum_{w=1}^{2 M} D\left(P_{\mathbf{Z} \mid W=w} \| P_{Z}^{n}\right)\right]
$$

Therefore, for at least $\frac{1}{3}$ of the codes in the ensemble both 39) and 40 hold simultaneously.

## Proof: Let

$$
\iota_{X, Z}(x, z) \triangleq \log \left(\frac{P_{X, Z}(x, z)}{P_{X}(x) P_{Z}(z)}\right), \quad \forall(x, z) \in \mathcal{X} \times \mathcal{Z}
$$

denote the information density function for the joint distribution $P_{X, Z}$ for the sake of brevity.

Using (13),

$$
\begin{align*}
& \min _{\hat{\mathrm{Q}}: A_{X, Z}(P ; \hat{\mathrm{Q}})=a} D\left(\hat{\mathrm{Q}} \| P_{X} \mid P\right) \\
&=a+\min _{\hat{\mathrm{Q}}: A_{X, Z}(P ; \hat{\mathrm{Q}})=a} D\left(\hat{\mathrm{Q}} \| P_{X \mid Z} \mid P\right) \tag{44}
\end{align*}
$$

Now, we have

$$
\begin{aligned}
& \min _{\hat{\mathbf{Q}}: A_{X, Z}(P ; \hat{\mathbf{Q}})=a} D\left(\hat{\mathrm{Q}} \| P_{X \mid Z} \mid P\right) \\
& \quad=\min _{\hat{\mathrm{Q}}}\left\{D\left(\hat{\mathrm{Q}} \| P_{X \mid Z} \mid P\right)+\max _{\rho \in \mathbb{R}} \rho\left(a-A_{X, Z}(P ; \hat{\mathrm{Q}})\right)\right\} \\
& \quad=\min _{\hat{\mathrm{Q}}} \max _{\rho \in \mathbb{R}}\left\{D\left(\hat{\mathbb{Q}} \| P_{X \mid Z} \mid P\right)+\rho\left(a-A_{X, Z}(P ; \hat{\mathrm{Q}})\right)\right\} \\
& \quad \stackrel{(*)}{=} \max _{\rho \in \mathbb{R}}\left\{\min _{\hat{\mathrm{Q}}}\left\{D\left(\hat{\mathrm{Q}} \| P_{X \mid Z} \mid P\right)-\rho A_{X, Z}(P ; \hat{\mathrm{Q}})\right\}+\rho a\right\}
\end{aligned}
$$

where (*) follows since $D\left(\hat{\mathrm{Q}} \| P_{X \mid Z} \mid P\right)$ is a convex function of $\hat{\mathrm{Q}}$ and $A_{X, Z}(P ; \hat{\mathrm{Q}})$ is a linear function of $\hat{\mathrm{Q}}$. Therefore, $D\left(\hat{\mathbf{Q}} \| P_{X \mid Z} \mid P\right)-\rho A_{X, Z}(P ; \hat{\mathbf{Q}})$ is also a convex function of $\hat{\mathrm{Q}}$ and we can swap the min and the max. Now,

$$
\begin{aligned}
& D\left(\hat{\mathrm{Q}} \| P_{X \mid Z} \mid P\right)-\rho A_{X, Z}(\hat{P} ; \hat{\mathrm{Q}}) \\
& \quad=\sum_{z \in \mathcal{Z}} P(z) \sum_{x \in \mathcal{X}} \hat{\mathrm{Q}}(x \mid z) \log \left(\frac{\hat{\mathrm{Q}}(x \mid z)}{P_{X \mid Z}(x \mid z) \exp \left[\rho \iota_{X, Z}(x, z)\right]}\right) \\
& \quad \geq \sum_{z \in \mathcal{Z}} P(z) \log \left(\frac{1}{\sum_{x \in \mathcal{X}} P_{X \mid Z}(x \mid z) \exp \left[\rho \iota_{X, Z}(x, z)\right]}\right)
\end{aligned}
$$

with equality iff $\hat{\mathrm{Q}}(x \mid z) \propto P_{X \mid Z}(x \mid z) \exp \left[\rho \iota_{X, Z}(x, z)\right]$ (using the concavity of logarithm). Therefore,

$$
\begin{aligned}
& \min _{\hat{\mathrm{Q}}}\left\{D\left(\hat{\mathrm{Q}} \| P_{X \mid Z} \mid P\right)-\rho A_{X, Z}(P ; \hat{\mathrm{Q}})\right\}+\rho a=\rho a \\
& \quad-\sum_{z \in \mathcal{Z}} P(z) \log \left(\sum_{x \in \mathcal{X}} P_{X \mid Z}(x \mid z) \exp \left[\rho \iota_{X, Z}(x, z)\right]\right) .
\end{aligned}
$$

Remark. It is easy to verify that $E_{b}\left(P_{X, Z}, P, a\right)$ is a convex function of $a$. Furthermore, (44) implies $E_{b}\left(P_{X, Z}, P, a\right) \geq a$ with equality at $a=D\left(P_{X \mid Z} \| P_{X} \mid P\right)$.
B. Derivation of $E_{1}$ and $E_{2}$

Proof of (35a): Using (31),

$$
\begin{align*}
& E_{1}\left(P_{X, Z}, R^{\prime}, P\right)=\min _{a \leq R^{\prime}}\left\{R^{\prime}+E_{b}\left(P_{X, Z}, P, a\right)-2 a\right\} \\
& =\min _{a \in \mathbb{R}}\left\{R^{\prime}+E_{b}\left(P_{X, Z}, P, a\right)-2 a+\max _{\lambda \leq 0} \lambda\left(R^{\prime}-a\right)\right\} \\
& =\min _{a \in \mathbb{R}} \max _{\lambda \leq 0}\left\{(1+\lambda) R^{\prime}+E_{b}\left(P_{X, Z}, P, a\right)-(2+\lambda) a\right\} \\
& \quad=\min _{a \in \mathbb{R}} \max _{\lambda \leq 1}\left\{\lambda R^{\prime}+E_{b}\left(P_{X, Z}, P, a\right)-(1+\lambda) a\right\} \\
& \stackrel{(*)}{=} \max _{\lambda \leq 1}\left\{\lambda R^{\prime}+\min _{a \in \overline{\mathbb{R}}}\left\{E_{b}\left(P_{X, Z}, P, a\right)-(1+\lambda) a\right\}\right\} \tag{45}
\end{align*}
$$

where (*) follows since $E_{b}\left(P_{X, Z}, P, a\right)$ is convex in $a$. Using (43) we have

$$
\begin{align*}
& \min _{a \in \mathbb{R}}\left\{E_{b}\left(P_{X, Z}, P, a\right)-(1+\lambda) a\right\} \\
& \quad=\min _{a \in \mathbb{\mathbb { R }}}\left\{\max _{\rho \in \mathbb{R}}\left\{\rho a-G_{0}\left(P_{X, Z}, P, \rho\right)\right\}-\lambda a\right\} \\
& \quad=\min _{a \in \mathbb{\mathbb { R }}} \max _{\rho \in \mathbb{R}}\left\{(\rho-\lambda) a-G_{0}\left(P_{X, Z}, P, \rho\right)\right\} \\
& \quad \stackrel{(*)}{=} \max _{\rho \in \mathbb{R}}\left\{\min _{a \in \overline{\mathbb{R}}}\{(\rho-\lambda) a\}-G_{0}\left(P_{X, Z}, P, \rho\right)\right\}, \tag{46}
\end{align*}
$$

where again (*) follows since $G_{0}\left(P_{X, Z}, P, \rho\right)$ is convex in $\rho$ (cf. Appendix E-A). We then note that the minimum of the linear term $(\rho-\lambda) a$ over the choices of $a$ is $-\infty$ unless $\rho=\lambda$. Therefore, the result of (46) is

$$
\begin{equation*}
\min _{a \in \overline{\mathbb{R}}}\left\{E_{b}\left(P_{X, Z}, P, a\right)-(1+\lambda) a\right\}=-G_{0}\left(P_{X, Z}, P, \lambda\right) \tag{47}
\end{equation*}
$$

Plugging the above into (45) completes the proof.
Proof of (35b): Similarly, using (32),

$$
\begin{align*}
& E_{2}\left(P_{X, Z}, R^{\prime}, P\right)=\min _{a>R^{\prime}}\left\{E_{b}\left(P_{X, Z}, P, a\right)-a\right\} \\
& =\min _{a \in \mathbb{R}}\left\{E_{b}\left(P_{X, Z}, P, a\right)-a+\max _{\lambda \geq 0} \lambda\left(R^{\prime}-a\right)\right\} \\
& =\min _{a \in \mathbb{R}} \max _{\lambda \geq 0}\left\{\lambda R^{\prime}+E_{b}\left(P_{X, Z}, P, a\right)-(1+\lambda) a\right\} \\
& \stackrel{(*)}{=} \max _{\lambda \geq 0}\left\{\lambda R^{\prime}+\min _{a \in \mathbb{R}}\left\{E_{b}\left(P_{X, Z}, P, a\right)-(1+\lambda) a\right\}\right\}, \tag{48}
\end{align*}
$$

where $\left(^{*}\right)$ follows since $E_{b}\left(P_{X, Z}, P, a\right)$ is convex in $a$. Using (47) in (48) completes the proof.

Appendix D
DERIVATION OF $E_{\mathrm{s}}$
Plugging (10) into (12) we have

$$
\begin{aligned}
& \min _{P \in \mathcal{P}(\mathcal{Z})}\left\{E_{t}\left(P_{X, Z}, P, R^{\prime}\right)+D\left(P \| P_{Z}\right)\right\} \\
& \quad=\min _{P \in \mathcal{P}(\mathcal{Z})}\left\{\max _{0 \leq \lambda \leq 1}\left\{\lambda R^{\prime}-G_{0}\left(P_{X, Z}, P, \lambda\right)\right\}+D\left(P \| P_{Z}\right)\right\} \\
& \quad \stackrel{(*)}{=} \max _{0 \leq \lambda \leq 1}\left\{\lambda R^{\prime}+\min _{P \in \mathcal{P}(\mathcal{Z})}\left\{D\left(P \| P_{Z}\right)-G_{0}\left(P_{X, Z}, P, \lambda\right)\right\}\right\}
\end{aligned}
$$

where $(*)$ follows since $G_{0}$, defined in (9) is a linear function of $P$ while $D\left(\hat{P} \| P_{Z}\right)$ is convex in $P$ and we can swap the $\min$ and the max. The claim follows then by observing that

$$
\begin{aligned}
& D\left(P \| P_{Z}\right)-G_{0}\left(P_{X, Z}, P, \lambda\right) \\
& =\sum_{z \in \mathcal{Z}} P(z)\left[\log \left(\frac{P(z)}{P_{Z}(z)}\right)-\right. \\
& \left.\quad-\log \left(\sum_{x \in \mathcal{X}} P_{X \mid Z}(x \mid z)^{1+\lambda} P_{X}(x \mid z)^{-\lambda}\right)\right] \\
& \geq \log \left[\frac{1}{\sum_{z \in \mathcal{Z}} P_{Z}(z) \sum_{x \in \mathcal{X}}\left(P_{X \mid Z}(x \mid z)^{1+\lambda} P_{X}(x)^{-\lambda}\right)}\right]
\end{aligned}
$$

with equality if

$$
P(z) \propto P_{Z}(z) \sum_{x \in \mathcal{X}}\left(P_{X \mid Z}(x \mid z)^{1+\lambda} P_{X}(x)^{-\lambda}\right)
$$

using the concavity of logarithm.

## Appendix E

Convexity Proofs
Lemma 5. Let $a_{i}>0$, and $b_{i} \geq 0, i=1, \ldots, k$ be arbitrary real numbers. Then the function

$$
f(s) \triangleq \log \left(\sum_{i=1}^{k} a_{i} b_{i}^{s}\right)
$$

is convex in $s$ for $\forall s \in \overline{\mathbb{R}}$.
Proof: Pick $s_{1}<s_{2}$ and $t \in(0,1)$. Let $\bar{t} \triangleq 1-t$ and $s \triangleq t s_{1}+\bar{t} s_{2}$. Then, Hölder's inequality implies
$\sum_{i=1}^{k} a_{i} b_{i}^{s}=\sum_{i=1}^{k}\left(a_{i}^{t} b_{i}^{t s_{1}} \times a_{i}^{\bar{t}} b_{i}^{\bar{t} s_{1}}\right) \leq\left(\sum_{i=1}^{k} a_{i} b_{i}^{s_{1}}\right)^{t}\left(\sum_{i=1}^{k} a_{i} b_{i}^{s_{2}}\right)^{\bar{t}}$.
Taking the $\log$ of both sides of the above concludes the proof.

Lemma 6. Suppose $f_{i}(s), i=1,2, \ldots, k$ are convex functions in $s$ and $a_{i}>0, i=1,2, \ldots, k$ is a sequence of real numbers. Then,
(i) $f(s) \triangleq \sum_{i=1}^{k} a_{i} f_{i}(s)$ is convex in $s$.
(ii) $g(s) \triangleq \log \left(\sum_{i=1}^{k} a_{i} \exp \left[f_{i}(s)\right]\right)$ is convex in $s$.

Proof: The convexity of $f(s)$ is trivial. To prove the convexity of $g(s)$, let $s_{1}<s_{2}$ and $s=t s_{1}+\bar{t} s_{2}$ for some $t \in(0,1)$ (where $\bar{t} \triangleq 1-t$ ). Then

$$
\begin{aligned}
& \sum_{i=1}^{k} a_{i} \exp \left[f_{i}(s)\right] \leq \sum_{i=1}^{k} a_{i} \exp \left[t f_{i}\left(s_{1}\right)+(1-t) f_{i}\left(s_{2}\right)\right] \\
& \quad=\sum_{i=1}^{k}\left(a_{i}^{t} \exp \left[t f_{i}\left(s_{1}\right)\right] \times a_{i}^{\bar{t}} \exp \left[\bar{t} f_{i}\left(s_{2}\right)\right]\right) \\
& \quad \leq\left(\sum_{i=1}^{k} a_{i} \exp \left[f_{i}\left(s_{1}\right)\right]\right)^{t}\left(\sum_{i=1}^{k} a_{i} \exp \left[f_{i}\left(s_{2}\right)\right]\right)^{\bar{t}}
\end{aligned}
$$

where the second inequality follows by Hölder's inequality. Taking the logarithm of both sides of the above proves (ii).

Convexity of the functions $F_{0}$ and $G_{0}$ is established using the above two lemmas as follows:
A. Convexity of $G_{0}$

Set $a_{i}=P_{X \mid Z}(x \mid z)$ and $b_{i}=\frac{P_{X \mid Z}(x \mid z)}{P_{X}(x)}$ in Lemma 5 and then use Lemma 6 part (i).
B. Convexity of $F_{0}$

Set $a_{i}=P_{X \mid Z}(x \mid z)$ and $b_{i}=\frac{P_{X \mid Z}(x \mid z)}{P_{X}(x)}$ in Lemma 5 and then use Lemma 6 part (ii).


[^0]:    ${ }^{1}$ The second assumption follows from the first together with the assumption that for $\forall z \in \mathcal{Z}$ there exist at least one $x$ such that $\mathrm{W}(z \mid x)>0$.

[^1]:    ${ }^{2}$ Since $\mathbf{z}$ and $w$ are assumed to be fixed throughout the proof, we drop them from the argument of $U_{n}$ for the sake of brevity.

[^2]:    ${ }^{3}$ This happens if for $\forall z \in \mathcal{Z}$, for every $x \in \mathcal{X}$ either $\mathrm{W}(z \mid x)=0$ or $\mathrm{W}(z \mid x)=\epsilon_{z}$ for some constant $\epsilon_{z}<1$ independent of $x$.

