

Chained Successive Cancellation Decoding of the Extended Golay code

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Abstract—The extended Golay code is shown to be representable as a chained polar subcode. This enables its decoding with the successive cancellation decoding algorithm and its stack generalization. The decoder can be further simplified by employing fast Hadamard transform. The complexity of the obtained algorithm is comparable with that of the Vardy algorithm.

I. INTRODUCTION

The $(24, 12, 8)$ extended Golay code is a quasi-perfect self-dual linear binary block code. It has found numerous applications in communication, storage and imaging systems [1], [2], [3], [4]. Rich algebraic structure of the Golay code admits very efficient decoding, see [5] and references therein. However, these algorithms are specific to the extended Golay code, and, in general, may not be used for decoding of other types of error correcting codes.

Polar codes is a novel class of capacity-achieving error correcting codes, which have very efficient construction, encoding and decoding algorithms [6]. Furthermore, the list and sequential successive cancellation decoding algorithms [7], [8] were shown to be applicable for decoding of short extended BCH codes [9]. Polar codes were adopted for use in 5G wireless, so many future communication systems are likely to have an implementation of a decoder for polar codes. It is tempting to explore application of the decoding techniques developed for polar codes for other types of error correcting codes. This would enable communication systems to support different channel coding schemes with the same hardware.

In this paper we show that the extended Golay code can be represented in the framework of chained polar subcodes [10], and suggest a low-complexity decoding algorithm based on this representation. The proposed algorithm can be considered as a generalization of sequential (stack) and block sequential decoding algorithms [8], [11], [12].

The paper is organized as follows. In section II we review polar codes, their generalizations and decoding algorithms. Section III introduces a representation of the extended Golay code as a chained polar subcode. This representation is used in Section IV to derive some new decoding algorithms. Simulation results are presented in Section V.

II. BACKGROUND

A. Dynamic frozen symbols

$(n = 2^m, k)$ polar code is a set of vectors $c_0^{n-1} = u_0^{n-1} A_m$, where $A_m = B_m \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{\otimes m}$ is the polarizing matrix, $u_i =$

$0, i \in \mathcal{F}$, B_m is the bit reversal permutation matrix, and $\mathcal{F} \subset 0, \dots, 2^m - 1$ is the set of $2^m - k$ frozen channel indices [6]. It is possible to show that A_m together with a binary input memoryless channel $\mathbf{W}(y|c)$ give rise to synthetic bit subchannels

$$\mathbf{W}_m^{(i)}(y_0^{n-1}, u_0^{i-1} | u_i) = \frac{1}{2^{n-1}} \sum_{u_{i+1}^{n-1}} \prod_{j=0}^{n-1} \mathbf{W}(y_j | (u_0^{n-1} A_m)_j),$$

and the capacities of these subchannels converge with m to 0 or 1 bits per channel use. The standard way to construct polar codes is to let \mathcal{F} be the set of low-capacity subchannels. However, the minimum distance of classical polar codes is quite low. It was suggested in [9] to select u_0^{n-1} in such way, so that the obtained vectors c_0^{n-1} are codewords of some linear block code with check matrix H . This corresponds to dynamic freezing constraints

$$u_i = \sum_{j < i} V_{s_i, j} u_j, i \in \mathcal{F}, \quad (1)$$

where $V = QHA_m^T$ is a $(n-k) \times n$ constraint matrix, and Q is an invertible matrix, such that last non-zero elements in rows of V are located in distinct columns, \mathcal{F} is the set of indices of such columns, and s_i is the index of the row having the last non-zero entry in column i . Alternatively, the codewords of a polar subcode can be obtained as $c_0^{n-1} = xWA_m$, where W is a $k \times n$ precoding matrix, such that $WV^T = 0$, and x is an information vector.

Decoding of such codes can be implemented with a straightforward generalization of the successive cancellation decoding algorithm, which makes decisions

$$\hat{u}_i = \begin{cases} \arg \max_{u_i \in \mathbb{F}_2} \mathbf{W}_m^{(i)}(y_0^{n-1}, \hat{u}_0^{i-1} | u_i), & i \notin \mathcal{F} \\ \sum_{j < i} V_{s_i, j} \hat{u}_j, & i \in \mathcal{F}. \end{cases} \quad (2)$$

Extended primitive narrow-sense BCH codes were shown to have particularly well-structured sets of frozen symbol indices, and admit efficient list/sequential SC decoding [9].

Representation of linear codes via the dynamic freezing constraints can be considered as a result of application of the generalized Plotkin decomposition introduced in [9].

Theorem 1 ([9]). *Any linear $(2n, k, d)$ code \mathcal{C} has a generator matrix given by*

$$G = \begin{pmatrix} I_{k_1} & 0 & \tilde{I} \\ 0 & I_{k_2} & 0 \end{pmatrix} \begin{pmatrix} G_1 & 0 \\ G_2 & G_2 \\ G_3 & G_3 \end{pmatrix}, \quad (3)$$

where I_l is a $l \times l$ identity matrix, $G_i, 1 \leq i \leq 3$, are $k_i \times n$ matrices, $k = k_1 + k_2$, and \tilde{I} is obtained by stacking a $(k_1 - k_3) \times k_3$ zero matrix and I_{k_3} , where $k_3 \leq k_1$.

In this paper we essentially present a generalization of this decomposition.

B. List successive cancellation decoding

In general, classical polar codes, polar subcodes and other codes represented by (1) require list successive cancellation decoding in order to obtain near-ML performance. Let

$$\mathcal{W}_m^{(i)}(u_0^i | y_0^{n-1}) = \max_{u_{i+1}^{n-1} \in \mathbb{F}_2^{n-i-1}} \mathbf{W}_m^{(n-1)}(u_0^{n-1} | y_0^{n-1})$$

be the probability of the most likely continuation of path u_0^i in the code tree, without taking into account freezing constraints on symbols $u_j, j > i$. It can be seen that for $\lambda > 0$

$$\begin{aligned} \mathcal{W}_\lambda^{(2i)}(u_0^{2i} | y_0^{n-1}) &= \max_{u_{2i+1} \in \mathbb{F}_2} \mathcal{W}_{\lambda-1}^{(i)}(u_{0,e}^{2i+1} \oplus u_{0,o}^{2i+1} | y_0^{\frac{n}{2}-1}) \\ &\quad \cdot \mathcal{W}_{\lambda-1}^{(i)}(u_{0,o}^{2i+1} | y_0^{\frac{n}{2}-1}), \end{aligned} \quad (4)$$

$$\begin{aligned} \mathcal{W}_\lambda^{(2i+1)}(u_0^{2i+1} | y_0^{n-1}) &= \mathcal{W}_{\lambda-1}^{(i)}(u_{0,e}^{2i+1} \oplus u_{0,o}^{2i+1} | y_0^{\frac{n}{2}-1}) \\ &\quad \cdot \mathcal{W}_{\lambda-1}^{(i)}(u_{0,o}^{2i+1} | y_0^{\frac{n}{2}-1}), \end{aligned} \quad (5)$$

and $\mathcal{W}_0^{(0)}(c | y_j) = \mathbf{W}(c | y_j)$. Let us define modified log-likelihood ratios

$$S_\lambda^{(i)}(u_0^{i-1}, y_0^{n-1}) = \log \frac{\mathcal{W}_\lambda^{(i)}(u_0^{i-1}, 0 | y_0^{n-1})}{\mathcal{W}_\lambda^{(i)}(u_0^{i-1}, 1 | y_0^{n-1})}.$$

It is possible to show that [8], [13]

$$\begin{aligned} S_\lambda^{(2i)}(u_0^{2i-1}, y_0^{N-1}) &= a \boxplus b = \text{sgn}(a) \text{sgn}(b) \min(|a|, |b|) \\ S_\lambda^{(2i+1)}(u_0^{2i}, y_0^{N-1}) &= (-1)^{u_{2i}} a + b, \end{aligned}$$

where $a = S_{\lambda-1}^{(i)}(u_{0,e}^{2i-1} \oplus u_{0,o}^{2i-1}, y_0^{\frac{N}{2}-1})$, $b = S_{\lambda-1}^{(i)}(u_{0,o}^{2i-1}, y_0^{\frac{N}{2}-1})$, $N = 2^\lambda$. Then the logarithm of the probability of the most likely continuation of a path u_0^i can be obtained as

$$\begin{aligned} R(u_0^i | y_0^{n-1}) &= \log \mathcal{W}_m^{(i)}(u_0^i | y_0^{n-1}) \\ &= R(u_0^{i-1} | y_0^{n-1}) + \tau \left(S_m^{(i)}(u_0^{i-1}, y_0^{n-1}), u_i \right), \end{aligned} \quad (6)$$

where

$$\tau(S, u) = \begin{cases} 0, & \text{sgn}(S) = (-1)^u \\ -|S|, & \text{otherwise.} \end{cases}$$

One can assume that $R(\epsilon | y_0^{n-1}) = 0$, where ϵ is an empty sequence. Observe that $R(u_0^i | y_0^{n-1})$ is equal up to the sign to the approximate path metric introduced in [14]. The above derivation shows that this value is not just an approximation to the path metric used by the Tal-Vardy list decoder, but reflects the likelihood of the most probable continuation of a path in the code tree, without taking into account not-yet-processed freezing constraints.

It can be also seen that $R(u_0^{n-1} | y_0^{n-1}) = -E(u_0^{n-1} A_m, y_0^{n-1})$, where

$$E(c_0^{n-1}, y_0^{n-1}) = - \sum_{j=0}^{n-1} \tau(S_0^{(0)}(y_j), c_j)$$

is the ellipsoidal weight or correlation discrepancy of vector c_0^{n-1} with respect to the noisy vector y_0^{n-1} .

C. Chained polar subcodes

Classical polar codes are limited to length 2^m . In order to obtain codes of arbitrary length, it was suggested in [10] to combine polarizing matrices of different size. That is, the codewords of chained polar subcodes are given by $c_0^{n-1} = xW \underbrace{\text{diag}(A_{m_0}, \dots, A_{m_{s-1}})}_A$, where $n = \sum_{i=0}^{s-1} 2^{m_i}$, and A is

the mixed polarizing transformation matrix. A generalization of the successive cancellation decoding algorithm and its derivatives to the case of chained polar subcodes is provided in [10]. Alternatively, the code can be described as a set of vectors $c_0^{n-1} = u_0^{n-1} A$, where $u_0^{n-1} V^T = 0$, and V is the constraint matrix, such that $WV^T = 0$.

In general, list or sequential decoding algorithm should be used for decoding of chained polar subcodes. These algorithms essentially operate by arranging the input symbols of polarizing transformations A_{m_i} in some order, called decoding schedule, and interleaving steps of conventional list/sequential successive cancellation for each A_{m_i} . The performance of such algorithm does depend on the ordering of symbols u_i . It was shown in [10] that the best performance is achieved by the greedy schedule, which aims on processing of frozen symbols as early as possible.

III. THE EXTENDED GOLAY CODE

(24, 12, 8) extended Golay code is a quasi-perfect self-dual binary linear block code [15]. One of many possible ways to describe it is given by the Turyn construction [16]. The codewords are obtained as

$$c = (u + v, u + w, u + v + w), v, w \in C', u \in C'',$$

where C' is the (8, 4, 4) extended Hamming code, and C'' is a code equivalent to C , such that $C' \cap C'' = \{(0, 0, 0, 0, 0, 0, 0, 0), (1, 1, 1, 1, 1, 1, 1, 1)\}$. Note that both C' and C'' are instances of extended BCH codes with generator polynomials $g'(x) = (x - \alpha)(x - \alpha^2)(x - \alpha^4) = x^3 + x + 1$ and $g''(x) = (x - \alpha^3)(x - \alpha^6)(x - \alpha^5) = x^3 + x^2 + 1$, where α is a primitive element of \mathbb{F}_{2^3} .

Their generator and check matrices are given by

$$G' = H' = \begin{pmatrix} 0 & \alpha^0 & \alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

B. Block decoding

The decoding complexity can be reduced by joint processing of some blocks of the input symbols of the polarizing transformation [12]. In order to exploit this approach, we observe that puncturing last 8 symbols transforms the extended Golay code into (16, 11, 4) extended Hamming code. It can be represented as a Plotkin concatenation of the (8, 4, 4) first-order Reed-Muller code, and a single-parity check code. Observe also, that puncturing all codeword symbols for the extended Golay code except those with indices 16, . . . , 23 results in (8, 7, 2) single parity check code, which can be obtained via Plotkin concatenation of the (4, 3, 2) first-order Reed-Muller code and (4, 4, 1) trivial code. Rows 6,7 of matrix V provide linear relations between the codewords of (8, 4, 4) and (4, 3, 2) codes.

The correlation metrics for the codewords of a first-order Reed-Muller code \mathcal{C} of length $N - 1$

$$\mathbf{C}(c^{(i)}, z_0^{N-1}) = \sum_{j=0}^N (-1)^{c_j^{(i)}} z_j, c^{(i)} \in \mathcal{C},$$

where z_i are the log-likelihood ratios, can be obtained via order- N fast Hadamard transform (FHT) with complexity $N \log_2 N$ summations. Given a correlation metric, the corresponding ellipsoidal weight can be computed as

$$E(c^{(i)}, z_0^{N-1}) = \frac{1}{2} \left(\sum_{j=0}^{N-1} |z_j| - \mathbf{C}(c^{(i)}, z_0^{N-1}) \right).$$

This implies that

$$2R(u_0^7, u_{16}^{19} | y_0^{23}) = - \sum_{j=0}^{11} |z_j| + \mathbf{C}(u_0^7 A_3, z_0^7) + \mathbf{C}(u_{16}^{19} A_2, z_8^{11}), \quad (7)$$

where $z_i = S_1^{(0)}(y_{2i}, y_{2i+1}) = S_0^{(0)}(y_{2i}) \boxplus S_0^{(0)}(y_{2i+1})$, $0 \leq i < 12$, and $u_0 = u_1 = u_2 = u_4 = u_{16} = 0$, $u_3 = u_{17}$, $u_5 = u_{18}$. Observe that the first summand does not depend on u_0^{23} , and can be neglected. With this simplification, one obtains $R(u_0^6, u_7 = 1, u_{16}^{18}, u_{19} = 1 | y_0^{23}) = -R(u_0^6, u_7 = 0, u_{16}^{18}, u_{19} = 0 | y_0^{23})$. Hence, the scores of 32 paths (u_0^7, u_{16}^{19}) can be computed via order-8 and order-4 FHTs and 16 additional summations. We propose to sort these paths in the descending order¹, and apply the below described second processing step until a stopping condition is satisfied.

For any path (u_0^7, u_{16}^{19}) with score $r = 2R(u_0^7, u_{16}^{19} | y_0^{23})$ one can compute $u_9 = u_{20} = u_3 + u_5 + u_6 + u_{19}$. Now one can compute $\tilde{z}_{8+i} = S_1^{(1)}(u_{16}^{19} A_3, y_{16+2i}^{16+2i+1})$, $0 \leq i < 3$. These can be considered as the LLRs for a codeword of the coset, given by the value of u_{20} , of (4, 3, 2) code. Hence, one can compute the corresponding correlation metrics using the order-4 FHT and obtain scores

$$\rho = 2R(u_0^7, u_{16}^{23} | y_0^{23}) = r - \sum_{i=8}^{11} |\tilde{z}_i| + \mathbf{C}(u_{20}^{23} A_3, \tilde{z}_8^{11}).$$

¹Observe that only 16 values need to be actually sorted.

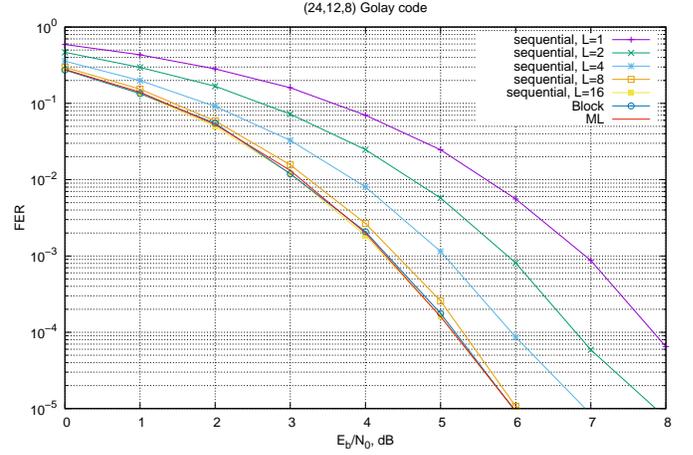


Fig. 1. Performance of the proposed decoding algorithms

Note that only vectors with $\mathbf{C}(u_{20}^{23} A_3, \tilde{z}_8^{11}) \geq 0$ need to be considered, since u_{23} is not frozen. Let the vectors u_{20}^{23} be ordered in the descending order of $\mathbf{C}(u_{20}^{23} A_3, \tilde{z}_8^{11})$. Now one can compute $u_{10} = u_3 + u_5 + u_{21}$ and $u_5 = u_{22}$. Let us further compute $\tilde{z}_i = S_1^{(1)}(u_0^7 A_4, y_{2i}^{2i+1})$, $0 \leq i < 7$. These can be considered as the LLRs for a coset, given by u_9, u_{10}, u_{12} , of the (8, 4, 4) first order Reed-Muller code. Hence, one can use order-8 FHT to compute the correlation metrics, and finally select the codeword with the highest value of

$$2R(u_0^{23} | y_0^{23}) = r - \sum_{i=0}^{11} |\tilde{z}_i| + \mathbf{C}(u_{20}^{23} A_3, \tilde{z}_8^{11}) + \mathbf{C}(u_8^{15} A_4, \tilde{z}_0^7).$$

Observe that coefficients 2 and 1/2 in the above equations can be omitted.

In order to avoid redundant calculations, one should keep the highest value R_{max} of $R(u_0^{23} | y_0^{23})$ obtained so far, and abort processing of vectors u_{20}^{23} as soon as one obtains the value of $\rho < R_{max}$, and abort processing of (u_0^7, u_{16}^{19}) as soon as one obtains $r < R_{max}$.

The best-case complexity of the above described algorithm corresponds to the case when the correct codeword has the highest values of $\mathbf{C}(u_0^7 A_3, z_0^7) + \mathbf{C}(u_{16}^{19} A_2, z_8^{11})$ and $\mathbf{C}(u_{20}^{23} A_3, \tilde{z}_8^{11})$, and exactly two FHTs of order 4 and 3 are computed. In this case the algorithm requires 111 summations and 45 comparisons.

At high signal-to-noise ratios one can further reduce the best-case decoding complexity by constructing the hard-decision vector for \tilde{z}_0^{11} corresponding to a given path (u_0^7, u_{16}^{19}) , and computing the values of u_{20}^{22} . If the obtained vector satisfies the constraints given by matrix V , one can skip computing FHTs in the second step of the algorithm.

V. NUMERIC RESULTS

Figure 1 illustrates the performance of the extended Golay code for the case of AWGN channel with BPSK modulation. We consider sequential decoding [8] using the schedule presented in Section IV-A, and the block algorithm introduced

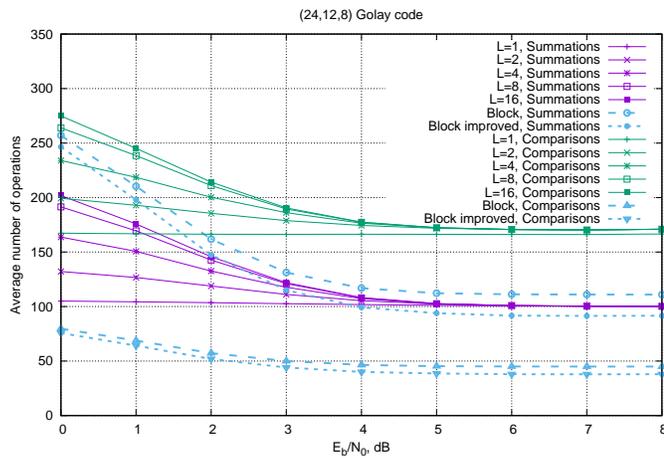


Fig. 2. Complexity of the proposed decoding algorithms

in Section IV-B. It can be seen that sequential decoding with $L = 16$ provides maximum likelihood decoding. This is the expected result, since the proposed decoding schedule requires one to process four unfrozen symbols (u_3, u_5, u_6, u_7), before one can process all freezing constraints which involve these symbols. Hence, one needs list size at least 16 in order to avoid killing the correct path at an early phase of decoding process. It can be seen that the proposed block algorithm also provides maximum likelihood decoding.

Figure 2 illustrates the average number of arithmetic operations for the proposed decoding algorithms. It can be seen that their complexity quickly decreases with SNR. At high SNR it approaches the complexity of the most efficient decoding algorithm for the Golay code [5], which requires 121 operations. The improved block decoding algorithm, which employs hard decisions to avoid computing FHTs at the second step, provides approximately 20% complexity reduction.

The maximal complexity of the block algorithm observed in our simulations was 1590 operations, which is close to the complexity of the FHT-based decoding algorithm suggested in [17].

VI. CONCLUSIONS

It was shown in this paper that the extended Golay code can be represented like a chained polar subcode. This enables one to decode it using the successive cancellation decoding algorithm and its list/sequential generalizations. With appropriate parameter selection, these algorithms can provide maximum likelihood decoding. The decoding complexity can be reduced by exploiting the fast Hadamard transform.

Although the complexity of these algorithms is slightly higher than the complexity of the Vardy algorithm, which was designed specifically for the extended Golay code, the proposed approach enables one to decode this code using the same techniques as polar codes. Since polar codes were recently adopted for use in 5G, many communication systems are likely to have an implementation of a decoder for polar codes. The proposed approach enables one to reuse the

corresponding hardware, and avoid implementing dedicated circuitry for decoder the extended Golay code, reducing thus the overall implementation complexity. It remains an open problem to identify other types of error-correcting codes, which can be decoded in the same way.

A similar representation of the extended Golay code as a punctured twisted polar code was independently derived in [18]. However, the authors considered only the straightforward implementation of the successive cancellation list decoder.

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