# New Repair strategy of Hadamard Minimum Storage Regenerating Code for Distributed Storage System 

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#### Abstract

The newly presented ( $k+2, k$ ) Hadamard minimum storage regenerating (MSR) code is the first class of high rate storage code with optimal repair property for all single node failures. In this paper, we propose a new simple repair strategy, which can considerably reduces the computation load of the node repair in contrast to the original one.


Index Terms - Distributed storage, MSR, Hadamard, repair strategy, computation load.

## 1 Introduction

In distributed storage systems, data is placed on a number of storage nodes with redundancy. Redundancy is the basis for distributed storage systems to provide reliable access service. Normally, there are two mechanisms of redundancy: replication and erasure coding. Compared with replication, erasure coding is becoming more and more attractive because of much better storage efficiency. Up to now, some famous storage applications, such as Google Colossus (GFS2) [3], Microsoft Azure [5], HDFS Raid [4, and OceanStore [6, have adopted erasure coding.

Due to the unreliability of individual storage nodes, node repair will be launched once node failures take place, so as to retain the same redundancy. With data growing much faster than before, node repair becomes a regular maintenance operation now. In general, there are several metrics to evaluate the cost of node repair, such as disk I/O, network bandwidth, number of accessed disks, etc. Among these metrics, the repair bandwidth, defined as the amount of data downloaded to repair a failed node, is the most useful. In [1], Dimakis et al. established a tradeoff between the storage and repair bandwidth where MBR (minimum bandwidth regenerating) code corresponding to minimum repair bandwidth and MSR (minimum storage regenerating) code corresponding to minimum storage are the most important.

In this study, we focus on MSR codes with high rate. So far, several explicit constructions of such MSR codes have been proposed based on the interference alignment technique [7, 8, , 9]. However, it should be noted that in all the aforementioned constructions except the one in [7], only the systematic nodes possess the optimal repair property. In [7], the first ( $k, k+2$ ) MSR code with optimal repair property for all storage nodes, including both $k$ systematic nodes and 2 parity nodes, was presented. Actually, the optimal repair property follows from Hadamard design with the help of lattice representation of the symbol extension technique. Therefore, we call this code Hadamard MSR code throughout this paper.

In this paper, we fully explore the fundamental properties of Hadamard design. As a result, we present a generic repair strategy for Hadamard MSR code only based on the elementary mathematics instead of the lattice knowledge. Further, the new generic repair strategy not only

[^0]includes the original repair strategy in [7, but also generates a much more simple but efficient one which can greatly reduce the computation load during the repair of failed nodes.

The remainder of this paper is organized as follows. In Section 2, the $(k+2, k)$ Hadamard MSR code is briefly reviewed. In Section 3, some fundamental properties of Hadamard deign are studied to help the optimal repair. In Section [4 the new repair strategy is proposed for systematic nodes, the first parity node and the second parity node respectively. The comparison of computation load between the original strategy in [7] and ours is given in Section [5. Finally, Section 6 concludes this paper.

## $2(k+2, k)$ Hadamard MSR code

The $(k+2, k)$ MSR code, consisting of $k$ systematic nodes and 2 parity nodes, is a typical high rate storage code in distributed storage system. Assume that the original data is of size $M=k N$, it can be equally partitioned into $k$ parts $\mathbf{f}=\left[\mathbf{f}_{1}^{T}, \mathbf{f}_{2}^{T}, \cdots, \mathbf{f}_{k}^{T}\right]^{T}$ and placed on $k$ systematic nodes, where $\mathbf{f}_{i}$ is a $N \times 1$ vector. In general, 2 parity nodes hold parity data, namely two $N \times 1$ vectors $\mathbf{f}_{k+1}$ and $\mathbf{f}_{k+2}$, of all the systematic nodes. Table 1 illustrates the structure of a $(k+2, k)$ MSR code.

Table 1: Structure of a $(k+2, k)$ MSR code

| Systematic node | Systematic data |
| :---: | :---: |
| 1 | $\mathbf{f}_{1}$ |
| $\vdots$ | $\vdots$ |
| $k$ | $\mathbf{f}_{k}$ |
| Parity node | Parity data |
| 1 | $\mathbf{f}_{k+1}=\mathbf{f}_{1}+\cdots+\mathbf{f}_{k}$ |
| 2 | $\mathbf{f}_{k+2}=A_{1} \mathbf{f}_{1}+\cdots+A_{k} \mathbf{f}_{k}$ |

Let $N=2^{k+1}$. The ( $k+2, k$ ) Hadamard MSR code [7] is characterized by the coding matrices $A_{1}, \cdots, A_{k}$ over finite field $\mathbb{F}_{q}(q \geq 2 k+3)$ as

$$
\begin{align*}
& A_{i}=a_{i} X_{i}+b_{i} X_{0}+I_{N}, 1 \leq i \leq k \\
& X_{j}=\operatorname{diag}(\underbrace{I_{2^{j}},-I_{2^{j}}, \cdots, I_{2^{j}},-I_{2^{j}}}_{2^{k+1-j}}), 0 \leq j \leq k, \tag{1}
\end{align*}
$$

where $I_{m}$ is the identity matrix of order $m$, the elements $a_{i} \neq 0$ and $b_{i} \neq 0$ over the finite field of odd characteristic and order $q \geq 2 k+3$ satisfy

$$
\begin{align*}
a_{i}^{2}-b_{i}^{2} & =-1,  \tag{2}\\
a_{i} \pm a_{j} & \neq b_{i}-b_{j}, \\
a_{i} \pm a_{i} & \neq-\left(b_{i}-b_{j}\right),
\end{align*}
$$

for all $1 \leq i \neq j \leq k$ [7]. In fact, the matrices in (1) are built on Hadamard design [2].

As the same as other $(k+2, k)$ MSR codes, this $(k+2, k)$ Hadamard MSR code can tolerate 2 arbitrary node failures [7]. Notably, recall that this Hadamard MSR code has an advantage over other $(k+2, k)$ MSR codes that both systematic nodes and parity nodes have optimal repair property. Indeed, to repair a failed node $1 \leq i \leq k+2$, the optimal repair property requires downloading $N / 2=2^{k}$ data from each surviving node $1 \leq l \neq i \leq k+2$ by multiplying its original data $\mathbf{f}_{l}$ with a $N / 2 \times N$ matrix [7], which will be discussed in detail in Section 4.

Example 1. For $k=2$, the $(4,2)$ Hadamard $M S R$ code has the following coding matrices over $\mathbb{F}_{7}$

$$
\begin{aligned}
& A_{1}=\operatorname{diag}(1,1,-1,-1,1,1,-1,-1)+3 \cdot \operatorname{diag}(1,-1,1,-1,1,-1,1,-1)+I_{8} \\
& A_{2}=\operatorname{diag}(1,1,1,1,-1,-1,-1,-1)+4 \cdot \operatorname{diag}(1,-1,1,-1,1,-1,1,-1)+I_{8}
\end{aligned}
$$

Its repair matrices will be elaborated in Section 4.
Example 2. For $k=3$, the $(5,3)$ Hadamard MSR code has the following coding matrices over $\mathbb{F}_{11}$

$$
\begin{aligned}
A_{1}= & 2 \cdot \operatorname{diag}(1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1)+ \\
& 7 \cdot \operatorname{diag}(1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1)+I_{16} \\
A_{2}= & 2 \cdot \operatorname{diag}(1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1)+ \\
& 4 \cdot \operatorname{diag}(1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1)+I_{16} \\
A_{3}= & 6 \cdot \operatorname{diag}(1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1)+ \\
& 2 \cdot \operatorname{diag}(1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1)+I_{16}
\end{aligned}
$$

## 3 Properties about Hadamard design

For $0 \leq i \leq k$, to characterize the diagonal matrix $X_{i}$ in (1) from Hadamard design, we define $\mathbf{x}_{i}=\left(x_{j}^{i}\right)_{j=0}^{N-1}$ to be the row vector of length $N$ formed by its elements of the main diagonal, i.e.,

$$
\mathbf{x}_{i}=(\underbrace{\mathbf{1}_{2^{i}},-\mathbf{1}_{2^{i}}, \cdots, \mathbf{1}_{2^{i}},-\mathbf{1}_{2^{i}}}_{2^{k+1-i}})
$$

where $\mathbf{1}_{2^{i}}$ is the all one row vector of length $2^{i}$. For example, when $k=2$,

$$
\begin{aligned}
& \mathbf{x}_{0}=(1,-1,1,-1,1,-1,1,-1) \\
& \mathbf{x}_{1}=(1,1,-1,-1,1,1,-1,-1) \\
& \mathbf{x}_{2}=(1,1,1,1,-1,-1,-1,-1)
\end{aligned}
$$

The following properties of $\mathbf{x}_{i}$ are obvious:

- Alternative Property: $x_{j}^{i}=-x_{j+2^{i}}^{i}$ for $0 \leq j<N-2^{i}$;
- Periodic Property: $x_{j}^{i}=x_{j+2^{i+1}}^{i}$ for $0 \leq j<N-2^{i+1}$, i.e., $\mathrm{x}_{i}$ has period $2^{i+1}$;
- Run Property: $x_{j}^{i}=(-1)^{\left\lfloor j / 2^{i}\right\rfloor}$ for $0 \leq j<N$, i.e., $x_{i}$ has $2^{k+1-i}$ runs of 1 or -1 of length $2^{i}$;
- Skew-symmetric Property: $x_{j}^{i}=-x_{N-1-j}^{i}$ for $0 \leq j<N$.

Based on the above properties, we derive the following useful lemmas, which are crucial to our repair strategy.

Lemma 1. For any $0 \leq i, l \leq k, j=\mu 2^{l+1}+\nu, 0 \leq \mu<2^{k-l}$, and $0 \leq \nu<2^{l}$,

$$
x_{j}^{i}=\left\{\begin{align*}
-x_{j+2^{l}}^{i}, & i=l  \tag{3}\\
x_{j+2^{l}}^{i}, & \text { otherwise }
\end{align*}\right.
$$

Proof: Firstly, when $i=l$, (3) holds due to the alternative property. Secondly, when $i<l$, (3) is true because of the periodic property. Thirdly, when $i>l$, write $\mu=\mu_{0} 2^{i-l-1}+\mu_{1}$ where $0 \leq \mu_{0}<2^{k+1-i}$ and $0 \leq \mu_{1} \leq 2^{i-l-1}-1$, then

$$
\left\lfloor\frac{j}{2^{i}}\right\rfloor=\left\lfloor\frac{j+2^{l}}{2^{i}}\right\rfloor=\mu_{0}
$$

since $0 \leq \mu_{1} 2^{l+1}+2^{l}+\nu \leq 2^{i}-2^{l+1}+2^{l}+\nu<2^{i}$, which results in (3) by the run property.
Lemma 2. For any $0 \leq i \leq k$ and $0 \leq j<N / 2$,

$$
x_{N-1-j-(-1)^{j}}^{i}=\left\{\begin{aligned}
x_{j}^{i}, & i=0 \\
-x_{j}^{i}, & 0<i \leq k
\end{aligned}\right.
$$

Proof: When $i=0$, the result directly follows from the periodic property that $\mathrm{x}_{0}$ has period 2 and $2 \mid\left(N-1-2 j-(-1)^{j}\right)$.

When $0<i \leq k$, let $j=\mu 2^{i}+\nu$ where $0 \leq \mu<2^{k+1-i}$ and $0 \leq \nu<2^{i}$. According to the run property, $x_{j}^{i}=(-1)^{\mu}$ and

$$
x_{N-1-j-(-1)^{j}}^{i}=(-1)^{\left\lfloor\frac{N-1-j-(-1)^{j}}{2^{i}}\right\rfloor}=(-1)^{2^{k+1-i}-\left\lceil\frac{1+j+(-1)^{j}}{2^{i}}\right\rceil}=(-1)^{\left\lceil\frac{1+j+(-1)^{j}}{2^{i}}\right\rceil}
$$

If $j$ is even, $1+j+(-1)^{j}=j+2=\mu 2^{i}+\nu+2$, which implies $\left\lceil\frac{1+j+(-1)^{j}}{2^{i}}\right\rceil=\mu+1$ since $0 \leq \nu \leq 2^{i}-2$ in this case. If $j$ is odd, $1+j+(-1)^{j}=j=\mu 2^{i}+\nu$, which still gives $\left\lceil\frac{1+j+(-1)^{j}}{2^{i}}\right\rceil=\mu+1$ since $1 \leq \nu \leq 2^{i}-1$. Therefore we always have

$$
x_{N-1-j-(-1)^{j}}^{i}=(-1)^{\mu+1}=-x_{j}^{i}
$$

Sylvester Hadamard matrices are one of the earliest infinite family of Hadamard matrices recursively defined by

$$
H_{1}=\left(\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right)
$$

and

$$
H_{k}=\left(\begin{array}{cc}
H_{k-1} & H_{k-1}  \tag{4}\\
H_{k-1} & -H_{k-1}
\end{array}\right), k \geq 2
$$

Normally, when a $2^{k} \times 2^{k}$ matrix, with each entry being 1 or -1 , is multiplied by a column vector of length $2^{k}$, we do not need multiplication and what we need are $2^{k}\left(2^{k}-1\right)$ additions. But for the Sylvester Hadamard matrix, we can reduce the number of additions by means of the recursive property.

Lemma 3. Let $H_{k}$ be the Sylvester Hadamard matrix in (4) and $\mathbf{z}$ be an arbitrary column vector of length $2^{k}$ where $k$ is a positive integer. Then,
(1) To compute $H_{k} \cdot \mathbf{z}, k \cdot 2^{k}$ additions are needed;
(2) To compute $\left(H_{k-1} H_{k-1}\right) \mathbf{z}$ or $\left(H_{k-1}-H_{k-1}\right) \mathbf{z}, k 2^{k}-2^{k-1}$ additions are needed.

Proof: Let $\mathcal{N}_{k}$ denote the number of additions of $H_{k} \cdot \mathbf{z}$.
(1) We prove the first assertion by induction. Obviously, it is true for $k=1$, i.e., $\mathcal{N}_{1}=2$.

Note that

$$
\begin{align*}
H_{k} \mathbf{z} & =\left(\begin{array}{cc}
H_{k-1} & H_{k-1} \\
H_{k-1} & -H_{k-1}
\end{array}\right)\binom{\mathbf{z}^{1}}{\mathbf{z}^{2}} \\
& =\binom{H_{k-1} \mathbf{z}^{1}+H_{k-1} \mathbf{z}^{2}}{H_{k-1} \mathbf{z}^{1}-H_{k-1} \mathbf{z}^{2}} \tag{5}
\end{align*}
$$

where $\mathbf{z}^{1}$ and $\mathbf{z}^{2}$ are two column vectors of length $2^{k-1}$. Then, we have

$$
\mathcal{N}_{k}=2 \mathcal{N}_{k-1}+2^{k}=2^{k-1} \mathcal{N}_{1}+(k-1) 2^{k}=k \cdot 2^{k} .
$$

(2) The second assertion follows directly from (5).

## 4 Optimal repair strategy

Let $\left\{\mathbf{e}_{0}, \cdots, \mathbf{e}_{2^{k}-1}\right\}$ be the basis of $\mathbb{F}_{q}^{2^{k}}$. For example, it can be simply chosen as the standard basis

$$
\begin{equation*}
\mathbf{e}_{i}=(\underbrace{0, \cdots, 0,1,0, \cdots, 0}_{2^{k}})^{T} \tag{6}
\end{equation*}
$$

with only the $i$ th entry being nonzero.
In this section, we present our repair strategy respectively for the systematic nodes, the first parity node, and the second parity node by giving the corresponding repair matrices, and then check the optimality.

### 4.1 Optimal repair of systematic nodes

In order to repair the $i$ th systematic node, $1 \leq i \leq k$, one downloads data $S_{i} \mathbf{f}_{l}, 1 \leq l \neq i \leq$ $k+2$, where the $N / 2 \times N$ repair matrix $S_{i}$ is

$$
S_{i}=(\underbrace{\mathbf{e}_{0}, \cdots, \mathbf{e}_{2^{i}-1}}_{2^{i}}, \underbrace{\mathbf{e}_{0}, \cdots, \mathbf{e}_{2^{i}-1}}_{2^{i}}, \cdots, \underbrace{\mathbf{e}_{2^{k}-2^{i}}, \cdots, \mathbf{e}_{2^{k}-1}}_{2^{i}}, \underbrace{\mathbf{e}_{2^{k}-2^{i}}, \cdots, \mathbf{e}_{2^{k}-1}}_{2^{i}})
$$

Let $s_{j}^{i}$ be the $j$ th column vector of $S_{i}$. Obviously, $\mathbf{s}_{j}^{i}=\mathbf{e}_{\mu 2^{i}+\nu}$ and

$$
\begin{equation*}
\mathbf{s}_{j+2^{i}}^{i}=\mathbf{s}_{j}^{i} \tag{7}
\end{equation*}
$$

where $j=\mu 2^{i+1}+\nu, 0 \leq \mu<2^{k-i}$ and $0 \leq \nu<2^{i}$.

Then, the data from two parity nodes are

$$
\begin{equation*}
\binom{S_{i}}{S_{i} A_{i}} \mathbf{f}_{i}+\sum_{l=1, l \neq i}^{k}\binom{S_{i}}{S_{i} A_{l}} \mathbf{f}_{l} \tag{8}
\end{equation*}
$$

where the second term is the interference resulted from systematic nodes except the failed one. To cancel the interference and recover the data $\mathbf{f}_{i}$, the optimal repair strategy requires 7 ]

$$
\begin{equation*}
\operatorname{rank}\binom{S_{i}}{S_{i} A_{i}}=N \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{rank}\binom{S_{i}}{S_{i} A_{l}}=\frac{N}{2} \tag{10}
\end{equation*}
$$

for $1 \leq i \neq l \leq k$.
Multiplying $A_{l}$ by $S_{i}, 1 \leq l \leq k$, we get

$$
\begin{equation*}
S_{i} A_{l}=\left(\left(a_{l} x_{0}^{l}+b_{l} x_{0}^{0}+1\right) \mathbf{s}_{0}^{i} \cdots\left(a_{l} x_{j}^{l}+b_{l} x_{j}^{0}+1\right) \mathbf{s}_{j}^{i} \cdots\left(a_{l} x_{N-1}^{l}+b_{l} x_{N-1}^{0}+1\right) \mathbf{s}_{N-1}^{i}\right) \tag{11}
\end{equation*}
$$

Consider the submatrix of $\binom{S_{i}}{S_{i} A_{l}}$ formed by columns $j$ and $j+2^{i}$ where $j=\mu 2^{i+1}+\nu$, $0 \leq \mu<2^{k-i}$ and $0 \leq \nu<2^{i}$, i.e.,

$$
\Delta_{j}=\left(\begin{array}{cc}
\mathbf{s}_{j}^{i} & \mathbf{s}_{j+2^{i}}^{i} \\
\left(a_{l} x_{j}^{l}+b_{l} x_{j}^{0}+1\right) \mathbf{s}_{j}^{i} & \left(a_{l} x_{j+2^{i}}^{l}+b_{l} x_{j+2^{i}}^{0}+1\right) \mathbf{s}_{j+2^{i}}^{i}
\end{array}\right)
$$

By Lemma (1) (2) and (7), we then have

$$
\operatorname{rank}\left(\Delta_{j}\right)= \begin{cases}2, & \text { if } i=l \\ 1, & \text { otherwise }\end{cases}
$$

which results in (9) and (10).
Example 3. When $k=2$, for the $(4,2)$ Hadamard MSR code determined by the coding matrices given in Example 1, the repair matrices of systematic nodes 1 and 2 are respectively

$$
S_{1}=\left(\begin{array}{llllllll}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{array}\right), S_{2}=\left(\begin{array}{llllllll}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
$$

### 4.2 Optimal repair of the first parity node

In order to repair the first parity node, we need the following transformation

$$
\begin{aligned}
\mathbf{y}_{1} & =\mathbf{f}_{1}+\cdots+\mathbf{f}_{k} \\
\mathbf{y}_{i} & =-\mathbf{f}_{i}, \quad 2 \leq i \leq k
\end{aligned}
$$

Let $\mathbf{y}=\left[\mathbf{y}_{1}^{T}, \cdots, \mathbf{y}_{k}^{T}\right]^{T}$. The storage code can then be described as

$$
\left(\begin{array}{c}
\mathbf{f}_{k+1} \\
-\mathbf{f}_{2} \\
\vdots \\
-\mathbf{f}_{k} \\
\mathbf{f}_{1} \\
\mathbf{f}_{k+2}
\end{array}\right)=\left(\begin{array}{cccc}
I_{N} & 0_{N} & \cdots & 0_{N} \\
0_{N} & I_{N} & \cdots & 0_{N} \\
\vdots & \vdots & \ddots & \vdots \\
0_{N} & 0_{N} & \cdots & I_{N} \\
I_{N} & I_{N} & \cdots & I_{N} \\
A_{1} & A_{1}-A_{2} & \cdots & A_{1}-A_{k}
\end{array}\right) \cdot \mathbf{y}
$$

where the first systematic node and the first parity node are exchanged.
Thus, it suffices to repair the new first systematic node by respectively downloading data $S \mathbf{f}_{i}, 1 \leq i \leq k$, and $\tilde{S} \mathbf{f}_{k+2}$, where the repair matrices $S$ and $\tilde{S}$ are

$$
\begin{aligned}
& S=(\underbrace{\mathbf{e}_{0}, \mathbf{e}_{1}, \cdots, \mathbf{e}_{2^{k}-2}, \mathbf{e}_{2^{k}-1}}_{2^{k}}, \underbrace{\mathbf{e}_{2^{k}-1}, \mathbf{e}_{2^{k}-2}, \cdots, \mathbf{e}_{1}, \mathbf{e}_{0}}_{2^{k}}) \\
& \tilde{S}=(\underbrace{\mathbf{e}_{0}, \mathbf{e}_{1}, \cdots, \mathbf{e}_{2^{k}-2}, \mathbf{e}_{2^{k}-1}}_{2^{k}}, \underbrace{-\mathbf{e}_{2^{k}-1},-\mathbf{e}_{2^{k}-2}, \cdots,-\mathbf{e}_{1},-\mathbf{e}_{0}}_{2^{k}})
\end{aligned}
$$

with the $j$ th columns $\mathbf{s}_{j}$ and $\tilde{\mathbf{s}}_{j}$ satisfying

$$
\begin{align*}
\mathbf{s}_{j} & =\mathbf{s}_{N-1-j} \\
\tilde{\mathbf{s}}_{j} & =-\tilde{\mathbf{s}}_{N-1-j} \tag{12}
\end{align*}
$$

for $0 \leq j<N$.
Then, the data from the new first parity node and the second parity node can be expressed as

$$
\binom{S}{\tilde{S} A_{1}} \mathbf{f}_{k+1}-\sum_{l=2}^{k}\binom{S}{\tilde{S}\left(A_{1}-A_{l}\right)} \mathbf{f}_{l} .
$$

The optimal repair strategy requires 7

$$
\begin{equation*}
\operatorname{rank}\binom{S}{\tilde{S} A_{1}}=N \tag{13}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{rank}\binom{S}{\tilde{S}\left(A_{1}-A_{l}\right)}=\frac{N}{2} \tag{14}
\end{equation*}
$$

for $2 \leq l \leq k$.
According to (13) and (14), we investigate
$\binom{S}{\tilde{S} A_{1}}=\left(\begin{array}{ccccc}\mathbf{s}_{0} & \cdots & \mathbf{s}_{j} & \cdots & \mathbf{s}_{N-1} \\ \left(a_{1} x_{0}^{1}+b_{1} x_{0}^{0}+1\right) \tilde{\mathbf{s}}_{0} & \cdots & \left(a_{1} x_{j}^{1}+b_{1} x_{j}^{0}+1\right) \tilde{\mathbf{s}}_{j} & \cdots & \left(a_{1} x_{N-1}^{1}+b_{1} x_{N-1}^{0}+1\right) \tilde{\mathbf{s}}_{N-1}\end{array}\right)$
and

$$
\begin{aligned}
& \binom{S}{\tilde{S}\left(A_{1}-A_{l}\right)} \\
= & \left(\begin{array}{ccc} 
& \mathbf{s}_{0} & \cdots \\
\left(a_{1} x_{0}^{1}+\left(b_{1}-b_{l}\right) x_{0}^{0}-a_{l} x_{0}^{l}\right) \mathbf{s}_{0} & \cdots & \left(a_{1} x_{j}^{1}+\left(b_{1}-b_{l}\right) x_{j}^{0}-a_{l} x_{j}^{l}\right) \tilde{\mathbf{s}}_{j} \\
\cdots & \mathbf{s}_{N-1} \\
\cdots & \left(a_{1} x_{N-1}^{1}+\left(b_{1}-b_{l}\right) x_{N-1}^{0}-a_{l} x_{N-1}^{l}\right) \tilde{\mathbf{s}}_{N-1}
\end{array}\right)
\end{aligned}
$$

The submatrices formed by columns $j$ and $N-1-j, 0 \leq j<N / 2$, are respectively

$$
\begin{aligned}
\Delta_{j} & =\left(\begin{array}{cc}
\mathbf{s}_{j} & \mathbf{s}_{N-1-j} \\
\left(a_{1} x_{j}^{1}+b_{1} x_{j}^{0}+1\right) \tilde{\mathbf{s}}_{j} & \left(a_{1} x_{N-1-j}^{1}+b_{1} x_{N-1-j}^{0}+1\right) \tilde{\mathbf{s}}_{N-1-j}
\end{array}\right) \\
& =\left(\begin{array}{cc}
\mathbf{s}_{j} & \mathbf{s}_{j} \\
\left(a_{1} x_{j}^{1}+b_{1} x_{j}^{0}+1\right) \tilde{\mathbf{s}}_{j} & \left(a_{1} x_{j}^{1}+b_{1} x_{j}^{0}-1\right) \tilde{\mathbf{s}}_{j}
\end{array}\right)
\end{aligned}
$$

and

$$
\begin{align*}
\Gamma_{j} & =\left(\begin{array}{cc}
\mathbf{s}_{j} & \mathbf{s}_{N-1-j} \\
\left(a_{1} x_{j}^{1}+\left(b_{1}-b_{l}\right) x_{j}^{0}-a_{l} x_{j}^{l}\right) \tilde{\mathbf{s}}_{j} & \left(a_{1} x_{N-1-j}^{1}+\left(b_{1}-b_{l}\right) x_{N-1-j}^{0}-a_{l} x_{N-1-j}^{l}\right) \tilde{\mathbf{s}}_{N-1-j}
\end{array}\right) \\
& =\left(\begin{array}{cc}
\mathbf{s}_{j} & \mathbf{s}_{j} \\
\left(a_{1} x_{j}^{1}+\left(b_{1}-b_{l}\right) x_{j}^{0}-a_{l} x_{j}^{l}\right) \tilde{\mathbf{s}}_{j} & \left(a_{1} x_{j}^{1}+\left(b_{1}-b_{l}\right) x_{j}^{0}-a_{l} x_{j}^{l}\right) \tilde{\mathbf{s}}_{j}
\end{array}\right) \tag{15}
\end{align*}
$$

by the skew-symmetric property and (12). In other words,

$$
\operatorname{rank}\left(\Delta_{j}\right)=2, \quad \operatorname{rank}\left(\Gamma_{j}\right)=1
$$

which leads to (13) and (14).
Example 4. When $k=2$, for the $(4,2)$ Hadamard MSR code determined by the coding matrices given in Example 1, the repair matrices of the first parity node are

$$
S=\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{array}\right), \tilde{S}=\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0
\end{array}\right)
$$

### 4.3 Optimal repair of the second parity node

Similar to the repair of the first parity node, the second parity node can be regarded as the first systematic node by the following transformation

$$
\begin{aligned}
\mathbf{y}_{1} & =A_{1} \mathbf{f}_{1}+\cdots+A_{k} \mathbf{f}_{k} \\
\mathbf{y}_{i} & =-A_{i} \mathbf{f}_{i}, 2 \leq i \leq k
\end{aligned}
$$

Let $\mathbf{y}=\left[\mathbf{y}_{1}^{T}, \cdots, \mathbf{y}_{k}^{T}\right]^{T}$. With this transformation, the storage code can be described as

$$
\left(\begin{array}{c}
\mathbf{f}_{k+2} \\
-A_{2} \mathbf{f}_{2} \\
\vdots \\
-A_{k} f_{k} \\
A_{1} \mathbf{f}_{1} \\
\mathbf{f}_{k+1}
\end{array}\right)=\left(\begin{array}{cccc}
I_{N} & 0_{N} & \cdots & 0_{N} \\
0_{N} & I_{N} & \cdots & 0_{N} \\
\vdots & \vdots & \ddots & \vdots \\
0_{N} & 0_{N} & \cdots & I_{N} \\
I_{N} & I_{N} & \cdots & I_{N} \\
A_{1}^{-1} & A_{1}^{-1}-A_{2}^{-1} & \cdots & A_{1}^{-1}-A_{k}^{-1}
\end{array}\right) \cdot \mathbf{y}
$$

where the three nodes, i.e., the first systematic node, the first parity node and the second parity node, are cyclically shifted.

Hence, it is sufficient to repair the new first systematic node by downloading data $S A_{i} \mathbf{f}_{i}$, $1 \leq i \leq k$, and $\tilde{S} \mathbf{f}_{k+1}$, where the two repair matrices $S$ and $\tilde{S}$ are

$$
\begin{aligned}
& S=(\underbrace{\mathbf{e}_{0}, \mathbf{e}_{1}, \cdots, \mathbf{e}_{2^{k}-2}, \mathbf{e}_{2^{k}-1}}_{2^{k}}, \underbrace{\mathbf{e}_{2^{k}-2}, \mathbf{e}_{2^{k}-1}, \cdots, \mathbf{e}_{0}, \mathbf{e}_{1}}_{2^{k}}) \\
& \tilde{S}=(\underbrace{\mathbf{e}_{0}, \mathbf{e}_{1}, \cdots, \mathbf{e}_{2^{k}-2}, \mathbf{e}_{2^{k}-1}}_{2^{k}}, \underbrace{-\mathbf{e}_{2^{k}-2},-\mathbf{e}_{2^{k}-1}, \cdots,-\mathbf{e}_{0},-\mathbf{e}_{1}}_{2^{k}})
\end{aligned}
$$

with the $j$ th columns $\mathbf{s}_{j}$ and $\tilde{\mathbf{s}}_{j}$ being

$$
\begin{aligned}
& \mathbf{s}_{j}= \begin{cases}\mathbf{e}_{j}, & 0 \leq j<N / 2 \\
\mathbf{e}_{N-1-j-(-1)^{j}}, & N / 2 \leq j<N\end{cases} \\
& \tilde{\mathbf{s}}_{j}= \begin{cases}\mathbf{e}_{j}, & 0 \leq j<N / 2 \\
-\mathbf{e}_{N-1-j-(-1)^{j}}, & N / 2 \leq j<N\end{cases}
\end{aligned}
$$

satisfying

$$
\begin{align*}
& \mathbf{s}_{j}=\mathbf{s}_{N-1-j-(-1)^{j}}  \tag{16}\\
& \tilde{\mathbf{s}}_{j}=-\tilde{\mathbf{s}}_{N-1-j-(-1)^{j}}
\end{align*}
$$

for $0 \leq j<N / 2$.
Then, the data from the new first parity node and the new second parity node can be expressed as

$$
\binom{S}{\tilde{S} A_{1}^{-1}} \mathbf{f}_{k+2}-\sum_{l=2}^{k}\binom{S}{\tilde{S}\left(A_{1}^{-1}-A_{l}^{-1}\right)} A_{l} \mathbf{f}_{l}
$$

The optimal repair strategy requires [7]

$$
\begin{equation*}
\operatorname{rank}\binom{S}{\tilde{S} A_{1}^{-1}}=N \tag{17}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{rank}\binom{S}{\tilde{S}\left(A_{1}^{-1}-A_{l}^{-1}\right)}=\frac{N}{2} \tag{18}
\end{equation*}
$$

for $2 \leq l \leq k$.

By (17) and (18), we need to discuss $\binom{S}{\tilde{S} A_{1}^{-1}}$ and $\binom{S}{\tilde{S}\left(A_{1}^{-1}-A_{l}^{-1}\right)}$ where

$$
\begin{aligned}
A_{i}^{-1} & =2^{-1}\left(I_{N}-a_{i}^{-1} b_{i} X_{0} X_{i}+a_{i}^{-1} X_{i}\right) \\
& =2^{-1} I_{N}+2^{-1} a_{i}^{-1} X_{i}\left(I_{N}-b_{i} X_{0}\right), 1 \leq i \leq k
\end{aligned}
$$

and

$$
\begin{aligned}
A_{1}^{-1}-A_{l}^{-1} & =2^{-1}\left(a_{l}^{-1} b_{l} X_{0} X_{l}-a_{1}^{-1} b_{1} X_{0} X_{1}+a_{1}^{-1} X_{1}-a_{l}^{-1} X_{l}\right) \\
& =2^{-1} a_{1}^{-1} X_{1}\left(I_{N}-b_{1} X_{0}\right)-2^{-1} a_{l}^{-1} X_{l}\left(I_{N}-b_{l} X_{0}\right), 2 \leq l \leq k
\end{aligned}
$$

according to [7]. For simplicity of the characterization of the matrices $A_{1}^{-1}$ and $A_{1}^{-1}-A_{l}^{-1}$, we define

$$
\begin{aligned}
p_{j}^{1} & =2^{-1}+2^{-1} a_{1}^{-1} x_{j}^{1}\left(1-b_{1} x_{j}^{0}\right) \\
q_{j}^{l} & =2^{-1} a_{1}^{-1} x_{j}^{1}\left(1-b_{1} x_{j}^{0}\right)-2^{-1} a_{l}^{-1} x_{j}^{l}\left(1-b_{l} x_{j}^{0}\right)
\end{aligned}
$$

where $1<l \leq k$ and $0 \leq j<N$. By Lemma 2, we have

$$
\begin{aligned}
p_{N-1-j-(-1)^{j}}^{1} & =2^{-1}-2^{-1} a_{1}^{-1} x_{j}^{1}\left(1-b_{1} x_{j}^{0}\right) \\
& =-p_{j}^{1}+1
\end{aligned}
$$

and

$$
q_{N-1-j-(-1)^{j}}^{l}=-q_{j}^{l}
$$

for $0 \leq j<N / 2$.
For $0 \leq j<N / 2$, consider the submatrices formed by columns $j$ and $N-1-j-(-1)^{j}$ in matrices $\binom{S}{\tilde{S} A_{1}^{-1}}$ and $\binom{S}{\tilde{S}\left(A_{1}^{-1}-A_{l}^{-1}\right)}$, i.e.,

$$
\begin{aligned}
\Delta_{j} & =\left(\begin{array}{cc}
\mathbf{s}_{j} & \mathbf{s}_{N-1-j-(-1)^{j}} \\
p_{j}^{1} \tilde{\mathbf{s}}_{j} & p_{N-1-j-(-1)^{j}} \tilde{\mathbf{s}}_{N-1-j-(-1)^{j}}
\end{array}\right) \\
& =\left(\begin{array}{cc}
\mathbf{s}_{j} & \mathbf{s}_{j} \\
p_{j}^{1} \tilde{\mathbf{s}}_{j} & p_{j}^{1} \tilde{\mathbf{s}}_{j}-\tilde{\mathbf{s}}_{j}
\end{array}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_{j} & =\left(\begin{array}{cc}
\mathbf{s}_{j} & \mathbf{s}_{N-1-j-(-1)^{j}} \\
q_{j}^{l} \tilde{\mathbf{s}}_{j} & q_{N-1-j-(-1)^{j}} \tilde{\mathbf{s}}_{N-1-j-(-1)^{j}}
\end{array}\right) \\
& =\left(\begin{array}{cc}
\mathbf{s}_{j} & \mathbf{s}_{j} \\
q_{j}^{l} \tilde{\mathbf{s}}_{j} & q_{j} \tilde{\mathbf{s}}_{j}
\end{array}\right)
\end{aligned}
$$

That is, $\operatorname{rank}\left(\Delta_{j}\right)=2$ and $\operatorname{rank}\left(\Gamma_{j}\right)=1$, which gives (17) and (18).
Example 5. When $k=2$, for the $(4,2)$ Hadamard MSR code determined by the coding matrices given in Example 1, the repair matrices of the second parity node are

$$
S=\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}\right), \tilde{S}=\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0
\end{array}\right)
$$

## 5 Comparison

In fact, in the original repair strategy [7] the basis $\left\{\mathbf{e}_{0}, \cdots, \mathbf{e}_{2^{k}-1}\right\}$ is chosen as the column vectors of the Sylvester Hadamard matrix in (4). Whereas, for our strategy, $\left\{\mathbf{e}_{0}, \cdots, \mathbf{e}_{2^{k}-1}\right\}$ can be any basis of $\mathbb{F}_{q}^{2^{k}}$. In this sense, our new repair strategy generalizes the previous one in [7].

Most importantly, by choosing the standard basis in (6), our new repair strategy can considerably reduce the computation, including both addition and multiplication, in contrast to the original repair strategy in [7]. Indeed, the decrease comes from the fact that in each row, our new repair matrices have 2 nonzero elements of 1 or -1 whereas the original matrices have $N$ nonzero elements of 1 or -1 .

The computation of node repair lies in 3 phases: download, interference cancellation and recover. In what follows, we investigate it case by case.

## Case 1. Computation load of the repair of systematic nodes

Since each $S_{i} \cdot \mathbf{f}_{l}$ needs $N / 2$ additions, the new strategy needs $(k+1) N / 2$ additions in the download phase. When $i \neq l$, note that in (11) $S_{i} A_{l}$ has only two nonzero elements in each row, which indicates that there exists an $N / 2 \times N / 2$ matrix

$$
\begin{equation*}
B_{l}=\operatorname{diag}\left(p_{0}^{l}, \cdots, p_{N / 2-1}^{l}\right) \tag{19}
\end{equation*}
$$

where $p_{\mu 2^{i}+\nu}^{l}=a_{l} x_{\mu 2^{i+1}+\nu}^{l}+b_{l} x_{\mu 2^{i+1}+\nu}^{0}+1,0 \leq \mu<2^{k-i}$ and $0 \leq \nu<2^{i}$ such that $S_{i} A_{l}=B_{l} S_{i}$. Hence, the new strategy needs $(k-1) N$ additions and at most $(k-1) N / 2$ multiplications to cancel the interference term in (8). In the recover phase, $N$ additions and at most $2 N$ multiplications are needed for the new strategy since the matrix $\binom{S_{i}}{S_{i} A_{i}}^{-1}$ still has only two nonzero elements in the each row. Therefore, totally $(3 k+1) N / 2$ additions and at most $(k+3) N / 2$ multiplications are needed for the new strategy.

For the original strategy, the download phase requires $(k+1)(2 k+1) N / 2$ additions by Lemma 3 since $S_{i}$ is equivalent to $\left(H_{k} H_{k}\right)$ with respect to columns permutation; The interference cancellation phase at most requires $(k-1)(N / 2+1) N / 2$ additions and $(k-1) N^{2} / 4$ multiplications; The recover phase requires $N(N-1)$ additions and at most $N^{2}$ multiplications. Thus, totally $(k+3) N^{2} / 4+\left(k^{2}+2 k-1\right) N$ additions and $(k+3) N^{2} / 4$ multiplications are needed at most.

## Case 2. Computation load of the repair of the first parity node

Similarly to case 1 , the new strategy needs $(3 k+1) N / 2$ additions and at most $(k+3) N / 2$ multiplications because (1) $\tilde{S} \cdot \mathbf{f}_{k+2}$ needs $N / 2$ additions, as the same as $S \cdot \mathbf{f}_{i}, 1 \leq i \leq k$; (2) For $2 \leq l \leq k$ there exists an $N / 2 \times N / 2$ matrix

$$
\begin{equation*}
B_{l}=\operatorname{diag}\left(a_{1} x_{0}^{1}+\left(b_{1}-b_{l}\right) x_{0}^{0}-a_{l} x_{0}^{l}, \cdots, a_{1} x_{N / 2-1}^{1}+\left(b_{1}-b_{l}\right) x_{N / 2-1}^{0}-a_{l} x_{N / 2-1}^{l}\right) \tag{20}
\end{equation*}
$$

such that $\tilde{S}\left(A_{1}-A_{l}\right)=B_{l} S$ by (15); (3) The matrix $\binom{S}{\tilde{S} A_{1}}^{-1}$ has only two nonzero elements in the each row.

For the original strategy, $(k+3) N^{2} / 4+\left(k^{2}+2 k-1\right) N$ additions and $(k+3) N^{2} / 4$ multiplications are required at most.

## Case 3. Computation load of the repair of the second parity node

During the download phase, the new strategy needs $(k+1) N / 2$ additions and at most $k N$ multiplications since (1) $S A_{i} \cdot \mathbf{f}_{i}, 1 \leq i \leq k$, needs $N / 2$ additions and at most $N$ multiplications; (2) $\tilde{S} \cdot \mathbf{f}_{k+1}$ needs $N / 2$ additions. In the interference cancellation phase and recover phase, the computation can be analyzed in the same fashion as that of Case 1 . Hence, totally $(3 k+1) N / 2$ additions and at most $(3 k+3) N / 2$ multiplications are needed for the new strategy.

For the original strategy, $(3 k+3) N^{2} / 4+(2 k-2) N / 2$ additions and $(3 k+3) N^{2} / 4$ multiplications are needed at most.

The above comparison is summarized in Table 2, where ADD and MUL respectively denote the numbers of addition and multiplication. The exact number of additions and multiplications depends on the concrete values of $a_{l}, b_{l}, 1 \leq l \leq k$, and the finite field $\mathbb{F}_{q}$. For the new strategy, the number of multiplications can be further reduced if set $a_{l} \pm b_{l}= \pm 2$ or $a_{1} \pm\left(b_{1}-b_{l}\right) \pm a_{l}= \pm 1$ such that there are some 1 or -1 in the diagonal matrix $B_{l}$ given by (19) or (20), which is feasible by the equations (81) and (82) in [7]. As for the old strategy, it seems hard to be analyzed because there are too many nonzeros in the Sylvester Hadamard matrix.

Table 2: Comparison between the original and new strategies for $(k, k+2)$ Hadamard MSR code

| Node <br> to repair | Repair <br> strategy | ADD | MUL |
| :---: | :---: | :---: | :---: |
| Systematic <br> node | New | $(3 k+1) N / 2$ | $\leq(k+3) N / 2$ |
|  | Original | $\leq(k+3) N^{2} / 4+\left(k^{2}+2 k-1\right) N$ | $\leq(k+3) N^{2} / 4$ |
| Parity | New | $(3 k+1) N / 2$ | $\leq(k+3) N / 2$ |
| node 1 | Original | $\leq(k+3) N^{2} / 4+\left(k^{2}+2 k-1\right) N$ | $\leq(k+3) N^{2} / 4$ |
| Parity <br> node 2 | New | $(3 k+1) N / 2$ | $\leq(3 k+3) N / 2$ |
|  | Original | $\leq(3 k+3) N^{2} / 4+(2 k-2) N / 2$ | $\leq(3 k+3) N^{2} / 4$ |

Finally, we give two examples to compare the computation load of our new strategy and the original strategy, by two concrete values $k=2$ and $k=3$. It can be seen our new repair strategy needs much less computation.

Example 6. When $k=2$, for the $(4,2)$ Hadamard MSR code determined by the coding matrices given in Example 1, the computation load is given in Table 3.

Example 7. When $k=3$, for the (5,3) Hadamard MSR code determined by the coding matrices given in Example 园, the computation load is given in Table 4.

## 6 Conclusion

In this paper, a new repair strategy of Hadamard MSR code was presented, which can be regarded as a generalization of the original repair strategy. By choosing the standard basis, our strategy can dramatically decrease the computation load in contrast to the original one.

Table 3: computation load of $(4,2)$ Hadamard MSR code

| Node <br> to repair | Repair <br> strategy | ADD | MUL |
| :---: | :---: | :---: | :---: |
| Systematic <br> node 1 | New | 28 | 17 |
|  | Original | 132 | 28 |
| Systematic <br> node 2 | New | 28 | 17 |
|  | Original | 132 | 28 |
| Parity <br> node 1 | New | 28 | 15 |
|  | Original | 132 | 24 |
| Parity <br> node 2 | New | 28 | 20 |
|  | Original | 152 | 120 |

Table 4: computation load of $(5,3)$ Hadamard MSR code

| Node to repair | Repair strategy | ADD | MUL |
| :---: | :---: | :---: | :---: |
| Systematic node 1 | New | 80 | 42 |
|  | Original | 528 | 128 |
| Systematic node 2 | New | 80 | 42 |
|  | Original | 528 | 128 |
| Systematic node 3 | New | 80 | 28 |
|  | Original | 528 | 256 |
| Parity node 1 | New | 80 | 44 |
|  | Original | 528 | 272 |
| Parity node 2 | New | 80 | 66 |
|  | Original | 736 | 576 |

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